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Asset and Liability Management for Banks and Insurance Companies

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Introduction

In recent years, the techniques known as asset and liability management (ALM) have become a cornerstone of risk management, not only for banks but also for insurance companies.

ALM can be defined as any continuous management process that defines, implements, monitors and back tests financial strategies to jointly manage a firm's assets and liabilities. More specifically, an ALM strategy is designed to achieve a financial goal for a given level of risk and under predefined constraints. Due to the increase of technicalities in the current bank and insurance regulation and the use of models which have become increasingly complex and powerful, ALM now plays a central part in any bank's or insurance company's financial strategy.

This book aims at covering the general concepts of ALM from the point of view of an insurance company or a bank. The core framework and the philosophy are the same, but regulations, products and models typically differ on numerous points in pratice because the business and the risks involved are quite different between an insurance company and a bank.

In this book, we have tried to draw a parallel between the uses of ALM in insurance companies and its uses in banks. To our knowledge, this is an innovative way to present and study ALM techniques. We hope that it will give the reader a better understanding of the commonalities and divergences between these two worlds.

Chapter 1 defines precisely what ALM is. After a brief history of the origins and the past developments of ALM, we describe the typical missions of an ALM department in a bank or in an insurance company.

In Chapter 2, we present the financial risks on which ALM classically focus. Indeed the actual regulations (Basel II, Basel III and Solvency II) specify that banks and insurance companies must identify, measure and manage the financial risks they are exposed to. In this chapter, we define the typical risks studied in ALM, and analyze the commonalities and differences between the problem of banks and insurance companies.

Chapter 3 describes the essential quantitative ALM tools and methods for banks. We introduce various mathematical concepts such as the actuarial value, embedded value or market-to-market values and, for the first time, we introduce extended duration and convexity concepts giving interesting new risk indicators. Techniques such as balance sheet immunization, endowment and sensitivity of financial flows are described. We also explain the use of deterministic and stochastic scenarios for dynamic financial analysis.

Conversely, Chapter 4 is devoted to insurance companies. From a pedagogical point of view, it gives the reader the keys to understand and develop an ALM internal model for life insurance products. In this chapter, we present basic ways for modeling an insurance company's assets and liabilities, and we describe a simple ALM model. This model is then used to realize, step-by-step, a typical ALM study, in the same way as it is concretely done in practice in ALM departments.

Lastly, Chapter 5 is the equivalent of Chapter 4 for banks as it explains how to build and use a specific ALM internal model for banks. We show how to proceed to gap reduction between asset and liability flows and we develop global stochastic models for equity time evolution. These models give the posibility to evaluate bankruptcy risks and so to compute VaR values. Let us also mention that we illustrate the introduced concepts with many numerical examples.

Finally, we can say that this book is written in a relatively self-contained form at least for our main audience formed by financial and risk managers of insurance companies and banks, particularly dealing with own risk solvency assessment (ORSA) techniques, enterprise risk management (ERM), graduate students in economic or actuarial masters and people involved in Solvency II for insurance companies and in Basel II and III for banks.

1

Definition of ALM in the Banking and Insurance Areas

1.1. Introduction

In recent years, the technique known as asset and liability management (ALM) has enjoyed remarkable popularity. Initially pioneered by English-speaking financial institutions during the 1970s as an actuarial and cash flow matching technique, ALM has grown into an essential framework for banks and insurance companies.

The objective of ALM is to ensure the proper coordination between assets and liabilities to achieve the financial targets for a specified level of risk and under predefined constraints. The ALM department, whether in an insurance company or in a bank, is therefore responsible for producing studies providing recommendations on marketing strategy and asset allocation.

In recent years, the ALM department has become increasingly important in a bank or an insurance company for three main reasons. First, modeling tools are increasingly sophisticated, facilitating making relevant cash flow projections. Second, accounting standards, which are central in ALM business, are constantly evolving. Last, but not least, financial communication is increasingly regulated.

This chapter is devoted to the definition of ALM in the banking and insurance areas. We will specifically focus on the history of ALM and the missions of an ALM department.

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Marine Corlosquet-Habart, William Gehin, Jacques Janssen and Raimondo Manca.
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1.2. Brief history of ALM for banks and insurance companies

Prior to the 1970s, interest rates in developed countries varied little and thus losses caused by asset and liability mismatches were low. The proceeds of their liabilities (for example deposits, life insurance policies or annuities) were invested in assets such as loans, bonds or real estate. All assets and liabilities were held at book value, hiding possible financial risks if assets and liabilities were to diverge suddenly.

In the 1970s, a period of volatile interest rates started and continued until the early 1980s. This volatility had dangerous implications for financial institutions. US regulation, which had capped the interest rates that banks could pay depositors, was abandoned to arise a migration overseas of the market for USD deposits. As most firms used accrual accounting, the emerging risk was slow to be recognized. Firms gradually accrued financial losses over the subsequent 5 or 10 years.

The most famous example is that of Equitable, a US mutual life insurance company. During the early 1980s, the USD yield curve was inverted. Equitable sold a number of long-term Guaranteed Interest Contracts (GICs) guaranteeing rates of around 16% for periods up to 10 years. During this period, GICs were routinely exchanged with a principal of USD 100MM or more. Equitable invested in short-term interest rates to pay the lower long-term high interest rates they guaranteed to their clients. But short-term interest rates soon collapsed. When Equitable had to reinvest, they could not get a sufficiently high interest rate to pay their GICs, and the firm was crippled. Ultimately, Equitable had to demutualize and was then acquired by the Axa Group.

Learning the lessons from Equitable, managers of financial firms focused on developing a sounder ALM. They sought ways to manage balance sheets in order to maintain a mix of loans and deposits consistent with the bank's goals for long-term growth and risk management. Thus, they started developing new financial techniques such as gap analysis, duration analysis or scenario analysis.

ALM practices have evolved since the early 1980s. Today, financial firms, particularly investment banks that enter trading operations daily, are increasingly using market-value accounting for certain business lines. For trading books, techniques of market risk management (for example

Value-at-Risk) are more appropriate than techniques of ALM. In financial firms, ALM is used for the management of assets and liabilities that must be accounted on an accrual basis. This includes bank lending, deposit taking and essentially all traditional insurance activities.

ALM techniques have also evolved. The growth of derivatives markets has facilitated a variety of hedging strategies. A significant development has been securitization, which facilitates firms to directly address asset and liability risk by substantially removing assets or liabilities from their balance sheets. This not only reduces asset and liability risk but also frees up the balance sheet for new business.

The scope of ALM activities has widened. Today, ALM departments are addressing a wider variety of risks, including foreign exchange risks. Also, ALM has extended to non-financial firms. Corporations have adopted some of the ALM techniques to manage interest-rate exposures, liquidity risks and foreign exchange risks. They also use related techniques to address commodities risks.

Nowadays, the process of ALM is at the crossroads between risk management and strategic planning. It not only offers solutions to mitigate or hedge the risks arising from the interaction of assets and liabilities, but also conducts the bank or the insurance company from a long-term perspective.

1.3. Missions of the ALM department

The objective of this chapter is to define the different missions of an ALM department in a bank or an insurance company. These two entities share the same goals, which are to analyze economic risks (mainly market risk), to produce studies providing recommendations on marketing strategy and asset allocation, and to monitor the implementation of those strategies. However, the underlying business is not the same between banks and insurance companies, and therefore the missions of their ALM department can differ.

1.3.1. *Missions of the ALM department for banks*

The first mission of ALM was essentially to manage interest risks and liquidity risks to prevent mismatches between the cash flows of the assets and the cash flows of the liabilities. This is why ALM uses concepts such as liquidity gap to quantify liquidity risks, and more mathematical indicators such as duration or convexity introduced a long time ago by McCauley. This led to the ALM policy of *immunization* which aimed to structure financial cash flows in a way that minimizes their sensitivity to small changes of the underlying interest rates. Thus, the ALM committee had to work hand in hand with the other departments of the bank and soon played a central role within the structure of the bank.

In 1988, the first Basel rules extended the field of application of ALM even more. They gave the ALM department supervision of other financial risks such as the equity risks, in addition to the traditional liquidity and interest rates risks. Therefore, progressively, ALM gained a central position in the management of the bank, often inside the risk management department. However, the ALM department must keep, as far as possible, a total independence within the firm.

In summary, ALM aims to coordinate the financial decisions of a firm so that the structure of its assets and liabilities optimizes both the financial benefits and the underlying risks, while respecting the prudential rules imposed by the regulators.

As we will see in Chapters 3 and 5, different deterministic or stochastic models exist which are particularly useful for risk managers.

1.3.1.1. *Deterministic models*

In a deterministic model, the evolution of the different financial variables (such as interest rates, equity volatility, etc.) during a given period of time is deterministic. This defines a single scenario, called the *central scenario.* This central scenario is used to project the cash flows generated by the firm’s assets and the cash flows generated by the firm’s liabilities and to study the discrepancies between these two series of cash flows. Of course, this initial study has to be complete with the consideration of other possible scenarios for the considered cash flows.

Nevertheless, deterministic models are useful to quickly detect the weak and strong points of hedging strategies and to have a basic understanding on how to reduce the *eventual mismatches.*

1.3.1.2. *Stochastic models*

To face the uncertainty of economic, financial and social evolution in the future, the use of stochastic models is necessary. However, stochastic models simple enough to be easily implemented and parameterized tend to rely on strong assumptions which are unfortunately sometimes unrealistic.

Nowadays, the models used in ALM are directly borrowed from quantitative finance. In Chapter 5, we will see how we can build such a model for the evolution of equities.

These models are also quite useful for the VaR computation. The VaR is an indicator of solvability recommended by the Basel authorities as well as by Solvency II for insurance companies.

Stochastic models are also used for the simulation of possible scenarios, and for each of them, basic *risk* and *profit indicators* can be computed.

1.3.1.3. *Mission of the bank ALM department*

The ALM department has to coordinate the management of assets and liabilities in such a way that benefits are optimized under an acceptable level of risk. This level of risk is now imposed by the regulatory authorities. This implies that the ALM department must have an overall, long-term view of the financial activity of the bank. Using different economic scenarios, the ALM department gives the necessary information to the Board of Directors of the bank so that they can soundly plan future financial investments.

The coordination of assets and liabilities was already considered long before the birth of ALM. However, nowadays the presence of a specific ALM department is crucial in a sector as competitive as banking. Indeed, the techniques used in ALM (for example to build generators of scenarios) are now increasingly specific and elaborate.

1.3.2. *Missions of the ALM department for insurance companies*

An insurance company must have good knowledge of its asset and liability risks to ensure its financial strength and honor its contractual commitments to its clients. To do so, the main tasks of the ALM department are the following:

– to ensure proper coordination of assets and liabilities to achieve a financial goal with an accepted level of risk under predefined constraints;

– to produce studies providing recommendations on marketing strategy and asset allocation;

– to calculate the capital requirement for market risks in the framework of the Solvency II regulations.

The following section details these three missions.

1.3.2.1. *To ensure proper coordination of assets and liabilities*

The Asset-Liability Manager's main function is to analyze the balance sheet of the company, and its likely evolution over a period of time. This analysis is based on a number of variables for which he anticipates the future evolution (interest rates, business development, macroeconomic indicators and other market variables). The main objective is to estimate and control the balance between resources (assets) and expenses (liabilities) to the risks taken by the insurance company, under the constraint of a level of profitability and a regulatory framework.

In order to model liabilities, policyholder behavior should be analyzed to determine all liability flows (for example deaths and lapses). The contract specifications must also be taken into account (for example in life insurance, profit sharing and the guaranteed minimum interest rate must be modeled).

Concerning the assets, the financial risks and their impact on the net worth of the company must be analyzed. An insurance company must, for example, manage interest risks (because assets mainly include bonds), liquidity risks (when the company is not able to sell an asset or a liability at the price it is valued), or currency risks (due to the variation of the value of an item after a change of the currency price).

If the asset manager invests without taking into account the expected behavior of insured people, then there will be a mismatch with liabilities. Without the analysis of future cash flow liabilities, it is impossible to determine the horizons of investments to be made. The insurance company can then no longer have enough reserves to pay out, or have a low profitability if the horizon is short. As the market is very competitive, poor financial performance will decrease the level of profit sharing, and thus weaken profitability. It is therefore essential to coordinate the assets and liabilities to achieve better efficiency for the company. Furthermore, life insurers are particularly exposed to asset and liability mismatching given the long-term nature of their engagements. The ALM department must rely on the skills of actuaries and asset managers to ensure proper management, hence a better profitability.

1.3.2.2. *To provide recommendations on marketing strategy and asset allocation*

ALM is responsible for establishing recommendations on the two main levers for controlling the activity: marketing strategy and asset allocation. On the one hand, marketing strategy can guide the composition of the portfolio of insurance contracts. On the other hand, strategic asset allocation limits financial risks arising from the reversal of the production cycle, in order to ensure the payment of benefits to policyholders and the future profits of the company.

The business of insurance is indeed particular: its production cycle is reversed. Life insurance premiums can be locked up to 30 years, so insurance companies must find investments with stable yields for the lifetime of these policies. Reinvestment risks are apparent, because future premiums must be invested even if their rate of return is lower than the rate necessary to cover present-day pricing. Likewise, insurance companies are exposed to disinvestment risk, when assets must be sold even at low prices to cover claims or other expenses. The impact of rising interest rates can be compounded by lost profits from suspended policies as consumers search for more lucrative financial instruments.

Thus, ALM studies are used to select the optimal strategic allocation of investments based on risk aversion of the insurance company.

1.3.2.3. *To calculate the Solvency II capital requirement for market risks*

The ALM department can also be in charge of different studies or regulatory calculations such as the solvency capital requirement (SCR) market for insurance. An asset and liability manager is involved in the innovation, development and improvement of the prospective cash flow model (or internal model). This includes investment strategy, credit risk modeling and policyholder behavior modeling. He can also work on integration in the ALM studies of elements of Solvency II’s Pillar 1 and 2: SCR and coverage ratio, risk profile and risk appetite.

1.4. Conclusion

This chapter has highlighted the fact that ALM has become a key indicator for risk management, not only for banks but also for insurance companies. It is a continuous process involving the formulation, implementation, monitoring and review of strategies related to assets and liabilities in order to achieve financial goals, taking into account a certain risk tolerance and constraints. Thus, ALM is crucial for any business, bank or insurance company that needs to invest capital to meet its contractual commitments and eager to ensure a well-balanced financial management.

The next chapter will describe the risks studied in ALM. After this definition of ALM in a bank and in an insurance company, we will now carry out a deep analysis of the different risks that the ALM department monitors.

2

Risks Studied in ALM

2.1. Introduction

In recent years, the financial and economic crises have increased the importance of risk management in banks and insurance companies. Indeed, banks, insurance and even reinsurance companies must now be able to understand the inherent risks of their business in order to allocate enough capital to cover it. This need has been initiated by Basel II and then Basel III for banks, and now has been extended to the insurance industry with Solvency II.

The purpose of this chapter is to list the key risks in asset and liability management (ALM), by differentiating banking and insurance problems. We describe how they are understood by today's standards: Basel II and III for banks and Solvency II for insurance companies. Last, but not least, we analyze the similarities and differences between the problems of banks and those of insurance companies.

2.2. Risks studied in a bank in the framework of Basel II and III

2.2.1. *Main risks for banks*

The main risks to which banks are exposed are the following: liquidity risk, credit risk, market risk (interest rate risk, foreign exchange risk and risk from change in market price of securities, financial derivatives and commodities), exposure risks, investment risks, risks related to the country of origin of the entity to which a bank is exposed, operational risk, legal risk, reputational risk and strategic risk. In this chapter, we will define all of these

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risks. Let us mention that some of these risks are similar for insurance companies (see section 2.3.2).

Liquidity risk is the risk caused by the inability of the banks to face their obligations, particularly for cash demands.

Credit risk is the risk caused by the eventual default of borrowers on their obligations to the bank and is also the risk of loss of present bond values due to the degradation of the issuer. This last one is also called risk of degradation.

Market risk can be subdivided as follows:

– interest rate risk, which is the risk caused by changes in interest rates;

– foreign exchange risk, which is the risk of financial losses caused by changes in exchange rates, especially for financial derivatives and commodities traded in the international market;

– asset risk, which is the risk of financial losses due to the depreciation of assets on stock exchanges.

Exposure risk is the risk of a bank's exposure to a single entity or a group of related entities.

Investment risk is the risk of bad investment outside the financial sector and in fixed assets.

Country risk is the risk from foreign activities due to political, economic or social condition in the considered country.

Operational risk is the risk caused by omissions in the work of employees, inadequate internal procedures and processes, inadequate management of information and other systems, failures in the computer system and unforeseeable external events.

Legal risk is the risk of loss caused by penalties or sanctions originating from court disputes due to a breach of contractual and legal obligations, and penalties and sanctions pronounced by a regulatory body.

Reputational risk is the risk of loss caused by a negative impact on the market positioning of the bank.

Strategic risk is the risk caused by a lack of a long-term development component in the bank's managing team.

2.2.2. *From Basel I to Basel III*

2.2.2.1. *Basel I and Basel II*

Inside the Bank for International Settlements (BIS), the Basel Committee, grouping the most important 27 central banks of the world, the prudential authorities established rules and norms to stabilize the banking system with a view to avoid bankruptcies and worldwide crises. Since 1988, three regulations have been elaborated: Basel I in 1988, Basel II in 2007 and Basel III after the subprime crisis. These rules are transformed into national rules in all the countries represented in the worldwide bank and all the banks have the obligation to apply them.

Let us summarize briefly the most important impacts of these successive Basel rules.

Basel I mainly concerns the market finance with the introduction of the Cooke ratio concerning the equities of the banks in conformity of the recommendations called capital adequacy directives (CA). Roughly, this ratio must ensure a minimal level of capital guaranteeing the solidity of the banks. Banks are required to maintain a liquidity buffer of 8% of their capital. This ratio is defined as follows:

$$\frac{\text{equities(Core Tier one + goodwill)}}{\text{credit risk + market risk}}(\geq 8\%).$$

To calculate this ratio, the equities are traditionally divided into three large bodies (the core or Tier 1, Tier 2 and Tier 3). The ratio of equity to risk-weighted assets must be equal to or greater than 8% with a minimum of 4% on Tier 1.

The credit risk calculation worked on a risk-weighted basis of the credits done by the bank. Each asset is placed into one of five categories and the total assets in each category are multiplied by a specific percentage. For example, loans to the national government in the bank's own country are considered so safe that the category total is multiplied by 0%, meaning those assets are effectively ignored. Riskier loans fall into the 10, 20, 50 and 100%

categories, meaning some or all of the asset's value is included in the overall total. The ratings were not considered.

This ratio was modified in 2007 with the introduction of the operational risk and becomes the McDonough ratio still with 8% but with a more refined definition of the equities and particularly the Tier one becoming core Tier one. This ratio is given by:

$$\frac{\text{equities(Core Tier one + goodwill)}}{\text{credit risk +market risk+}\mathit{operational\ risk}}(\geq 8\%).$$

Moreover, the banks must also control their own risks and noted the qualities of their assets.

Basel II is articulated on three pillars:

– pillar 1 defines the minimum capital requirements, which are devoted to bank's core capital requirement;

– pillar 2 focuses on supervisory review. It allows for supervisor discretion to adjust this requirement to allow for additional risk and particular circumstances;

– pillar 3 focuses on market discipline.

The main risks considered in the first pillar are: credit risk, operational risk and market risk.

For the first time, the credit risk component can be computed by the bank using an internal rating-based approach (IRB) instead of the standardized approach.

The Advanced IRB is more flexible than the IRB approach. For example, in the last one, only notations and default probabilities are established by the bank while in the Advanced IRB, which is also the case for the recuperation rate and the risk exposure.

The possibility to use internal models was a significant advance for big banks, while the other banks in general use the standard approach.

Three different approaches can be used for computing the operational risk component: the basic indicator approach (BIA), the standardized approach

(TSA) and the most sophisticated called advanced measurement approach (AMA).

The computation of the market risk grouping rate, exchange rate and asset risk components is treated by Value-at-Risk (VaR) techniques, in general with a VaR on 10 days and at a level of 99%. The bank has to periodically proceed to back testings and stress testings (see section 2.3). The aim of the back testings is to control a *posteriori* the validity of the VaR values on the observed results.

Let us briefly say that the second pillar supervisory review gives tools for regulators' systemic risk, pension risk, concentration risk, strategic risk, reputational risk, liquidity risk and legal risk. This leads to what is called capital adequacy assessment process (CAAP) in the core of the risk management system of the bank for which the ALM approach is recommended.

The aim of pillar 3 is to allow market discipline to operate by requiring institutions to disclose details on the scope of application, capital, risk exposures, risk assessment processes and the capital adequacy of the institution. It must be consistent with how the senior management, including the board, assesses and manages the risks of the institution.

Figure 2.1 gives the main risks considered in Basel II[1].

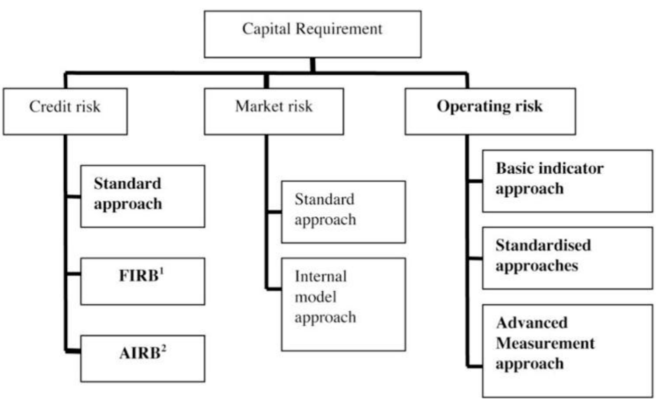

Figure 2.1. *Structure of capital requirement in Basel II (source: APRA 2004, press.anu.edu.au/agenda/016/01/.../ch02s04.html)*

1 The new features (under Basel II) are shown in bold.

FIRB means foundation internal ratings based approach and AIRB means advanced internal ratings based approach. These approaches are new with respect to Basel II as well as a standard approach for credit risk.

Basel II did not change the two methods that can be used for assessing the capital requirement for market risk introduced in 1996 but as mentioned above, it introduced a new important risk called operational risk needed to be covered by capital.

An important dimension of Basel II is its treatment of credit-risk mitigation techniques such as the use of collateral, guarantees (by a third party) and other credit-risk reduction measures such as credit derivatives, which reduce the amount of loss in cases of default. Credit-default swaps (CDSs) are the most extensively used credit derivative. This is one of the reasons that obliged the Basel Committee to revise these rules after the subprimes crisis.

2.2.2.2. *Basel III*

We will present a summary of Basel III rules, but for a more complete presentation, the readers can refer to Basel III regularity consistency assessment (Level 2), preliminary report: European Union, Basel Committee on Banking Supervision (October 2012).

Basel III rules focus on:

– improving the banking sector's ability to absorb shocks arising from financial and economic stress;

– improving risk management and governance;

– strengthening banks' transparency and disclosures;

– Basel III mainly concerns capital adequacy, stress testing and market liquidity risk to strengthen bank capital requirements by increasing bank liquidity and decreasing bank leverage. Faced with the systemic risk amplitude, the Basel III authorities propose to increase the quality of equities particularly with a proportion of Tier 1 of 4.5%. They maintain 8% but it will grow up to 10.5% in 2019 with 6% of Tier one;

– they also ask for strong equities for the so-called trading book related to short-time assets used for negotiation such as credit derivatives. They maintain the expected shortfall (ES) to replace the VaR introduced after the subprime crisis in the so-called Basel 2.5 in 2012;

– we also find norms on liquidity risk by asking for a selection of easy liquid assets to be used, for example, if customers want to remove their money and also the necessity to weight the assets following their quality. The need for liquidity is measured by two leverage ratios: the liquidity coverage ratio (LCR) for short term and the net stable funding ratio (NSFR) for medium term;

– reinforcement of risk models used by the bank and particularly those related to the VaR with more back testings and stress testings;

– extension of the regulation adapted by other financial institutions such as hedge funds, pension funds and special purpose vehicles (SPVs) dangerous for the systemic risk.

Let us note that the new Basel III rules are still being established.

2.3. Stress tests

2.3.1. *What is a stress test?*

A stress test is an analysis of the balance sheet of a bank after crashes or economic crisis. Several scenarios are simulated to see how robust the bank is. They are the consequences of past crises such as the Russian crisis of 1998 (debt default) and of course the consequences of the 9/11 attack.

In order to do that, banks must use scenario generators [CLÉ 15] concerning, for example, an increase or decrease in the basic interest rates, an increase in the unemployment rate in the year, a big change in the EURO value faced with the dollar, a big perturbation in the oil market, a degradation for real estate values, a degradation of the economic growth of several percent, etc.

It is clear that stress tests are only as good as the scenarios on which they are constructed.

They were imposed by the European Bank Authority and the International Monetary Fund to verify that the equities and capital allocation allow us to cover potential heavy losses in the case of the realization of a catastrophic scenario. After that, they were required by the Basel Accord of 1996 for unexpected market events.

Since 2009, successive financial stress tests conducted by the European Banking Authority and the Committee of European Banking Supervisors are done on a yearly basis. In general, they select the scenario to be used for the banks themselves.

It is clear that if a bank does not succeed in these stress tests, its reputation as well as its market value will rapidly go down. Moreover, the shareholders have to increase the capital following the result obtained. Otherwise, their activities can be interrupted.

2.3.2. *The stress tests of 2014*

These concern the most important 130 European banks which are representing more than 85% of the European bank activity. The catastrophic scenario for the French banks was the following: two consecutive years of recession (2014 and 2015), in 2016, unemployment will grow up to 12.2%, the interest rate will go up from 1.2 to 3.8% in 2016 and, moreover, the real estates prices fall by 30%.

Unfortunately, the retained scenario does not include deflation.

If the first results show that, globally, the European banking system is valid, nevertheless 25 banks fall, mainly in Greece, Italy and Spain for which there is a lack of 25 billion for their equities.

This global result is better than for the stress test in 2011, for which, of 90 banks, 20 of them failed.

It must be said that this stress test also permits us to discover some deficiencies, for example, for the so-called non-performant exposition (NPE), credits or loans evaluated by the European Central Bank (ECB) to 900 billion. This new estimation with respect to the old estimation of 743 billion will lead to a reinforcement of the equities of the concerned banks as well as by prudential adjustments.

REMARK 2.1.– The shock for the interest rate of 2.6% starting from 1.2% is a very big one. We will see in Chapter 3 how the simple ALM indicators can evaluate its impact on financial flows.

2.4. Risks studied in an insurance company in the framework of Solvency II

This section defines the fundamental concepts of Solvency II. We will voluntarily not describe in detail all the calculations. For that, the readers can refer to the documents available on the website of the European Insurance and Occupational Pensions Authority (EIOPA)[2], particularly the Technical Specifications.

2.4.1. *Solvency II in a nutshell*

The initial Solvency I directive was introduced in 1973 and was aimed at revising and updating the current European Union (EU) Solvency legislation. Solvency II has a much wider scope because it reflects the new risk management practices to define the required capital and manage risks.

In fact, the aim of Solvency II project is to review the prudential legislation for insurance and reinsurance undertakings in the EU. It introduces new, harmonized EU-wide insurance regulatory rules.

More precisely, the key objectives of Solvency II are the following:

– better protecting consumers and rebuilding trust in the financial system;

– ensuring a high, effective and consistent level of regulation and supervision by taking into account the varying interests of all Member States and the different nature of financial institutions;

– giving a greater harmonization and coherent application of rules for financial institutions and markets across the EU;

– promoting a coordinated EU supervisory response.

As a first step, the Solvency II directive was adopted by the Council of the European Union and the European Parliament in November 2009.

2 EIOPA website: https://eiopa.europa.eu.

Following an EU Parliament vote on the Omnibus II Directive on 11 March 2014, Solvency II is scheduled to come into effect on 1 January 2016. This date has been previously pushed back many times.

Just like in Basel II for banking, Solvency II also has a three pillar structure, with each pillar governing a different aspect of the Solvency II requirements and approach. As shown in Figure 2.2, the three pillars are the following: quantitative requirements, supervisor review and disclosure requirement.

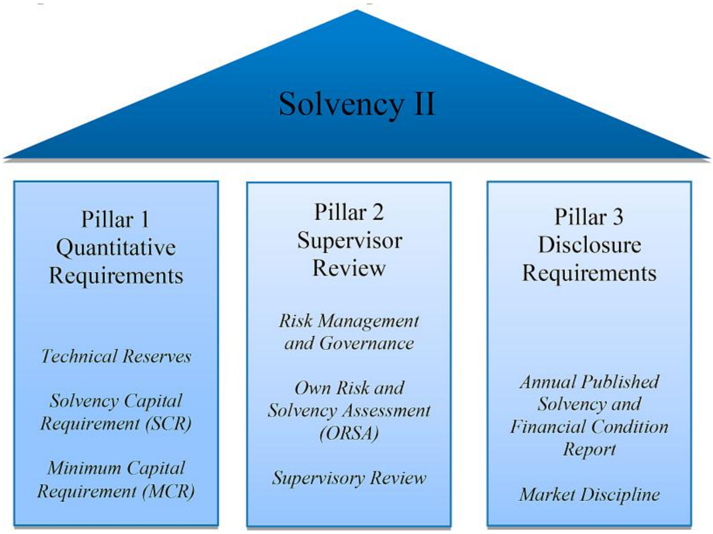

Figure 2.2. *The three pillar structure of Solvency II*

Pillar 1 defines the quantitative requirements for the calculation of technical reserves and own funds:

– technical reserves are assessed on a current exit value basis, as the sum of:

- the best estimate of liabilities (BELs), which is the expected present value of future cash-flows, discounted with the risk-free interest rate,

- the risk margin, which is the cost of providing the required capital until the end of all the insurers' obligations;

– two levels of capital requirements capturing all quantifiable risks affecting the balance sheet are:

- the solvency capital requirement (SCR), which is the target capital requirement. It corresponds to a VaR of the basic own funds of an insurance or reinsurance company subject to a confidence level of 99.5% over a 1-year period,

- the minimum capital requirement (MCR), which is a threshold below which an insurance company cannot drop. It corresponds to a VaR of 80–90% of own funds over a 1-year time horizon;

– to calculate those two levels of capital requirements, the insurance company can choose to use either the standard formula, a full or a partial internal model:

- in a full model, all risk categories are quantified using the internal model. It does not mean that all elements of the SCR have to be statistically/stochastically modeled,

- in a partial model, one or more modules of the SCR as laid out under Solvency II are calculated using the standard formula.

Pillar 2 aims to set qualitative norms for the internal risk management and governance. It defines how supervisors must use their powers of monitoring in this context. Moreover, it introduces the concept of own risk and solvency assessment (ORSA).

The ORSA can be defined as the entirety of the processes and procedures employed to identify, assess, monitor, manage and report the short- and long-term risks a firm faces or may face, and to determine the own funds necessary to ensure that overall solvency needs are met at all times.

The ORSA must be conducted as a part of the risk management system and, as a regular "assessment", it is also the key output from that system. It should include an assessment of overall capital needs, taking into account the risk profile, approved risk tolerance limits and business strategy of the company. While the calculation of the SCR will look at regulatory capital requirements arising over a 1-year time horizon, the ORSA will consider economic capital requirements over a business planning timeframe. The ORSA should also be an integral part of the business strategy, taken into account in strategic decisions and should be used to help identify and manage risk. The ORSA should also make the link between actual reported results and the capital assessment.

The ORSA is a requirement for all firms whether using the standard formula or an internal model to calculate regulatory capital requirements. An ORSA does not require an internal model but, where an internal model is used, it is an integral tool to the ORSA process.

Pillar 3 aims to define all the detailed information available for the public and supervisors. It focuses on risk management and capital disclosure.

After having pointed out what the three pillars of structure are, let us now focus on the risks.

2.4.2. *Focus on the risks*

The calculation of the SCR, according to the standard formula of Solvency II, is divided into modules as shown in Figure 2.3.

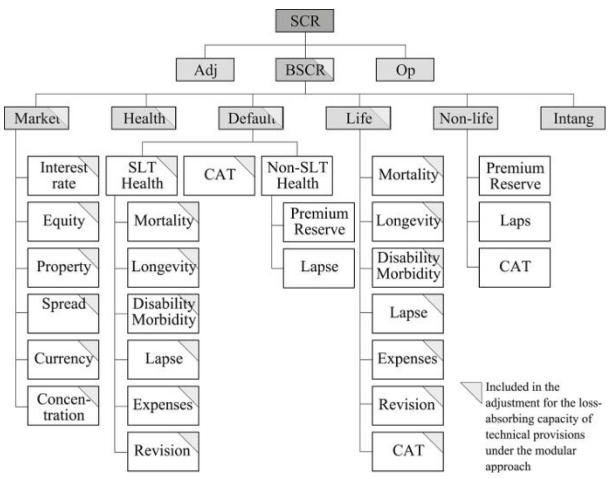

Figure 2.3. *Overall structure of the SCR (source: Technical Specifications for the Solvency II Preparatory Phase – Part I of the 30.04.2014*[3]*)*

3 Source available at: https://eiopa.europa.eu/fileadmin/tx_dam/files/publications/technical_specifications/A_-_Technical_Specification_for_the_Preparatory_Phase__Part_I_.pdf.

Where:

– BSCR is the basic solvency capital requirement, that is the SCR before any adjustments, combining capital requirements for the six major risk categories (market risk, health underwriting risk, counterparty default risk, life underwriting risk, non-life underwriting risk and intangible assets risk);

– Op is the capital requirement for operational risk, that is the risk of loss arising from inadequate or failed internal processes, from human and IT errors or from external events;

– Adj is the adjustment for the risk absorbing effect of technical provisions and deferred taxes.

Such that SCR = BSCR + Adj + Op.

The calculation of the SCR is done by a bottom-up approach in four steps. First of all, submodules are aggregated into risk modules. Second, risk modules are aggregated into the BSCR. Third, operational risk is added and a fourth adjustment is performed. Aggregations in the standard formula are based on a defined correlation matrix.

Let us now describe the different modules, which are: market risk, health underwriting risk, counterparty default risk, life underwriting risk, non-life underwriting risk and intangible asset risk.

2.4.2.1. *Market risk*

Market risk arises from the level or volatility of market prices of financial instruments. Exposure to market risk is measured by the impact of movements in the level of financial variables such as stock prices, interest rates, immovable property prices and exchange rates.

In the standard formula cartography, the market risk module encompasses six submodules, which are:

– interest rate risk, which exists for all assets and liabilities which are sensitive to changes in the term structure of interest rates or interest rate volatility, whether valued by mark-to-model or mark-to-market techniques;

– equity risk, which arises from the level or volatility of market prices for equities. Exposure to equity risk refers to all assets and liabilities whose value is sensitive to changes in equity prices;

– property risk, which arises as a result of sensitivity of assets, liabilities and financial investments to the level or volatility of market prices of property;

– spread risk, which results from the sensitivity of the value of assets, liabilities and financial instruments to changes in the level or in the volatility of credit spreads over the risk-free interest rate term structure;

– currency risk, which arises from changes in the level or in the volatility of currency exchange rates;

– concentration risk, the scope of which extends to assets considered in the equity, spread risk and property risk submodules, and excludes assets covered by the counterparty default risk module in order to avoid any overlap between both elements of the standard calculation of the SCR. The definition of market risk concentrations regarding financial investments is restricted to the risk regarding the accumulation of exposures with the same counterparty. It does not include other types of concentration (e.g. geographical area, industry sector, etc.).

2.4.2.2. *Health underwriting risk*

The health underwriting risk module captures the risk of health (re)insurance obligations.

In the standard formula cartography, the health underwriting risk module encompasses three submodules, which are:

– similar to life (SLT) health risk, which refers to health insurance obligations pursued on a similar technical basis to that of life insurance. SLT health risk arises from the underwriting of health (re)insurance obligations, pursued on a similar technical basis to life insurance, and is associated with both the perils covered and processes used in the conduct of the business;

– catastrophe (CAT) risk, which covers the risk of loss, or adverse changes in the value of insurance liabilities, resulting from the significant uncertainty of pricing and provisioning assumptions related to the outbreaks of major epidemics, as well as the unusual accumulation of risks under such extreme circumstances;

– non-SLT health risk, which refers to health insurance obligations not pursued on a similar technical basis to that of life insurance. Non-SLT health risk arises from the underwriting of health (re)insurance obligations, not pursued on a similar technical basis to that of life insurance, following from both the perils covered and processes used in the conduct of business. Non-SLT health underwriting risk also includes the risk resulting from uncertainty included in assumptions about the exercise of policyholder options such as renewal and termination options.

2.4.2.3. *Counterparty default risk*

The counterparty default risk module should reflect possible losses due to unexpected default of the counterparties and debtors of undertakings over the forthcoming 12 months. The scope of the counterparty default risk module includes risk-mitigating contracts, such as reinsurance arrangements, securitizations and derivatives, and receivables from intermediaries, as well as any other credit exposures which are not covered in the spread risk submodule.

2.4.2.4. *Life underwriting risk*

The life underwriting risk module captures the risk of life (re)insurance obligations other than health (re)insurance obligations.

In the standard formula cartography, the life underwriting risk module encompasses seven submodules, which are:

– mortality risk, which is the risk of loss, or adverse changes in the value of insurance liabilities, resulting from changes in the level, trend or volatility of mortality rates, where an increase in the mortality rate leads to an increase in the value of insurance liabilities;

– longevity risk, which is the risk of loss, or adverse changes in the value of insurance liabilities, resulting from changes in the level, trend or volatility of mortality rates, where a decrease in the mortality rate leads to a decrease in the value of insurance liabilities;

– disability-morbidity, which is the risk of loss, or adverse changes in the value of insurance liabilities, resulting from changes in the level, trend or volatility of disability and morbidity rates;

– lapse risk, which is the risk of loss, or adverse changes in liabilities due to a change in the expected exercise rates of policyholder options. The relevant options are all legal or contractual policyholders' rights to fully or partly terminate, surrender, decrease, restrict or suspend insurance cover or permit the insurance policy to lapse;

– expense risk, which arises from the variation in the expenses incurred in servicing insurance and reinsurance contracts;

– revision risk, which is the risk of loss, or adverse changes in the value of insurance and reinsurance liabilities, resulting from fluctuations in the level, trend or volatility of revision rates applied to annuities, due to changes in the legal environment or in the state of health of the person insured;

– CAT risk, which arises from extreme or irregular events, whose effects are not sufficiently captured in the other life underwriting risk submodules. Examples could be a pandemic event or a nuclear explosion.

2.4.2.5. *Non-life underwriting risk*

The non-life underwriting risk module captures the risk of non-life (re)insurance obligations other than health (re)insurance obligations.

In the standard formula cartography, the non-life underwriting risk module encompasses three submodules, which are:

– premium reserve risk, which combines a treatment for the two main sources of underwriting risk, premium risk and reserve risk;

– lapse risk, the capital requirement of which should be equal to the loss in basic own funds of undertakings that would result from the combination of two shocks:

- discontinuance of 40% of the insurance policies for which discontinuance would result in an increase in technical provisions without the risk margin,

- decrease in 40% of the number of future insurance or reinsurance contracts used in the calculation of technical provisions associated with reinsurance contracts cover;

– CAT risk, which is the risk of loss, or adverse changes in the value of insurance liabilities, resulting from significant uncertainty of pricing and provisioning assumptions related to extreme or exceptional events.

2.4.2.6. *Intangible assets risk*

Intangible assets risks are exposed to two risks:

– market risks, as for other balance sheet items, derived from the decrease in prices in the active market, and also from an unexpected lack of liquidity of the relevant active market, that may result in an additional impact on prices, even impeding any transaction;

– internal risks, inherent to the specific nature of these elements (e.g. linked to either failures or unfavorable deviations in the process of finalization of the intangible asset, or any other features in such a manner that future benefits are no longer expected from the intangible asset or its amount is reduced; and risks linked to the commercialization of the intangible asset, triggered by a deterioration of the public image of the undertaking).

2.5. Commonalities and differences between banks and insurance companies' problems

After having described the Basel II, III and Solvency II directives, this chapter aims at clarifying their common points and differences.

2.5.1. *Commonalities*

Often called "Basel for insurers", Solvency II is somewhat similar to the banking regulations of Basel II-III.

In fact, both directives share the same aim, which is to better adapt the capital required with the risks inherent to the business. Both directives are based on a three pillar structure, as described above. Both of them aim at understanding the significant risks in the business in a quantifiable way, and at enhancing the risk management capability. Both of them aim at understanding internal controls by developing an integrated risk management control system. Both of them encourage banks or insurance companies to well understand their internal model and use it.

2.5.2. *Differences*

In spite of these common points, there also exist significant differences, and here are some examples.

First of all, durations are different between banks and insurance companies. While insurance companies buy and hold long-term investments, financial risks assumed by banks are short or medium terms.

Moreover, there is no charge for liabilities under Basel II, whereas Solvency II is mainly about liability capital charge.

But the main difference comes from the nature of the activities: for an insurance company, the premiums are transformed into reserves to pay the observed claims which are coming under a probability distribution combining the number of claims and their amounts so that the main absolute risk comes from this situation. For the banks which transform deposits of their customers to loans and credits for the economic agents, the two main risks are the consequent credit and default risk as well as the liquidity risk. This last one conditions its financial need to the Central Bank. So, the main aim of the bank is to find the equilibrium between its financial sources to guarantee its liquidity, its interest rate management faced with its commercial activities and to ensure that unexpected losses will be absorbed by the equities.

2.6. Conclusion

This chapter has highlighted the fact that risk management is nowadays of paramount importance, whether it be in a bank or in an insurance company. These entities are fully aware of this, since the Basel II, III and Solvency II directives ask them to quantify and control the risks inherent to their business.

This chapter described the different risks modules defined by Basel and Solvency directives. They concern, at the same time, about the assets and liabilities, and are thus in direct link with ALM risks. A cashflow projection model developed to answer the expectations of Basel II, III or Solvency II can also be used for all the ALM studies described in Chapter 1.

On this subject, after having described the aims essential for a quantitative approach of ALM studies, the next chapter will make a quantitative study of some indicators of ALM management both for banks and insurance companies.

3

Durations (Revisited) and Scenarios for ALM

3.1. Introduction

The purpose of this chapter is to present the essential tools and methods for a quantitative approach of asset and liability management (ALM) studies. Those tools are based on various mathematical concepts:

– actuarial values, embedded values and market-to-market value;

– balance sheet immunization, endowment and sensitivity of financial flows;

– deterministic and stochastic scenario analysis;

– dynamic financial analysis for balance sheet.

We already know that ALM must be seen as an important part of risk management. Indeed, ALM techniques are used to hedge the main risks that banks and insurance companies must manage every day. These companies seek the optimal management of their equities to guarantee a good financial rentability with a very small probability of bankruptcy, for example with a level of 0.005 for Solvency II (see [FAB 95, BES 95, JAN 96, JAN 98]).

In this chapter, we present the concept of duration for risks other than interest risks, and we also introduce the concept of financial scenarios.

Asset and Liability Management for Banks and Insurance Companies
Marine Corlosquet-Habart, William Gehin, Jacques Janssen and Raimondo Manca.
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3.2. Duration and convexity risk indicators

Duration analysis and gap management are the main theoretical principles recommended for the ALM. We have already presented gap analysis in Chapter 1, so this section will focus on the concept of duration and its main application: to match the cash flows from assets and the cash flows from liabilities. We will follow the presentation of Bergendhal and Janssen [BER 99].

Let us consider an asset or liability paying the financial future cash flows:

$$F = \left\{ \left(F_j, t_j \right), j = 1,...,n \right\} \quad [3.1]$$

where F_j, t_j represent respectively the *amount* and the *time* of the *j*th movement of this flow, on a time scale $\left[0, \infty\right)$ with non-negative amounts.

With a fixed annual rate *i*, the present value of the flow *F* is given by:

$$C(i) = \sum_{j=1}^{n} F_j \left(1+i\right)^{-t_j} . \quad [3.2]$$

EXAMPLE 3.1. (from [BER 99]).– Let us consider a bank having customer loans for a total of € 100,000,000 generating an interest income of 5% per year for three years. The administrative costs are estimated at 1% per year and, due to credit losses, the loans will be repaid at 98%. These loans are refunded on the interbank at a rate of 3.5%. Table 3.1 gives the detailed successive flows in million €.

Year	Interest income	Adm. costs	Repayment
1	0.05×100	-0.01×100	4
2	0.05×100	-0.01×100	4
3	0.05×100	-0.01×100	98 + 4 = 102

Table 3.1. *Flows for example 3.1*

It follows that the market value of the flow at time 0 is given by:

$$C = \frac{4}{1.035} + \frac{4}{(1.035)^2} + \frac{102}{(1.035)^3} = € \, 99{,}596.933$$

So, there is a loss of € 403,067, that is 4.03%.

Now let us consider that the interbank rate decreases by 0.025; then the new market value becomes:

$$C = \frac{4}{1.025} + \frac{4}{(1.025)^2} + \frac{102}{(1.025)^3} = € \, 102{,}426{,}836$$

producing a profit of € 2,426,386 that is 2.42% now.

This example shows that the movements of the interest rate can have a strong influence on the profit.

Fortunately, it is possible to construct an indicator to measure this influence called the *duration* of the considered flow *F*.

Let us assume that the interest rate has a variation (positive or negative) Δi; using the Taylor formula to the value *C(i)* defined by relation [3.2], we obtain:

$$\Delta C = C(i + \Delta i) - C(i) \cong C'(i)\Delta i + \frac{1}{2} C''(i)(\Delta i)^2$$

and for the relative increment:

$$\frac{C(i + \Delta i) - C(i)}{C(i)} \cong -D_m(i)\Delta i + \frac{1}{2} CV(i)\left(\Delta i\right)^2, \qquad [3.3]$$

with

$$-D_m(i) = \frac{C'(i)}{C(i)}, (1 + \mathrm{i})D_m(i) = D(i), \; CV(i) = \frac{C''(i)}{C(i)}. \qquad [3.4]$$

It follows that the absolute variation of the market value is given by the two following formulas, respectively, at *first- and second-order approximations:*

$$C(i+\Delta i)-C(i)\cong -D_m(i)\Delta i\times C(i). \qquad [3.5]$$

$$C(i+\Delta i)-C(i)\cong -D_m(i)C(i)\Delta i+\frac{1}{2}CV(i)C(i)\left(\Delta i\right)^2. \qquad [3.6]$$

The three indicators $D(i), D_m(i), CV(i)$ are respectively called *duration* (of Mc Cauley), *modified duration* and *convexity*.

From relation [3.2], we can compute the values of these indicators with the cash amounts as follows:

$$\begin{aligned} D(i) &= \frac{1}{C(i)}\sum_{j=1}^{n} t_j F_j \left(1+i\right)^{-t_j}, \\ D_m(i) &= \frac{1}{(1+i)C(i)}\sum_{j=1}^{n} t_j F_j \left(1+i\right)^{-t_j}, \\ CV(i) &= \frac{1}{C(i)}\sum_{j=1}^{n} t_j \left(t_j+1\right) F_j \left(1+i\right)^{-t_j-2}. \end{aligned} \qquad [3.7]$$

Let us remark that they are independent *of the monetary unit of the device.*

Let us come back to example 3.1.

Relations [3.5] give the following results:

D(i)	2.876110396
D_m(i)	2.778850624
CV(i)	12.32384544

Table 3.2. *Duration and convexity for example 3.2*

And so, with results [3.5] and [3.6], we obtain for the interest rate scenario Δi = -0.01:

Results			
Relative variation		Absolute variation	
First approx.	*Second approx.*	*First approx.*	*Second approx.*
0.027788506	0.028404699	102.425954	102.4259539

Table 3.3. *Results for example 3.2*

REMARK 3.1 (LIQUIDITY AND DEFAULT RISKS).– The liquidity risk will be the risk that funds will not be available when needed. Assume that in example 3.1, the average customer makes a delay of 10 days to pay interest and the final instalment. Then, the decision maker will have to use a central bank funding at say 6% to pay interest and depreciation on his own interbank borrowing. This will cause an extra cost of 2% over 10 days, which will become the liquidity risk.

Let us recall these three risks – interest, liquidity and default – are the main targets for the ALM in banks and in insurance companies. Out of them, the interest rate risk is assumed to be the most important risks in the medium term that is over a planning period of one month to one year.

EXAMPLE 3.2 (CASE OF BONDS).– Let us consider two bonds A and B with the following characteristics:

obligation	maturity	nominal	coupon
A	7	1150	50
B	7	1060	60

Table 3.4. *Bond data*

So, with bond A, the investor receives a coupon of € 50 at the end of each year during the 6 years and of course he receives the amount of 1,060 at the end of the seventh year. For bond B, the situation of the cash flows is similar except that now the coupon only has a value of € 60.

The following table gives the computation of present value (PV):

i	0.02	
Bond A		
Year	Cash	PV
1	50	49.0196078
2	50	48.0584391
3	50	47.1161167
4	50	46.1922713
5	50	45.2865405
6	50	44.3985691
7	1,050	1,001.14421
C(i.A)		1,203.96
Bond B		
Year	Cash	PV
1	60	58.8235294
2	60	57.6701269
3	60	56.5393401
4	60	55.4307256
5	60	54.3438486
6	60	53.2782829
7	1,060	922.793789
C(i.B)		1,258.87964

The modified durations are of 5.995 for bond A and of 5.932 for bond B showing, that bond B is not only better for rentability but also against interest risk.

3.3. Scenario on the cash amounts of the flow

In our example, 3.1, we finally see that under the interest scenario $\Delta i = 0.01$, the relative loss will be of 2.778850624 × 0.01= 0.02778850624 and for the absolute loss will have as amount of 0.02778850624 × $99.596.933 =$ € 2.76764999.

But of course other scenarios are possible and particularly concern the considered flow amounts. We will now consider that the market value of the considered flow is also function of the cash amounts $F_j, j = 1,...,n$.

So, a scenario will be defined not only by a variation of the interest rate Δi but also by the cash amount fluctuations respectively noted by $\Delta F_1, \Delta F_2, ..., \Delta F_n$.

Consequently, the variation of C is given by:

$$C(F_1 + \Delta F_1, F_2 + \Delta F_2, ..., F_n + \Delta F_n; i + \Delta i) - C(F_1, F_2, ..., F_n; i) \quad [3.8]$$

with from relation [3.2]:

$$C(F_1, ..., F_n; i) = \sum_{j=1}^{n} F_j \left(1+i\right)^{-t_j}. \quad [3.9]$$

A Taylor development of order 2 gives here:

$$\begin{aligned} &C(F_1 + \Delta F_1, F_2 + \Delta F_2, ..., F_n + \Delta F_n; i + \Delta i) - C(F_1, F_2, ..., F_n; i) \\ &\approx \sum_{j=1}^{n} (1+i)^{-t_j} \Delta F_j + \frac{\partial C}{\partial i}(F_1, F_2, ..., F_n; i)\Delta i + \frac{1}{2}\frac{\partial^2 C}{\partial i^2}(F_1, F_2, ..., F_n; i)(\Delta i)^2. \end{aligned} \quad [3.10]$$

and so with relations [3.3] and [3.4], we obtain:

$$\begin{aligned} &C(F_1 + \Delta F_1, F_2 + \Delta F_2, ..., F_n + \Delta F_n; i + \Delta i) - C(F_1, F_2, ..., F_n; i) \\ &\approx \sum_{j=1}^{n} (1+i)^{-t_j} \Delta F_j - D_m(i) C(F_1, F_2, ..., F_n; i)\Delta i \\ &+ \frac{1}{2} CV(i) C(F_1, F_2, ..., F_n; i)(\Delta i)^2. \end{aligned} \quad [3.11]$$

Particular case: Let us suppose that:

$$\Delta F_j = \alpha F_j, \alpha \in \mathbb{R}, j = 1, ..., n \quad [3.12]$$

Then, the previous relation becomes:

$$\begin{aligned} &C(F_1 + \alpha F_1, F_2 + \alpha F_2, ..., F_n + \alpha F_n; i + \Delta i) - C(F_1, F_2, ..., F_n; i) \\ &\approx \alpha C(F_1, F_2, ..., F_n; i) - D_m(i) C(F_1, F_2, ..., F_n; i)\Delta i \\ &+ \frac{1}{2} CV(i) C(F_1, F_2, ..., F_n; i)(\Delta i)^2. \end{aligned} \quad [3.13]$$

and so:

$$\begin{aligned}&C(F_1+\alpha F_1,F_2+\alpha F_2,...,F_n+\alpha F_n;i+\Delta i)-C(F_1,F_2,...,F_n;i)\\&\approx\left[\alpha-D_m(i)\Delta i+\frac{1}{2}CV(i)(\Delta i)^2\right]C(F_1,F_2,...,F_n;i)\end{aligned} \quad [3.14]$$

This relation gives relevant information on the possible scenario $(\Delta i,\alpha)$: if $\alpha-D_m(i)\Delta i=0$, then there is at first order *immunization* that is the present value is not affected by this scenario. If $\alpha-D_m(i)\Delta i>0$, then this scenario is good as the new present value is larger than the initial value. However, if $\alpha-D_m(i)\Delta i<0$, this scenario is bad for the bank as it produces a loss of the present value. The consideration of the convexity shows a slight improvement in case of immunization.

EXAMPLE 3.3.– Let us return to example 3.1, the next table gives results for several scenarios.

		Absolute variation (in million €)		Relative variation (in million €)
α	Δi	First-order	Second-order	in %
0.1	0.01	7.192043311	7.253414172	7.282768603
-0.1	0.01	-12.7273433	-12.66597244	-12.7172314
0.1	-0.01	12.7273433	12.78871417	12.84046985
-0.1	-0.01	-7.192043311	-7.130672451	-7.159530148

Table 3.5. *Results four scenarios for example 3.1*

The worst scenario clearly is (-0.1; 0.01) giving a loss of 12.8%. Consequently, if we can attribute a subjective probability of 99.5%, the amount of € 12.66 million will represent the VaR value for this investment.

3.4. Scenario on the time maturities of the flow

We will now consider that the market value of the considered flow is function of the time maturities $t_j, j=1,...,n.$ So, a scenario will be defined not only by variation of the interest rate Δi but also by the cash amount fluctuations, respectively, noted by $\Delta t_1, \Delta t_2, ..., \Delta t_n$.

Consequently, the variation of C is given by:

$$C(t_1 + \Delta t_1, t_2 + \Delta t_2, ..., t_n + \Delta t_n; i + \Delta i) - C(t_1, t_2, ..., t_n; i) \quad [3.15]$$

From relation [3.2]:

$$C(F_1, ..., F_n; i) = \sum_{j=1}^{n} F_j \left(1 + i\right)^{-t_j}. \quad [3.16]$$

A Taylor development of order 2 here gives:

$$\begin{aligned} & C(t_1 + \Delta t_1, t_2 + t_2, ..., t_n + \Delta t_n; i + \Delta i) - C(t_1, t_2, ..., t_n; i) \\ & \approx -\sum_{j=1}^{n} (1+i)^{-t_j} F_j \ln(1+i) \Delta t_j + \frac{\partial C}{\partial i}(t_1, t_2, ..., t_n; i)\Delta i \\ & + \frac{1}{2}\frac{\partial^2 C}{\partial i^2}(t_1, t_2, ..., t_n; i)(\Delta i)^2. \end{aligned} \quad [3.17]$$

If all the Δt_j are equal to Δt, we obtain:

$$\begin{aligned} & C(t_1 + \Delta t, t_2 + \Delta t, ..., t_n + \Delta t; i + \Delta i) - C(t_1, t_2, ..., t_n; i) \\ & \approx -\ln(1+i)\Delta t C(i) + \frac{\partial C}{\partial i}(t_1, t_2, ..., t_n; i)\Delta i + \frac{1}{2}\frac{\partial^2 C}{\partial i^2}(t_1, t_2, ..., t_n; i)(\Delta i)^2. \end{aligned} \quad [3.18]$$

So, with relations [3.3] and [3.4], we can write that:

$$\begin{aligned} & C(t_1 + \Delta t, t_2 + \Delta t, ..., t_n + \Delta t; i + \Delta i) - C(t_1, t_2, ..., t_n; i) \\ & \approx -\ln(1+i)\Delta t C(t_1, t_2, ..., t_n; i) - D_m(i) C(t_1, t_2, ..., t_n; i)\Delta i \\ & + \frac{1}{2} CV(i) C(t_1, t_2, ..., t_n; i)(\Delta i)^2. \end{aligned} \quad [3.19]$$

In function of the scenario ($\Delta t.\Delta i$), we can finally write that:

$$\Delta C(\Delta t, \Delta i) \approx -\left[\ln(1+i)\Delta t + D_m(i)\Delta i\right] C(i) + \frac{1}{2} CV(i) C(i)(\Delta i)^2. \quad [3.20]$$

EXAMPLE 3.4.– Let us return to example 3.1. Table 3.6 gives results for several scenarios.

Δt	Δi	ΔC	in %
0.1	0.01	-3.048906796	-0.030612457
-0.1	0.01	-2.363651477	-0.023732171
0.1	-0.01	2.486393198	0.024964556
-0.1	-0.01	3.171648517	0.031844841

Table 3.6. *Results of four scenarios (Δt.Δi) for example 3.1*

The worst scenario clearly is (0.1; 0.01) giving a loss of 3%. Consequently, if we can attribute a subjective probability of 99.5%, the amount of € 3.049 million will represent the VaR value for this investment.

3.5. Matching asset and liability

Let us consider two flows: an *asset flow A* and a *liability flow B* represented by the following future cash flows:

$$A = \{(A_j, t_j),\ \ j = 1,\ldots,n\}), \qquad B = \{(B_j, t_j),\ \ j = 1,\ldots,n\}. \qquad [3.21]$$

With a constant interest rate, the present market value called *S(i)* is given by:

$$S(i) = \sum_{j=1}^{n} (A_j - B_j)(1+i)^{-t_j} \qquad [3.22]$$

In many cases, the asset flow can be considered as *hedging* for covering the liability flow that is giving sufficient cash flows to pay the liability dues at determined maturities.

We can now introduce the duration and convexity indicators of these two flows, respectively noted as:

$D_{m,A}(i), D_{m,B}(i), CV_A(i), CV_B(i)$ so that we can write with a rate scenario of Δi:

$$\Delta S(i) \cong -\left[A(i)\, D_{m,A}(i) - B(i)\, D_{m,B}(i)\right]\Delta i + \frac{1}{2}\left[CV_A(i)\, A(i) - CV_B\, B(i)\right]\left(\Delta i\right)^2 \quad [3.23]$$

where $\Delta S(i) = S(i+\Delta i) - S(i)$.

Introducing the following indicators:

$$D^G{}_{m,S(i)} = \left[A(i)\, D_{m,A}(i) - B(i)\, D_{m,B}(i)\right], \quad [3.24]$$

$CV^G{}_{S(i)} = \left[CV_A(i)\, A(i) - CV_B\, B(i)\right]$, respectively called *global modified duration* and *global convexity* of the surplus S.

So, we obtain for the first- and second-order approximations:

$$\begin{aligned} &S(i+\Delta i) - S(i) \cong -D^G{}_{m,S}(i)\Delta i, \\ &S(i+\Delta i) - S(i) \cong -D^G{}_{m,S(i)}\Delta i + \frac{1}{2} CV^G{}_{S(i)}\left(\Delta i\right)^2 \end{aligned} \quad [3.25]$$

Let us now point out that the two new indicators may be negative.

At the first-order, the position $\{A, B\}$ will be immunized against small variations of the interest rate if and only if:

$$A(i) D_{m,A}(i) = B(i) D_{m,B}(i). \quad [3.26]$$

We also say that the liability flow hedges well the asset flow.

For a null position at time 0, i.e. *A (i)=B(i)*, this last condition becomes:

$$D_{m,A}(i) = D_{m,B}(i). \quad [3.27]$$

Always from relation [3.23], this immunization is reinforced if:

$$\left[CV_A(i)\, A(i) - CV_B\, B(i)\right] \geq 0.$$

Of course a non-immunized position is favorable to the investor against the interest rate increasing provided that:

$$A(i)D_{m,A}(i) - B(i)D_{m,B}(i) < 0 \quad [3.28]$$

or if and only if:

$$\frac{A(i)D_{m,A}(i)}{B(i)D_{m,B}(i)} < 1. \quad [3.29]$$

Consequently, the ratio $\frac{A(i)D_{m,A}(i)}{B(i)D_{m,B}(i)}$ can be used as an indicator of matching *durations*.

Of course, if inequalities [3.28] or [3.29] are reversed, then the situation is unfavorable for the firm.

EXAMPLE 3.5.– Let us consider two flows A and B giving the following results with a rate of 3%:

$A(0.03)$ = 105 million. $B(0.03)$ = 100 million. $D_{m,A} = 3$, and $D_{m,B} = 3.15$.

We have a surplus of 5 million and formula [3.24] gives $D^G{}_{m,S} = 0$ and so there is immunization at first-order meaning that the investor is protected against weak variations of the interest rate.

Let us suppose now that $CV_A = 85$, and $CV_B = 60$; it follows from relation [3.24] that the value of the global convexity is 2,925 $\times$ $(10)^6$.

From result [3.23], we obtain:

$$S(0.03 + \Delta i) - S(0.03) \cong 1462.5 \times (10)^6 (\Delta i)^2.$$

For instance, with a decrease or an increase of 25BP of the rate, the margin will increase by € 9,140, and with a variation of 50BP, the increase will be of € 36,562.5.

In percentage, we obtain: $1.828 \times (10)^{-3}$ or $.3125 \times (10)^{-3}$.

Let us point out that we have similar but decreasing values with a negative global convexity.

REMARK 3.2.–

1) In relative variations, result [3.23] gives:

$$\frac{S(i+\Delta i)-S(i)}{S(i)} \cong -\frac{\left[A(i)\,D_{m,A}(i)-B(i)\,D_{m,B}(i)\right]}{S(i)}\Delta i + \frac{1}{2}\frac{\left[CV_A(i)\,A(i)-CV_B\,B(i)\right]}{S(i)}\left(\Delta i\right)^2. \qquad [3.30]$$

With the following definitions of relative global duration and convexity, respectively:

$$D^R{}_{m,S}(i) = \frac{\left[A(i)\,D_{m,A}(i)-B(i)\,D_{m,B}(i)\right]}{S(i)}, \quad CV^G{}_S(i) = \frac{\left[CV_A(i)\,A(i)-CV_B\,B(i)\right]}{S(i)}, \qquad [3.31]$$

result becomes:

$$\frac{S(i+\Delta i)-S(i)}{S(i)} \cong -D^R{}_{m,S(i)}\Delta i + \frac{1}{2}CV^R{}_{S(i)}\left(\Delta i\right)^2. \qquad [3.32]$$

EXAMPLE 3.5 (continued).– The relative indicators are given by:

$$D^R{}_{m,S(0.03)} = 0, \qquad CV^R{}_{S(0.03)} = \frac{2925\times 10^6}{5\times 10^6} = 585,$$

and so

$$\frac{S(0.03+\Delta i)-S(0.03)}{S(0.03)} \cong 292.5\left(\Delta i\right)^2.$$

3.6. Matching with flow scenarios

For the flows given by relation [3.21] for which the surplus is given by:

$$S(A_1,...A_n;B_1,...B_n;i)=\sum_{j=1}^{n}\left(A_j-B_j\right)\left(1+i\right)^{-t_j}.$$

Let us consider an interest rate scenario *Δi* and flow amount variations given so the following flow scenario:

$$\Delta A_1,\Delta A_2,...,\Delta A_n;\Delta B_1,\Delta B_2,...,\Delta B_n;\Delta i.$$

If this scenario occurs, the new value of the surplus is given by:

$$\begin{aligned}&S(A_1+\Delta A_1,...A_n+\Delta A_n;B_1+\Delta B_1,...B_n+\Delta B_n;i+\Delta i)\\&=\sum_{j=1}^{n}\left(A_j+\Delta A_j-B_j-\Delta B_j\right)\left(1+i+\Delta i\right)^{-t_j}\end{aligned}\qquad [3.33]$$

or

$$\begin{aligned}&\Delta S=S(A_1+\Delta A_1,...A_n+\Delta A_n;B_1+\Delta B_1,...B_n+\Delta B_n;i+\Delta i)\\&-S(A_1,...A_n;B_1,...;i+\Delta i).\end{aligned}$$

We can approximate this value by the Taylor formula giving here:

$$\begin{aligned}&\Delta S\approx\sum_{j=1}^{n}(1+i)^{-t_j}\Delta A_j-\sum_{j=1}^{n}(1+i)^{-t_j}\Delta B_j\\&-\left[A(i)D_{m,A}(i)-B(i)D_{m,B}(i)\right]\Delta i+\frac{1}{2}\left[CV_A(i)A(i)-CV_B\,B(i)\right]\left(\Delta i\right)^2\end{aligned}$$

and so:

$$\begin{aligned}&\Delta S\approx\sum_{j=1}^{n}(1+i)^{-t_j}\Delta A_j-\sum_{j=1}^{n}(1+i)^{-t_j}\Delta B_j-D^{G}{}_{m,S(i)}\Delta i\\&+\frac{1}{2}CV^{G}{}_{S(i)}\left(\Delta i\right)^2\end{aligned}\qquad [3.34]$$

At the first-order, it gives:

$$S(i+\Delta i)-S(i)\approx\sum_{j=1}^{n}(1+i)^{-t_j}\Delta A_j-\sum_{j=1}^{n}(1+i)^{-t_j}\Delta B_j-D^G{}_{m,S(i)}\Delta i\,. \quad [3.35]$$

This relation gives the result of the considered scenario $(\Delta A_1,\Delta A_2,...,\Delta A_n;\Delta B_1,\Delta B_2,...,\Delta B_n;\Delta i)$.

If different scenarios are considered, it is possible to select the most favorable ones to compute a VaR value with this proxy with a subjective probability of an historical one of 99.5%.

Particular case

Let us assume that:

$$\Delta A_j=\alpha A_j,\Delta B_j=\beta B_j,\alpha,\beta\in\mathbb{R},j=1,...,n. \quad [3.36]$$

So that from relation [3.35], we can write:

$$S(i+\Delta i)-S(i)\approx\alpha\sum_{j=1}^{n}(1+i)^{-t_j}A_j-\beta\sum_{j=1}^{n}(1+i)^{-t_j}B_j-D^G{}_{m,S(i)}\Delta i \quad [3.37]$$

We have at the first-order:

$$S(i+\Delta i)-S(i)\approx\alpha A(i)-\beta B(i)-D^G{}_{m,S(i)}\Delta i \quad [3.38]$$

and at the second-order:

$$S(i+\Delta i)-S(i)\approx\alpha A(i)-\beta B(i)-D^G{}_{m,S(i)}\Delta i+\frac{1}{2}CV^G{}_{S(i)}\left(\Delta i\right)^2. \quad [3.39]$$

Using relations [3.40], we obtain:

$$\begin{aligned}S(i+\Delta i)-S(i)&\approx\alpha A(i)-\beta B(i)-\left[A(i)\,D_{m,A}(i)-B(i)\,D_{m,B}(i)\right]\Delta i\\&+\frac{1}{2}\left[CV_A(i)\,A(i)-CV_B\,B(i)\right]\left(\Delta i\right)^2\end{aligned} \quad [3.40]$$

Or finally:

$$S(i+\Delta i)-S(i)\approx[(\alpha-D_{m,A}(i)\Delta i+\frac{1}{2}CV_A(i)A(i)(\Delta i)^2]A(i)$$
$$-[(\beta-D_{m,B}(i)\Delta i+\frac{1}{2}CV_B(i)(\Delta i)^2]B(i) \quad [3.41]$$

So, the first-order approximation gives:

$$S(i+\Delta i)-S(i)\approx[(\alpha-D_{m,A}(i)\Delta i]A(i)-[(\beta-D_{m,B}(i)\Delta i]B(i) \quad [3.42]$$

The first-order condition of immunization is thus given by:

$$[(\alpha-D_{m,A}(i)\Delta i]A(i)-[(\beta-D_{m,B}(i)\Delta i]B(i)=0$$
or
$$\alpha A(i)-\beta B(i)=[A(i)D_{m,A}(i)-B(i)D_{m,B}(i)]\Delta i. \quad [3.43]$$

Of course, the position of the investor is positive if:

$$[(\alpha-D_{m,A}(i)\Delta i]A(i)-[(\beta-D_{m,B}(i)\Delta i]B(i)>0. \quad [3.44]$$

It follows that in the plane (*α,β*), the couples preserving immunization giving only the interest rate scenario are on the straight line of equation:

$$\alpha A(i)-\beta B(i)=D_m^G\Delta i$$
or
$$\beta=\frac{A(i)}{B(i)}\alpha-\frac{D_m^G}{B(i)}\Delta i. \quad [3.45]$$

For a flow amounts scenario, the first-order immunization condition becomes:

$$\frac{\alpha A(i)-\beta B(i)}{A(i)D_{m,A}(i)-B(i)D_{m,B}(i)}=\Delta i. \quad [3.46]$$

Of course, the position of the investor is positive if:

$$\frac{\alpha A(i) - \beta B(i)}{A(i)D_{m,A}(i) - B(i)D_{m,B}(i)} > \Delta i \qquad [3.47]$$

EXAMPLE 3.5 (continued).– Let us assume that the liability flow increases with $\beta = 0.02$; the immunization condition [3.43] becomes

$\alpha A(i) - \beta B(i) = [A(i)D_{m,A}(i) - B(i)D_{m,B}(i)]\Delta i$
with $\beta = 0.01$, $A(0.03) =$ 105 millions, $B(0.03) =$ 100 millions, $\Delta i = 0.03$, $D_{m,A} = 3$ and $D_{m,B} = 3.15$.

So, we obtain for α the following condition:

$$\alpha = \frac{\beta B(i) + [A(i)D_{m,A}(i) - B(i)D_{m,B}(i)]\Delta i}{A(i)} \qquad [3.48]$$

This gives $\alpha = 0.01$.

Now we have $D_{m,B} = 2.8$. Indeed, we find a larger value as now $\alpha = 0.02$.

3.7. ALM with the yield curve

3.7.1. *Yield curve*

Up to now, we have assumed that the market rate *i* was constant that it takes the same value whatever the maturity of the investment is. When this last situation is correct, the annual interest rate for a maturity *T* will be represented by *i(T)*.

As in Figure 3.1, the graph of this function in a plane *T-i* is called the *yield curve* (YC).

More generally, if at time *t*, the annual interest rate for a maturity *T*, that is for an investment from t to *t* + *T* is *R(t.T)*. With *t* fixed the function *R:* $[0,\infty) \mapsto R^+$:

$$T \mapsto R(t,T) \qquad [3.49]$$

is called the *YC* at time *t* for the considered financial market. Up to now, we have considered the special case of a flat YC for which for all t and T: $R(t,T) = i$.

In the general case of a YC given by [3.49], the market value *MV(t.T)* at time *t* of the financial flow

$F = \left\{ \left(F_j, t_j \right), \quad j = 1, \ldots, n \right\}$ is given by:

$$MV(F,t) = \sum_{j=1}^{n} F_j \left(1 + R\left(t, t_j - t\right)\right)^{-\left(t_j - t\right)} \quad [3.50]$$

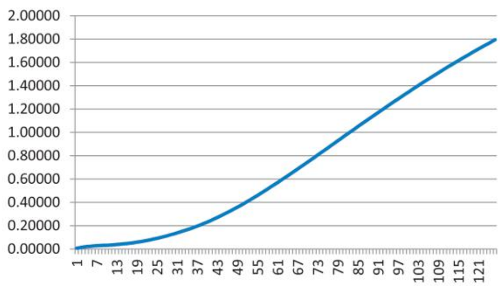

Figure 3.1. *Yield curve in % on July 14, 2014 month by month (source: French Institute of Actuaries)*

REMARK 3.3.– To obtain a scenario for the YC movements, we can select a new YC or also used stochastic models generating YC in function of the parameters of the considered models such as, for example, the Ornsteirn-Uhlenebeck-Vasicek and Cox-Ingersoll-Roll models (see, for example [JAN 09]). Some of them will be used in the next chapter.

3.7.2. *ALM with the equivalent constant rate*

The technique of *equivalent constant rate* consists of defining at time *t*, a *fictive constant annual rate i(t)* such that:

$$MV(F,t)) = \sum_{j=1}^{n} F_j \left(1 + i(t)\right)^{-\left(t_j - t\right)}, \quad [3.51]$$

and to compute the ALM indicators of the considered flow F with this rate *i(t)*. Here, an interest rate scenario consists of assuming a modification of the YC becoming the function $T \mapsto R_S(t,T)$ so that the new market value becomes:

$$VM_S(F,t) = \sum_{j=1}^{n} F_j \left(1 + R_S\left(t, t_j - t\right)\right)^{-(t_j - t)} \qquad [3.52]$$

Then, we can define a new fictive *equivalent fixed annual rate* i_s with the following algebraic equation:

$$VM_S(F,t)) = \sum_{j=1}^{n} F_j \left(1 + i_s\right)^{-(t_j - t)}. \qquad [3.53]$$

Now the scenario giving that the YC moves from R to R_S corresponds to a shift:

$$\Delta i = i_S - i \qquad [3.54]$$

where i is the equivalent rate corresponding to the initial YC R. With this constant interest rate scenario, we can proceed as in the previous sections.

EXAMPLE 3.6.– Let us come back to example 3.1 but now with a YC defined in Table 3.7.

With YC		Cash	PV
1	0.03	4	3.883495146
2	0.035	4	3.734042801
3	0.045	102	89.38225361
C(YC)	96.9997916		
1	0.0445	4	3.829583533
2	0.0445	4	3.666427509
3	0.0445	102	89.51067637
C(i equiv)	97.0066874		
	i=0.0445		

Table 3.7. *Computation of equivalent constant rate*

3.8. Matching with two rates

For evident reasons, banks and insurance companies work with interest rate i_A for assets and another rate i_B for liabilities.

Now the surplus *S* becomes a function of these two rates and proceeding as before. With assumptions [3.36], we obtain:

$$\begin{aligned} S(i_A+\Delta i_A, i_B+\Delta i_B) - S(i_A, i_B) \approx [(\alpha - D_{m,A}(i_A)\Delta i_A \\ +\frac{1}{2}CV_A(i_A)\left(\Delta i_A\right)^2]A(i) \\ -[(\beta - D_{m,B}(i_B)\Delta i_B + \frac{1}{2}CV_B(i_B)\left(\Delta i_B\right)^2]B(i_B) \end{aligned} \quad [3.55]$$

For a *given rate scenario* Δi_A and Δi_B, the first-order immunization condition takes now the form:

$$\begin{aligned} &[(\alpha - D_{m,A}(i_A)\Delta i_A]A(i_A) - [(\beta - D_{m,B}(i_B)\Delta i_B]B(i_B) = 0 \\ &\text{or} \\ &\alpha A(i_A) - \beta B(i_B) = [A(i_A)D_{m,A}(i_A)\Delta i_A - B(i_B)D_{m,B}(i_B)\Delta i_B]. \end{aligned} \quad [3.56]$$

If we introduce the new indicator of general modified duration defined by:

$$[A(i_A)D_{m,A}(i_A)\Delta i_A - B(i_B)D_{m,B}(i_B)\Delta i_B] = D_m^G(i_A, i_B; \Delta i_A, \Delta i_B)(= D_m^G), \quad [3.57]$$

We see that all the couples α. β admissible for first-order immunization are in the positive quadrant of the plane Oαβ on the straight line:

$$\begin{aligned} &\alpha A(i_A) - \beta B(i_B) = D_m^G \\ &\text{or} \\ &\beta = \frac{A(i_A)}{B(i_B)}\alpha - \frac{D_m^G}{B(i_B)}. \end{aligned} \quad [3.58]$$

Inversely, for a *given flow scenario* (*A.B*), the first-order immunization condition becomes:

$$[(\alpha - D_{m,A}(i_A)\Delta i_A]A(i_A) - [(\beta - D_{m,B}(i_B)\Delta i_B]B(i_B) = 0$$

or [3.59]

$$\frac{\alpha A(i_A) - \beta B(i_B)}{B(i_B)D_{m,B}(i_B)} = \frac{A(i_A)D_{m,A}(i_A)}{B(i_B)D_{m,B}(i_B)}\Delta i_A - \Delta i_B.$$

And so the admissible couples $\Delta i_A, \Delta i_B$ are, in the plane (O) $\Delta i_A, \Delta i_B$ on the straight line:

$$\Delta i_B = \frac{A(i_A)D_{m,A}(i_A)}{B(i_B)D_{m,B}(i_B)}\Delta i_A - \frac{\alpha A(i_A) - \beta B(i_B)}{B(i_B)D_{m,B}(i_B)}. \quad [3.60]$$

More particularly, assuming a dependence of the two rates of type:

$$\frac{\Delta i_B}{\Delta i_A} = \gamma,$$

Relation [3.60] gives:

$$\gamma \Delta i_A = \frac{A(i_A)D_{m,A}(i_A)}{B(i_B)D_{m,B}(i_B)}\Delta i_A - \frac{\alpha A(i_A) - \beta B(i_B)}{B(i_B)D_{m,B}(i_B)},$$

or

$$\Delta i_A = \frac{\dfrac{\alpha A(i_A) - \beta B(i_B)}{B(i_B)D_{m,B}(i_B)}}{\gamma - \dfrac{A(i_A)D_{m,A}(i_A)}{B(i_B)D_{m,B}(i_B)}} \quad [3.61]$$

$$= \frac{\alpha A(i_A) - \beta B(i_B)}{B(i_B)D_{m,B}(i_B)\gamma - A(i_A)D_{m,A}(i_A)}.$$

3.9. Equity sensitivity

3.9.1. *Presentation of the problem*

It is well known that the equities represent not only the capital of the shareholders but also the measure and the security of the customers of the bank or insurance company and so their sensitivity to usual risks and particularly the interest rate risk is crucial for the perennity of the firm.

Bergendhal and Janssen have shown the possibility to study the sensitivity of equities using the property of linearity of durations [BER 99].

The problem is the following: giving a scenario for both the YC and asset and liability flows on a finite time horizon of length T years and using as time unit the year, the trimester of even the month, what will be the impact of this scenario on the equities?

Then, the investor must select a set of scenarios with the worse of them being used to compute the VaR value.

It is clear that this problem is the core of ORSA.

3.9.2. *Formalization of the problem*

We start from a balance sheet with m sections for the asset and n sections for the liability. Let us call respectively $A_1,...,A_m$, and $B_1,...,B_n$ the present market values at time t=0 of these m+n sections for all the activity of the considered firm on the time horizon $[0,T]$. It follows that the present value of the equities is given by:

$$\begin{aligned} &E = A - B \\ &\text{with} \\ &A = A_1 + ... + A_m, \\ &B = B_1 + ... + B_n. \end{aligned} \qquad [3.62]$$

Given a financial-economic scenario, each section has a variation of $\Delta A_1,...,\Delta A_m;\Delta B_1,...,\Delta B_n$ and the corresponding equity change value in function of the scenario change is given by:

$$\begin{aligned} &\Delta E = \Delta A - \Delta B, \\ &\text{where} \\ &\Delta A = \Delta A_1 + ... + \Delta A_m; \Delta B = \Delta B_1 + ... + \Delta B_n. \end{aligned}$$

We can measure the impact of scenario variations of the equities with the value of ΔE, that is the variation of the present value at t=0 of the equities.

From the first equality of relations [3.62], we can write that:

$$ED_{m,E} = AD_{m,A} - BD_{m,B} \quad [3.63]$$

or

$$D_{m,E} = \frac{AD_{m,A} - BD_{m,B}}{E}. \quad [3.64]$$

3.9.3. *Time dynamic of asset and liability flows*

3.9.3.1. *Basic data of the considered scenario*

For section i of the asset and for section j the liability, let us consider their successive values of the time horizon [0,T] by:

$$A_{i1},...A_{iT}, i = 1,...,m; \quad \text{and} \quad B_{j1},...,B_{jT}, j = 1,...,n. \quad [3.65]$$

From these values, we can deduce the unrolling of the successive flow sections represented respectively by $(a_{ik}, i = 1,...,m, k = 1,...,T-1)$, as follows:

$$\begin{aligned} &a_{i1} = A_{i1} - A_{i0}, \quad a_{ik} = A_{i(k+1)} - A_{ik}, \\ &b_{j1} = B_{j1} - B_{j0}, \quad b_{jk} = B_{i(k+1)} - B_{ik},. \end{aligned} \quad [3.66]$$

A_{i0} and B_{j0} being the initial values respectively of the section i of the asset and the section j of the liability.

Of course, we have:

$$\begin{aligned} &A(k)(= A_k) = A_{1k} + ... + A_{mk}, \\ &B(k)(= B_{k)} = B_{1k} + ... + B_{nk}, \\ &E(k)(= E_k) = A(k) - B(k). \end{aligned} \quad [3.67]$$

The unrolling dynamics can be represented by the following table:

	1	2			T	
A(1)	a(11)	a(12)			a(1T)	
A(2)						
A(m)	a(m1)				b(mT)	
A	a(11)+...a(m1)				a(1T)+...a(mT)	
B(1)	b(11)	b(12)			b(1T)	
B(2)						
B(n)	b(n1)				b(nT)	
B	b(11)+...b(n1)				b(1T)+...b(nT)	
E	a(11)+...a(m1)				a(1T)+...a(mT)	
	b(11)-...b(n1)				b(1T)-...b(nT)	

Table 3.8. *Dynamics of asset and liability flows*

Of course, the given scenario also contains YCs for assets and liabilities eventually *m* + *n* YCs for the *m* + *n* sections.

3.9.3.2. *Equivalent constant rates*

The *m* + *n* constant equivalent rates $i_1,...,i_m;\ j_1,...,j_n$ and the equivalent constant rates i_A, i_B satisfy the following relations:

$$A_k = \sum_{\nu=1}^{T} \frac{a_{k\nu}}{(1+i_k)^{\nu}}, \qquad B_l = \sum_{\nu=1}^{T} \frac{b_{l\nu}}{(1+j_l)^{\nu}},$$

$$(A_1 + ... + A_m =)\ A = \sum_{\nu=1}^{T} \frac{a_{1\nu} + ... + a_{m\nu}}{(1+i_A)^{\nu}},\ (B_1 + ... + B_n =) B = \sum_{\nu=1}^{T} \frac{b_{1\nu} + ... + b_{n\nu}}{(1+i_B)^{\nu}}$$

$$\begin{aligned} \sum_{k=1}^{m}\left(\sum_{\nu=1}^{T} \frac{a_{k\nu}}{(1+i_k)^{\nu}}\right) &= \sum_{\nu=1}^{T} \frac{a_{1\nu} + ... + a_{m\nu}}{(1+i_A)^{\nu}} \quad (= A), \\ \sum_{l=1}^{n}\left(\sum_{\nu=1}^{T} \frac{b_{l\nu}}{(1+j_l)^{\nu}}\right) &= \sum_{\nu=1}^{T} \frac{b_{1\nu} + ... + b_{n\nu}}{(1+i_B)^{\nu}} \quad (= B). \end{aligned} \qquad [3.68]$$

Let us recall that the present values $A_1,...A_m;\quad B_1,...B_n$ have been computed with the *m* + *n* YCs for the *m* + *n* sections.

For the equities, the equivalent rate on i_E is solution of the following equation:

$$\sum_{s=1}^{T} \frac{E_s}{(1+i_E)^s} = A - B, \qquad [3.69]$$

with

$$\begin{aligned} E_1 &= A(1) - B(1) \quad = (a_{11} + ... + a_{m1}) - (b_{11} + ... + b_{n1}), \\ &\vdots \\ E_s &= A(s) - B(s) = (a_{1s} + ... + a_{ms}) - (b_{1s} + ... + b_{ns}), \\ &\vdots \\ E_T &= A(T) - B(T) = (a_{1T} + ... + a_{mT}) - (b_{1T} + ... + b_{nT}); \end{aligned}$$

3.9.4. *Sensitivity of equities and VaR indicator*

To test the sensitivity of equities, it is necessary to select different scenarios and to see how the equities react particularly if the worst of them will be realized. This is a very big task. In general, as we will see in the two next chapters, we start from a *central scenario* and we construct several other scenarios taking into account the economics and financial environment. As an example, let us simply consider the interest rate risk by assuming some variations of the consider YCs respectively noted $\Delta i_1,...,\Delta i_m$, and $\Delta j_1,...,\Delta j_n$. As:

$$\Delta E = \Delta A_1 + ... + \Delta A_m - \Delta B_1 - ... - \Delta B_n . \qquad [3.70]$$

It suffices to express the variations of the *m*+*n* balance sheet sections with their respective durations to obtain at the first-order:

$$\begin{aligned} \Delta E &\approx -\left[D_m(A_1)A_1\Delta i_1 + ... + D_m(A_m)A_m\Delta i_m\right] \\ &+\left[D_m(B_1)B_1\Delta j_1 + ... + D_m(B_m)B_m\Delta j_n\right] \end{aligned} \qquad [3.71]$$

or in relative value:

$$\begin{aligned} \frac{\Delta E}{E} &\approx \frac{-\left[D_m(A_1)A_1\Delta i_1 + ... + D_m(A_m)A_m\Delta i_m\right]}{E} \\ &+ \frac{\left[D_m(B_1)B_1\Delta j_1 + ... + D_m(B_m)B_m\Delta j_n\right]}{E} \end{aligned} \qquad [3.72]$$

These two results give the answer for the impact on interest rate theory on the equities.

Of course, we can select simpler scenarios: for example, let us assume that there is only one variation ΔA_1 in section 1 of the asset so that the variation of the equities is given by:

$$\frac{\Delta E}{E} \simeq \frac{-D_m(A_1)A_1\Delta i_1}{E}. \qquad [3.73]$$

3.9.5. *Duration of equities*

From relation [3.71], the duration of the equities satisfies the following relation:

$$\begin{aligned} -D_m(E)E\Delta i_E &\approx -\left[D_m(A_1)A_1\Delta i_1 + ... + D_m(A_m)A_m\Delta i_m\right] \\ &+\left[D_m(B_1)B_1\Delta j_1 + ... + D_m(B_n)B_n\Delta j_n\right] \end{aligned} \qquad [3.74]$$

and so:

$$\begin{aligned} -D_m(E)\Delta i_E &\approx \frac{-\left[D_m(A_1)A_1\Delta i_1 + ... + D_m(A_m)A_m\Delta i_m\right]}{E} \\ &\frac{+\left[D_m(B_1)B_1\Delta j_1 + ... + D_m(B_n)B_n\Delta j_n\right]}{E}. \end{aligned} \qquad [3.75]$$

3.9.6. *Special case of the aggregated balance sheet*

This special case corresponds to $m = n = 1$ and so the flows are given by Table 3.9.

	0	1	2	T
A	A(0)	a(1)	a(2)	a(T)
B	B(0)	b(1)	b(2)	b(T)
E	E(0)	A(1)-B(1)	A(2)-B(2)	A(T)-B(T)

Table 3.9. *Dynamics of the flows for the aggregated balance sheet*

The three steps of the general method exposed above become:

1) Compute the present market values with the adequate YC of the selected scenario

$$A = \sum_{\nu=1}^{T} \frac{a_\nu}{(1+i_A(0,\nu))^\nu},$$
$$B = \sum_{\nu=1}^{T} \frac{b_\nu}{(1+i_B(0,\nu))^\nu}, \qquad [3.76]$$
$$E = A - B.$$

2) Compute the equivalent constant rates i_A, i_B and i_E by solving the three following equations:

$$A = \sum_{\nu=1}^{T} \frac{a_\nu}{(1+i_A)^\nu},$$
$$B = \sum_{\nu=1}^{T} \frac{b_\nu}{(1+i_B)^\nu}, \qquad [3.77]$$
$$\sum_{\nu=1}^{T} \frac{A(\nu)-B(\nu)}{(1+i_E)^\nu} = A - B,$$

A and B being given by relations [3.76].

3) Compute the modified durations of A and B: $D_m(A), D_m(B)$. The reply of the interest rate risk question what if the two YC variations correspond to the following variations $\Delta i_A, \Delta i_B$ is given by:

$$\Delta E \approx -D_m(E)E\Delta i_E$$
where: [3.78]
$$D_m(E)\Delta i_E = \frac{D_m(A)A\Delta i_A - D_m(B)B\Delta i_B}{E}.$$

Of course, the relative variation is given by:

$$\frac{\Delta E}{E} \approx -D_m(E)\Delta i_E. \qquad [3.79]$$

3.9.7. *A VaR approach*

The presented approach given above allows us to give a new VaR approach concerning the equities of the considered firm.

Let us consider n scenarios including very bad ones that are of the *stress test* type.

For each scenario k, we can compute the impact on E: $\Delta E_k, k = 1,...,n$ and we can give as VaR value on the considered time horizon:

$$VaR = \max_{k=1,...,n} \Delta E_k . \qquad [3.80]$$

The probability to have a loss of such an amount is smaller than or equal to the realization of the worst scenario. If the probability of this realization can be evaluated by α, we have thus computed a VaR at this level.

EXAMPLE 3.7.– Let us consider the following dynamics data on a balance sheet of a given firm [JAN 98] in 12 years.

year	assets	liab.	equity	asset	liab.	ΔE	yield	YC
	A	B	E=A-B	flow	flow		E	equities
0	24406	22631	1775					
1	27805	25714	2091	3399	3083	316	0,17802817	0,17802817
2	31379	28894	2485	3574	3180	394	0,18842659	0,18321596
3	36546	33661	2885	5167	4767	400	0,16096579	0,17575225
4	40162	37083	3079	3616	3422	194	0,06724437	0,14763248
5	44853	40674	4179	4691	3591	1100	0,35725885	0,18679253
6	49939	44911	5028	5086	4237	849	0,20315865	0,18950467
7	56753	50283	6470	6814	5372	1442	0,28679395	0,20293927
8	64461	55671	8790	7708	5388	2320	0,35857805	0,2213744
9	73461	59999	13462	9000	4328	4672	0,53151308	0,25247168
10	76683	63137	13546	3222	3138	84	0,00623979	0,22535315
11	82567	68370	14197	5884	5233	651	0,04805847	0,20806624

Table 3.10. *Balance sheet data in €*

To simplify the presentation, we take the same YC for both assets and liabilities given by Table 3.11.

Year	Rate
1	0.003595
2	0.008533
3	0.0018486
4	0.0034254
5	0.0072551
6	0.0078573
7	0.010291
8	0.0126732
9	0.0149198
10	0.0169824
11	0.0179363

Table 3.11. *YC*

Table 3.12 gives the first results for corresponding fixed rates found to be $i_A = 0.1223; i_B = 0.0121$.

	Data									
Year	Asset	Liability	YC		Present as.	Present liab.	Fixed rate for	Present as.	Fixed rate for	Present liab.
	flows	flows			values	values	assets	values	liabilities	values
							0.01223		0.0121	0.0121
1	3399	3083	0.0036	0.9964	3386.8244	3071.9563	0.9879	3357.9325	0.9880	3046.14169
2	3574	3180	0.0085	0.9831	3513.7780	3126.4169	0.9760	3488.1579	0.9762	3104.41855
3	5167	4767	0.0018	0.9945	5138.4505	4740.6606	0.9642	4981.9670	0.9646	4598.06258
4	3616	3422	0.0034	0.9864	3566.8764	3375.5119	0.9525	3444.3843	0.9530	3261.26662
5	4691	3591	0.0073	0.9645	4524.4736	3463.5226	0.9410	4414.3769	0.9416	3381.41349
6	5086	4237	0.0079	0.9541	4852.6848	4042.6318	0.9297	4728.2576	0.9304	3942.01175
7	6814	5372	0.0103	0.9308	6342.7375	5000.4676	0.9184	6258.1749	0.9193	4938.23815
8	7708	5388	0.0127	0.9042	6969.2681	4871.6161	0.9073	6993.7172	0.9083	4893.73207
9	9000	4328	0.0149	0.8752	7876.9299	3787.9281	0.8964	8067.3271	0.8974	3883.97502
10	3222	3138	0.0170	0.8450	2722.6459	2651.6645	0.8855	2853.2084	0.8867	2782.3944
11	5884	5233	0.0179	0.8224	4838.8933	4303.5229	0.8748	5147.5600	0.8761	4584.51145
tot	58161	45739		total PV.	53733.5624	42435.8993		53735.0638		42416.1658

Table 3.12. *Results for finding equivalent fixed rates*

We obtain the following results:

	Assets	Liability	Equity	
Present value	53423.27304	42416.16577	11007.10728	
Duration	6.49561973	6.214145831		
Modified dur.	6.417138131	6.139853603		
Abs. mod. dur.	342824.5225	260429.0482		
Rel. convexity	50.18227142	52.97711264		
Abs. convexity	1340450.594	1123542.996		
Rate scenario	**0.009**	**-0.008**		
Variation	-3085.420703	-0.057754243	-5168.853088	
Rel. var. first-order	-0.057754243	0.049118829	-0.469592324	
Var with conv.	-3031.132454	-3031.132454	-5114.564839	
Rel. var with conv.	-0.056738052	0.049118829	-0.464660216	
E value after scen.	50337.85234	42416.10801	5838.254188	
id. with conv.	50392.14059	42416.10801	5892.542437	
Initial E/A				0.206035809
id. afterscen.				0.115981392
id. with conv.				0.116933759

Table 3.13. *Matching results*

Of course, for the variation of E, we use the formula:

$$\Delta E \approx -D_m(A)A\Delta i_1 + D_m(B)B\Delta j_1.$$

As it is clear that the bad scenario consists of an increase in the asset rate and a decrease in the liability rate. We can compute the VaR value assuming that the scenario: $\Delta i_A = 0.01, \Delta i_B = -0.01,$ for which we can attribute by proxy or from historical data that its probability of realization has value 0.005 so we can compute the VaR value from the following results:

	Assets	**Liability**	**Equity**	
Present value	53423.27304	42416.16577	11007.10728	
Duration	6.49561973	6.214145831		
Modified dur.	6.417138131	6.139853603		
Abs. mod. dur.	342824.5225	260429.0482		
Rel. convexity	50.18227142	52.97711264		
Abs. convexity	1340450.594	1123542.996		
Rate scenario	**0.01**	**-0.01**		
Variation	-3428.245225	-0.064171381	-6032.535707	
Relative var.	-0.064171381	0.061398536	-0.548058228	
Var with conv.	-3361.222695	-3361.222695	-5965.513177	
Rel. var with conv.	-0.062916825	0.061398536	-0.541969205	
Present val. after scen.	49995.02782	42416.10159	4974.571569	
id. with conv.	50062.05035	42416.10159	5041.594099	
Initial E/A				0.206035809
id. after scen.				0.099501326
id. with conv.				0.100706904
VaR value	**6032.535707**			

Table 3.14. *VaR computation*

REMARK 3.4. (DURATION WITHOUT EQUIVALENT INTEREST RATES).– Instead of using the method given the previous section and based on the computation of equivalent interest rates, it is still possible to directly work with the given YCs (see, for example, [DEV 12]).

3.10. ALM and management of the bank

3.10.1. *Basic principles*

In a mismatching situation, the firm must adjust assets and liabilities in such a way to reduce it.

This implies taking a series of measures preconized by the internal ALM Committee. Actions on the amount and/or the maturities of the flows can be taken using as main financial tools:

– new maturity times for example with swaps;

– new flow amounts using optional products like calls, puts and synthetic options and credit derivatives;

– new investment portfolio; etc.

REMARK 3.5.– Let us also remark that in case of first-order immunization, it is still possible to improve the result using the following relation:

$$S(i_A + \Delta i, i_B + \Delta i) - S(i_A, i_B) \approx \frac{1}{2}[A(i_A)CV_A(i_A) - B(i_B)CV_B(i_B)](\Delta i)^2 \quad [3.81]$$

which shows that the largest absolute weighted convexity shows the side of the leverage effect.

Indeed, if the first factor of the second member of this last relation is positive, the firm receives a supplementary benefit and a loss in the opposite situation.

3.10.2. *ALM and shares*

3.10.2.1. *Infinite horizon*

The definition of duration for a share portfolio can be given under several assumptions concerning the flow generated by the share.

The simplest way is to assume, as a basic scenario, that all the amounts are known and that the investor will keep his share all the time.

Under this scenario, the present value A of the flow in $t = 0$ is given by:

$$A = \sum_{n+1}^{\infty} \frac{\beta_n}{(1+i_n)^n} \quad [3.82]$$

β_n and i_n representing respectively the dividend of the share for year n and the maturity rate on $[0.n]$.

Of course, if: $\beta_n = \beta, i_n = i$, then the last result becomes:

$$\begin{aligned} A &= \sum_{n=1}^{\infty} \frac{\beta}{(1+i)^n}, \\ &= \frac{\beta}{i}. \end{aligned} \quad [3.83]$$

However, with the assumptions of Gordon-Shapiro, we can introduce an increasing rate for the dividends c ($<i$) and in this case:

$$\begin{aligned} A &= \sum_{n+1}^{\infty} \frac{\beta(1+c)^n}{(1+i)^n}, \\ &= \beta \frac{1+c}{i-c} (c < i) \end{aligned} \quad [3.84]$$

With the introduction of a constant risk premium λ, the last result becomes:

$$\begin{aligned} A &= \sum_{n=1}^{\infty} \frac{\beta(1+c)^n}{(1+i+\lambda)^n}, \\ &= \beta \frac{1+c}{i+\lambda-c}. \end{aligned} \quad [3.85]$$

Now let us suppose that the interest rate has a small variation Δi . Then, the variation of the present value is given by:

$$A(i+\Delta i) - A(i) \approx A'(i)\Delta i$$

where

$$\begin{aligned} A'(i) &= -\beta \frac{(1+c)}{(i+\lambda-c)^2}, \\ &= -\frac{1}{i+\lambda-c} A(i), \end{aligned} \quad [3.86]$$

Thus, we can define the modified duration for this case, simply noted as *D(i)* as follows:

$$D(i) = \frac{1}{i+\lambda-c}. \qquad [3.87]$$

From relation [3.85], we also get:

$$D(i) = \frac{A}{\beta(1+c)}, \qquad [3.88]$$

that is the price earning ratio (PER) giving the percentage of benefit by action.

So, we have:

$$A(i+\Delta i) - A(i) \simeq -D(i)A(i)\Delta i\ . \qquad [3.89]$$

More generally, we can consider a more general scenario given by the quadruplet $(\Delta i, \Delta\beta, \Delta c, \Delta\lambda)$ for which:

$$\begin{aligned} &\mathrm{A}(i+\Delta i, \beta+\Delta\beta, \mathrm{c}+\Delta c, \lambda+\Delta\lambda) - \mathrm{A}(\Delta \mathrm{i}, \Delta\beta, \Delta c, \Delta\lambda) \approx \\ &\frac{\partial A}{\partial i}\Delta i + \frac{\partial A}{\partial \beta}\Delta\beta + \frac{\partial A}{\partial c}\Delta c + \frac{\partial A}{\partial \lambda}\Delta\lambda. \end{aligned} \qquad [3.90]$$

As

$$\frac{\partial A}{\partial i} = -\beta\frac{(1+c)}{(i+\lambda-c)^2}, \frac{\partial A}{\partial \beta} = \frac{1+i}{i+\lambda-c}, \frac{\partial A}{\partial c} = \beta\frac{i+\lambda+1}{(i+\lambda-c)^2}, \frac{\partial A}{\partial \lambda} = -\beta\frac{1+c}{(i+\lambda-c)^2},$$

We can also write using still relation [3.85] that:

$$\begin{aligned} &\frac{\partial A}{\partial i} = -D_i A(i), \frac{\partial A}{\partial \beta} = \frac{1}{\beta}A(i), \frac{\partial A}{\partial c} \\ &= \frac{i+\lambda+1}{(i+\lambda-c)(1+c)}A(i), \frac{\partial A}{\partial \lambda} = -\frac{1}{(i+\lambda-c)}A(i). \end{aligned} \qquad [3.91]$$

Defining the four *partial modified durations* as follows:

$$D_i = \frac{1}{i+\lambda-c}, D_\beta = \frac{1}{\beta}, D_c = \frac{(i+\lambda-1)}{(i+\lambda-c)(1+c)}, D_\lambda = \frac{1}{(i+\lambda-c)}, \quad [3.92]$$

We can write:

$$\Delta A \approx (-D_i \Delta i + \mathrm{D}_\beta \, \Delta\beta + D_c \Delta c - D_\lambda \Delta\lambda) A(i). \quad [3.93]$$

The next table gives a numerical example for one share of unit 1 at time 0:

Data		Durations			Scenario
i	0.02	D(i)	33.3333333	Δi	0.005
β	0.04	D(β)	25	Δβ	0.01
c	0	D(c)	-32.3333333	Δc	0
λ	0.01	D(λ)	33.3333333	Δλ	0.01
Results					
A	1.333333	ΔA	-0.25	ΔA/A	-0.1875

Table 3.15. *Example for one share*

3.10.2.2. *VaR approach*

Assuming that we can select from historical data or by proxy, a stress scenario with a probability (historical or subjective) of occurrence of 0.05 for example the following one:

$(\Delta i, \Delta\beta, \Delta c, \Delta\lambda)$=(-0.01;-0.02;0.005;0.01), we obtain the following results:

Data		Durations			Scenario
i	0.02	D(i)	33.3333333	Δi	0.01
β	0.04	D(β)	25	Δβ	-0.02
c	0	D(c)	-32.3333333	Δc	-0.005
λ	0.01	D(λ)	33.3333333	Δλ	0.01
Results					
A	1.333333	ΔA	-1.005	ΔA/A	-0.75375

Table 3.16. *Stress scenario*

From these results, we deduce that the unit VaR value for the share section represents 75% of the market value.

3.10.2.3. *Finite horizon (the Janssen-Manca model)*

Let us now consider the case of a finite horizon time T for which relation [3.85] becomes:

$$A=\sum_{n=1}^{T}\frac{\beta(1+c)^n}{(1+i+\lambda)^n}. \qquad [3.94]$$

Now if we define:

$$y=\frac{1+c}{1+i+\lambda} \; : \qquad [3.95]$$

the previous result becomes:

$$\begin{aligned}A&=\beta\sum_{n=1}^{T}y^n,\\ &=\beta y\frac{1-y^T}{1-y},\\ &=\beta\frac{1+c}{1+i+\lambda}[1-(\frac{1+c}{1+i+\lambda})^T]\frac{1+i+\lambda}{i+\lambda-c}\\ &=\beta\frac{1+c}{i+\lambda-c}[1-(\frac{1+c}{1+i+\lambda})^T]\end{aligned} \qquad [3.96]$$

From the second equality of relations [3.96], it can be easily proved that:

$$A'(y)=\beta\frac{1-y^T[T+1-Ty]}{(1-y)^2}$$

and so:

$$\begin{aligned}A'(i)&=-\beta\frac{1-y^T[T+1-Ty]}{(1-y)^2}\frac{(1+c)}{(1+i+\lambda)^2}\\ &=A'(i)=-\beta\frac{1-y^T[T+1-Ty]}{(1-y)^2}\frac{y^2}{(1+c)}.\end{aligned} \qquad [3.97]$$

Dividing this last result by *A(i)*, we obtain the modified duration:

$$D_A(i) = \frac{\dfrac{1-y^T[T+1-Ty]}{(1-y)^2}}{\dfrac{y-y^{T+1}}{(1-y)}} \frac{(1+c)}{(1+i+\lambda)^2},$$

$$= \frac{1-y^T[T+1-Ty]}{(1-y)(y-y^{T+1})} \frac{(1+c)^2}{(1+i+\lambda)^2(1+c)}, \qquad [3.98]$$

$$= \frac{1-y^T[T+1-Ty]}{(1-y)(y-y^{T+1})} \frac{y^2}{(1+c)}.$$

We also have:

$$\text{A} = \beta \frac{1+c}{i+\lambda-c}[1-(\frac{1+c}{1+i+\lambda})^T],$$

$$\frac{\partial A}{\partial i} = -\beta \left[\frac{(1+c)}{(i+\lambda-c)^2} \right] [1-(\frac{1+c}{1+i+\lambda})^T] - \beta \frac{1+c}{i+\lambda-c} \frac{1+c}{1+i+\lambda})^{T-1}$$

$$\frac{\partial A}{\partial \beta} = \frac{1+c}{i+\lambda-c}[1-(\frac{1+c}{1+i+\lambda})^T] = \frac{A}{\beta},$$

$$\frac{\partial A}{\partial c} = \beta \left[\frac{i+\lambda-c+(1+c)}{(i+\lambda-c)^2} \right] [1-(\frac{1+c}{1+i+\lambda})^T] - \beta T \frac{1+c}{i+\lambda-c} (\frac{1+c}{1+i+\lambda})^{T-1},$$

$$\frac{\partial A}{\partial \lambda} = \beta \left[-\frac{(1+c)}{(i+\lambda-c)^2} \right] [1-(\frac{1+c}{1+i+\lambda})^T] - \beta T \frac{1+c}{i+\lambda-c} (\frac{1+c}{1+i+\lambda})^{T-1}.$$

EXAMPLE 3.8.– Data

Data			Scenario	New data
β	0.04	Δβ	-0.01	0.03
λ	0.01	Δλ	0.01	0.02
i	0.02	Δi	0.005	0.025
c	0	Δc	-0.005	0

Table 3.17. *Stress scenario*

T	Duration	A	Old A value	ΔA	ΔA/A
3	1.8858	0.082	0.1131	-0.0311	-0.379268293
5	2.7866	0.1317	0.1832	-0.0515	-0.391040243
10	4-916	0.2374	0.3451	-0.1077	-0.453664701
20	8.665	0.3902	0.591	-0.2008	-0.514607893
50	16.265	0.5928	1.0292	-0.4364	-0.736167341
100	21.03	0.658	1.264	-0.606	-0.920972644
200	22.19	0.6665	1.3297	-0.6632	-0.995048762
500	22.222	0.6666	1.3333	-0.6667	-1.000150015
1000	22.222	0.6666	1.333	-0.6664	-0.99969997

Table 3.18. *Results after scenario*

3.10.2.4. *Scenario: Selling of the share at time T at price T*

In the two previous sections, we assume that the firm never sells back its share. To be more realistic, we will now assume the contrary: the firm considers the following scenario: to keep its share on [0.*T*] and to sell at time *T* for a sum of *B*.

The present value of this investment is given by:

$$\tilde{A} = \sum_{n=1}^{T-1} \frac{\beta(1+c)^n}{(1+i+\lambda)^n} + \frac{(\beta(1+c)^T + B)}{(1+i+\lambda)^T} \quad [3.99]$$

From relation [3.94], we can also write:

$$\tilde{A} = A + \frac{B}{(1+i+\lambda)^T} \quad [3.100]$$

so that:

$$\begin{aligned}
&\tilde{A} = A + \frac{B}{(1+i+\lambda)^T}, \mathrm{A} = \sum_{n=1}^{T} \frac{\beta(1+c)^n}{(1+i+\lambda)^n} \\
&= \ \mathrm{A} = \beta \frac{1+c}{i+\lambda-c}[1-(\frac{1+c}{1+i+\lambda})^T], \\
&\frac{\partial \tilde{A}}{\partial i} = \frac{\partial A}{\partial i} - \frac{BT}{(1+i+\lambda)^{T+1}}, \frac{\partial \tilde{A}}{\partial \beta} = \frac{\partial A}{\partial \beta}, \frac{\partial \tilde{A}}{\partial c} = \frac{\partial A}{\partial c} \\
&\frac{\partial \tilde{A}}{\partial \lambda} = \frac{\partial A}{\partial \lambda} - \frac{BT}{(1+i+\lambda)^{T+1}}, \frac{\partial \tilde{A}}{\partial B} = \frac{1}{(1+i+\lambda)^T}.
\end{aligned} \quad [3.101]$$

So, we obtain:

$$\Delta\tilde{A} \approx \frac{\partial \tilde{A}}{\partial i}\Delta i + \frac{\partial \tilde{A}}{\partial \beta}\Delta\beta + \frac{\partial \tilde{A}}{\partial c}\Delta i + \frac{\partial \tilde{A}}{\partial \lambda}\Delta i + \frac{\partial \tilde{A}}{\partial B}\Delta B$$

And by relation [3.109]:

$$\begin{aligned}
&\Delta\tilde{A} \approx (\frac{\partial A}{\partial i} - \frac{BT}{(1+i+\lambda)^{T+1}})\Delta i + \frac{\partial A}{\partial \beta}\Delta\beta + \frac{\partial A}{\partial c}\Delta c + \frac{\partial \tilde{A}}{\partial \lambda}\Delta i + \frac{1}{(1+i+\lambda)^T}\Delta B, \\
&= \frac{\partial A}{\partial i}\Delta i + \frac{\partial A}{\partial \beta}\Delta\beta + \frac{\partial A}{\partial c}\Delta c + \frac{\partial A}{\partial \lambda}\Delta\lambda + \frac{1}{(1+i+\lambda)^T}\Delta B - \frac{BT}{(1+i+\lambda)^{T+1}}(\Delta i + \Delta\lambda), \\
&= \Delta A - \frac{BT}{(1+i+\lambda)^{T+1}}(\Delta i + \Delta\lambda) + \frac{1}{(1+i+\lambda)^T}\Delta B.
\end{aligned} \quad [3.102]$$

Introducing what we call the selling correction:

$$\Delta S = -\frac{BT}{(1+i+\lambda)^{T+1}}(\Delta i + \Delta\lambda) + \frac{1}{(1+i+\lambda)^T}\Delta B,$$

Finally, we have:

$$\Delta\tilde{A} = \Delta A + \Delta S,$$

with, of course, the same scenario for the four parameters $(\Delta i, \Delta\beta, \Delta c, \Delta\lambda)$ for both A and $\tilde{A}$.

EXAMPLE 3.9.– Let us consider the following data:

Par.	New value	Scen	Scen. values	Old value
β	0.03	Δβ	-0.01	0.02
λ	0.02	Δλ	0.01	0.01
i	0.025	Δi	0.005	0.02
c	0	Δc	-0.005	0
T	3	ΔT		
A	0.08246893	ΔA		
B	1	ΔB	0.25	

Table 3.19. *Data for example 3.10*

The results are given in the table below.

T	Duration	New A	Old A value	ΔA	ΔA/A	$\Delta\tilde{A}$		$\Delta\tilde{A}/A$
3	1.8858	0.0824689	0.1131	-0.030631	0.3714256	0.9893966	0.1507078	0.1523230
5	2.7866	0.1316993	0.1832	-0.05150	0.39104761	0.9856511	0.0915199	0.0928522
10	4-916	0.2373816	0.3451	-0.1077185	0.4537777	0.9890277	-0.0391663	-0.039601
20	8.665	0.3902381	0.591	-0.2007619	0.5144600	1.0056429	-0.2161374	-0.2149246
50	16.265	0.5928602	1.0292	-0.4363398	-0.735991	1.1399097	-0.4881190	-0.4282085
100	21.03	0.6584956	1.264	-0.6055044	-0.9195269	1.2762566	-0.620034	-0.4858220
200	22.19	0.6665665	1.3297	-0.6631335	-0.9948497	1.3298502	-0.6635272	-0.4989488
500	22.222	0.6666667	1.3333	-0.6666333	-0.9999500	1.3333	-0.6666333	-0.4999875
1000	22.222	0.6666667	1.333	-0.6663333	-0.9995	1.333	-0.6663333	-0.4998750

Table 3.20. *Results for example 3.10*

3.10.3. *Stochastic duration [JAN 08]*

As the previous scenario involves the knowledge of the price of the asset at time T, sometimes, it could be more useful to use the classical Black and Scholes model of trend tendance μ and volatility σ. This means that if *S(t)* represents the asset value at time, the stochastic process $S=\left\{S(t),0\leq t\leq T\right\}$ is governed by the stochastic differential equation:

$$\begin{aligned}&dS(t)=\mu S(t)dt+\sigma S(t)dW(t),\\&S(0)=S_0\end{aligned} \qquad [3.103]$$

$W=(W(t),0\leq t\leq T)$ is a standard Brownian process (see, for example [JAN 09]). Then, it is possible to prove that:

$$\begin{aligned}&S(T)=S_0e^{(\mu-\frac{\sigma^2}{2})T}e^{\sigma W(T)}\\&E\left[S(T)\right]=S_0e^{\mu T}.\end{aligned} \qquad [3.104]$$

3.10.3.1. *Mean approach*

We assume that the instantaneous rate without risk is μ, the annual corresponding rate i and that the risk premium is v so that we can write that:

$$\begin{aligned}&e^{\mu T}=e^{\nu T}e^{rT},\\&e^{r}=1+i,\\&e^{\nu}=1+\delta.\end{aligned} \qquad [3.105]$$

Here, the present value of the investment noted now $\tilde{A}$ becomes:

$$\tilde{A}=\sum_{n+1}^{T}\frac{\beta}{(1+i)^n}+S_0e^{\nu T} \ . \qquad [3.106]$$

or

$$\tilde{A}=A+S_0e^{\nu T} \qquad [3.107]$$

Consequently:

$$\begin{aligned}&\tilde{A}'(i)=A'(i),\\&D_{\tilde{A}}(i)=\frac{A'(i)}{A(i)+S_0e^{\nu T}}.\end{aligned} \qquad [3.108]$$

In opposition to the *deterministic duration* $D_A(i)$, the duration $D_{\tilde{A}}(i)$ will be called *duration in mean.*

As we know that $D_A(i) = \frac{A'(i)}{A(i)}$, we have:

$$D_{\tilde{A}}(i) = \frac{A(i)}{A(i) + S_0 e^{\nu T}} D_A(i). \quad [3.109]$$

Without risk premium, we see that:

$$D_{\tilde{A}}(i) = \frac{A(i)}{A(i) + S_0} D_A(i). \quad [3.110]$$

Clearly, the duration in mean is strictly smaller than the deterministic duration and without risk premium. For $T \to \infty$, we see that $D_{\tilde{A}}(i) = D_A(i)$.

Moreover, under the assumption of absence of opportunity arbitrage (AOA), we now have $S_0 = A(i)$ and so $D_{\tilde{A}}(i) = \frac{D_A(i)}{2}$.

3.10.3.2. *Stochastic duration [JAN 08]*

In relation [3.105], let us replace the last term by the stochastic value of the asset at time *T* given by the second equality of [3.104] so that the present value of the investment becomes:

$$\hat{A}(i) = \sum_{n+1}^{T} \frac{\beta}{(1+i)^n} + e^{-rT} S_0 e^{\left(\mu - \frac{\sigma^2}{é}\right)T + \sigma W(t)}, \quad [3.111]$$

or from the first equality of [3.104]:

$$\hat{A}(i) = \sum_{n+1}^{T} \frac{\beta}{(1+i)^n} + S_0 e^{\left(\nu - \frac{\sigma^2}{é}\right)T + \sigma W(t)}, \quad [3.112]$$

As $\hat{A}'(i) = A'(i)$, it follows that the duration takes the following form:

$$D_{\hat{A}}(i) = \frac{A'(i)}{A(i) + S_0 e^{\left(\nu - \frac{\sigma^2}{2}\right)T + \sigma W(t)}}. \quad [3.113]$$

From the definition of the classical duration, we can finally write:

$$D_{\hat{A}}(i) = \frac{A(i)}{A(i) + S_0 e^{\left(\nu - \frac{\sigma^2}{2}\right)T + \sigma W(t)}} D_A(i). \quad [3.114]$$

Here the presence of *W(t)* shows that the duration has a stochastic value; that is why, it will be called *stochastic duration* of the considered share. Clearly, its support is $\left[0, D_A(i)\right]$.

Janssen and Manca [JAN 08] have given the distribution function of this random variable under the following form:

$$P\left[D_{\hat{A}}(i) < u\right] = 1 - \varphi\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{A(i)}{S_0}\left(\frac{D_A(i)}{u} - 1\right)\right) - \left(\nu - \frac{\sigma^2}{2}\right)T\right]\right), 0 \le u \le D_A(i). \quad [3.115]$$

They also give the d.f. of the ratio $\frac{D_{\hat{A}}(i)}{D_A(i)}$ with $\left[1, \infty\right)$ as support:

$$P\left[\frac{D_A(i)}{D_{\hat{A}}(i)} \le z\right] = \varphi\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{A(i)}{S_0}(z-1)\right) - \left(\nu - \frac{\sigma^2}{2}\right)T\right]\right), \quad [3.116]$$

EXAMPLE 3.10.– For the following data:

Data				
β	50			
λ	0		μ	0.04
i	0.04		ν	0.02
c	0		σ	0.2
T	10		σ^2	0.04
$S(0)$	1200			

Table 3.21. *Data for stochastic duration example*

The deterministic duration is $D(A)$ = 4.97813838 and the distribution function [3.114] by the following table:

1	0.04
1.5	0.04658199
2	0.16386842
2.5	0.36871338
3	0.62146272
3.5	0.84465113
4	0.96979049
4.5	0.99930243
4.97	1

Table 3.22. *Distribution function of stochastic duration*

If we discretize this function on the set {0; 1; 2; 3; 4.97}, we obtain the following results:

Discretization of $F_{\hat{D}_A(i)}$	
1	0.040917901
2	0.322131389
3	0.475937752
4	0.154651301
4.97	0.000697566
Mean	2.735066041
Order 2 mom.	8.104534545
Variance	0.623948298
Standard-dev.	0.78990398

Table 3.23. *Discretized results*

3.11. Duration of a portfolio

Let us consider a portfolio with n assets (shares, obligations, real estate, etc.) such that there are $x_j (\text{j} = 1,...,n)$ numbers of the asset is considered on the time horizon T.

If $A(i)$ represents the present value of this portfolio at time 0, we have:

$$A(i)=\sum_{j=1}^{n}x_j A_j(i)\,. \qquad [3.117]$$

where $A_j(i), i=1,...,n$ represents the present value for the flow generated by asset *j (j=1.....n).*

If $D_A(i), D_{A_j}(i)(j=1,...,n)$ respectively represent the durations of the portfolio and of the *n* assets, the linearity of durations gives the following result:

$$A(i)D_A(i)=\sum_{j=1}^{n}x_j A_j(i)D_{A_j}(i) \qquad [3.118]$$

and so:

$$D_A(i)=\sum_{j=1}^{n}\frac{x_j A_j(i)}{A(i)}D_{A_j}(i) \qquad [3.119]$$

This duration is well a weighted mean of all the "individual" assets, relation proving once more the interest of diversification.

3.12. Conclusion

In this chapter, we have seen how the quantitative methodology of ALM concepts like duration and convexity can be extended to interact with economic and financial scenario generators. These generators influence the future cash flows of the bank or the insurance company. In this way, we have proposed a new type of generators giving the possibility to construct risk indicators extending the classical ones.

In the next chapter, we will consider a simplified case for banks.

4

Building and Use of an ALM Internal Model in Insurance Companies

4.1. Introduction

When we use an asset and liability management (ALM) internal model in an insurance company, we are essentially interested in modeling life-insurance products. Indeed, life-insurance products, unlike non-life products, are very sensitive to market risks, as well as underwriting risks. Moreover, life products have, generally speaking, a much longer maturity than non-life products.

Therefore, a life insurance company faces the challenge to meet the liabilities of its policyholder over a long period of time while, at the same time, trying to maximize its profit. However, to complicate matters further, liabilities react to changes in financial market conditions as well. For instance, policyholder behaviors can respond to market changes: if interest rates increase for a significantly long period of time, customers are more likely to surrender their life contracts to open a new one and benefit from the increased expected return. We called this behavior “dynamic surrenders”. Other examples of correlation between assets and liabilities are mechanisms such as profit sharing.

Therefore, a life insurance company must define a long-term strategic asset allocation that reflects its liabilities (and the interactions between its liabilities and its assets) and its risk tolerance which maximizes its profit. These are the goals of an ALM internal model.

Asset and Liability Management for Banks and Insurance Companies, First Edition.
Marine Corlosquet-Habart, William Gehin, Jacques Janssen and Raimondo Manca.
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4.2. Asset model

First, we will see how to model, in a simplified way, the basic asset classes that an insurer can have in its portfolio. Above all, our goal here is to be pedagogical. More sophisticated ways to model these assets exist, and we can always refine the distinction we made between the different assets and multiply the asset classes we use. However, by experience, complex ALM models are a lot less robust and intuitive than simplified models. The search for an ever increasing sophistication should not be made to the detriment of the robustness and clarity of the results.

4.2.1. *Equity portfolio*

4.2.1.1. *A simple equity model*

The simplest way to model an equity portfolio is to consider a single asset X_t that shares the same properties (expected return $r_{t,portfolio}$ and volatility $\sigma_{t,portfolio}$) as the global portfolio. The return of the asset X_t during the period $[t, t+1[$, defined as $r_t = \frac{X_t}{X_{t-1}} - 1$, can therefore be modeled by a random variable drawn from a normal distribution $N\left(r_{t,portfolio}, \sigma_{t,porfolio}\right)$. An example of this very simple model is shown in Figure 4.1.

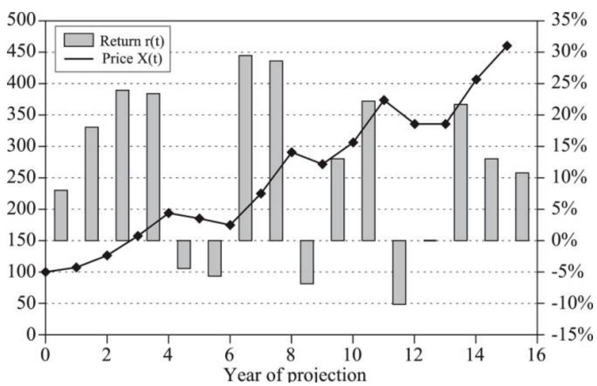

Figure 4.1. *Example of a Gaussian scenario for a stock price*

The determination of the expected return $r_{t,portfolio}$ and the volatility $\sigma_{t,portfolio}$ of the overall portfolio are generally based on historical data. It is, respectively, the average return of the portfolio and its standard deviation over a given period of time.

A common mistake and source of confusion is to use the implied volatility (and not the historical volatility) to project the future returns r_t of the portfolio. The implied volatility is defined as the volatility we must use in the Black and Scholes formula to equalize the Black and Scholes prices and the market prices of vanilla options (such as call and put options). The implied volatility is not the "market's view of the volatility" as we may quickly (and wrongly) conclude. The implied volatility is only a parameter in the Black and Scholes framework and should not be used anywhere else other than in the Black and Scholes framework.

To determine the expected return and volatility we must use our simulation; it is therefore best to use the historical estimators. Nonetheless, insurance companies usually incorporate the opinion of experts (such as macroeconomists or asset managers) in their estimators, concerning the future evolution of these indicators.

4.2.1.2. *Limitations of the model*

To describe a portfolio with many assets by a single, fictive, aggregated index, X_t is, nonetheless, a very limited model. It is only a helpful model when we do not have to distinguish between the different equities that composed the portfolio. For instance, some reserves are calculated equity by equity (like the French liquidity risk reserve), and not globally on the whole portfolio (like the French capitalization reserve).

Moreover, by considering only the aggregated index X_t, we lose the individual properties of the equities (their individual expected return $r_{t,i}$ and volatility $\sigma_{t,i}$) and the correlations $\rho_{i,j}$ between them that may be statistically significant. Thus, a more realistic model would be to use a multivariate normal distribution that takes into account the whole structure of the portfolio. However, in practice, the calibration of a multivariate model may be challenging.

Indeed, the correlation $\rho_{i,j}$ between two equities i and j is not a very robust statistic and may depend heavily on the historic used. We can use the market price of structured assets (such as basket options) to compute an implied correlation (the correlation that we must use in our pricing to equalize theoretical and market prices), the same way we can compute an implied volatility for a single equity. But not all, the couple of equities has their counterpart in the basket options market (only the biggest indexes and stocks), and the implied correlation, like the implied volatility, is generally not homogeneous to a statistical correlation but simply a parameter of a model for pricing derivatives.

When we need to distinguish different kinds of equities, a good compromise between the too simple "single index model" and the too complicated to calibrate "multivariate model", is a model where the return $r_{t,i}$ of each equity is drawn independently of its own normal distribution $N(r_{t,i}, \sigma_{t,i})$.

4.2.1.3. *Dividends*

So far we have left aside the issue of modeling dividends.

The easiest way to model dividends is by adding a fixed annual rate in the financial return of our equity portfolio that represents the average annual dividend yield. This average dividend yield is usually defined by macroeconomist studies, but can be estimated by averaging the historical dividend yields over a significant period of time.

However, under the term "equity" is hidden a large variety of financial instruments: single stocks that may or may not pay dividends, indexes that do not pay dividends, funds that usually reinvest the dividends of their shares, and so on. Therefore, the estimation of a global dividend yield must take into account the variety of the dividends management of each of the components we have in our portfolio. A global dividend yield can do the trick if the dividends management is globally homogeneous over our different assets, but can be deceptive if not.

4.2.2. *Bond portfolio*

4.2.2.1. *The CIR model*

To project a bond portfolio, we need to define a model that describes the evolution of interest rates. Among the various interest rate models that exist, we will focus on the Cox-Ingersoll-Ross (CIR) model.

The CIR model can be classified in the "one-factor model" group. The one factor models suppose that there is only one source of randomness that drives the interest rates movements. It is, in spirit, similar to a simple Gaussian equity model, where only one risk factor drives the equity price (that is, its return).

This one source of randomness in the one-factor models is called the "instantaneous interest rate", noted as r_t, and is defined as the interest rate at which we can borrow money from t to $t+\varepsilon$, ε being infinitesimally small. In the CIR model, the evolution of the instantaneous rate r_t is described by the following stochastic differential equation:

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t$$

where W_t is a Wiener process and a, b and σ are the CIR parameters. These parameters can be interpreted, respectively, as the speed of adjustment of the process (a), the mean (b) and the volatility (σ).

The CIR model can be calibrated using the least squares method to match the bond prices given by the model $P_{CIR}(t,T)$ and the observed market prices. The model price, at a time t and for a bond of maturity T, is given by the following formula:

$$P_{CIR}(t,T) = A(t,T)e^{-B(t,T)r_t}$$

where:

$$A(t,T) = \left(\frac{2h * \exp((\mathrm{a}+\mathrm{h})(\mathrm{T}-\mathrm{t})/2)}{2h + (a+h)\big(\exp\big((T-t)h\big) - 1\big)}\right)^{2ab/\sigma^2}$$

$$B(t,T) = \frac{2h * \exp((\mathrm{a}+\mathrm{h})(\mathrm{T}-\mathrm{t})/2)}{2h + (a+h)\big(\exp\big((T-t)h\big) - 1\big)}$$

and:

$$h = \sqrt{a^2 + 2\sigma^2}.$$

By discretizing the continuous differential equation of the CIR model, we obtain the recurrence relation we will use during our simulations as:

$$r_{t+\Delta t} = r_t + a(b - r_t)\Delta_\mathrm{t} + \sigma\sqrt{\mathrm{r_t}}\epsilon_\mathrm{t}$$

where ϵ_t is a random variable that follows a standard normal distribution $N(0,1)$.

The CIR recurrence relation allows us to simulate at each step t the instantaneous interest rate r_t. By inversing the bond price at time t for different maturity, we can then determine the term structure at time *t* (the zero-coupon interest rates curve) implied by the instantaneous rate r_t. Finally, we can evaluate the new value at t of our bonds portfolio.

The main advantages of the CIR model are that, on the one hand, it is a simple model that gives a closed formula to the bond prices (and therefore simplify the calibration process), and on the other hand, the factor $\sqrt{r_t}$ in the stochastic differential equation prevents the simulated rates being negative.

Nonetheless, in the economical crisis of the 2010s, the impossibility of the CIR rates being negative is paradoxically problematic. Since the subprime crisis, in European countries such as France or Germany, we have observed historically low interest rates. Therefore, the starting point of the simulated interest rates (especially short-term interest rates) is close to 0%. It is sufficiently close to the "forbidden zone" (negative rates) of the CIR model to produce undesirable effects during the projections.

Let us take an example to illustrate these edge effects that appear during low interest rate situations. Let us suppose that the calibration of a CIR model gives the following parameters: $a = 10\%$ and $b = 3\%$. First, let us take a volatility $\sigma = 5\%$ and let us suppose that the starting point of our simulation (i.e. the current interest rate level observed today in the markets) is $r_0 = 1\%$. Figure 4.2 represents 30 simulated rates over a projection of 20 years.

The simulated paths we see on the graph are not very satisfactory. Indeed, over a period of 20 years, the dispersion of our simulated rates appears to be too small: the interest rates rarely exceed 5%. Unfortunately, over a 20-year timeframe, we expect the interest rates to vary more.

The only way to increase the dispersion of our simulations, to be more realistic, is by increasing the volatility of the model. Let us now take $\sigma = 10\%$ and see what happens.

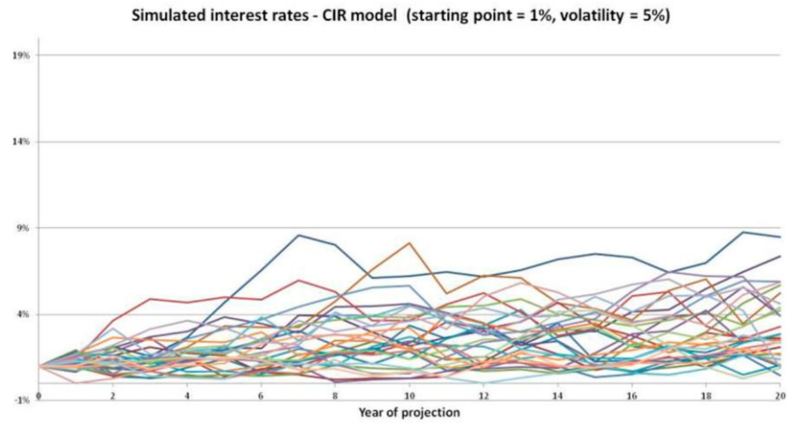

Figure 4.2. *Simulated interest rates – CIR model (starting point = 1%, volatility = 5%). For a color version of this figure, see www.iste.co.uk/corlesquet/ALM.zip*

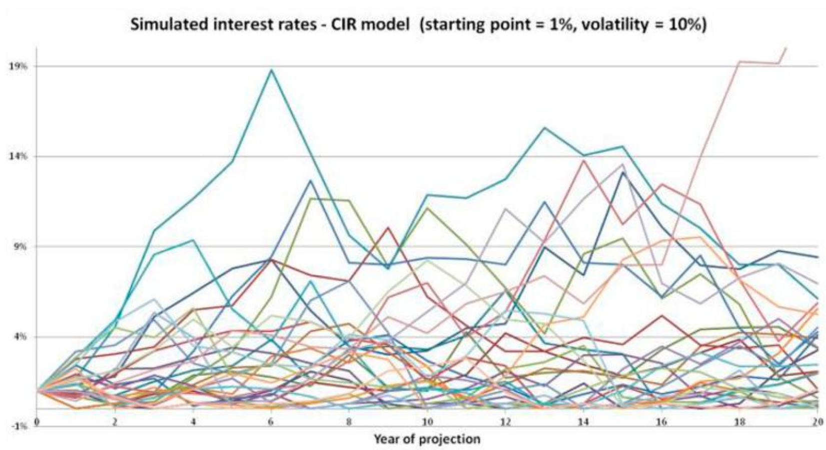

Figure 4.3. *Simulated interest rates – CIR model (starting point = 1%, volatility = 10%). For a color version of this figure, see www.iste.co.uk/corlesquet/ALM.zip*

The dispersion is now better, but notice how the majority of the scenarios crash against the *X* axis. We see here an edge effect: the simulated rates have the unfortunate tendency to be close (or equal) to 0%.

Now, if we define another starting point, for instance, $r_0 = 7\%$, and we do not change the other parameters, we obtain the situation in Figure 4.4.

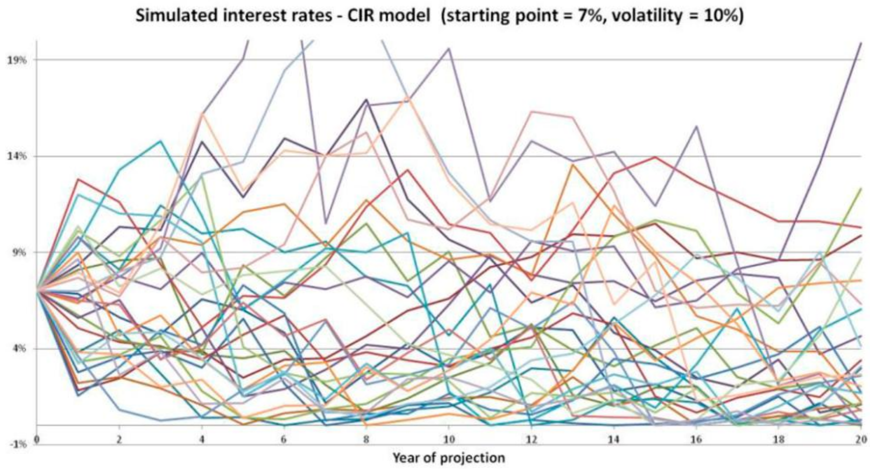

Figure 4.4. *Simulated interest rates – CIR model (starting point = 7%, volatility = 10%). For a color version of this figure, see www.iste.co.uk/corlesquet/ALM.zip*

Now, the distribution of the scenarios is more homogeneous. The edge effect has almost disappeared (but we still notice a little tendency to stay close to the *X* axis).

As we usually cannot change the starting point of our simulations (as it is normally defined as the present situation, i.e. with low interest rates), the edge effect that appears in a CIR model (and, more generally, with every other interest rate model that prevents negative rates) can be a real issue. That is why, in a low interest rate environment, and if we want a set of scenarios with a good dispersion but without an edge effect around the *X* axis, the use of models that allow negative rates can be a solution. However, this calls for a discussion on the legitimacy and the economical implications of negative interest rates that, unfortunately, far exceeds our subject.

As we have seen for the equity model, the parameters of bond models can be adjusted to incorporate the views of macroeconomists concerning the economic situation we may face in the future. Particularly, the estimation of quantities such as the future levels of the short-term rate (1-year maturity) and the long-term rate (10-year maturity), as well as their future volatilities, can be based on economical hypotheses and macroeconomical studies.

4.2.2.2. *Other interest rate models*

The CIR model, as we have seen, is widely used in insurance companies to model interest rates. But many other models exist, of which we will give a non-exhaustive introductory list in this section.

Basically, interest rates models can be divided between the "short-rate models" (such as the CIR model described before) and the "forward-rate models". Short-rate models can be further divided into one-factor models and the multifactor models.

One-factor short-rate models suppose that only one risk factor drive the level of interest rates. The difference between the one-factor models is the form of the stochastic differential equation that describes the diffusion of the instantaneous interest rate. The main one-factor short-rate models are:

– the *Vasicek model* is a very simple model that allows negative rates and has the advantage to lead to explicit formulas for pricing bonds and simple derivatives on interest rates. Its stochastic differential equation is the following:

$$dr_t = (\theta - \alpha r_t)dt + \sigma dW_t$$

– the *CIR model*, which prevents negative rates;

– the *Hull-White model*, which is a generalization of the Vasicek model by facilitating the parameters to vary with time as:

$$dr_t = (\theta_t - \alpha_t r_t)dt + \sigma_t dW_t$$

The main disadvantage of one-factor short-rate models is their lack of control over the form of the yield curve (YC) they produce. We cannot precisely calibrate the entire YC we observe at $t = 0$ with the Hull-White model (and not with the Vasicek model). Nonetheless, even with the Hull-While model, we cannot control the evolution of the entire YC, we will obtain during our simulations.

Multifactor short-rate models introduce two (or more) risk factors to model interest rates. The main advantage of these models is that they generate more realistic YC deformations than one-factor models. Nonetheless, they are a lot more difficult to calibrate.

One of the most famous multifactor short-rate models is the Longstaff-Schwartz model which supposes that the instantaneous rate follows the following dynamic:

$$dr_t = (\mu X_t + \theta Y_t)dt + \sigma_t \sqrt{Y_t} dW_t$$

where X_t and Y_t follow two independent CIR models.

The second group of interest rate models is the forward-rate models, usually referred to as the Heath-Jarrow-Morton (HJM) framework. Without entering the complex mathematics beyond the HJM framework, let us just say that it is a general framework which can model the full dynamic and deformations over time of an entire forward YC (whereas the short-rate models only describe the evolution of a single point: the instantaneous rate). Thus, HJM models facilitate a very fine parametrization and control over the generated scenarios, but the downside is that they are very hard (if not impossible) to calibrate in practice.

An insurance company, during an ALM study, usually focuses essentially on the few important maturities that are important for its management, i.e. the duration of its liabilities. The precise calibration and projection of the entire YC is not a main concern for an insurance company. This is why we tend to prefer a simple, robust and easy-to-calibrate model, such as the CIR model, over a more advanced, but more complex and tricky-to-fit, model.

4.2.3. *Real estate*

Real estate is the third major type of asset (with equities and bonds) that an insurance company manages.

Basically, real estate behaves like equities. However, it is usually less volatile and less liquid. An insurance company can quite easily buy and sell equities every day, and thus change the composition of its portfolio to match a new strategy, but it cannot do the same with its real estate. The purchase or sale of a 10-floor building on the Champs-Elysées is not something you want to do every other day. This is why, usually, we do not want an ALM study to let the real estate composition of our portfolio fluctuate too much (except if the study precisely focuses on real estate strategy).

Often, the real estate portfolio of the insurance company is considered fixed during the simulations. A fixed global dividends yield (that corresponds to the rents received from properties owned by the insurance company) is then add to the financial return of the portfolio to take into account the real estate.

4.2.4. *Central scenario and simulated scenarios*

We have seen different ways to model our classes of asset (bonds and equities). With these models, we can draw as many scenarios as we need or desire. However, there is one particular scenario that is of paramount importance in ALM: the so-called “central scenario”.

The central scenario, sometimes referred to as the “determinist scenario” as well, represents the average scenario. Let us take an example. Let us say that in our simple equity model, we used an expected average of 4% and a volatility of 14%. In the first scenario we draw, the equity price may increase by 2% the first year, and decrease by 1% the second year. In the next scenario we draw, the price may decrease by 5% the first year and by 2% the second year and so on. However, in the central scenario, the average scenario, the price will simply increase by 4% each year. All of the other simulated scenarios will revolve around the central scenario. If we take a sufficiently large number of simulated scenarios, they will converge to the central scenario.

The chart below shows an example of a simple Gaussian equity model with an expected return of 4% and a volatility of 14%. The central scenario and four different simulated scenarios are displayed. The dispersion between the simulated scenarios and the central scenario is proportional to the volatility (be careful: the standard deviation of the equity prices is not equal to the volatility, which is the standard deviation of the equity returns). Therefore, the higher the volatility is (the more randomness is introduced in the model), the bigger will be the dispersion between the simulated scenarios and the central scenario.

We can see the central scenario as a scenario where we have removed all the randomness (that is why, it is called “determinist”). In our previous example, in the central scenario, the price of the equities increases each year by 4%. It is as if we set the volatility equal to zero.

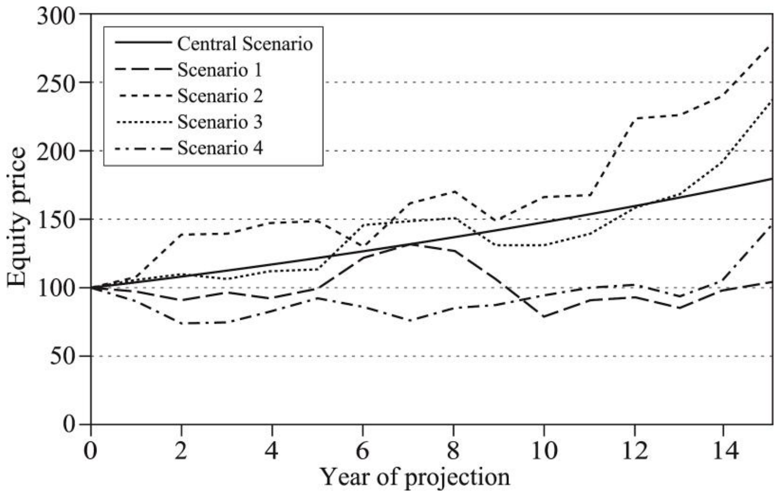

Figure 4.5. *Central scenario and simulated scenarios*

The central scenario is heavily used in ALM because it allows an insurance company to work out, with a single scenario, the average behavior of its future cash flows, according to the economic hypotheses used to determine the central scenario. That explains why the central scenario is strategic for an insurance company and is usually discussed and defined during committees that involve the board of directors, the actuarial departments, the risk managers, the asset managers and the macroeconomists.

4.3. Liability model

Let us see now how we can model the liabilities of a life insurer. For that purpose, we must first make some hypotheses to simplify the problem, without significant loss of generalization. As a consequence, we assume that there is no reinsurance. The insurer takes all the risks of its liabilities.

Moreover, we suppose that the premiums the insurer receives from its policyholders occur at the beginning of the year (on January 1st). The premiums are annual. The cash flows the insurer pays to its policyholders occur at the end of the year (on December 31st). These cash flows are either the annual benefits paid to the policyholders or the final cash flow paid when a contract ends (because of its surrender or because of the death of its owner).

Throughout this book, we only consider one kind of life insurance contract, but the model and the concepts we present here can easily be extended to other kinds of contract. Therefore, we focus on the case of a general fund.

A general fund is a life insurance contract where the net premiums are 100% guaranteed. The customers can enter and exit the fund at any moment, and they will at least receive their net premiums.

Each year, the policyholders receive an annual benefit. This benefit is usually expressed as an interest rate applied to the value of the contract. The agreement of the contract and the legislation of the country can be defined as a minimal benefit rate (called Minimal Guaranteed Rate and noted as TMG) that the insurer must give to its customers. This guarantee introduces an asymmetry of the profit and loss share between the insurer and policyholders when financial markets change.

To simplify our model, we consider that $TMG = 0$. Thus, as we will see in the next sections, the annual benefit is only a function of the asset portfolio's return and of the insurer's commercial strategy.

4.3.1. *Model points*

It would be time-consuming and memory-consuming (and not very useful in fact) to model each and every single policyholder of an insurance company. It is therefore common to use "model points".

Model points are a set of homogenous groups of contracts. The goal of model points is to define a more manageable number of aggregated contracts, whose behavior mimics that of the global portfolio.

To construct a model point, the first step is to define the discriminating variables that describe as efficiently and as completely as possible, the liabilities. For instances, such variables may be the term of the contract, the annual management fees, the profit sharing agreement, the age of the policyholders and so on.

These discrimating variables define a set of homogeneous groups. We can then put each policyholder into one of these groups and determine, for each group, the average properties that define our liabilities.

Let us take a very simple and straightforward example. Let us suppose we have the policyholders listed in Table 4.1 in our portfolio.

# Policyholder	Age	Annual Fees	Math. reserves
1	32	1,00%	16 700 €
2	78	1,00%	450 789 €
3	65	1,50%	35 309 €
4	25	1,00%	397 097 €
5	45	1,00%	87 090 €
6	37	1,50%	154 879 €
7	60	1,50%	9 864 €
8	56	1,00%	76 543 €
9	23	1,00%	34 875 €
10	49	1,50%	12 498 €

Table 4.1. *Example of portfolio*

Let us suppose that our discriminating variables are the annual fees and if the policyholder is younger or older than 50 years old. Thus, these discriminating variables define four categories: the young policyholders that pay 1.00% of annual fees (cat.1), the young policyholders that pay 1.50% of annual fees (cat.2), the senior policyholders that pay 1.00% of annual fees (cat.3) and finally the senior policyholders that pay 1.50% of annual fees (cat.4).

For instance, the first category contains four policyholders whose average mathematical reserve is € 133,941. Thus, the complete model points are simply represented in Table 4.2.

Age	Annual Fees	Frequency	Average math. reserves
< 50	1,00%	4	133 941 €
< 50	1,50%	2	83 689 €
>= 50	1,00%	2	263 666 €
>= 50	1,50%	2	22 587 €

Table 4.2. *Example of model points*

And so with this model points, we have described our liabilities in a much simpler way without significant loss of information.

4.3.2. *Mathematical reserves and annual policyholder benefits*

The liability of an insurer is divided into different reserves, such as the mathematical reserves and the profit sharing. Other reserves may exist too, depending on the legislation of the country.

The mathematical reserve is usually the biggest reserve for a life insurance company. By definition, this reserve equals the insurer's future liabilities.

In the case of the general fund, we consider in this section, the insurance company guarantees 100% of the net premiums. Therefore at the inception of the contract, when the premium is paid by the policyholder, and under the assumption that TMG = 0, the mathematical reserves equal the net premium.

The policyholder can, at any time, partially withdraw or surrender his policy. If we consider a model with new business (see section 4.3.7), then the policyholder can pay complementary premiums at any time, which must be taken into account in the mathematical reserves. We made the hypothesis that these cash flows from the policyholders (premiums, withdrawals and surrender) occur the first day of the year.

Moreover, at the end of each year, the value of the contract is re-evaluated with the annual policyholder benefits paid by the insurer and annual fees are deducted (see section 4.4).

Finally, the mathematical reserves at the end of the year N (denoted as PM_N) are given by the following formula:

$$PM_N = PM_{N-1} + Annual\ benefits + Complementary\ premiums \\ - Withdrawals - Annual\ management\ fees$$

4.3.3. *Annual policyholder benefits and crediting rate*

As we saw in the introduction, we consider a general fund that paid annual benefits to the policyholders. These benefits can be expressed as a rate, called the crediting rate, which revalues the mathematical reserves. This crediting rate is defined each year and depends on different factors, such as the financial return of the assets of the fund or the quantity of profit sharing, the insurer must redistribute to its policyholders. However, ultimately, the crediting rate is a management decision taken by the shareholders.

Strategic elements must be taken into consideration when defining the crediting rate. For instance, does the insurer want to be competitive and pay a higher rate than its competitors? Given the current financial and economical context, is it optimal to give all the financial return to the policyholders or, instead, to increase the profit sharing?

Counterintuitively, it is not always an optimal strategy for an insurer to be competitive. Let us take, for example, a general fund with a 5% bond portfolio. If the bonds available today on the markets give a lower rate than the bond portfolio of the general fund, let us say 3%, it could be a better strategy for the insurer to be less competitive than its competitors and to pay a lower crediting rate. Indeed, if the insurer pays a higher crediting rate, many customers could subscribe to new contracts and the insurer could be compelled to invest in the 3% bonds, which will lower its global bond portfolio return.

As we have just seen, the crediting rate encompasses a strategic dimension. More precisely, the crediting rate is very dependent on the competitive rate CR_t^{comp}, i.e. the average crediting rate of the competitors over a given year. To take this competitive rate into account in our model, we must find a way to model it so we can project it during our simulations. How can we model the competitive rate? The easiest way is to find a simpler variable with the same behavior.

Figure 4.6 shows the history of the French average crediting rate between 2006 and 2013.

We have considered the annual crediting rates of 68 different life insurance contracts. For each year, we have also computed the average rate of the French 10Y *Obligations Assimilables au Trésor* (OAT). OATs are government bonds issued by the French Treasury, with maturities of 7–50 years. It is usual to consider the 10-year OAT as the reference indicator in the French market for medium- and long-term instruments (such as life insurance products).

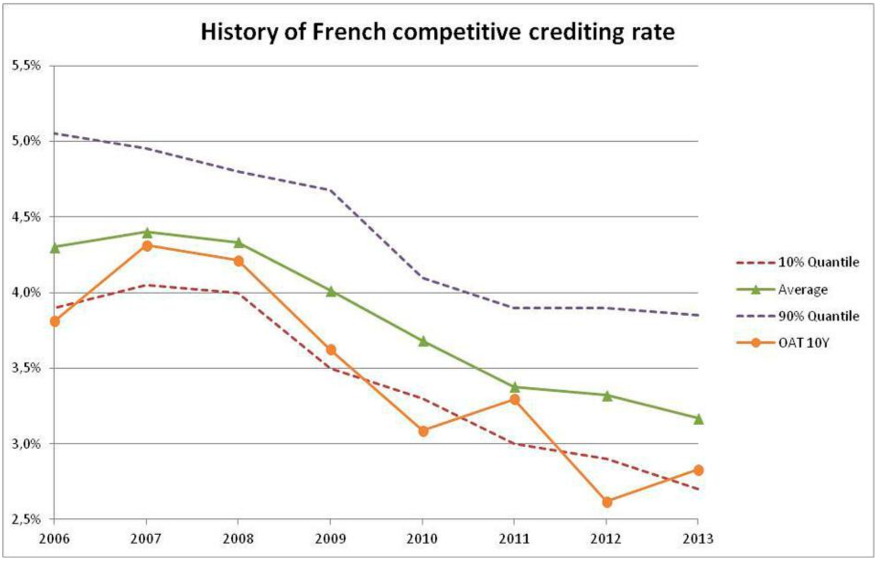

Figure 4.6. *History of French competitive crediting rate. For a color version of this figure, see www.iste.co.uk/corlesquet/ALM.zip*

In the previous graph, we have seen that the averaged 10Y OAT seems to be correlated with the average crediting rate (precisely, the correlation between the two series is 0.93). Between 2006 and 2013, the difference between the French competitive rate and the French 10Y OAT is, on average, 0.35%. We can then use a very simple model to project the competitive rate during our simulations:

$$\widehat{CR}_t^{comp} = R_t^{10Y\ OAT} + 0.35\%$$

We now introduce a new rate, named the target rate denoted as CR_t^{target}. The target rate is defined as the target, the insurer will aim to pay its policyholders during the simulations.

The way we compute the target rate corresponds to the management strategy we want to model. If we define $CR_t^{target} = \widehat{CR}_t^{comp}$, we will model the strategy to be as close as possible (in terms of policyholder benefits) to our competitors. If we model CR_t^{target} such as $CR_t^{target} > \widehat{CR}_t^{comp}$, we will aim to be more competitive than our competitors. Likewise, if CR_t^{target} is such that $CR_t^{target} < \widehat{CR}_t^{comp}$, we want to be less competitive than our competitors (and this strategy can trigger dynamic lapses).

For our model, we choose to define CR_t^{target} such that $\mathrm{CR}_t^{\mathrm{target}} = \widehat{\mathrm{CR}}_t^{\mathrm{comp}}$. We will now see how an insurer can use the profit sharing to follow as closely as possible, the strategy defined by $\mathrm{CR}_t^{\mathrm{target}}$.

4.3.4. *Profit sharing*

At the end of each year, the asset portfolio gives the insurer a nett financial return denoted by FRA_t. Let us suppose that in a given year, $CR_t^{target} = 3\%$ but $FRA_t = 4\%$. To be consistent with its commercial strategy, the insurer would want to pay its policyholders 3%, but it cannot keep for itself a part of the netted financial return, no matter what its commercial strategy is.

Now suppose that the next year, the asset portfolio has not performed as we would have expected and that $FRA_{t+1} = 2\%$. We suppose that we still have $CR_{t+1}^{target} = 3\%$. In this case the financial return is not enough to pay the policyholders the target rate, whereas the previous year, the insurer had a financial surplus that it would be glad to use now.

With this simple example, we see that the volatility of the asset portfolio's return translates into the same volatility in the crediting rate, leaving the insurer with very poor control over what it can pay the policyholders.

Profit sharing was introduced to avoid the situation we have just described. It is a reserve that the insurer can use to smooth its crediting rates over the years. The first year of our example, all the insurer can put in provision is the 1% financial surplus it has got. Thus, it pays the target rate and follows its commercial strategy, but moreover it puts money into its profit sharing that it could use during the years where the financial return is too low to pay the target rate. So, the next year in our example where the insurer lacks 1% to pay the target rate, it can use its profit sharing to supplement the financial rate.

The profit sharing is not the insurer's, but is owed to the policyholders. In some countries, the legislation can define a maximum number of years the insurer must pay what it puts into the profit sharing a given year. For instance, French insurers must pay the profit sharing to its policyholders within 8 years.

Usually insurance companies want to set min and max limits to their profit sharings. They want a minimal amount of profit sharing any year to protect themselves against an unexpected bad financial return, without exceeding a certain threshold. In France, the amounts of profit sharing of insurance companies are public and too high profit sharing is the sign that an insurance company remunerates its policyholders poorly.

If we denote by PS_t the profit sharing for the year t, by PS^{Min} and PS^{Max}, respectively, the minimal and maximal level of profit sharing defined by the insurance company, then the equations that for a given year t give the crediting rate CR_t paid by the insurance company to its policyholders and the amount of profit sharing are the following:

$$CR_t = \begin{cases} CR_t^{target} + max\left(PS_{t-1} + FRA_t - CR_t^{target} - PS^{max}; 0\right), if\ FRA_t \geq CR_t^{target} \\ FRA_t + \min\left(CR_t^{target} - FRA_t; PS_{t-1} - PS^{min}\right), if\ FRA_t \leq CR_t^{target} \end{cases}$$

$$PS_t = PS_{t-1} + FRA_t - CR_t$$

4.3.5. *Policyholder demography and behavior*

Now that we have seen how we can model some of the financial mechanisms (such as the crediting rates and the profit sharing) that lie beyond a standard life-insurance product, we want to model the population changes of our liability portfolio.

Basically, a life-insurance contract closes for two reasons: either because the policyholder decides to surrender his contract or because the policyholder has died. When a contract closes, the mathematical reserves of our portfolio decrease by the value of this contract. We will now see how we can model these two kinds of event.

Static laws: mortality and structural surrenders

The first step to model these events (mortality and surrender) is usually to define statistical laws based upon the historical data of the insurer.

These laws give the average annual rate of surrender $q_{surrender}^{static}(x)$ of a contract given its seniority x, and the average annual rate of mortality $q_{mortality}^{static}(x)$ of a policyholder given its age x. Figure 4.7 represents an illustrated example of a contract held by a 55-year-old man.

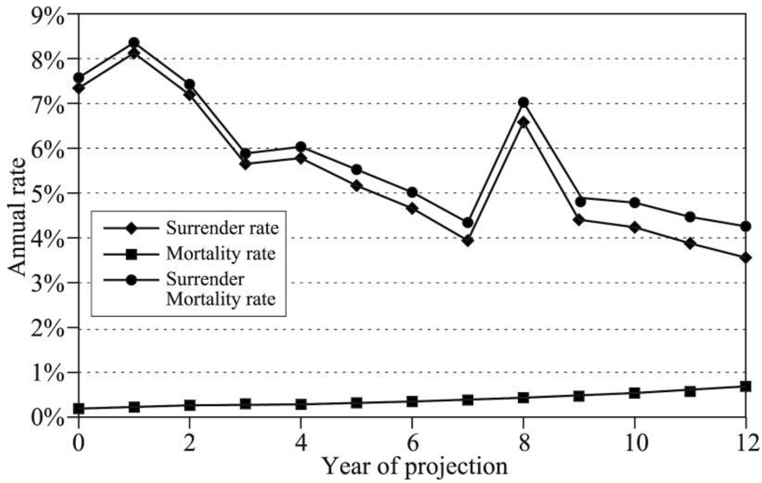

Figure 4.7. *Annual surrender and mortality rate*

At an individual level, these rates give the probability that a given year a policyholder will surrender his contract or pass away. However, at an aggregated level, these laws give the rate at which the mathematical reserves annually decrease.

It is useful to distinguish between the mortality rates and the surrender rates. Indeed, usually life insurance products have some Guaranteed Minimum Death Benefit (GMDB) options that guarantees a minimal contract value for the beneficiary in case of the death of the policyholder. Therefore, in case of the death of an insurer, the insurer may pay an additional cash flow. Moreover, insurers, in general, evaluate distinctly the management costs of surrender and the management costs of mortality.

However, for our purposes, as we want to keep things simple, we will use a global annual rate of reduction of the mathematical reserves $q_i^{static}(t)$ for our year of projection t. These rates are simply the sum for each year of our simulation of the mortality rates and the surrender rates (see Figure 4.7) for a given row i of our model point.

Dynamic surrenders

When we estimated the static laws in the previous section, we did not consider the interactions that may exist between the financial markets and our cash flows of liabilities. Indeed, there may be a correlation between the changes in interest rates and the surrenders, an insurer might face.

If the interest rates increase significantly and substantially, a new fund that was benefited from the high return bonds can outperform the old funds, the insurers have offered so far to their policyholders. Therefore, it may be optimal for a policyholder to surrender its old contract and invest in the new fund to benefit from the better market conditions. This behavior is called dynamic surrenders and expressed the fact that the rates of surrender should not only be static, estimated by averaging the surrenders we observed in the past, but should also depend on the difference between what an insurer can pay to its policyholder (the return of its fund) and the level of interest rates observed in the market.

Dynamic surrenders raise the problem that, so far, they are only theoretical and we have yet to observe in practice a sufficiently high increase of interest rates that might trigger them. Therefore, insurers do not have any statistics to model them. We do not know what the behavior of the policyholders would be, if the interest rates were to suddenly increase.

The French prudential supervisory authority has defined a modelization for the dynamic surrenders $\boldsymbol{q_{surrender}^{dynamic}(t)}$ that insurers should use:

$$q_{surrender}^{dynamic}(t) = \begin{cases} 40\%, & CR_t^{paid} - R_t^{10Y\ OAT} < -4\% \\ 40\% \times \dfrac{CR_t^{paid} - R_t^{10Y\ OAT}}{-4\%}, & -4\% \leq CR_t^{paid} - R_t^{10Y\ OAT} < 0\% \\ 0\%, & 0\% \leq CR_t^{paid} - R_t^{10Y\ OAT} < 1\% \\ -4\% \times \dfrac{CR_t^{paid} - R_t^{10Y\ OAT} - 1\%}{3\%}, & 1\% \leq CR_t^{paid} - R_t^{10Y\ OAT} < 4\% \\ -4\%, & CR_t^{paid} - R_t^{10Y\ OAT} \geq 4\% \end{cases}$$

To keep our model simple, we would not model dynamic lapses.

4.3.6. *Other reserves*

In this section, we will give a description of the main reserves a French insurance company must have. This list is non-exhaustive. Other reserves exist and must be calculated by an insurance company, but we present here the reserves that are usually the most important in cash amounts for a life insurance company.

The definitions, evaluation and mechanisms of allowance of these reserves, which we present here, correspond to the French regulatory rules. For other countries, differences in regulatory rules may be observed. We will not model these reserves in our model.

Reserve for permanent depreciation

French accounting rules require insurance companies to record their assets by acquisition value (and not by market value, as Solvency II norms recommend for instance). Therefore, there may be a notable difference between the market value of an asset (its real value at a time t) and the recorded value (its value when the company acquired it) that can lead to significant losses for the company.

If an asset depreciates (i.e. its market value decreases significantly and durably), reserves such as the reserve for permanent depreciation or the liquidity risk reserve facilitate an insurance company to account for these unrealized losses.

The reserve for permanent depreciation (in French *Provision pour Dépréciation Durable* or PDD) corresponds to a durable depreciation of an asset defined by the article R. 332-20 of the French insurance code. Such assets are for instance equities, real estate or loans. A durable depreciation is defined as an unrealized loss of at least 20% (or 30% during periods of high volatility on the markets) over at least 6 months. It is the French regulator that decides if we are in a high volatile period or not.

Therefore, the reserve for permanent depreciation evaluates asset by asset and, if there is a durable depreciation of an asset, it is equal to the difference between the acquisition price and the recovery value. The way to estimate the recovery value of an asset is defined, for each kind of asset, by the insurance code.

For instance, the recovery value of a stock is defined as its market price revalued at the risk-free rate during the estimated time of detention by the company. For a property, like a building, the recovery value must be estimated by expertise. Each class of assets has its own definition of a recovery value.

If the reserve for permanent depreciation is to be included in an ALM model, as it is evaluated asset by asset, each asset present in the portfolio of the company should be modeled and simulated separately. However, as it is a time-consuming and fastidious task, it is usual to aggregate the assets in homogeneous classes of assets, in the same way that we aggregate the liabilities in a model points.

Liquidity risk reserve

Whereas the reserve for permanent depreciation is evalued asset by asset, the liquidity risk reserve is evalued globally, over all the R 332-20 assets owned by an insurance company. A liquidity risk reserve is established if a company observes a global unrealized loss over all its R 332-20 assets.

If the insurance company satisfies various criteria of solvency defined by the regulator, the amount of the liquidity risk reserve is equal to a third of the global unrealized loss. If not, the reserve then represents simply the totality of the unrealized loss.

Capitalization reserve

The capitalization reserve focuses on the assets defined by article R 332-19 of the French insurance code. These assets are mainly the different kinds of bonds.

When an insurance company sells a bond, the realized loss or profit impacts the capitalization reserve. If the selling price is higher than the accounted price, which corresponds to a profit, the company must put in the capitalization reserve the profit made. If the selling price is lower than the accounted price, the loss must be covered by the capitalization reserve.

The capitalization reserve aims at smoothing the financial results when an insurance company sells bonds before their maturity.

If there was no a capitalization reserve, insurance companies would be tempted to sell their high-coupon bonds to realize profits when interest rates decrease, and then buy low-coupon bonds which would result in a lower crediting rate for the policyholders. However, as the capitalization reserve exists, insurance companies cannot use interest rates movements to make profits to the detriment of its policyholders.

The accounting rules for the capitalization reserve are different than those of the other reserves we have seen. Indeed, the capitalization reserve is a part of the own funds of the company and is taken into account in the solvency margin.

4.3.7. *Future new business*

One choice of modelization we need to make is to decide if we take into account in our projections future business (new clients that enter the fund or old clients that pay complementary premiums) or if we simply project our portfolio in run-off (i.e. without future new business). In fact, this decision depends on the finality of our simulations.

If we want to determine whether a new product or a new business will be profitable and if we need to assess its risks, then we should consider new productions and future clients. But if, for instance, we want to determine the current situation of our products (with its current management strategy and its current liability structure), then the portfolio we project must be in a situation of run-off. This is typically the case in the Solvency II framework.

The future new business is generally quantified with the sales and marketing department of the insurance company. In the case of insurance products commercialized for a significant number of years, the future new business can easily be estimated by looking at the sales history. However, if we consider a product that is new to the company, the estimation and projection of its business plan can be tricky and deceptive.

We should be careful when using estimations of future new business. It is not unusual to find out years later that sales or marketers had been a little too optimistic when estimating future production. It is therefore necessary for the ALM department to determine how the risks and rentability of a product are sensible to future production. This is done by “shocking” the business plan (i.e. by significantly increasing or decreasing the future production), running the simulations with this new business plan and examine how the results vary.

4.3.8. *Fees and business costs*

Policyholders pay fees to open, own and close an insurance contract. Symmetrically, there are costs for an insurance company to market, sell and manage insurance products.

Fees and costs are, in the end, what determines whether an insurance product is profitable or not. Unfortunately, the results of an ALM study are therefore very sensitive to the fees and costs used. The problem is that, as we will see, the estimation of fees and costs is not an easy and straightforward task for an insurance company, as we would have thought at first sight.

Fees

The usual fees stipulated by a life insurance contract are the subscription fees, the fees on complementary premiums, the annual management fees and the transaction fees (when the policyholder arbitrates between different assets).

One of the main reglementary constraints when writing up a life insurance contract is that the fees must be clearly defined and visible. The fees must also be listed in the summary of the contract (the summary of a life-insurance contract is a strictly regulated one-page description of the contract that must be delivered to the policyholder before the subscription). Any hidden fee can make the contract invalid.

Although the fees are explicitly defined in the contracts, it is common, especially for the private bank customers, that the policyholder at the subscription of the contract has negotiated the fees. Thus, the real fees are often very different from the fees originally stipulated by the contracts.

As the profitability of a product is very sensitive to its fees, fees must be taken into account when describing the liabilities of an insurance company. Therefore, the commercial practice to negotiate fees complexifies the creation of model points because we may find out that there are as many fee structures as there are policyholders.

Moreover, it may sometimes be very difficult, if not impossible, to know precisely what the real fees paid by the policyholders are. With some databases or management systems used by insurance companies, sometimes the actuarial departments simply do not have this information. In this case, the actuary has two options. The first option is to use the fees stipulated by the contracts, the other option is to ask the accounting department for the raw data of the payments of every policyholder. The first option is the easiest and fastest solution, but can lead to an overestimation of the real profitability of a product. The second option can be very tedious but is a lot more accurate.

Business costs

Business costs include all the expenses an insurance company must face to market, sell and manage a specific product. The main costs are the following:

– The *conception costs*: represent all the expenses tied to the creation of the product, such as the costs of preparatory studies (marketing, actuarial, ALM, etc.), the costs of writing and designing a new kind of contract, the costs of IT setup to be able to manage a new product, training of financial advisers and insurance brokers, advertising costs, and so on.

– The *acquisition costs*: these are the costs an insurance company must pay for every new contract. To sell a contract, you must pay an adviser of a broker that will call a customer or receive him in their office. For private bank products, the acquisition costs can be quite high.

– The *closing costs*: represent the management costs when a contract closes (either by lapse or by the death of its policyholder). For instance, when a policyholder dies, the insurance company is legally compelled to search for any living beneficiaries.

The evaluation of business costs tied to a specific product is a real challenge for an insurance company. Indeed, all the expenses a company makes are global and must be broken down between the different products and sales networks the company manages. This is generally not straightforward, and often arbitrary.

An insurance company can find various ways of splitting up its costs between its products and departments. But the profitability of a product is so cost sensitive that you can make a profitable product non-profitable (and vice versa) just by changing the way the company breaks down its costs.

When studying the profitability of a product or a business, costs are essential inputs of any ALM model, sometimes even more important than the financial inputs such as the central scenario, the equity volatilities, and so on. Therefore, to produce a profitability study as complete as possible, it is necessary to assess the sensitivity of the results to the costs, by running the model with different cost levels.

4.4. Structure of an ALM study

The elements present in an ALM study may depend on the goal of the study (Solvency II study, allocation study, profitability study, and so on) and may vary from one insurance company to another. However, there are a certain number of elements that are almost always present in an ALM study.

In this section, we will describe the main structure of an ALM study. Typically, ALM studies are divided into two parts: the deterministic study and the stochastic study.

4.4.1. *Determinist study*

In the deterministic part of an ALM study, we use only one scenario to do our calculation: the central scenario.

Results with the central scenario

It is usual to begin an ALM study by taking the current situation of the considered business as the starting point, and by projecting the assets and liabilities by using the central hypothesis. The horizon of projection of the study depends on the kind of products studied. For a life-insurance product, it is usual to project until the product expires (until every policyholder has closed its contract) or, at least, to use a projection frame sufficiently large to assess the majority of the liabilities outflows.

The indicators that describe the results of an ALM study vary greatly from one company to another. Even more, it is usual for an insurance company to define its own indicators of risk and profitability, based on its lines of business and the preferences of its board of directors.

Nonetheless, classic ALM indicators are almost always shown in any ALM study. Some of these classic indicators include:

– The *present value of future profits* (PVFP): it is the sum of the discounted annual profit and loss of the considered business:

$$PVFP = \sum_{t=1}^{T} \frac{PL_t}{(1+r)^t}$$

where PL_t is the profit of the loss of the business during the year t, and r is the discounted rate used.

Let us suppose that for the first 3 years of business, the profits and losses on a given insurance product are the following: € -10 million, € -1 million, € 5 million, and let us suppose that the discounted rate is 3%, then the PVFP over these 3 years is € -6,08 million (not a very profitable product).

– The *internal rate of return* (IRR): is defined as the discounted rate we must use to make the PVFP equal to the value of cash invested. The IRR is an indicator of the profitability of a product. It is usual to add to this IRR the return the company supposes to make on the own funds that corresponds to this product.

Each company defines its own minimum acceptable rate of return to discriminate between profitable and non-profitable businesses.

– The *cost of capital*: represents the solvency capital requirement for the considered business. The cost of capital is calculated differently under Solvency I and Solvency II norms. The two measures are usually shown in an ALM study. The cost of capital is often expressed as a percentage of the mathematical reserves. If, for instance, a business has a cost of capital of 3%, then for every € 1 of mathematical reserve, the insurance company must constitute a provision of 3 cents.

– The *client value*: each company has its own definition of how to quantify the client value of a product. The client value of a life-insurance product lies mainly in the expected future crediting rates. Therefore, it is common to define the client value as an average of the expected crediting rates over a given number of years.

Stress tests

One of the main concerns of an ALM study is usually to identify and quantify the main risks on an insurance product or a business. One simply, but very effective, way to do that is by stressing the central hypotheses of the study.

The usual stress tests that are relevant to an insurance company are:

– a rise in interest rates (e.g. +2%);

– a drop in interest rates (e.g. -1%);

– a stock market crash (e.g. -20%);

– an increase of the business costs (e.g. costs x2);

– an increase of lapses (e.g. lapses x 3);

– bad selling (e.g. -200% over the business plan of future production).

These stress tests are useful to determine the sensitivity of the profitability and the client value to these main risks.

Duration matching and cash flows matching

The main financial risk of a life-insurance product is typically the interest rate risk. To protect themselves against such risk, insurance companies want their assets to hedge the interest rate risk of their liabilities. This is called *immunization*.

Immunization describes a situation where a liability and a supportive asset portfolio are insensitive to (small) changes in interest rates. The two common techniques of immunization are duration matching and cash flow matching.

As we have seen in Chapter 3, duration matching simply corresponds to the situation where the global duration of the asset portfolio matches the global duration of the liabilities. The duration is a measure of the sensitivity of an instrument to interest rate changes. Mathematically, the duration D of an instrument which value $V(r)$ depends on the interest rate r is defined as follows:

$$D = -\frac{1+r}{V}\frac{dV}{dr}$$

The sensitivity S to the interest rate of this instrument is then defined as:

$$S = \frac{D}{1+r} = -\frac{1}{V}\frac{dV}{dr}$$

The duration matching technique implies investing in assets such as $\boldsymbol{D}^{\boldsymbol{assets}} = \boldsymbol{D}^{\boldsymbol{liabilities}}$. By definition, duration matching is thus a first order hedge against the interest rate risks. It protects the insurance company against a translation of the entire YC, or against very small local changes in interest rates.

Cash flow matching is the second main technique of immunization. Cash flow matching means matching the cash outflows (the payments to the policyholders) with the cash inflows (the cash flows of the assets such as bond coupons). This implies that the frequency and amount of cash inflows match those of the cash outflows.

Insurance companies usually never exactly match cash inflows and cash outflows. Instead, they monitor their liquidity gaps. A liquidity gap $Gap(t)$ at a time t is defined as the difference between the total cash inflow $CF^{In}(t)$ received and the total cash outflow $CF^{out}(t)$ paid at time t:

$$Gap(t) = CF^{in}(t) - CF^{out}(t)$$

A positive liquidity gap means that the asset inflows are sufficient to cover the company's liabilities. A negative liquidity gap implies that the asset inflows would not fully fulfill the liabilities and the insurance company would then have to use its own funds. An example of a cash flow projection, with the corresponding liquidity gaps, is shown in Figure 4.8.

For new products, or after applying a stress test, it is usual to observe negative gaps during the first years of the projection. A useful measure is therefore the cumulative liquidity gaps, which are simply the sum of the gaps of the previous years. Indeed, it is important to know when the equilibrium is reached, i.e. when the cumulative gap becomes positive. In the above example, from the eighth year, the cumulative gap is positive and therefore the previous cash inflows have finally covered the cash outflows.

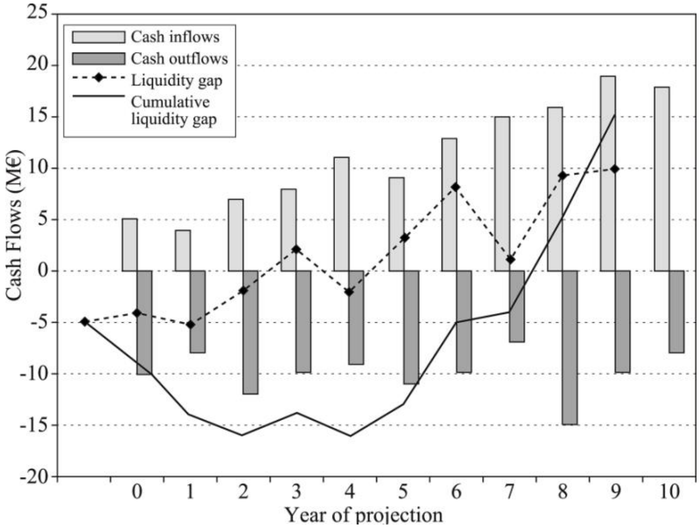

Figure 4.8. *Cash flow projection*

4.4.2. *Stochastic study*

So far we have only considered one scenario: the central scenario. With the deterministic study, we have projected the current business under the central hypothesis and we have obtained various indicators of risk and profitability. We have also determined the sensitivities to different risks of our product by applying a set of stress tests. Finally, with duration matching and cash flow matching techniques, we have defined the duration and the cash inflow structure; our portfolio must hedge us against changes in interest rates.

After the deterministic study, the second time of an ALM study is usually the stochastic study.

During the stochastic study, we use the asset models to generate a set of asset scenarios. The number of scenarios used is a compromise between a satisfying convergence of the results and the time required to make the computations.

By running the ALM model with numerous asset scenarios, we can assess the risk distribution the company faces. And so, we can determine if this risk does not exceed the risk limits defined by the company.

Usually, the risk is measured by the Value-at-Risk (VaR) of the distribution of the PVFP at a certain level. For instance, the 90%-VaR of the PVFP corresponds to the minimal loss the insurance company faces in the worst 10% of scenarios. Each company defines a risk appetite (its risk target) and a risk tolerance (the maximum limit of risk the company is willing to take).

One of the main goals of an ALM study is to define the best portfolio allocation that will maximize the profits (i.e. the mean of the PVFP distribution) of the insurance company, maximize the client value (i.e. the mean of the crediting rates distribution) and still be in line with the risk policy of the company.

To do so, we test different portfolio allocations. Each allocation is then usually represented in a graph, with the profits of the company (mean PVFP) on the *X* axis and the risk taken (90%-VaR PVFP) on the *Y* axis. Such a graph typically has the form displayed in Figure 4.9, where the equity part varies from 0% to 50% (see labels beside each point).

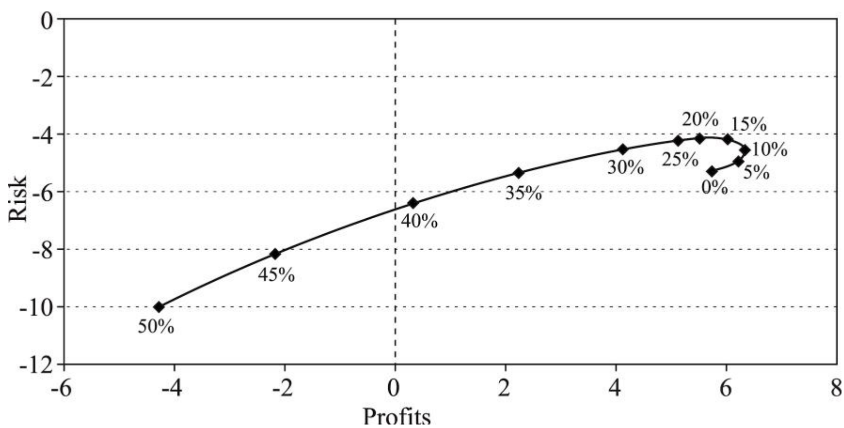

Figure 4.9. *Portfolio allocations*

From 0 to 10% of investment in equity, the profits increase and the risk decreases. But when there is more than 10% of equity in the portfolio, we observe that with the increase of equity the risk increases but, nonetheless, the profits decrease.

This inflection of the risk-expected return plot is called the *efficient frontier*. The efficient frontier regroups the allocations that maximize the risk/profit ratios. In our example, the efficient frontier is composed only of a single point that corresponds to a portfolio invested at 10% in equity.

Now that we have seen how to theoretically construct an ALM study, it is high time to put all these concepts in practice and to run our model in real situation. In the next section, we will do so, by constructing, step by step, an ALM study of a very simple life product.

4.5. Case study

4.5.1. *Goal of the study*

An insurance company wants to launch a new life-insurance contract. The product is a general fund that guarantees 100% of the net premiums. There is no other guarantee (no GMDB, for instance). The product will be sold over 5 years. The main distribution network will be the private banks.

The company wants to know the profitability of this product and to define the best investment strategy for its asset portfolio. The ALM department will therefore do a study to answer these questions.

First, we explain the hypothesis we will use during our study. Then, step-by-step, we describe how we have implemented in Excel the model we have developed during this chapter. Finally, we give the results of our study.

To simplify the display of our results, the horizon of our projections is limited to 20 years.

4.5.2. *Business plan and other liability inputs*

Business plan

The marketing department expects the following business plan during the next 5 years production:

–2,000 new subscribers per year;

– average gross entry premium: € 500,000 per subscriber;

– average gross complementary premiums: € 50,000 per subscriber per year.

Mortality and surrender laws

The actuarial department has estimated the statistical mortality and surrender laws to be expected with this new product, given the private bank clients' observed behavior and the expected average age of the subscribers.

To simplify our numerical applications, these laws can be resumed into a single annual decreasing rate of -10% to be applied to the mathematical reserves.

Profit sharing and other reserves

The insurance company wants to restrict the profit sharing between 0 and 4%. Therefore, in our model, we will use $\mathrm{PS}^{\min} = 0\%$ and $\mathrm{PS}^{\max} = 4\%$.

We would not model the complete mechanism of the liquidity risk reserve. But we must nonetheless take into account the market risk on this product. To do that, we define a "pseudo" liquidity risk reserve: if in a given year, the equity return is negative, the company must put in reserve a third of the unrealized loss. Therefore, a third of the unrealized annual equity loss is recorded as a loss for the company and influences its annual profit.

4.5.3. *Central scenario and other asset inputs*

The company's macroeconomists' financial forecasts are the following:

– *Equity*: the macroeconomists anticipate an expected return on equity of 6%, with an annual volatility of 17%. We suppose that the annual dividends rate is 3%.

– *Interest rates*: the company will only invest on French 10Y OATs. Therefore, we will only model the 10-year interest rate. For this maturity, the macroeconomists' forecasts are an expected rate over the long-term of 3% (parameter a of the CIR model) with an annual volatility of 10%. The parameter b of the CIR model is calibrate using the data over the three previous years and is estimated equal at 3%. The French 10Y OAT was 2.4% January 1st, 2014, which will be our starting point for our CIR projections. We suppose furthermore that the 10Y OAT pays a 3% coupon.

To simplify its management mandate, and to limit costs, the company does not want to invest in real estate.

Due to the everyday management transactions of the company's portfolio managers, we suppose that each year, 10% of the profit and loss on equities is realized. This is called the *equity turn-over*.

Finally, the company's preliminary studies defines a strategic investment goal of a portfolio invests at 20% in equities. This will be our central allocation hypothesis.

4.5.4. *Fee and cost hypotheses*

The fees stipulated by the contract are 2% on premium (entry premiums and complementary premiums) and 1.5% annually (management fees).

The accounts department has estimated the following costs concerning this new product:

– *Conception cost*: the costs for the marketing, the advertising and the training of the private bank advisers add up to € 1 million. This conception cost will be uniformly redeemed over the 5 years of marketing.

– *Acquisition costs*: € 1,000 per new subscriber.

– *Closing costs*: € 500 per closing contract.

4.5.5. *Step-by-step model*

Now that we have defined the hypotheses and parameters we need to run our model, it is time to implement it step-by-step in Excel.

Production

The first thing to do is to assess the movements of our policyholders: the new subscribers, the complementary premiums, the closing contracts, the fees and costs. Ultimately, we want to know how your mathematical reserves change. The formulas used in Excel are shown in Figure 4.10.

The names of the variables are self-explanatory. When not shown, the formulas have been dragged horizontally.

To estimate the annual benefits, we must first assess the annual financial return of our asset portfolio.

	D	F	G	H
3	Production	Year 1	Year 2	Year 3
4	Number of clients	=new_subscribers	=F4*(1-decreasing_PM_rate)+new_subscribers	5 420
5	Gross entry premiums	=new_subscribers*entry_premium	1 000 000 000 €	1 000 000 000 €
6	Gross complementary premiums	=new_subscribers*complementary_premium	190 000 000 €	271 000 000 €
7	Total new gross premiums	=F5+F4	1 190 000 000 €	1 271 000 000 €
8	Fees on premiums	=F6*premium_fees	23 800 000 €	25 420 000 €
9	Total new net premiums	=F6-F7	1 166 200 000 €	1 245 580 000 €
10	Conception costs	=conception_costs/5	200 000 €	200 000 €
11	Acquisition costs	=acq_costs*new_subscribers	2 000 000 €	2 000 000 €
12	Closing costs	=F4*decreasing_PM_rate*closing_costs	190 000 €	271 000 €
13	Math. Reserves beginning of year	=F8	=F15+G8	3 216 751 290 €
14	Management fees	=F11*management_fees	32 296 635 €	48 251 269 €
15	Annual benefits	=F11*F29	65 669 825 €	98 110 914 €
16	Withdrawals	=F11*decreasing_PM_rate	215 310 900 €	321 675 129 €
17	Math. Reserves end of year	=F11-F12+F13-F14	1 971 171 290 €	2 944 935 806 €

Figure 4.10. *Excel spreadsheet for the production*

Assets

We present here how to simulate the annual performances of our asset portfolio under the central scenario hypotheses. The way we compute it in Excel is displayed in Figure 4.11.

	D	F	G	H
21	Assets	Year 1	Year 2	Year 3
22	Equity scenario	=eq_return	6,00%	6,00%
23	10Y OAT scenario	=ir_expected	2,70%	2,70%
24	Dividends (€)	=F13*eq_allocation*dividends	12 918 654 €	19 300 508 €
25	Coupon (€)	=F13*(1-eq_allocation)*ir_coupon	51 674 616 €	77 202 031 €
26	Realized equity profit and loss	=F13*(F22+dividends)*eq_allocation*eq_turnover	3 875 596 €	5 790 152 €
27	Financial return	=(F22*eq_turnover+dividends)*eq_allocation+ir_coupon*(1-eq_allocation)	3,12%	3,12%

Figure 4.11. *Excel spreadsheet for assets*

"Eq" stands for "Equity" and "IR" for "Interest Rate". The nominal of the fund is represented by the mathematical reserves (nonetheless, this is true only because TMG = 0). Therefore, the investments the company can make in a given year amount to the level of mathematical reserves of the fund.

The investments are split between equities and bonds by the proportion we have defined in our central scenario. We will test different allocations (i.e. different proportions bonds/equities) during the stochastic study.

Liabilities

Now that we have the annual financial returns of our assets portfolio, we can estimate the annual benefits. We first define the crediting rate target, and then we deduct the actual crediting rate and profit sharing from it.

	D	F	G	H
29	**Liabilities**	**Year 1**	**Year 2**	**Year 3**
30	Crediting rate target	=E23+0,35%	3,05%	3,05%
31	Crediting rate	=IF(F27>F30;F30+MAX(E32+F27-F30-PSmax;0);F27+MIN(F30-F27;E32-PSmin))	3,05%	3,05%
32	Profit sharing	=E32+F27-F31	0,14%	0,21%
33	Pseudo liquidity risk reserve	=IF(F22<0;-1/3*F22*eq_allocation;0)	0,00%	0,00%

Figure 4.12. *Excel spreadsheet for liabilities*

We also compute the “pseudo” liquidity risk reserve we have introduced to take into account the equity market risk for the company. With the crediting rate, we can compute the annual benefits and therefore evaluate the mathematical reserves at the end of the year.

4.5.6. *The ALM study*

Now that we have defined the different hypotheses we needed and implemented our model in Excel, we have everything in hand to do our ALM study. The first step of any ALM study is usually the deterministic study where we analyze our product under the central hypotheses.

The deterministic study

– Central scenario

Let us begin our study by projecting the future cash flows of our product under the central scenario and by the analysis of the liquidity gap we obtain. The cash flows projection for the central scenario is represented in Figure 4.13.

The liquidity gap is positive for all the projected years. This is not surprising, as the central scenario supposes that every year equities grow at 6%. Therefore, with such good market conditions, the cash inflows (coupons, dividends and realized equity profits) can easily cover the cash outflows (annual benefits).

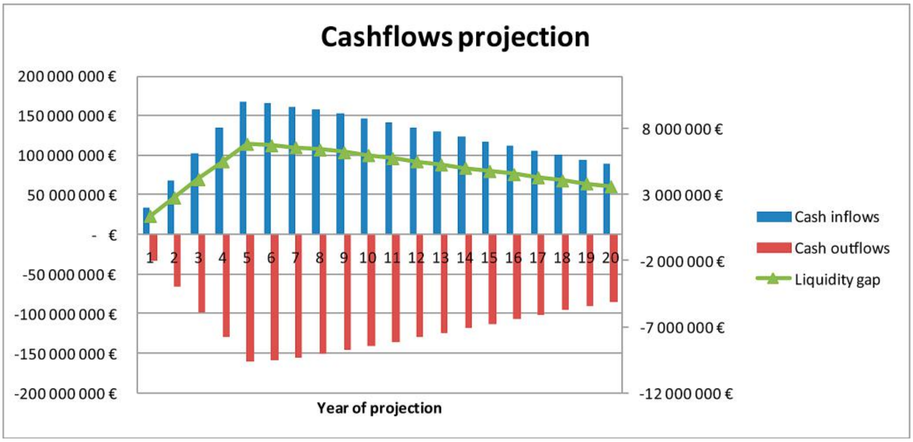

Figure 4.13. *Cash flow projections*

As the hypotheses of the central scenario are constant, with an equity allocation of 20%, the annual financial return for any given year is 3.12%, whereas the crediting rate target is 3.05%. Therefore, the profit sharing increases each year by 0.07%.

The PVFP of the company is € 1,027 million and the profitability is 8.9%. In the central scenario, the product is thus quite profitable.

– Stress tests

Let us now see how our model reacts when we stress some of the hypotheses we used.

Let us suppose first that the equity market performances drop: the expected return on equity is not 6% but 3%, and the dividend rate is not 3% but 2%. The annual financial return of our asset portfolio is now 2.86%, less than the crediting rate target of 3.05%. If we want to be competitive, we must therefore inject cash into our fund, which results in a loss for the company. The PVFP of the product is then € 907 million and the profitability drops to 6.5%. The cash flow projection is shown in Figure 4.14.

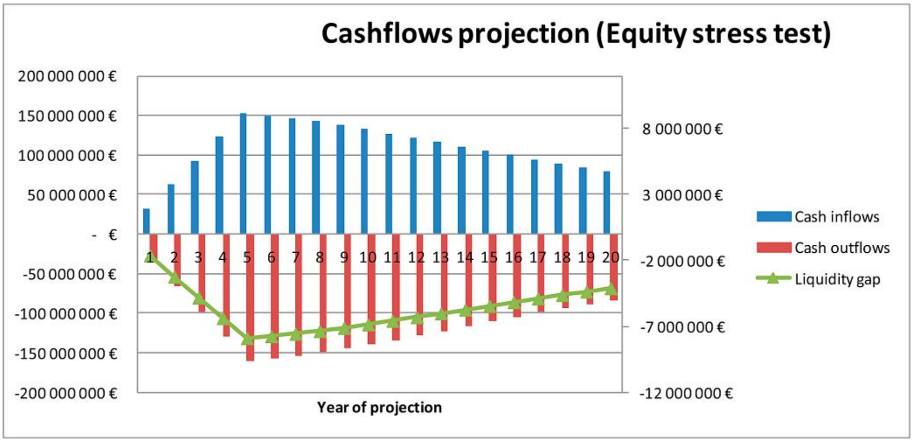

Figure 4.14. *Cash flow projection (equity stress test)*

The liquidity gap is now always negative. This shows that market risk is a major risk for a new general fund if an insurance company wants to be competitive with historical general funds (i.e. if the company wants to pay at least the market crediting rate) that has older and higher-yield bonds.

Let us consider now a stress test on the business plan. Let us suppose that the production is half what has been expected (there are still 2,000 clients a year, but the average premium is only € 250,000) and the different business costs are two times more important. In this case, the PVFP of the company drops by almost 50% and is now € 695 million. The profitability is 3.7%. The profit of the company is therefore very sensitive to the business plan (costs and production projections).

We only consider here a few stress tests, to show how such a test is concretely done, but any exhaustive ALM study should present the stress tests corresponding to the risks that the company faces.

Now that we have done the main parts of the determinist study (the results under the central scenario and the stress tests), it is time to do the stochastic study. Indeed, usually with each new insurance fund (and also periodically for older funds), the company must define its investment strategy. This is precisely one of the main goals of the stochastic study.

The stochastic study

Let us now run our model with simulated scenarios to define the equity allocation that maximize the ratio risk/return for the insurance company. To do that, we will simulate 1,000 scenarios of equity and interest rates, using the models we have described, and we will assess for each scenario the PVFP of the company. The risk will be measured as the 90%-VaR of the PVFP, and the return as the average PVFP.

We have tested different equity allocations, from 0% to 50% (by step of 5%), and we have plotted, as we have seen, the results in Figure 4.15

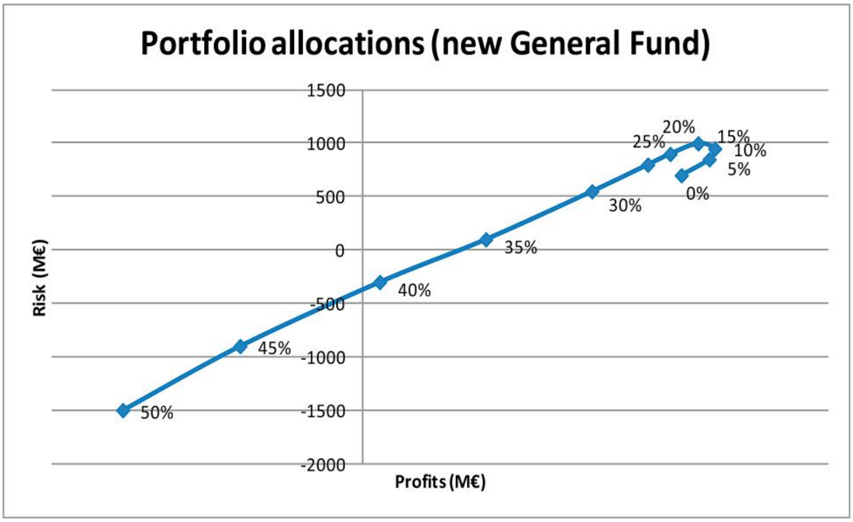

Figure 4.15. *Portfolio allocations (new General Fund)*

The efficient frontier seems to be for an equity allocation between 10 and 15%. We have only considered here the risk/return of the insurance company. We could also have studied the client value for different allocations, that is the average and 90%-VaR of the annual crediting rates.

SCR calculation

With the allocation study, the calculation of the SCR is the other main goal of the stochastic ALM study.

To correctly estimate the SCR of an insurance product, we need to work in the Solvency II framework. This means that we must use different hypothesis. The two main differences are, first, that we cannot anymore take into account future business and, second, that we must use asset scenarios generated under a risk-neutral hypothesis.

For our example, we suppose that we compute the SCR of our new product next year, thus after one year of business. For the sake of simplicity, we will use the same scenarios that we have used for the allocation study. Therefore, we suppose that the risk-neutral hypotheses are the same as the hypotheses we have used so far.

Furthermore, we will only consider the risk market module of Solvency II by applying market shocks on equities and interest rates (which are the main market risks of our product).

The EIOPA's Technical Specifications for Solvency II give the following shocks we must apply: a shock of -46.5% on equities, and the worst case between a shock of +42% and a shock of -31% on 10 year interest rates (which is the maturity we use in our example). We must apply these shocks, separately, at t = 0, and compute the new profits of the company. The difference between the initial averaged PVFP and the shocked averaged PVFP is the SCR of the considered market risk. We must take into account a correlation between the equity risk and the interest rate risk when computing the global market SCR. This correlation is 0.5 if the worst case of the interest rate is a decrease of interest rates. The correlation is 0 otherwise.

We consider an equity allocation of 15%, which is an optimal portfolio according to our stochastic ALM study. The averaged PVFP we obtain, without applying any market shock, is now € 75.6 million. With the equity shock, the PVFP becomes € 62.3 million. The equity SCR is therefore € 13.3 million. The initial mathematical reserves (which correspond to only one year of production) are € 538 million. The equity SCR thus represents 2.5% of the initial mathematical reserves.

Similarly, if we now apply the interest rate shocks, we obtain an SCR of 0% for the down shock (with a decrease in interest rates, the PVFP is better for the company because the discount factors are higher), and for the up shock, we obtain an SCR of 0.9% of the initial mathematical reserves, that is to say € 4.8 million. As the worst case is the up shock, to sum up the equity

SCR and the interest rate SCR, we must use a correlation of 0. And so, finally, the global market SCR is 3.4% of the initial mathematical reserves, that is to say € 18.2 million.

We have only computed here the market SCR. However, basically, the other risks can be estimated the same way, by applying the corresponding shocks (defined by the EIOPA), by comparing the initial and the shocked PVFP, and by aggregating (using correlation matrices) the different submodules and modules.

4.6. Conclusion

We have presented throughout this chapter a simple ALM model for insurance companies, and we have seen the different parts of a usual ALM study. Finally, we have used this model to study an insurance product, a new general fund a company wants to sell.

The model we have exposed here is a very simple model, based on various simplifications and assumptions. Furthermore, the indicators and risk measures we have presented can differ from company to company. But, behind all this, what is important is the spirit of an ALM study, its goals and its methodology.

The robustness and clarity of an ALM study depends heavily on the robustness and clarity of the model it is based upon. There is a compromise to find between a too simple model that reflects poorly on the reality and a too complex model that can lead to mischievous, non-intuitive and model-dependent results.

5

Building and Use of ALM Internal Models in Banks

5.1. Introduction

In Chapter 3, we have already shown how the asset and liability management (ALM) approach can be used to know the robustness of the bank concerning its equities.

In this chapter, we will now use the ALM approach to analyze a bond portfolio devoted to hedge one of its activities for which the surplus is negative and we will see how to restructure this portfolio to improve the situation using different scenarios. Doing so, we construct a *partial* internal model for the bank that can be included in a *global* internal model. It is clear that the construction of such a global model is outside the scope of this book as this task needs the collaboration of all the departments of the bank under the responsibility of its risk management, but often this model is built with several partial internal models that are connected.

Finally, we present an internal model based on stochastic models whose general presentation is given by Deelstra and Janssen [DEE 01].

5.2. Case 1: Reduction of gaps

5.2.1. *Basic numerical data*

We consider one asset flow and one liability flow whose values are given yearly over 20 years in columns 2 and 3 of Table 5.1.

Asset and Liability Management for Banks and Insurance Companies, First Edition.
Marine Corlosquet-Habart, William Gehin, Jacques Janssen and Raimondo Manca.
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We use the yield curve (YC) given by column 6, which is the zero-bond curve of the French Institute of Actuaries (IA, Paris) on October 2014.

Columns 3 and 5 give the present values for the asset and liability for the given YC.

It follows that the total present values of the two flows are, respectively, given by:

$$TAPV = 640181774.1;\ TLPV = 640445513.6$$

Time	*Asset flow*	*Present assets*	*Liability flow*	*Present liabilities*	*Yield curve*
1	24007001	24008040.55	25659879	25660990.12	*-0.00004*
2	23506015	23509959.81	24406087	24410182.86	*-0.00008*
3	23005145	23010936.37	24005145	24011188.11	*-0.00008*
4	21546713	21553945.59	22872221	22879898.53	*-0.00008*
5	23005145	23014798.09	23584721	23594617.28	*-0.00008*
6	21009873	20849412.5	25009873	24818862.96	*0.00128*
7	18973870	18600204.12	17669870	17321884.72	*0.00285*
8	201355467	193945702.7	201355467	193945702.7	*0.00470*
9	132578143	124881102.8	143589013	135252718.8	*0.00667*
10	78452199	71984420.64	65321447	59936197.8	*0.00864*
11	56152192	50033361.37	48779210	43463803.54	*0.01054*
12	23568971	20345778.22	33568971	28978220.52	*0.01233*
13	9548703	7972532.419	4587101	3829924.486	*0.01397*
14	6817521	5499596.503	3817632	3079629.032	*0.01546*
15	4532998	3530965.448	2231047	1737867.493	*0.01679*
16	2532548	1904571.728	1532008	1152127.866	*0.01797*
17	4532998	3291731.469	1032666	749892.0514	*0.01900*
18	3200001	2244713.837	8014287	5621804.781	*0.01989*
19	0	0	0	0	*0.02066*

20	0	0	0	0	*0.02132*
	TAPV	640181774.1	TLPV	640445513.6	
	Surplus PV	-263739.485			

Table 5.1. *ALM for bond portfolio of a bank*

Table 5.2 gives the surplus and present surplus values and also the equivalent constant rates for both asset and liability. These equivalent constant rates are thus given by:

$$i_A = 0.00713, i_B = 0.00695$$

which gives a small difference in the surplus.

			Constant rate Asset	Constant rate Liability
Surplus	*Present surplus*	*Global surplus*	0.00713	0.00695
-1652878	-1652949.573	-1652949.573	23837042.88	25482773.72
-900072	-900223.0511	-2553172.624	23174370.37	24070346.53
-1000000	-1000251.742	-3553424.366	22519999.51	23511515.02
-1325508	-1325952.934	-4879377.3	20943000.15	22247269.39
-579576	-579819.1933	-5459196.493	22202266.49	22781966.65
-4000000	-3969450.458	-9428646.951	20133080.53	23991867.21
1304000	1278319.403	-8150327.548	18053324.62	16833639.05
0	0	-8150327.548	190230068.2	190502279.1
-11010870	-10371616	-18521943.54	124366134.7	134911836.5
13130752	12048222.83	-6473720.711	73071798	60950424.44

7372982	6569557.833	95837.12175	51930900.57	45200974.07
-10000000	-8632442.3	-8536605.178	21642839.01	30891795.7
4961602	4142607.933	-4393997.246	8706276.578	4192137.526
2999889	2419967.471	-1974029.775	6172044.175	3464841.52
2301951	1793097.955	-180931.8197	4074764.709	2010898.387
1000540	752443.8615	571512.0418	2260420.104	1371306.496
3500332	2541839.417	3113351.459	4017274.153	917963.6249
-4814286	-3377090.944	-263739.485	2815856.537	7074937.026
0	0	-263739.485	0	0
0	0	-263739.485	0	0
			640151461.3	640408771.9
		Surplus	-257310.5683	

Table 5.2. *Surplus with constant rates*

5.2.2. *Basic ALM indicators*

Using the basic ALM indicators of Chapter 3, we can summarize the results in Table 5.3 with as a rate scenario an increase of 1% of both rates which is in the context of quasi null rates decided by the BCE in 2014 very improbable.

From these results, we can see that there is a mismatching after realization of the rate scenario, a loss of €743710.4312, more or less three times than the loss of the initial situation, which can be taken as VaR value by proxy as this scenario is very improbable.

	Asset	**Liability**	**Result**	
Present value	64015146.3	640408771.9	-257310.5683	
Duration	8.120525191	7.998873984		
Mod. Duration	8.063035746	7.943665508		
Abs. Mod. duration	5161564116	5087193073		
Rel. Convexity	69.58882882	80.00632681		
Absolute Convexity	22273695229	25618376748		
Rate scenario	**0.01**	**0.01**		
Variation	-51615641.16	-50871930.73	-743710.4312	
Relative variation	-0.080630357	-0.079436655	2.890322135	
Variation with conv.	-50501956.4	-50501956.4	-910944.5072	
Rel. Var. with conv.	-0.078890637	-0.079436655	3.540252983	
Value after scenario.	588535820.2	589536841.2	-1001021	
Id. with conv.	589649504.9	589536841.2	-1168255.075	
Result/asset				-0.000401953
Id. after scen.				-0.001700867
Id. with conv.				-0.00198127
VaR value	743710.4312			

Table 5.3. *Basic ALM indicators*

5.2.3. *Scenario for loss reduction*

As we see from Table 5.2, negative gaps are especially present in the first few years. To reduce these gaps, the bank decides to use the interests of produce by an investment of € 10,000,000 over 5 years at 3%. Thus, the bank annually receives the amount of € 900,000 at each end of the first five years that are injected in the asset flow. It follows that the five new values of the asset flow are mentioned in Table 5.4.

24407001
23906015
23405145
21946713
23405145

Table 5.4. *New asset values*

Table 5.4 gives a global positive present value for the new surplus of € 1,700,611.128.

Let us now consider our catastrophic scenario again; the results after the scenario are given in Table 5.5.

	Asset	**Liability**	**Result**	
Present value	642109383	640408771.9	1700611.128	
Duration	8.104868345	7.998873984		
Mod. Duration	8.047489743	7.943665508		
Abs. mod. duration	5167368674	5087193073		
Rel. Convexity	69.41842612	80.00632681		
Absolute Convexity	22287111382	25618376748		
Rate scenario	**0.01**	**0.01**		
Variation	-51673686.74	-50871930.73	-801756.0142	
Relative variation	-0.080474897	-0.079436655	-0.471451704	
Variation with conv.	-50559331.17	-50559331.17	-968319.2824	
Rel. Var. with conv.	-0.078739437	-0.077436497	-0.569394888	
Value after scenario.	590435696.3	589536841.2	898855.1138	
Id. with conv.	591550051.8	589536841.2	732291.8455	
Result/asset				0.002648
Id. after scen.				0.001522
Id. with conv.				0.001238
VaR value	**801756.0142**			

Table 5.5. *Basic ALM indicators after scenario*

Comparing both Tables 5.3 and 5.5, we have seen that if the durations are more or less the same, there is an increase in the VaR value of 580046 that is of 7.8%.

In conclusion, we have seen that the reduction of the gaps and of the surplus has led to a riskier situation from the point of view of VaR hedging.

5.3. Case 2: A stochastic internal model

We will present indicators of the risk that the value of the liabilities of a bank becomes larger than the value of the assets. We will also give the probability of mismatching, called bankruptcy probability in case of the consideration of all the balance sheet. These results are not only important to determine ALM-strategies for banks but also for insurance companies.

5.3.1. *Probability of bankruptcy*

Let $E=\left(E(t),t\in\left[0,\infty\right)\right)$ E be the stochastic process of equities of a bank defined on a filtered probability space $\left(\Omega,\Im,(\Im_t),P\right)$ satisfying the usual assumptions.

As mentioned in Chapter 3, we can describe the process as the difference of the asset process $A=\left(A(t),t\in\left[0,\infty\right)\right)$ and the liability process $B=\left(B(t),t\in\left[0,\infty\right)\right)$ as:

$$E(t)=A(t)-B(t),0\leq t.$$

Of course, the most dangerous risk for the bank is bankruptcy; that is why we introduce the random variable – also called stopping time – defined by:

$$\alpha=\inf\left\{t:E(t)<0\right\} \qquad [5.1]$$

which represents the first time at which the equities become negative.

Now we can introduce two *probabilities of bankruptcy*: the first probability is related to the event of having bankruptcy before the time *t*, and the second probability anywhere in the future:

$$\Psi(u,t) = P\left[T < t \middle| \alpha(0) = u\right],$$
$$\Psi(u) = P\left[T < \infty \middle| \alpha(0) = u\right], \qquad [5.2]$$

with of course:

$$\Psi(\text{u,t}) \leq \Psi(\text{u}).$$

The probabilities of non-bankruptcy are thus given by:

$$\text{N}(u,t) = P\left[T > t \middle| \alpha(0) = u\right] = 1 - \Psi(u,t),$$
$$\text{N}(u,t) = P\left[T = \infty \middle| \alpha(0) = u\right] = 1 - \Psi(u).$$

5.3.2. *Presentation of the first model (Model I) [JAN 07]*

Let us assume that the stochastic dynamics of the processes *A* and *B* are governed by the following two stochastic differential equations:

$$dA(t) = \mu_A dt + \sigma_A dW_A(t),$$
$$dB(t) = \mu_B dt + \sigma_B dW_B(t), \qquad [5.3]$$

with initial conditions as:

$$A(0) = A_0, B(0) = B_0, (A_0, B_0 > 0) \qquad [5.4]$$

where $\mu_A, \mu_B, \sigma_A, \sigma_B$ and u are strictly positive constants, and $W_A = (W_A(t), t \geq 0), W_B = (W_B(t), t \geq 0)$, are two independent standard Brownian motions (SBM).

It is clear that the solution of these equations is given by:

$$A(t) = A_0 + \mu_A t + \sigma_A W_A(t),$$
$$B(t) = B_0 + \mu_B t + \sigma_B W_B(t). \qquad [5.5]$$

So, we have:

$$E(t) = u + (\mu_A - \mu_B)t + (\sigma_A W_A(t) - \sigma_B W_B(t)),$$
$$\text{with } u = A_0 - B_0. \qquad [5.6]$$

From the independence assumption of the two SBM A and B, it can be deduced that the stochastic process $(\sigma_A W_A(t) - \sigma_B W_B(t), t \geq 0)$ is equivalent to the process

$$\left(\sqrt{\sigma_A^2 + \sigma_B^2} W(t), t \geq 0\right)$$

where the process W is still an SBM.

Setting:

$$\mu = \mu_A - \mu_B, \sigma = \sqrt{\sigma_A^2 + \sigma_B^2} \quad [5.7]$$

we obtain:

$$E(t) = u + \mu t + \sigma W(t) \quad [5.8]$$

Using the well-known result of Cox and Miller [COX 65], we can write the expressions of bankruptcy probabilities as:

$$\Psi(u,t) = 1 - \varphi\left(\frac{u + (\mu_A - \mu_B)t}{\sqrt{\sigma_A^2 + \sigma_B^2}\sqrt{t}}\right) + e^{-\frac{2(\mu_A - \mu_B)u}{\sqrt{\sigma_A^2 + \sigma_B^2}}} \varphi\left(\frac{-u + (\mu_A - \mu_B)t}{\sqrt{\sigma_A^2 + \sigma_B^2}\sqrt{t}}\right) \quad [5.9]$$

and for $t \to \infty$, we have the value of the bankruptcy probability on $[0, \infty)$ as:

$$\Psi(u) = \lim_{t \to \infty} \Psi(u,t) = \begin{cases} 1, \mu_A - \mu_B \leq 0, \\ e^{-\frac{2(\mu_A - \mu_B)}{\sigma_A^2 + \sigma_B^2} u}, \mu > 0. \end{cases} \quad [5.10]$$

REMARK 5.1.–

1) Interpretation of the parameters of the model.

From relations [5.5], we have:

$$E[A(t)] = A_0 + \mu_A t + \sigma_A W_A(t), \mathrm{var}[A(t)] = \sigma_A^2,$$
$$E[B(t)] = B_0 + \mu_B t, \mathrm{var}[B(t)] = \sigma_B^2$$

and so:

$$E[E(t)] = u + (\mu_A - \mu_B)t, \ \operatorname{var}[E(t)] = \sigma_A^2 + \sigma_B^2.$$

It also follows that for every *t*, the random variables *A(t)* and *B(t)*, respectively, have normal distributions of parameters $(u + \mu_A t, \sigma_A^2 t), (\mu_B t, \sigma_B^2 t)$ and consequently *E(t)* has a normal distribution $(u + (\mu_A - \mu_B)t, (\sigma_A^2 + \sigma_B^2)t)$.

2) Basic parameter of the model.

From equation [5.10], we can see that this basic parameter is given by:

$$R = 2\frac{\mu_A - \mu_B}{\sigma_A^2 + \sigma_B^2}. \qquad [5.11]$$

If $R \leq 0$, we can see that bankruptcy is a certain event on $[0,\infty)$ and if $R > 0$, there is a non-null probability to have no bankruptcy on $[0,\infty)$. Of course, the larger the R, the smaller will be the no bankruptcy probability.

This result leads to an important rule of ALM: we can correct a slight positive trend of the equity by decreasing their volatility and, inversely, we can correct an excess of volatility of the equity by increasing its trend and with results [5.9] and [5.10], measuring the effect of these corrections on $[0,t)$ and $[0,\infty)$.

5.3.3. *Presentation of the model with correlations (Model Ibis) [JAN 03]*

We will keep the stochastic dynamics [5.2] but we now suppress the assumption of the independence of the two SBM W_A and W_B to have a correlation between the asset and liability given by the correlation coefficient ρ between W_A and W_B.

Using the Cholesky transformation (see [JAN 09a]), it is possible to show that the result [5.6]:

$$E(t) = E_0 + (\mu_A - \mu_B)t + (\sigma_A W_A(t) - \sigma_B W_B(t))$$

can be written under the form:

$$E(t) = E_0 + (\mu_A - \mu_B)t + [\sigma_A(\sqrt{1-\rho^2}W_1(t) + (\sigma_A\rho - \sigma_B)W_B(t)] \quad [5.12]$$

where the two SBM W_1, W_B are now independent.

It can be easily proven that the process defined by: $\tilde{W}(t) = \sigma_A\sqrt{1-\rho^2}W_1 t) + (\sigma_A\rho - \sigma_B)W_B(t)$ is stochastically equivalent to the process $(\sqrt{\sigma_A^2(1-\rho^2) + (\sigma_A\rho - \sigma_B)^2}W(t), t \geq 0)$ with W still an SBM and, consequently, we can write that:

$$\begin{aligned} &E(t) = u + (\mu_A - \mu_B)t + \sigma W(t), \\ &\sigma = \sqrt{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}. \end{aligned} \quad [5.13]$$

This last result is identical to relation [5.8] and we can use results [5.9] and [5.10] to obtain the two bankruptcy probabilities under the following forms:

$$\Psi(u,t) = 1 - \varphi\left(\frac{u + (\mu_A - \mu_B)t}{\sqrt{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}\sqrt{t}}\right) + e^{-\frac{2(\mu_A - \mu_B)u}{\sqrt{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}}}\varphi\left(\frac{-u + (\mu_A - \mu_B)t}{\sqrt{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}\sqrt{t}}\right) \quad [5.14]$$

$$\Psi(u) = \lim_{t\to\infty}\Psi(u,t) = \begin{cases} 1, \mu_A - \mu_B \leq 0, \\ e^{-\frac{2(\mu_A - \mu_B)}{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}u}, \mu_A - \mu_B > 0. \end{cases} \quad [5.15]$$

REMARK 5.2.–

1) Interpretation of the parameters of the model.

From relations [5.12], we have:

$$E[E(t)] = u + (\mu_A - \mu_B)t, \text{var}[E(t)] = (\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B)t.$$

It also follows that for every t, the random variable $E(t)$ has a normal distribution $(u + (\mu_A - \mu_B)t, (\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B)t)$.

2) Basic parameters of the model.

From result [5.10], we can see that this basic parameter is given by:

$$R = 2\frac{(\mu_A - \mu_B)}{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}. \qquad [5.16]$$

If $R \leq 0$, we can see that bankruptcy is a certain event on $[0,\infty)$ and if $R > 0$, there is a non-null probability to have no bankruptcy on $[0,\infty)$. Of course, the larger the R is, the smaller will be the bankruptcy probability.

This result gives the possibility to complete our ALM rule of section 5.3.2: the more the asset and liability are correlated, the better it is. The best case is attained for ρ=1 for which:

$$R = 2\frac{(\mu_A - \mu_B)}{(\sigma_A - \sigma_B)^2}.$$

We can also see that in this case the more σ_A and σ_B are closer, the better it is.

5.3.4. *Presentation of the model with correlations and non-negative values for assets and liabilities (Model II) [JAN 03]*

5.3.4.1. *The model ALM II*

It is clear that in the two previous models, the processes A and B may take negative values. To avoid that, we must modify the model [5.3] "à la Black and Scholes", that is, to start with the following stochastic differential system:

$$\begin{aligned} dA(t) &= \mu_A A dt + \sigma_A A dW_A(t), \\ dB(t) &= \mu_B B dt + \sigma_B B dW_B(t), \end{aligned} \qquad [5.17]$$

with initial conditions as:

$$A(0) = A_0, B(0) = B_0, (A_0, B_0 > 0)$$

and where $\mu_A, \mu_B, \sigma_A, \sigma_B$ and u are strictly positive constants and $W_A = (W_A(t), t \geq 0), W_B = (W_B(t), t \geq 0)$ are two correlated SBM with correlation coefficient φ.

From well-known results related to the Black and Scholes model, we know the explicit form of the two processes *A* and *B* indeed given by:

$$A(t) = A(0)\exp(\mu_A - \frac{\sigma_A^2}{2})t + \sigma_A W_A(t),$$
$$B(t) = B(0)\exp(\mu_B - \frac{\sigma_B^2}{2})t + \sigma_B W_B(t), \quad [5.18]$$

From results [5.18], it follows that, for every *t*, the random variables $\frac{A(t)}{A(0)}$ and $\frac{B(t)}{B(0)}$ have a lognormal distribution as parameters respectively has a lognormal distribution with parameters $\left(\left(\mu_A - \frac{\sigma_A^2}{2}\right), \sigma_A^2 t\right), \left(\left(\mu_B - \frac{\sigma_B^2}{2}\right), \sigma_B^2 t\right)$, and so:

$$E\left(\ln\frac{A(t)}{A(0)}\right) = \left(\mu_A - \frac{\sigma_A^2}{2}\right)t, E\left(\ln\frac{B(t)}{B(0)}\right) = \left(\mu_B - \frac{\sigma_B^2}{2}\right)t$$
$$\text{var}\left(\ln\frac{A(t)}{A(0)}\right) = \sigma_A^2 t, \text{var}\left(\ln\frac{B(t)}{B(0)}\right) = \sigma_B^2 t. \quad [5.19]$$

From result [5.18], we obtain the explicit form of the trajectories of the processes *A* and *B* [JAN 09]:

$$A(t) = A_0 e^{\left(\mu_A - \frac{\sigma_A^2}{2}\right)t} e^{\sigma W_A(t)}, B(t) = B_0 e^{\left(\mu_B - \frac{\sigma_B^2}{2}\right)t} e^{\sigma W_B(t)}. \quad [5.20]$$

These processes are called geometric Brownian motions.

From properties of the lognormal distribution, we obtain:

$$E\left(\frac{A(t)}{A_0}\right) = e^{\mu_A t}, \mathrm{var}\left(\frac{A(t)}{A_0}\right) = e^{2\mu_A t}(e^{\sigma_A^2 t} - 1),$$
$$E\left(\frac{B(t)}{B_0}\right) = e^{\mu_B t}, \mathrm{var}\left(\frac{B(t)}{B_0}\right) = e^{2\mu_B t}(e^{\sigma_B^2 t} - 1). \quad [5.21]$$

So, we have seen that the mean value of the asset at time t is given as if the initial amount A_0 was invested at instantaneous interest rate μ_A, and that its value is above or below A_0 following the "hazard" variations modeled with the Brownian motion and similarly for liability.

Using stochastic calculus (see, for example, [JAN 09b]), it is also possible to give the value of bankruptcy probabilities but it is more difficult than the previous model. This time, we have to start with the ratio asset on liability *A(t)/B(t)* and the following result can be obtained [JAN 09a]:

Under the assumptions of the model II, the asset liability ratio is given by:

$$\frac{A(t)}{B(t)} = \frac{A_0}{B_0}\exp(\mu t + \sigma \tilde{W}(t)) \quad [5.22]$$

where:

$$\mu = \mu_A - \mu_B - \frac{1}{2}(\sigma_A^2 - \sigma_B^2),$$
$$\sigma = \sqrt{\sigma_{\mathrm{A}}^2 + \sigma_{\mathrm{B}}^2 - 2\rho\sigma_A\sigma_B} \quad [5.23]$$

where the process $(\tilde{W}(t), t \geq 0)$ *is an SBM.*

REMARK 5.3.–

1) From this result, we can give three consequences for the ALM:

i) we can simulate the impact of possible scenarios given in function of the choice of the basic parameters $\mu_A, \mu_B, \sigma_A, \sigma_B$ and ρ , $\mu_A, \mu_B, \sigma_A, \sigma_B$ and ρ ;

ii) we can construct confidence interval of the asset liability ratio;

iii) we can compute VaR values.

Indeed, from relation [5.22], we obtain:

$$\ln \frac{A(t)}{B(t)} - \ln \frac{A_0}{B_0} = \mu t + \sigma \tilde{W}(t). \quad [5.24]$$

So, if $\left[W_1, W_2\right]$ is a confidence interval at level $1-\varepsilon$, the confidence interval for A(t)/B(t) is given by:

$$\frac{A_0}{B_0} e^{\mu t + \sigma W_1} \leq \frac{A(t)}{B(t)} \leq \frac{A_0}{B_0} e^{\mu t + \sigma W_2} \quad [5.25]$$

2) It is clear that the minimization of risk implies those of σ. At the limit, this risk component *t* becomes null when:

$$\sigma_A = \sigma_B, \ \rho = 1, \quad [5.26]$$

conditions of course never realized.

5.3.4.2. *The lifetime of the bank and non-bankruptcy probabilities*

The random variable *T*, called lifetime, represents the time length between 0 and the time of bankruptcy, so *T* is the first time such that A*(t)* < *B(t)* in *t* = *T*, or, equivalently, the first time that the equity *E(t)=A(t)-B(t)* becomes negative.

To use the result [5.24], we can express the definition of *T* the first time at which the process $\left(\ln \frac{A(t)}{B(t)}, t \geq 0\right)$ attains $\left(-\infty, 0\right)$ and so:

$$T = \inf \left\{T : a + \mu t + \sigma \tilde{W}(t) < 0\right\} \quad [5.27]$$

where:

$$a = \ln \frac{A_0}{B_0}, a > 0.$$

It follows that we can use results of the model I to compute the probability of never being bankrupt given by:

$$\Psi(a) = P\left[T = \infty \middle| a\right],$$
$$= \begin{cases} 0, & \mu \leq 0, \\ 1 - \exp\left(-\dfrac{2a\mu}{\sigma^2}\right), & \mu > 0. \end{cases} \quad [5.28]$$

So, from relation [5.23], bankruptcy on an infinite horizon is certain if, and only if:

$$(\mu_A - \mu_B) \leq \frac{1}{2}(\sigma_A^2 - \sigma_B^2).$$

REMARK 5.4.–

1) The term $\mu = \mu_A - \mu_B - \frac{1}{2}(\sigma_A^2 - \sigma_B^2)$ can be used as an ALM risk indicator as indeed if $\mu \leq 0$, then bankruptcy is certain on an infinite time horizon.

2) On the other side, if $\mu > 0$, the probability of being never ruined is given by:

$$\Psi(a) = 1 - \exp\left(-\frac{2a\mu}{\sigma^2}\right) \quad [5.29]$$

where probability is larger as the ratio μ / σ is smaller and so as for the previous model, this ratio can still be used as second ALM risk indicator.

Ars and Janssen [ARS 94] also give the distribution of the lifetime *T*.

Let us now define the probability *P(.,.)* as follows:

$$P(x,t) = P\left[T > t, e^{a-x} < \frac{A(t)}{B(t)}\right], x < a \quad [5.30]$$

Using results of Cox and Miller [COX 65] on absorption problems in diffusion processes, it can be proven that:

$$P(x,t) = \phi\left(\frac{x + \mu t}{\sigma\sqrt{t}}\right) - \exp\left(-\frac{2\mu a}{\sigma^2}\right).\phi\left(\frac{x - 2a + \mu t}{\sigma\sqrt{t}}\right). \quad [5.31]$$

Setting *x=a* in relation [5.31], we obtain the *non-bankruptcy probability* on time horizon [0,t]:

$$N(a,t) = P(T > t \mid a) = P(t,a)$$

given by:

$$P(a,t) = \phi\left(\frac{a+\mu t}{\sigma\sqrt{t}}\right) - \exp\left(-\frac{2\mu a}{\sigma^2}\right).\phi\left(\frac{-a+\mu t}{\sigma\sqrt{t}}\right)$$

and so

$$\Psi(a,t) = 1 - N(a,t)$$
$$=1-\phi\left(\frac{a+\mu t}{\sigma\sqrt{t}}\right) + \exp\left(-\frac{2\mu a}{\sigma^2}\right).\phi\left(\frac{-a+\mu t}{\sigma\sqrt{t}}\right). \quad [5.32]$$

For $\mu > 0,$ enabling $t \to \infty,$ we obtain result [5.29] again.

Cox and Miller [COX 65] also give a result concerning the lifetime of the bank when $\mu \le 0$.

In this case, the law of *T* is an inverse Gaussian, this means that its density function f_T is given by:

$$f_T(t) = \begin{cases} 0, & t < 0, \\ \dfrac{a}{\sigma\sqrt{2\pi t^3}}\exp\left(-\dfrac{(a+\mu t)^2}{2\sigma^2 t}\right), & t \ge 0 \end{cases} \quad [5.33]$$

and so:

$$E[T] = \frac{a}{|\mu|}, \mathrm{var}[T] = \frac{a\sigma^2}{|\mu|^3}. \quad [5.34]$$

This result enables us, as we will see in the following examples, to measure the security of an insurance company even if it is in a temporary difficult situation.

5.3.5. *Consequences for ALM*

As the conclusions are the same for both ALM models, in the following, we will only work with model ALMII.

1) Let us recall that from relation [5.24], the random variable $\ln \frac{A(t)}{B(t)} \frac{B_0}{A_0}$ has a normal distribution of mean μt and variance $\sigma^2 t$ so that its mean and its variance are given by:

$$
\begin{aligned}
&E\left[\frac{A(t)}{B(t)}\right] = \frac{A_0}{B_0} \exp\left(\mu + \frac{\sigma^2}{2}\right) t, \\
&\operatorname{var}\left[\frac{A(t)}{B(t)}\right] = \left(\frac{A_0}{B_0}\right)^2 (\exp(2\mu + \sigma^2)t)(e^{\sigma^2 t} - 1).
\end{aligned}
\qquad [5.35]
$$

where from relation [5.23]:

$$
\begin{aligned}
&\mu = \mu_A - \mu_B - \frac{1}{2}(\sigma_A^2 - \sigma_B^2), \\
&\sigma = \sqrt{\sigma_{\mathrm{A}}^2 + \sigma_{\mathrm{B}}^2 - 2\rho\sigma_A\sigma_B}.
\end{aligned}
\qquad [5.36]
$$

These results confirm that the larger the risk, the larger the gain measured by the ratio asset on liability will be but at the same time, the variance of this ratio will increase too so that we are in conformity with the Markowitz portfolio theory.

2) Influence of φ

To go deeper, let us look at the influence of the correlation coefficient φ in the expression of σ given by the last relation of [5.36].

As in model I, we can see that for fixed σ_A, σ_B, the maximum risk is attained for *ρ=1*, in which case $\sigma^2 = (\sigma_A + \sigma_B)^2$ and $\mu + \frac{\sigma^2}{2} = (\mu_A + \mu_B) + (\sigma_A + \sigma_B)^2$.

However, for *ρ=-1*, the risk is minimum for value $\sigma^2 = (\sigma_A - \sigma_B)^2$ and $\mu + \frac{\sigma^2}{2} = (\mu_A + \mu_B) + (\sigma_A - \sigma_B)^2$. Moreover, the closer σ_A and σ_B are, the better it is.

It follows that an optimal risk management consists of a maximum anti-correlation of the asset and liability and with their dispersions more or less identical. Of course, the price to pay will also decrease the trend of the gain given by μ. This management has to be complete by taking a μ as large as possible which is quite intuitive.

As a summary, we have:

– *"maximum" risk* $\rho = -1$ (perfect anti-correlation)

$$\mu + \frac{\sigma^2}{2} = (\mu_A + \mu_B) + (\sigma_A + \sigma_B)^2$$

and for identical asset and liability risks

$$\mu + \frac{\sigma^2}{2} = (\mu_A - \mu_B) + 2\sigma_A^2$$

– *"minimum" risk:* $\rho = 1$ (perfect correlation):

$$\mu + \frac{\sigma^2}{2} = (\mu_A - \mu_B) + (\sigma_A - \sigma_B)^2$$

and for identical asset and liability risks:

$$\mu + \frac{\sigma^2}{2} = \mu_A - \mu_B$$

3) Influence of μ

Whatever the correlation is, danger for the bank appears when $\mu \leq 0$, that is when:

$$(\mu_A - \mu_B) \leq \frac{1}{2}(\sigma_A^2 - \sigma_B^2). \qquad [5.37]$$

In this case, the excess of risk must be reduced using two possible actions:

1) increasing μ_A or increasing μ_B or both if possible.

These actions imply decisions for example from the marketing department, the department of credit, etc. in coordination with the risk department including the ALM support.

2) If $\mu > 0$, we know that the situation is less dangerous and the security of the bank can be measured by the non-bankruptcy probability leading to the VaR computation as we will show in our case study below.

REMARK 5.5.–

1) ALM models I and II have the same risk parameters μ and σ given by relations [5.23] and [5.36], their meanings are still different from one model to the other as indeed the first model considers Brownian model for the *difference* of asset and liability and the second model uses a geometric model for the *ratio* asset on liability. As a result, the values of parameters for model I are larger than the values for model II.

2) Though we present our models with the year as a time unit, these models can be used with other time units for example the trimester or even the month, provided the data are available.

3) We can make the same type of remark concerning the fact that we take the entire balance sheet. Indeed, we can use these models for just one section of the asset related to one section of the liability. The non-bankruptcy probability on $[0,t]$ becomes the probability of perfect hedging of the considered liability by the considered asset.

4) Use of the non-bankruptcy probabilities on $[0,t]$ is equivalent to computing the probability of a perfect hedging of B by A on this time interval.

5) We can demand much less. For example, we can ask that A covers B at the end of the year. In this case, we must only compute the probability of the following event $\{A(1) > B(1)\}$ or $\{A(1) - B(1) > \theta\} (\theta > 0)$ if we want the equities to be larger than θ at time 1. Let us see how we can give these probabilities, for example, for the ratio A(1)/B(1). We know that:

$$\ln \frac{A(t)}{B(t)} \frac{B_0}{A_0} \prec N(\mu t, \sigma^2 t) \qquad [5.38]$$

and so $P\left\{\frac{A(1)}{B(1)}>\theta\right\}$ $=P\left\{\ln\frac{A(1)}{B(1)}>\ln\theta\right\}$. From [5.38], we obtain the desired result:

$$P\left\{\frac{A(1)}{B(1)}>\theta\right\}=\frac{1}{\sqrt{2\pi}\sigma}\int_{\theta}^{\infty}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx. \quad [5.39]$$

6) In the models presented, we only consider the global asset and liability of the balance sheet. This is called by Deelstra and Janssen [DEE 01] an *unsegmented model* by opposition to a *multidimensional model* in which A and B are subdivided in subaccounts. The authors also extend these models using the interest rate models of Ornstein-Uhlenbeck-Vasicek and Cox, Ingersoll and Ross in case of assets represented by zero-coupons.

5.4. Calibration of the models

The two ALM models I and II depend on five parameters $\mu_A,\mu_B,\sigma_A,\sigma_B$ and ρ the initial parameters b $A(0)=A_0,B(0)=B_0,(A_0,B_0>0)$ being known.

5.4.1. *Historical method*

Let us note by $A(i),B(i),i=1,...,n$ the past asset and liability values in the inverse time order represented in Table 5.6.

A(1)	A(2)			A(n-1)	A(n)
B(1)	B(2)			B(n-1)	B(n)

Table 5.6. *Historical data at t=0*

5.4.1.1. *Model I*

If

X(i)=A(i+1)- A(i), =B(i+1)- lnB(i), i=1,…,n-1, we deduce from remark 5.1 that:

$$X(i) \prec N((\mu_A, \sigma_A^2),$$
$$Y(i) \prec N((\mu_B, \sigma_B^2),$$
$$X(1),...,X(n) \text{ independent},$$
$$Y(1),...,Y(n) \text{ independent},$$
$$\rho(\mathrm{X(i),Y(i)}) = \rho, i = 1,...,n.$$

Using classical results of mathematical statistics, we obtain the following unbiased estimators of our five parameters:

$$\mu_A \approx \frac{\sum_{i=1}^{n} X(i)}{n} (= \overline{X}),$$

$$\mu_B \approx \frac{\sum_{i=1}^{n} Y(i)}{n} (= \overline{Y}),$$

$$\sigma_A^2 \approx \frac{\sum_{i=1}^{n} (X(i) - \overline{X})^2}{n-1},$$

$$\sigma_B^2 \approx \frac{\sum_{i=1}^{n} (Y(i) - \overline{Y})^2}{n-1},$$

$$\rho \approx \frac{\sum_{i=1}^{n} (X(i) - \overline{X})(Y(i) - \overline{Y})}{\sqrt{\sum_{i=1}^{n} (X(i) - \overline{X})^2 \sum_{i=1}^{n} (Y(i) - \overline{Y})^2}}.$$

5.4.1.2. *Model II*

As from relation [5.19], we know that:

$$E\left(\ln\frac{A(t)}{A(0)}\right)=\left(\mu_A-\frac{\sigma_A^2}{2}\right)t, E\left(\ln\frac{B(t)}{B(0)}\right)=\left(\mu_B-\frac{\sigma_B^2}{2}\right)t$$
$$\text{var}\left(\ln\frac{A(t)}{A(0)}\right)=\sigma_A^2 t, \text{var}\left(\ln\frac{B(t)}{B(0)}\right)=\sigma_B^2 t,$$

we have:

$$E\left\{\ln\frac{A(i+1)}{A(0)}-\ln\frac{A(i)}{A(0)}\right\}=\left(\mu_A-\frac{\sigma_A^2}{2}\right), E\left\{\ln\frac{B(i+1)}{B(0)}-\ln\frac{B(i)}{B(0)}\right\}=\left(\mu_B-\frac{\sigma_B^2}{2}\right)$$
$$\text{var}\ln\left\{\ln\frac{A(i+1)}{A(0)}-\ln\frac{A(i)}{A(0)}\right\}=\sigma_A^2, \text{var}\left\{\ln\frac{B(i+1)}{B(0)}-\ln\frac{B(i)}{B(0)}\right\}=\sigma_B^2,$$
$$\rho\left\{(\ln\frac{A(i+1)}{A(0)}-\ln\frac{A(i)}{A(0)}), \ln(\frac{B(i+1)}{B(0)}-\ln\frac{B(i)}{B(0)})\right\}=\rho$$

From the data of Table 5.6, we can take the logarithms of the ratio of two successive asset or liability values

$$\tilde{X}(i)=\ln\frac{A(i+1)}{A(i)}, \tilde{Y}(i)=\ln\frac{B(i+1)}{B(i)}, i=1,...,n-1,$$

Always from relation [5.19], we know that:

$$X(i)\prec N((\mu_A-\frac{\sigma_A^2}{2},\sigma_A^2), Y(i)\prec N((\mu_B-\frac{\sigma_B^2}{2},\sigma_B^2),$$
$$X(1),...,X(n) \text{ independent, } Y(1),...,Y(n) \text{ independent,}$$
$$\rho(X(i),Y(i))=\rho, i=1,...,n,$$

and so from the last relations above, we can write that:

$$E(\tilde{X}(i))=\left(\mu_A-\frac{\sigma_A^2}{2}\right), E(\tilde{Y}(i))=\left(\mu_B-\frac{\sigma_B^2}{2}\right)$$
$$\text{var}(\tilde{X}(i))=\sigma_A^2, \text{var}(\tilde{Y}(i))=\sigma_B^2,$$
$$\rho(\tilde{X}(i),\tilde{Y}(i)=\rho.$$

It follows that the unbiased estimators are given by:

$$\left(\mu_A - \frac{\sigma_A^2}{2}\right) \approx \frac{\sum_{i=1}^{n} \tilde{X}(i)}{n} (= \bar{\tilde{X}}),$$

$$\left(\mu_B - \frac{\sigma_B^2}{2}\right) \approx \frac{\sum_{i=1}^{n} \tilde{Y}(i)}{n} (= \bar{\tilde{Y}}),$$

$$\sigma_A^2 \approx \frac{\sum_{i=1}^{n} (\tilde{X}(i) - \bar{\tilde{X}})^2}{n-1},$$

$$\sigma_B^2 \approx \frac{\sum_{i=1}^{n} (\tilde{Y}(i) - \bar{\tilde{Y}})^2}{n-1},$$

$$\rho \approx \frac{\sum_{i=1}^{n} (\tilde{X}(i) - \bar{\tilde{X}})(\tilde{Y}(i) - \bar{\tilde{Y}})}{\sqrt{\sum_{i=1}^{n} (\tilde{X}(i) - \bar{\tilde{X}})^2 \sum_{i=1}^{n} (\tilde{Y}(i) - \bar{\tilde{Y}})^2}}.$$

REMARK 5.6.–

1) The choice of the time horizon of the historic events depends on a lot of things. For example, do we include crisis periods or not? Moreover, there are constraints from regulators generally asking for a period that is not too small.

2) As we already mention above, though we present our models with the year a time unit, these models can be used with other time units, for example, the trimester or even the month provided the data are available.

3) We can make the same type of remark concerning the fact that we take the entire balance sheet. Indeed, we can use these models for just one section of the asset related to one section of the liability. The non-bankruptcy probability on $[0,t]$ becomes the probability of perfect hedging of the considered liability by the considered asset.

5.4.2. *Scenario generator*

As a complement to the historical method, the use of scenario generators (see [CLÉ 15]) based for example on economic indicators and also political events can lead to changes of the values of the main parameters of the model and after some simulations can show the possible impact of the realization of such scenarios.

We can also combine both approaches with the consideration of historical shocks.

Finally, global stress testing or microstress testing may be considered in the same way.

5.5. Example

5.5.1. *Model Ibis*

Let us consider the following data over a horizon of 12 years given by Table 5.7.

Year	Asset	Liability
1	12001	11110
2	13024	12555
3	14002	12000
4	15007	13000
5	16662	13250
6	16665	14002
7	17444	15602
8	17555	16000
9	19980	16200
10	19000	17845
11	19550	18000
12	19550	17500

Table 5.7. *Balance sheet data*

Table 5.8 gives the minimum steps for calibration.

Year	Asset	Liability	X	Y	E
1	12001	11110			891
2	13024	12555	1023	1445	469
3	14002	12000	978	-555	2002
4	15007	13000	1005	1000	2007
5	16662	13250	1655	250	3412
6	16665	14002	3	752	2663
7	17444	15602	779	1600	1842
8	17555	16000	111	398	1555
9	19980	16200	2425	200	3780
10	19000	17845	-980	1645	1155
11	19550	18000	550	155	1550
12	19550	17500	0	-500	2050
Means	16703,3333	14755.3333	686,272727	580,909091	
Variances	7210866.06	5781519.7	833408.618	605049.891	
Standard Dev.	2685.30558	2404.47909	912.912163	777.84953	
Cor. coef.			-0.2506806		

Table 5.8. *Calibration*

It follows that the main indicators for model I are given by Table 5.8.

μ	1948
σ square	1794479.25
σ	1339.58175
R	0.0021711

Table 5.9. *Main indicators*

Tables 5.10 and 5.11 give the values of bankruptcy probability and the values of equities to satisfy this probability at level ε.

E(0)	ψ
200	0.64776929
300	0.52135139
400	0.41960505
500	0.33771542
600	0.27180727
700	0.21876168
800	0.1760684
891	0.14450327
1000	0.1140517
1500	0.03851702
2000	0.01300779

Table 5.10. *Bankruptcy probability*

ε	E(0)	E/A
0.1	1060.55985	0.08837262
0.05	1379.82018	0.11497543
0.005	2440.38003	0.20334806

Table 5.11. *Computation of VaR values*

Table 5.11 shows that this bank is not in a very good situation with respect to the VaR value on $[0,\infty)$ as this value costs 20% of their assets and three times their actual equities.

To compute the VaR value in one year, we must use result [5.32] for t=1. Table 5.12 gives the result.

t	1			
u	α		β	$\psi(u,1)$
900	2.12603673	0.141707096	0.78233374	0.12770545
1000	2.2006869	0.114051703	0.70768358	0.10060733
1500	2.57393773	0.038517019	0.33443274	0.02933064
1800	2.79788823	0.020080901	0.11048225	0.01349564
2000	2.94718856	0.013007791	-0.03881809	0.00790589
2500	3.32043939	0.004392932	-0.41206892	0.00194361
2100	3.02183873	0.010469206	-0.11346825	0.00601793
2200	3.09648889	0.008426048	-0.18811842	0.00456351
2150	3.05916381	0.009392232	-0.15079334	0.00524301

Table 5.12. *Computation of VaR value*

5.5.2. *ALM II*

We start with the same data as the previous example given by Table 5.8.

Using statistical results from section 5.6.1.2, we obtain the results as mentioned in Table 5.13.

	Mean	0.044362301		0.041305025
	Stand. dev.	0.053149669		0.055463728
	Variance	0.002824887		0.003076225
	Cor. coef.	-0.128344305		
Trend A	0.045774744		a(0)	0.077144376
Trend B	0.042843138		ψ(a(0))	0.346182153
μ	0.003057275			
σ square	0.006657799			
σ	0.081595339		$\alpha = \ln \frac{A_0}{B_0}, \alpha > 0.$	
R	13.75071319			

Table 5.13. *Main indicators*

The value of bankruptcy probabilities are given in Table 5.14.

A/B	a	ψ(a)
1.01005017	0.01	0.87152813
1.03045453	0.03	0.66197903
1.0512711	0.05	0.50281365
1.07250818	0.07	0.38191778
1.09417428	0.09	0.29008996
1.10517092	0.1	0.25282156
1.22140276	0.2	0.06391874
1.34985881	0.3	0.01616004
1.64872127	0.5	0.00103293
1.8221188	0.6	0.00026115

Table 5.14. *Bankruptcy probability*

Table 5.14 shows that the VaR on the time horizon $[0, \infty)$ is larger than 2,000 and from Table 5.15 we deduce that the VaR value is given by 2,440.

ε	a	A/B
0.95	0.217860138	1.243413149
0.99	0.334904097	1.397806325
0.995	0.385312187	1.47007319

Table 5.15. *Computation of VaR values*

The ratio VaR value of 1.47 means that the asset at time 0 is 47% higher than the liability.

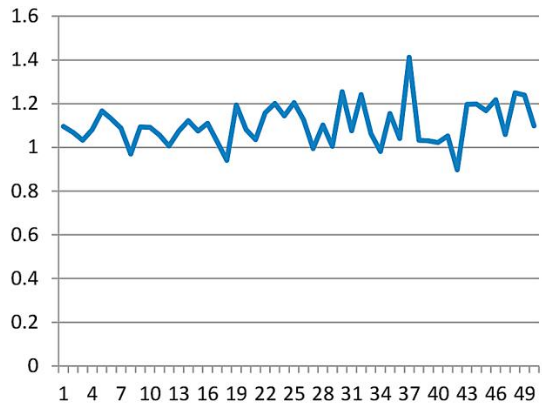

Figure 5.1. *Simulation 1 of the asset liability ratio as a function of time (time unit: 0.1)*

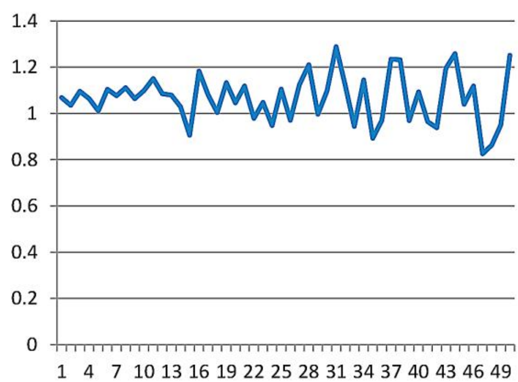

Figure 5.2. *Simulation 2 of the asset liability ratio as a funcion of time (time unit: 0.1)*

Table 5.16 gives the non-bankruptcy and bankruptcy probabilities over a time horizon of one year.

A/B	a (lnA/B)	N(a,1)	Ψ(a,1)
1,08	0.07696104	0.672548	0.327452002
1,1	0.09531018	0.76033827	0.239661732
1,2	0.18232156	0.95306532	0.046934675
1,3	0.26236426	0.9884404	0.011559598
1,4	0.,33647224	0.99627559	0.003724411
1,5	0.40546511	0.99865461	0.001345392
1,6	0.47000363	0.99948046	0.000519536
1,7	0.53062825	0.99978792	0.000212084
1,8	0.58778666	0.99990906	9.09364E-05
1,9	0.64185389	0.99995926	4.07412E-05
2	0,69314718	0,99998101	1.8987E-05

Table 5.16. *Non-bankruptcy and bankruptcy probabilities*

From these results, it follows that the VaR value for one year gives a report of more or less 1.2 for the asset liability ratio instead of 1.41 for the VaR on an infinite horizon. The value 1.2 means that the asset at time is 20% higher than the liability at time 0.

As a conclusion, we can say that the presented models are simple but efficient for the asset liability management for banks.

The model "*à la Black & Scholes*" leads to fundamental results from which we can follow the stochastic evolution of the asset and the liability of the company as a function of a limited number of basic parameters. These parameters represent the policy followed by the company in its ALM. As a result, simulation shows the influence of them. Moreover, they give other numerical parameters measuring the reliability of the company, for example the value of its mean lifetime.

Let us also add that such models can be used to compare at the macroeconomic level different societies to test their solvability [REF 00].

5.6. Key points for building internal models

Up to now, we have seen how to use standard tools of ALM to build partial internal models for specific situations. Now we will give some key points related to internal models.

5.6.1. *How to present an internal model?*

To build an internal model, it is important to have in mind the following points to present this model first in the bank, i.e. to people who will use it, to the Board and also to the regulators.

1) The first point is the *mathematical description* of the model. Indeed, as we have seen above, a model is always based on a mathematical construction using tools more or less sophisticated. Thus, it is important to describe in detail what this model contains, essentially for those who will install this model on the computer tools for the bank and also for those who will apply it.

2) It is well known that such a mathematical description is forbidden for a presentation to the Board. On the contrary, a *non-mathematical description*, using literary terms, is welcome including the meaning of basic definitions and parameters and key risk indicators. It must also include the main assumptions giving a sense to the considered model.

3) Of course, the *results of the first tests* are very important and they must include a presentation of simulations based on several possible scenarios for the future.

4) Another very important point is the use of scenario generators. From our point of view, the best thing is to start with a central scenario and for example, have another one representing optimistic and pessimistic evolutions.

5) It is clear that the results of *back testing* and *stress-testing* must be presented.

Here the selection of the database is essential as indeed false data must be avoided and the selected time horizon is fundamental. Sometimes, the database may be replaced by proxy, particularly for stress testing.

6) The *flexibility* of the model must be shown for example by showing how the values of basic parameters can be modified to adapt the model to new situations.

7) Finally, the model must be inserted in the *reporting tools*.

5.6.2. *Validation of the model*

It must be clear that the working of the model depends on a lot of things, for example, the security rules, the computer system, the accounting department, the internal control and so on.

In this environment, the bank has to proceed to the *internal validation* of the model with a staff of independent experts including the validation of the documentation related to the considered model. The experts may be external consultants and anyway, they have to create a written report for the external audit of the regulators.

In this step, the *back testing* and *stress testing* are done on internal and external data over a time horizon of last three months.

Finally, the risk management has to do a permanent *follow-up* of the model.

5.6.3. *Partial and global internal models*

It is clear that a global internal model taking into account the entire activity of the bank is of course a "formidable" task. In our opinion, this cannot be done without the use of several partial internal models for the main risks (market risk, credit risk, operational risk, liquidity risk, etc.) in which the ALM approach can play an important role in minimizing some losses by the technique of immunization.

In any case, the Basel rules ask for the computation of the VaR for the equities and that is what we have done in our two examples in section 5.5.

5.7. Conclusion

The two case studies in section 5.2 show that there are two possibilities in ALM methodology: the first possibility is a flow approach, for example, of a product of the bank and the second possibility concerns the supervision of the balance sheet leading of course to teatime evolution of equities.

The stochastic models are particularly used for the second case. The first model we present, i.e. the model I treats the asset and liability independently thus without interaction between them, whereas the last two models, model Ibis and model II, take into account to this interaction with a correlation between them.

The consequences on an ALM policy inside the bank and the flexibility of the models are shown with our numerical examples.

Conclusion

In recent years, asset and liability management (ALM) has become a key indicator for risk management, not only for banks but also for insurance companies. ALM now plays a central part in these firms' financial strategy, as they constantly need to invest their capital to guarantee their contractual commitments as well as to protect their financial results. By controlling the financial risks from the potential mismatches between the firm's assets and liabilities, ALM focuses on a long-term perspective as well as on a sound and global management.

In this book, we have tried to highlight the main challenges of an ALM department, both for a bank and for an insurance company. We have seen how classical and recent advanced methods in ALM can be used to improve management of insurance companies and banks which faces increasing risks and regulation rules of Solvency II and Basel III.

We have described and explained the commonalities and divergences of techniques and uses between these two worlds. We have seen that the exact role and perimeter of an ALM department, as well as the methods used, vary significantly because the business and the risks are quite different between an insurance company's activity and a bank's activity.

Regulation rules, particularly their technicalities and their rapid evolution through recent years, play a central part in the development of innovative ALM techniques and missions. Throughout this book, we hope

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that we have equipped the reader with the necessary tools and methodologies to understand and implement robust, practical and flexible mathematical models to analyze and manage the risks insurance companies and banks have to face every day.

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