TIMOTHY J. BIEHLER

the mathematics of money

Math for Business and Personal Finance Decisions

The Mathematics *of* Money

MATH *for* BUSINESS
and PERSONAL FINANCE DECISIONS

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The Mathematics *of* Money

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Dedication

To Teresa, Julia, and Lily

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About the Author

Timothy Biehler is an Assistant Professor at Finger Lakes Community College, where he has been teaching full time since 1999. He is a 2005 recipient of the State University of New York Chancellor's Award for Excellence in Teaching. Before joining the faculty at FLCC, he taught as an adjunct professor at Lemoyne College, SUNY–Morrisville, Columbia College, and Cayuga Community College.

Tim earned his B.A. in math and philosophy and M.A. in math at the State University of New York at Buffalo, where he was Phi Beta Kappa and a Woodburn Graduate Fellow. He worked for 7 years as an actuary in the life and health insurance industry before beginning to teach full time. He served as Director of Strategic Planning for Health Services Medical Corp. of Central New York, Syracuse, where he earlier served as Rating and Underwriting Manager. He also worked as an actuarial analyst for Columbian Financial Group, Binghamton, New York.

Tim lives in Fairport, New York, with his wife and two daughters.

Preface to Student

"Money is the root of all evil"—so the old adage goes. Whether we agree with that sentiment or not, we have to admit that if money is an evil, it is a *necessary* one. Love it or hate it, money plays a central role in the world and in our lives, both professional and personal. We all have to earn livings and pay bills, and to accomplish our goals, whatever they may be, reality requires us to manage the financing of those goals.

Sadly, though, financial matters are often poorly understood, and many otherwise promising ventures fail as a result of financial misunderstandings or misjudgments. A talented chef can open an outstanding restaurant, first rate in every way, only to see the doors closed as a result of financial shortcomings. An inventor with a terrific new product can nonetheless fail to bring it to market because of inadequate financing. An entrepreneur with an outstanding vision for a business can still fail to profit from it if savvier competition captures the same market with an inferior product but better management of the dollars and cents. And, on a more personal level, statistics continually show that "financial problems" are one of the most commonly cited causes of divorce in the United States.

Of course nothing in this book can guarantee you a top-rated restaurant, world-changing new product, successful business, or happy marriage. Yet, it is true that a reasonable understanding of money matters can certainly be a big help in achieving whatever it is you want to achieve in this life. It is also true that mathematics is a tool essential to this understanding. The goal of this book is to equip you with a solid understanding of the basic mathematical skills necessary to navigate the world of money.

Now, unfortunately (from my point of view at least), while not everyone would agree that money is root of all evil, it is not hard to find people who believe that mathematics is. Of course while some students come to a business math course with positive feelings toward the subject, certainly many more start off with less than warm and cozy feelings. Whichever camp you fall into, it is important to approach this book and the course it is being used for with an open mind. Yes, this is mathematics, but it is mathematics being put to a specific use. You may not fall in love with it, but you may find that studying math in the context of business and finance makes skills that once seemed painfully abstract do fall together in a way that makes sense.

Those who do not master money are mastered by it. Even if the material may occasionally be frustrating, hang in there! There is a payoff for the effort, and whether it comes easily or not, it will come if you stick with it.

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WALKTHROUGH

The Mathematics of Money: Math for Business and Personal Finance is designed to provide a sound introduction to the uses of mathematics in business and personal finance applications. It has dual objectives of teaching both mathematics and financial literacy. The text wraps each skill or technique it teaches in a real-world context that shows you the *reason* for the mathematics you're learning.

HOW TO USE THIS BOOK

This book includes several key pedagogical features that will help you learn the skills needed to succeed in your course. Watch for these features as you read, and use them for review and practice.

FORMULAS

Core formulas are presented in formal, numbered fashion for easy reference.

EXAMPLES

Examples, using realistic businesses and situations, walk you through the application of a formula or technique to a specific, realistic problem.

DEFINITIONS

Core concepts are called out and defined formally and numbered for easy reference.

Throughout the text, key terms or concepts are set in color boldface italics within the paragraph and defined contextually.

The same logic applies to discount. If a $500 note is discounted by $20, it stands to reason that a $5,000 note should be discounted by $200. If a 6-month discount note is discounted by $80, it stands to reason that a 12-month note would be discounted by $160. Thus, modeling from what we did for interest, we can arrive at:

FORMULA 2.1
The Simple Discount Formula

$$D = MdT$$

where
D represents the amount of simple DISCOUNT for a loan,
M represents the MATURITY VALUE
d represents the interest DISCOUNT RATE (expressed as a decimal)
and
T represents the TERM for the loan

The simple discount formula closely mirrors the simple interest formula. The differences lie in the letters used (D rather than I and d in place of R, so that we do not confuse discount with interest) and in the fact that the discount is based on maturity value rather than on principal. Despite these differences, the resemblance between simple interest and simple discount should be apparent, and it should not be surprising that the mathematical techniques we used with simple interest can be equally well employed with simple discount.

Example 8.3.1 ***Ampersand Computers bought 12 computers from the manufacturer. The list price for the computers is $895.00, and the manufacturer offered a 25% trade discount. How much did Ampersand pay for the computers?***

As with markdown, we can either take 25% of the price and subtract, or instead just multiply the price by 75% (found by subtracting 25% from 100%). The latter approach is a bit simpler: (75%)($895.00) = $671.25 per computer. The total price for all 12 computers would be (12)($671.25) = $8,055.

Even though it is more mathematically convenient to multiply by 75%, there are sometimes reasons to work things out the longer way. When the manufacturer bills Ampersand for this purchase, it would not be unusual for it to show the amount of this discount as a separate item. (The bill is called an ***invoice***, and the net cost for an item is therefore sometimes called the ***invoice price***.) In addition, the manufacturer may add charges for shipping or other fees on top of the cost of the items purchased (after the discount is applied). The invoice might look something like this:

International Difference Engines Invoice No. 1207
Box 404
Marbleburg, North Carolina 20252
INVOICE

Sold To:
Ampersand Computers
4539 North Henley Street
Olean, NY 14760

Date: May 28, 2007
Order #: 90125
Shipped: May 17, 2007

Quantity	Product #	Description	MSRP	Total
12	87435-G	IDE-Model G Laptop	$895.00	$10,740.00
			Subtotal	$10,740.00
			LESS: 25% discount	($2,685.00)
			Net	$8,055.00
			PLUS: Freight	$350.00
			Total due	$8,405.00

The discount may sometimes be written in parentheses (as it is in the example above) because this is a commonly used way of indicating a negative or subtracted number in

Definition 1.1.1
Interest *is what a borrower pays a lender for the temporary use of the lender's money.*

Or, in other words:

Definition 1.1.2
Interest *is the "rent" that a borrower pays a lender to use the lender's money.*

Interest is paid in addition to the repayment of the amount borrowed. In some cases, the amount of interest is spelled out explicitly. If we need to determine the total amount to be repaid, we can simply add the interest on to the amount borrowed.

One question that may come up here is how we know whether that 8½% interest rate quoted is the rate per year or the rate for the entire term of the loan. After all, the problem says the interest rate is 8½% for 3 years, which could be read to imply that the 8½% covers the entire 3-year period (in which case we would not need to multiply by 3).

The answer is that *unless it is clearly stated otherwise, interest rates are always assumed to be rates per year*. When someone says that an interest rate is 8½%, it is understood that this is the rate *per year*. Occasionally, you may see the Latin phrase ***per annum*** used with interest rates, meaning *per year* to emphasize that the rate is per year. You should not be confused by this, and since we are assuming rates are per year anyway, this phrase can usually be ignored.

The Simple Interest Formula

EXERCISES THAT BUILD BOTH SKILLS AND CONFIDENCE

Each section of every chapter includes a set of exercises that gives you the opportunity to practice and master the skills presented in the section. These exercises are organized in three groupings, designed to build your skills and your confidence so that you can master the material.

BUILDING FOUNDATIONS

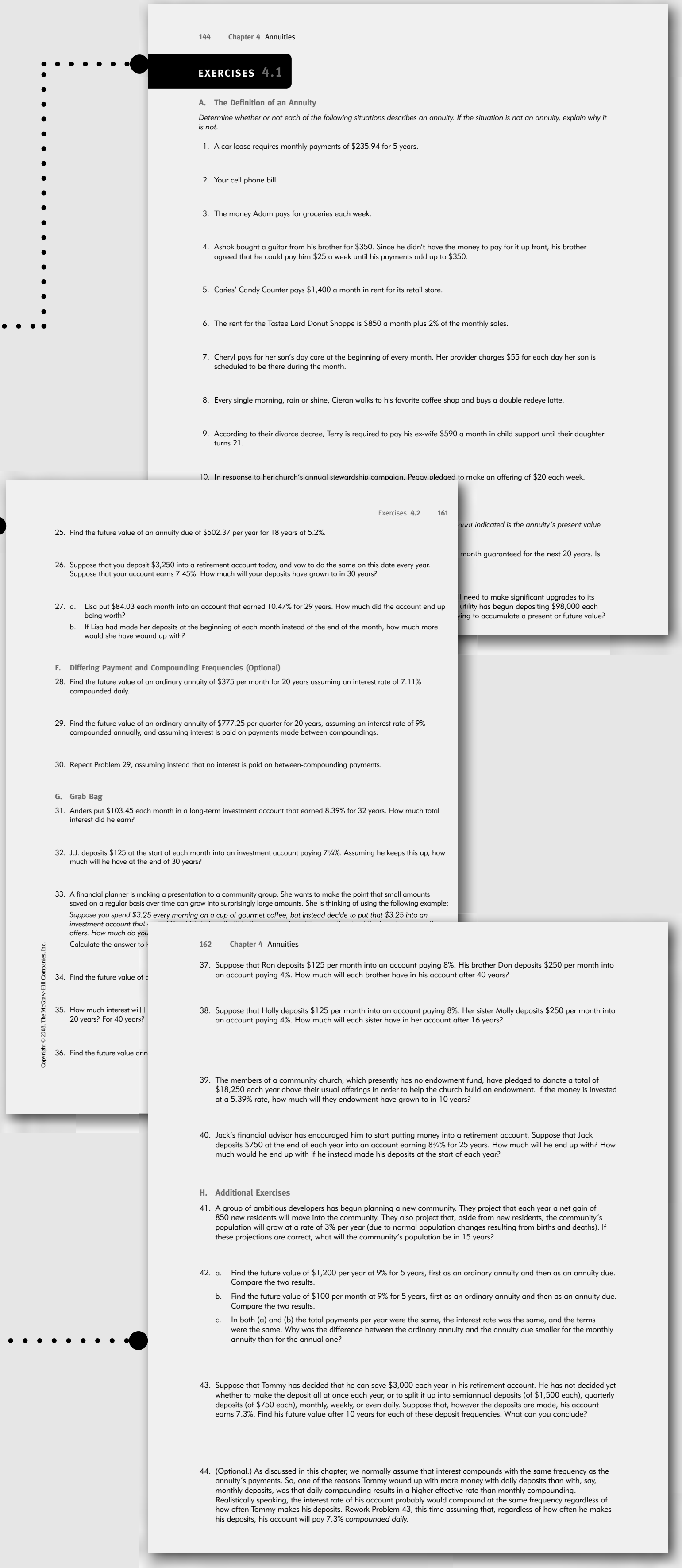
In each exercise set, there are several initial groupings of exercises under a header that identifies the type of problems that will follow and gives a good hint of what type of problem it is.

BUILDING CONFIDENCE

In each set there is also a grouping of exercises labeled "Grab Bag." These sections contain a mix of problems covering the various topics of the section, in an intentionally jumbled order. These exercises add an additional and very important layer of problem solving: identifying the type of problem and selecting an appropriate solution technique.

EXPANDING THE CONCEPTS

Each section's exercise set has one last grouping, labeled "Additional Exercises." These are problems that go beyond a standard problem for the section in question. This might mean that some additional concepts are introduced, certain technicalities are dealt with in greater depth, or that the problem calls for using a higher level of algebra than would otherwise be expected in the course.



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EXERCISES 4.1

A. The Definition of an Annuity

Determine whether or not each of the following situations describes an annuity. If the situation is not an annuity, explain why it is not.

1. A car lease requires monthly payments of $235.94 for 5 years.
2. Your cell phone bill.
3. The money Adam pays for groceries each week.
4. Ashok bought a guitar from his brother for $350. Since he didn't have the money to pay for it up front, his brother agreed that he could pay him $25 a week until his payments add up to $350.
5. Caries' Candy Counter pays $1,400 a month in rent for its retail store.
6. The rent for the Tastee Lard Donut Shoppe is $850 a month plus 2% of the monthly sales.
7. Cheryl pays for her son's day care at the beginning of every month. Her provider charges $55 for each day her son is scheduled to be there during the month.
8. Every single morning, rain or shine, Cieran walks to his favorite coffee shop and buys a double redeye latte.
9. According to their divorce decree, Terry is required to pay his ex-wife $590 a month in child support until their daughter turns 21.
10. In response to her church's annual stewardship campaign, Peggy pledged to make an offering of $20 each week.

Exercises 4.2 161

25. Find the future value of an annuity due of $502.37 per year for 18 years at 5.2%.
26. Suppose that you deposit $3,250 into a retirement account today, and vow to do the same on this date every year. Suppose that your account earns 7.45%. How much will your deposits have grown to in 30 years?
27. a. Lisa put $84.03 each month into an account that earned 10.47% for 29 years. How much did the account end up being worth?
    b. If Lisa had made her deposits at the beginning of each month instead of the end of the month, how much more would she have wound up with?

F. Differing Payment and Compounding Frequencies (Optional)

28. Find the future value of an ordinary annuity of $375 per month for 20 years assuming an interest rate of 7.11% compounded daily.
29. Find the future value of an ordinary annuity of $777.25 per quarter for 20 years, assuming an interest rate of 9% compounded annually, and assuming interest is paid on payments made between compoundings.
30. Repeat Problem 29, assuming instead that no interest is paid on between-compounding payments.

G. Grab Bag

31. Anders put $103.45 each month in a long-term investment account that earned 8.39% for 32 years. How much total interest did he earn?
32. J.J. deposits $125 at the start of each month into an investment account paying 7¼%. Assuming he keeps this up, how much will he have at the end of 30 years?
33. A financial planner is making a presentation to a community group. She wants to make the point that small amounts saved on a regular basis over time can grow into surprisingly large amounts. She is thinking of using the following example: *Suppose you spend $3.25 every morning on a cup of gourmet coffee, but instead decide to put that $3.25 into an investment account that ...*

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37. Suppose that Ron deposits $125 per month into an account paying 8%. His brother Don deposits $250 per month into an account paying 4%. How much will each brother have in his account after 40 years?
38. Suppose that Holly deposits $125 per month into an account paying 8%. Her sister Molly deposits $250 per month into an account paying 4%. How much will each sister have in her account after 16 years?
39. The members of a community church, which presently has no endowment fund, have pledged to donate a total of $18,250 each year above their usual offerings in order to help the church build an endowment. If the money is invested at a 5.39% rate, how much will they endowment have grown to in 10 years?
40. Jack's financial advisor has encouraged him to start putting money into a retirement account. Suppose that Jack deposits $750 at the end of each year into an account earning 8¾% for 25 years. How much will he end up with? How much would he end up with if he instead made his deposits at the start of each year?

H. Additional Exercises

41. A group of ambitious developers has begun planning a new community. They project that each year a net gain of 850 new residents will move into the community. They also project that, aside from new residents, the community's population will grow at a rate of 3% per year (due to normal population changes resulting from births and deaths). If these projections are correct, what will the community's population be in 15 years?
42. a. Find the future value of $1,200 per year at 9% for 5 years, first as an ordinary annuity and then as an annuity due. Compare the two results.
    b. Find the future value of $100 per month at 9% for 5 years, first as an ordinary annuity and then as an annuity due. Compare the two results.
    c. In both (a) and (b) the total payments per year were the same, the interest rate was the same, and the terms were the same. Why was the difference between the ordinary annuity and the annuity due smaller for the monthly annuity than for the annual one?
43. Suppose that Tommy has decided that he can save $3,000 each year in his retirement account. He has not decided yet whether to make the deposit all at once each year, or to split it up into semiannual deposits (of $1,500 each), quarterly deposits (of $750 each), monthly, weekly, or even daily. Suppose that, however the deposits are made, his account earns 7.3%. Find his future value after 10 years for each of these deposit frequencies. What can you conclude?
44. (Optional.) As discussed in this chapter, we normally assume that interest compounds with the same frequency as the annuity's payments. So, one of the reasons Tommy wound up with more money with daily deposits than with, say, monthly deposits, was that daily compounding results in a higher effective rate than monthly compounding. Realistically speaking, the interest rate of his account probably would compound at the same frequency regardless of how often Tommy makes his deposits. Rework Problem 43, this time assuming that, regardless of how often he makes his deposits, his account will pay 7.3% *compounded daily.*

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ICONS

Throughout the **core chapters**, certain icons appear, giving you visual cues to examples or discussions dealing with several key kinds of business situations.

r — **retail**

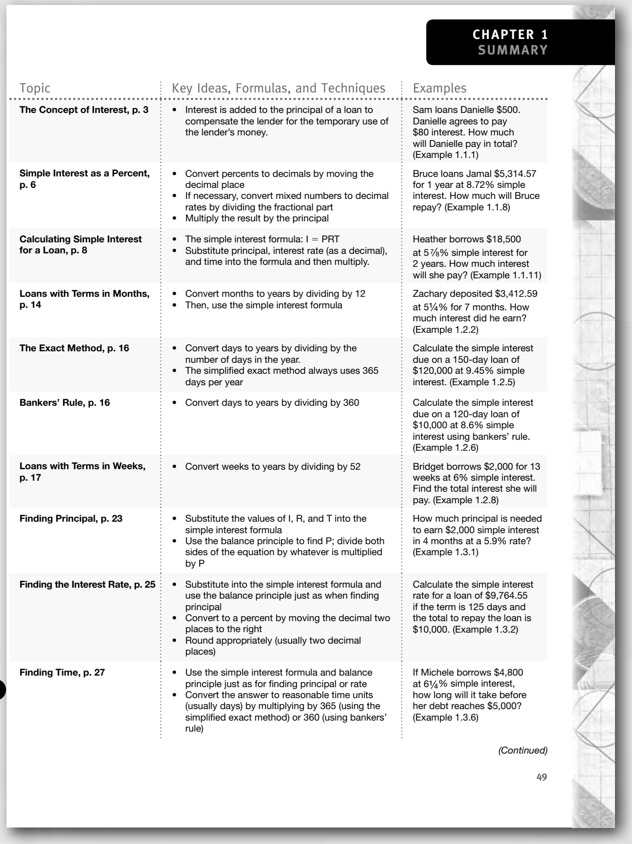
i — **insurance**

f — **finance**

b — **banking**

END-OF-CHAPTER SUMMARIES

Each chapter ends with a table summarizing the major topics covered, the key ideas, formulas, and techniques presented, and examples of the concepts. Each entry in the table has page references that point you back to where the material was in the chapter, making reviewing the key concepts easier.



CHAPTER 1 SUMMARY

Topic	Key Ideas, Formulas, and Techniques	Examples
The Concept of Interest, p. 3	• Interest is added to the principal of a loan to compensate the lender for the temporary use of the lender's money.	Sam loans Danielle $500. Danielle agrees to pay $80 interest. How much will Danielle pay in total? (Example 1.1.1)
Simple Interest as a Percent, p. 6	• Convert percents to decimals by moving the decimal place • If necessary, convert mixed numbers to decimal rates by dividing the fractional part • Multiply the result by the principal	Bruce loans Jamal $5,314.57 for 1 year at 8.72% simple interest. How much will Bruce repay? (Example 1.1.8)
Calculating Simple Interest for a Loan, p. 8	• The simple interest formula: I = PRT • Substitute principal, interest rate (as a decimal), and time into the formula and then multiply.	Heather borrows $18,500 at $5\frac{7}{8}$% simple interest for 2 years. How much interest will she pay? (Example 1.1.11)
Loans with Terms in Months, p. 14	• Convert months to years by dividing by 12 • Then, use the simple interest formula	Zachary deposited $3,412.59 at $5\frac{1}{4}$% for 7 months. How much interest did he earn? (Example 1.2.2)
The Exact Method, p. 16	• Convert days to years by dividing by the number of days in the year. • The simplified exact method always uses 365 days per year	Calculate the simple interest due on a 150-day loan of $120,000 at 9.45% simple interest. (Example 1.2.5)
Bankers' Rule, p. 16	• Convert days to years by dividing by 360	Calculate the simple interest due on a 120-day loan of $10,000 at 8.6% simple interest using bankers' rule. (Example 1.2.6)
Loans with Terms in Weeks, p. 17	• Convert weeks to years by dividing by 52	Bridget borrows $2,000 for 13 weeks at 6% simple interest. Find the total interest she will pay. (Example 1.2.8)
Finding Principal, p. 23	• Substitute the values of I, R, and T into the simple interest formula • Use the balance principle to find P; divide both sides of the equation by whatever is multiplied by P	How much principal is needed to earn $2,000 simple interest in 4 months at a 5.9% rate? (Example 1.3.1)
Finding the Interest Rate, p. 25	• Substitute into the simple interest formula and use the balance principle just as when finding principal • Convert to a percent by moving the decimal two places to the right • Round appropriately (usually two decimal places)	Calculate the simple interest rate for a loan of $9,764.55 if the term is 125 days and the total to repay the loan is $10,000. (Example 1.3.2)
Finding Time, p. 27	• Use the simple interest formula and balance principle just as for finding principal or rate • Convert the answer to reasonable time units (usually days) by multiplying by 365 (using the simplified exact method) or 360 (using bankers' rule)	If Michele borrows $4,800 at $6\frac{1}{4}$% simple interest, how long will it take before her debt reaches $5,000? (Example 1.3.6)

(Continued)

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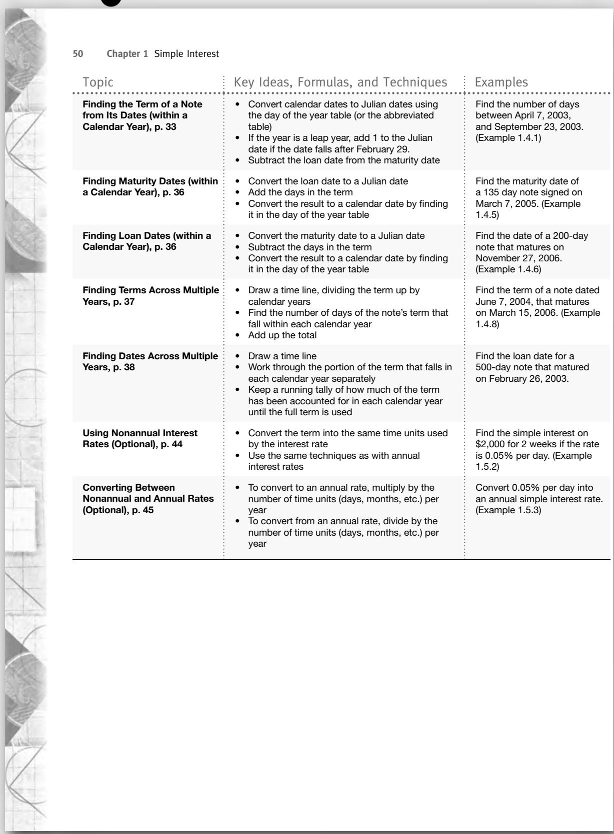

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Topic	Key Ideas, Formulas, and Techniques	Examples
Finding the Term of a Note from Its Dates (within a Calendar Year), p. 33	• Convert calendar dates to Julian dates using the day of the year table (or the abbreviated table) • If the year is a leap year, add 1 to the Julian date if the date falls after February 29. • Subtract the loan date from the maturity date	Find the number of days between April 7, 2003, and September 23, 2003. (Example 1.4.1)
Finding Maturity Dates (within a Calendar Year), p. 36	• Convert the loan date to a Julian date • Add the days in the term • Convert the result to a calendar date by finding it in the day of the year table	Find the maturity date of a 135 day note signed on March 7, 2005. (Example 1.4.5)
Finding Loan Dates (within a Calendar Year), p. 36	• Convert the maturity date to a Julian date • Subtract the days in the term • Convert the result to a calendar date by finding it in the day of the year table	Find the date of a 200-day note that matures on November 27, 2006. (Example 1.4.6)
Finding Terms Across Multiple Years, p. 37	• Draw a time line, dividing the term up by calendar years • Find the number of days of the note's term that fall within each calendar year • Add up the total	Find the term of a note dated June 7, 2004, that matures on March 15, 2006. (Example 1.4.8)
Finding Dates Across Multiple Years, p. 38	• Draw a time line • Work through the portion of the term that falls in each calendar year separately • Keep a running tally of how much of the term has been accounted for in each calendar year until the full term is used	Find the loan date for a 500-day note that matured on February 26, 2003.
Using Nonannual Interest Rates (Optional), p. 44	• Convert the term into the same time units used by the interest rate • Use the same techniques as with annual interest rates	Find the simple interest on $2,000 for 2 weeks if the rate is 0.05% per day. (Example 1.5.2)
Converting Between Nonannual and Annual Rates (Optional), p. 45	• To convert to an annual rate, multiply by the number of time units (days, months, etc.) per year • To convert from an annual rate, divide by the number of time units (days, months, etc.) per year	Convert 0.05% per day into an annual simple interest rate. (Example 1.5.3)

Acknowledgments

Any project of this scope involves more people than the one whose name is printed on the cover, and this book is no exception.

For their support and the many helpful suggestions they offered, I would like to particularly thank Len Malinowski, Joe Shulman, and Mike Prockton. I would also like to thank my current colleagues and predecessors in the Math Department at Finger Lakes Community College. I owe a debt of gratitude to John Caraluzzo and the other faculty who preceded me at FLCC, for their work to develop the business math course that led to this book.

This book has undergone several rounds of reviews by instructors who are out there in the trenches, teaching this material. Each of them, with their thoughts and insights, helped improve this book.

Yvonne Alder, *Central Washington University–Ellensburg*

Kathy Boehler, *Central Community College*

Julliana R. Brey, *Cardinal Stritch University*

Bruce Broberg, *Central Community College*

Kelly Bruning, *Northwestern Michigan College*

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Dr. Kelly Bruning has been involved in this book since her initial review. In addition to all the useful feedback she's given me, she has also provided error checking on the manuscript and created the test bank that accompanies this book. I thank her for her support and contribution.

While I'm thanking people, I'd like to take a moment to acknowledge my book team at McGraw-Hill: Executive Editor Dick Hercher, Developmental Editor Cynthia Douglas, Senior Marketing Manager Sankha Basu, Marketing Coordinator Dean Karampelas, Senior Project Manager Susanne Riedell, Designer Artemio Ortiz, Copy Editor George F. Watson, Media Technology Producer Xin Zhu, Media Project Manager Matthew Perry, Production Supervisor Gina Hangos, and Editorial Director Stewart Mattson.

Above all, I'd like to thank my family for their love and support.

Tim Biehler

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Brief Contents

Table of Contents

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CORE MATHEMATICAL TOOLS 1

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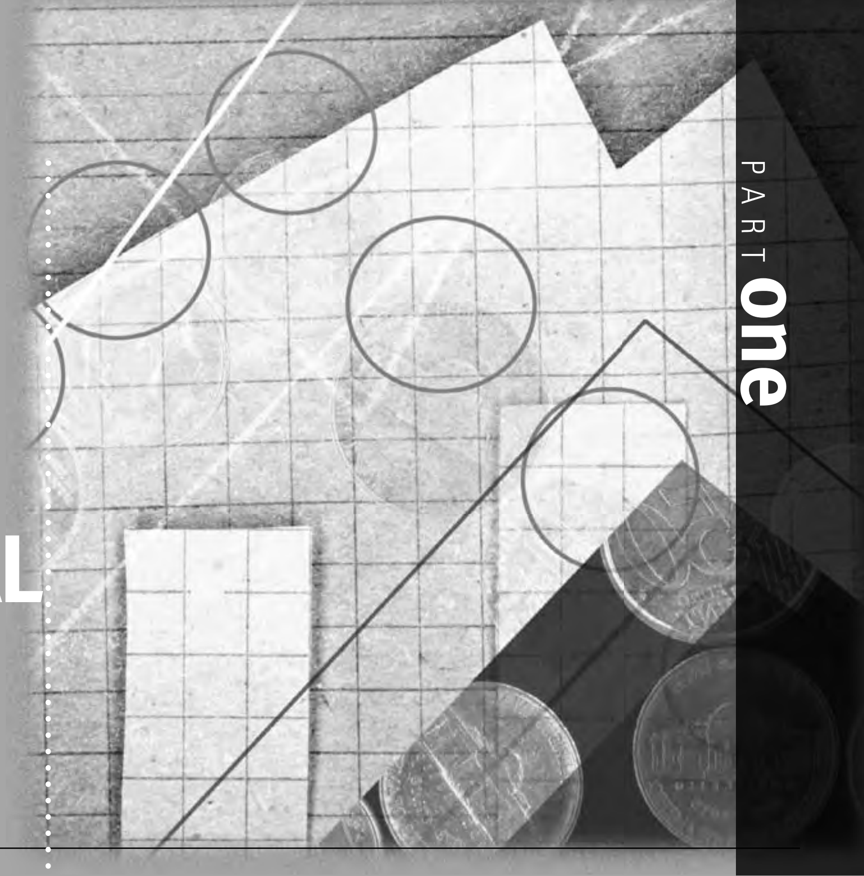
PART one



CORE MATHEMATICAL TOOLS

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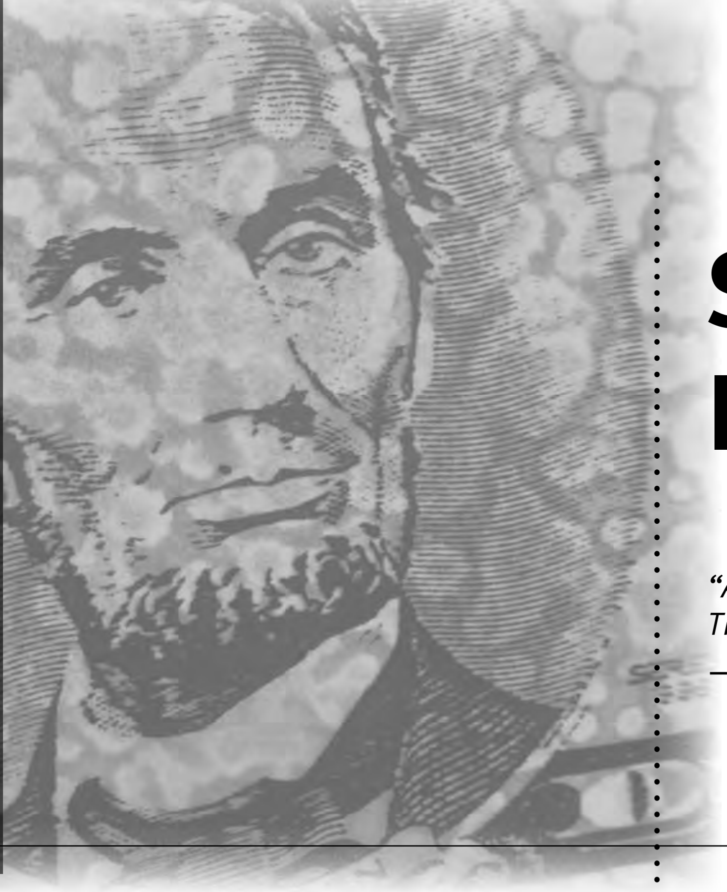
CHAPTER 1



Simple Interest

"And why do they call it ***interest?*** *There's nothing interesting about it!"*

—Coach Ernie Pantuso, "Cheers"

Learning Objectives

LO 1 Understand the concept of the time value of money, and recognize the reasoning behind the payment of interest.

LO 2 Calculate the amount of simple interest for a given loan.

LO 3 Use the simple interest formula together with basic algebra techniques to find the principal, simple interest rate, or term, given the other details of a loan.

LO 4 Determine the number of days between any two calendar dates.

LO 5 Apply these skills and concepts to real-world financial situations such as promissory notes.

Chapter Outline

1.1 Simple Interest and the Time Value of Money

Suppose you own a house and agree to move out and let me live there for a year. I promise that while I'm living there I will take care of any damages and make any needed repairs, so at the end of the year you'll get back the exact same house, in exactly the same condition, in the exact same location. Now since I will be returning your property to you exactly the same as when you lent it to me, in some sense at least you've lost nothing by letting me have it for the year.

Despite this, though, you probably wouldn't be willing to let me live there for the year for free. Even though you'll get the house back at the end just as it was at the start, you'd still expect to be paid something for a year's *use* of your house. After all, though you wouldn't actually give up any of your property by lending it to me, you nonetheless would be giving up something: the opportunity to live in your house during the year that I am there. It is only fair that you should be paid for the property's temporary use. In other, ordinary terms, you'd expect to be paid some rent. There is nothing surprising in this. We are all familiar with the idea of paying rent for a house or apartment. And the same idea applies for other types of property as well; we can rent cars, or party tents, or construction equipment, and many other things as well.

Now let's suppose that I need to borrow $20, and you agree to lend it to me. If I offered to pay you back the full $20 one year from today, would you agree to the loan under those terms? You would be getting your full $20 back, but it hardly seems fair that you wouldn't get any other compensation. Just as in the example of the house, even though you will eventually get your property back, over the course of the year you won't be able to use it. Once again it only seems fair that you should get some benefit for giving up the privilege of having the use of what belongs to you.

We ordinarily call the payment for the temporary use of property such as houses, apartments, equipment, or vehicles *rent*. In the case of money, though, we don't normally use that term. Instead we call that payment ***interest***.

Definition 1.1.1

***Interest** is what a borrower pays a lender for the temporary use of the lender's money.*

Or, in other words:

Definition 1.1.2

***Interest** is the "rent" that a borrower pays a lender to use the lender's money.*

Interest is paid in addition to the repayment of the amount borrowed. In some cases, the amount of interest is spelled out explicitly. If we need to determine the total amount to be repaid, we can simply add the interest on to the amount borrowed.

Example 1.1.1 Sam loans Danielle $500 for 100 days. Danielle agrees to pay her $80 interest for the loan. How much will Danielle pay Sam in total?

Interest is added onto the amount borrowed. $500 + $80 = $580. Therefore Danielle will pay Sam a total of $580 at the end of the 100 days.

In other cases, the borrower and lender may agree on the amount borrowed and the amount to be repaid without explicitly stating the amount of interest. In those cases, we can determine the amount of interest by finding the difference between the two amounts (in other words, by subtracting.)

Example 1.1.2 Tom loans Larry $200, agreeing to repay the loan by giving Larry $250 in 1 year. How much interest will Larry pay?

The interest is the difference between what Tom borrows and what he repays. $250 – $200 = $50. So Larry will pay a total of $50 in interest.

It is awkward to have to keep saying "the amount borrowed" over and over again, and so we give this amount a specific name.

Definition 1.1.3

*The **principal** of a loan is the amount borrowed.*

So in Example 1.1.1 the principal is $500. In Example 1.1.2 we would say that the principal is $200 and the interest is $50.

There are a few other special terms that are used with loans as well.

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Definition 1.1.4

*A **debtor** is someone who owes someone else money. A **creditor** is someone to whom money is owed.*

In Example 1.1.1 Sam is Danielle's creditor and Danielle is Sam's debtor. In Example 1.1.2 we would say that Tom is Larry's creditor and Larry is Tom's debtor.

Definition 1.1.5

*The amount of time for which a loan is made is called its **term**.*

In Example 1.1.1 the term is 100 days. In Example 1.1.2 the term of the loan is 1 year.

Interest Rates as Percents

Let's reconsider Tom and Larry's loan from Example 1.1.2 for a moment. Tom and Larry have agreed that the interest Tom will charge for a loan is $50. Now suppose Larry decides that, instead of borrowing $200, he needs to borrow $1,000. He certainly can't expect that Tom will still charge the same $50 interest! Common sense screams that for a larger loan Tom would demand larger interest. In fact, it seems reasonable that for 5 times the loan, he would charge 5 times as much interest, or $250.

By the same token, if this loan were for $200,000 (one thousand times the original principal) we could reasonably expect that the interest would be $50,000 (one thousand times the original interest.) The idea here is that, as the size of the principal is changed, the amount of interest should also change in the same proportion.

For this reason, interest is often expressed as a percent. The interest Tom was charging Larry was ¼ of the amount he borrowed, or 25%. If Tom expresses his interest charge as a percent, then we can determine how much he will charge Larry for *any* size loan.

Example 1.1.3 ***Suppose that Larry wanted to borrow $1,000 from Tom for 1 year. How much interest would Tom charge him?***

Tom is charging 25% interest, and 25% (or ¼) of $1,000 is $250. So Tom would charge $250 interest. Note that $250 is also 5 times $50, and so this answer agrees with our commonsense assessment!

Of course, the situation here is simplified by the fact that 25% of $1,000 is not all that hard to figure out. With less friendly numbers, the calculation becomes a bit trickier. What if, for example, we were trying to determine the amount of interest for a loan of $1835.49 for 1 year at 11.35% simple interest? The idea should be the same, though the calculation requires a bit more effort.

Working with Percents

When we talk about percents, we usually are taking a percent of *something*. The mathematical operation that translates the "of" in that expression is multiplication. So, to find 25% of $1,000, we would *multiply 25% times $1,000*.

However, if I simply multiply 25 times 1,000 on my calculator, I get 25,000, which is far too big and also does not agree with the answer of $250 which we know is correct. The reason for this discrepancy is that 25% is not the same as the number 25. The word *percent* comes from Latin, and means "out of 100." So when we say "25%," what we really mean is "25 out of 100"—or in other words 25/100.

If you divide 25/100 on a calculator, the result is 0.25. This process of converting a percent into its real mathematical meaning is often called **converting the percent to a decimal.**

It is not necessary, though, to bother with dividing by 100 every time we need to use a percent. Notice that when we divided 25 by 100, the result still had the same 25 in it, just with a differently placed decimal. Now we don't normally bother writing in a decimal place with whole numbers, but we certainly can. 25 can be written as "25."; now 0.25 is precisely what you would have gotten by moving that decimal two places to the left. This is not a coincidence, and in fact we can always convert percents into their decimal form simply by moving the decimal place.

So, when using percents, we can either go to the trouble of actually dividing by 100, or instead we can just move the decimal place.

Example 1.1.4 ***Convert 25% to a decimal.***

By dividing: 25% = 25/100 = 0.25

By moving the decimal: 25% = 0.25

Why did we place that extra zero to the left of the decimal? The zero placed to the left of the decimal place is not really necessary. It would be just as good to have written ".25". Tacking on this zero does not change the numerical value in any way. It only signifies that there is nothing to the left of the decimal. There is no mathematical reason to prefer "0.25" over ".25" or vice versa; they both mean exactly the same thing. However, we will often choose to tack on the zero because the decimal point is so small and easy to miss. It is not hard to miss that tiny decimal point on the page and so .25 can be easily mistaken for 25. This tiny oversight can lead to enormous errors; 0.25 is far less likely to be misread.

Example 1.1.5 ***Convert 18.25% to a decimal.***

Moving the decimal two places to the left we see that 18.25% = 0.1825.

Example 1.1.6 ***Convert 5.79% to a decimal.***

Here, there aren't two numbers to the left of the decimal. Simply moving the decimal point two places to the left would leave us with "0._579". The blank space is obviously a problem. We deal with it by placing a 0 in that position to "hold the space." So 5.79% = 0.0579.

Let's put this all together to recalculate the interest on Larry's $1,000 loan once again.

Example 1.1.7 ***Rework Example 1.1.3, this time by converting the interest rate percent to a decimal and using it.***

We have seen that 25% = 0.25, and that to use it we multiply it by the principal. Thus:

*Interest = Principal * Interest Rate as a decimal*
*Interest = $1,000 * 0.25*
Interest = $250

This answer agrees with our previous calculations.

Notation for Multiplication

There are a number of different ways to indicate multiplication. Probably the most familiar is the × symbol, though the asterisk * that we used above is also widely used, especially with computers. It is also a standard mathematical convention that, when no symbol is written between two quantities, multiplication is assumed. From this point forward, we will be following that convention. To indicate "1,000 times 0.25" we will write:

(1,000)(0.25)

The parentheses are used to make the separation between the numbers clear. If we simply wrote the two numbers next to each other without them, "1,000 0.25" could be easily misread as the single number "10,000.25". However, we don't really need both sets of parentheses to avoid this, and so we could equally well put parentheses around only one of the numbers. So, to indicate "1,000 times 0.25" we may write any of the following:

(1000)(0.25) *or* (1000)0.25 *or* 1000(0.25).

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Back to Percents

So now let's return to the problem proposed a while back of determining 11.35% interest on an $1,835.49 loan. We must convert 11.35% to a decimal, which gives us 0.1135, and then multiply by the amount borrowed. So we get:

Interest = (Principal)(Interest Rate as a decimal)
Interest = ($1,835.49)(0.1135)
Interest = $208.33

Actually, multiplying these two numbers yields $208.32811. Since money is measured in dollars and cents, though, it's pretty clear that we should round the final answer to two decimal places. We will follow the usual rounding rules, standard practice in both mathematics and in business. To round to two decimal places, we look at the third. If the number there is 5 or higher, we "round up," moving the value up to the next higher penny. This is what we did above. Since the number in the third decimal place is an 8, we rounded our final answer up to the next penny. If the number in the third decimal place is 4 or lower, though, we "round down," leaving the pennies as is and throwing out the extra decimal places.

Example 1.1.8 ***Suppose Bruce loans Jamal $5,314.57 for 1 year. Jamal agrees to pay 8.72% interest for the year. How much will he pay Bruce when the year is up?***

First we need to convert 8.72% into a decimal. So we rewrite 8.72% as 0.0872. Then:

Interest = (Principal)(Interest Rate as a decimal)
Interest = ($5,314.57)(0.0872)
Interest = $463.43

Actually, the result of multiplying was 463.4305, but since the number in the third decimal place was not five or higher, we threw out the extra decimal places to get $463.43.

We are not done yet. The question asked how much Jamal will pay Bruce in the end, and so we need to add the interest to the principal. So Tom will pay $5,314.57 + $463.43 = $5,778.00.

Mixed Number and Fractional Percents

It is not unusual for interest rates to be expressed as mixed numbers or fractions, such as 5¾% or 8⅜%. Decimal percents like those in 5.75% and 8.375% might be preferable, and they are becoming the norm, but for historical and cultural reasons, mixed number percents are still quite common. In particular, rates are often expressed in terms of halves, quarters, eighths, or sixteenths of a percent.[1]

Some of these are quite easy to deal with. For example, a rate of 4½% is easily rewritten as 4.5%, and then changed to a decimal by moving the decimal two places to the right to get 0.045.

However, fractions whose decimal conversions are not such common knowledge require a bit more effort. A simple way to deal with these is to convert the fractional part to a decimal by dividing with a calculator. For example, to convert 9⅝% to a decimal, first divide 5/8 to get 0.625. Then replace the fraction in the mixed number with its decimal equivalent to get 9.625%, and move the decimal two places to get 0.09625.

Example 1.1.9 ***Rewrite 7 13/16% as a decimal.***

13/16 = 0.8125, and so 7 13/16% = 7.8125% = 0.078125.

[1]The use of these fractions is supposed to have originated from the Spanish "pieces of eight" gold coin, which could be broken into eight pieces. Even though those coins haven't been used for hundreds of years, tradition is a powerful thing, and the tradition of using these fractions in the financial world has only recently started to fade. Until only a few years ago, for example, prices of stocks in the United States were set using these fractions, though stock prices are now quoted in dollars and cents. It is likely that the use of fractions will continue to decline in the future, but for the time being, mixed number rates are still in common use.

The Impact of Time

Let's return again to Tom and Larry. Suppose that Larry returns to the original plan of borrowing $200, but instead of paying it back in 1 year, he offers to pay it back in 2 years. Could he reasonably expect to still pay the same $50 interest, even though the loan is now for twice as long?

The answer is obviously no. Of course, Tom should receive more interest for letting Larry have the use of his money for a longer term. Once again, though, common sense suggests the proper way to deal with this. If the loan is for twice as long, it seems reasonable that Larry would pay twice as much interest. Thus, if the loan is extended to 2 years, Larry would pay (2)($50) = $100 in interest.

Example 1.1.10 ***Suppose that Raeshawn loans Dianne $4,200 at a simple interest rate of 8½% for 3 years. How much interest will Dianne pay?***

We have seen that to find interest we need to multiply the amount borrowed times the interest rate, and also that since this loan is for 3 years we then need to multiply that result by 3. Combining these into a single step, we get:

Interest = (Amount Borrowed)(Interest Rate as a decimal)(Time)
Interest = ($4,200)(0.085)(3)
Interest = $1,071.00

One question that may come up here is how we know whether that 8½% interest rate quoted is the rate per year or the rate for the entire term of the loan. After all, the problem says the interest rate is 8½% for 3 years, which could be read to imply that the 8½% covers the entire 3-year period (in which case we would not need to multiply by 3).

The answer is that *unless it is clearly stated otherwise, interest rates are always assumed to be rates per year*. When someone says that an interest rate is 8½%, it is understood that this is the rate *per year*. Occasionally, you may see the Latin phrase ***per annum*** used with interest rates, meaning *per year* to emphasize that the rate is per year. You should not be confused by this, and since we are assuming rates are per year anyway, this phrase can usually be ignored.

The Simple Interest Formula

It should be apparent that regardless of whether the numbers are big, small, neat, or messy, the basic idea is the same. To calculate interest, we multiply the amount borrowed times the interest rate (as a decimal) times the amount of time. We can summarize this by means of a formula:

FORMULA 1.1
The Simple Interest Formula

$$I = PRT$$

where
I represents the amount of simple INTEREST for a loan
P represents the amount of money borrowed (the PRINCIPAL)
R represents the interest RATE (expressed as a decimal)
and
T represents the TERM of the loan

Since no mathematical operation is written between these letters, we understand this to be telling us to multiply. The parentheses that we put around numbers for the sake of clarity are not necessary with the letters.

At this point, it is not at all clear why the word *simple* is being thrown in. The reason is that the type of interest we have been discussing in this chapter is not the only type. Later on, in Chapter 3, we will see that there is more to the interest story, and at that point it will become clear why we are using the term "simple interest" instead of just "interest." In the

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meantime, though, we will do a bit of sweeping under the rug and simply not worry about the reason for the addition of the word *simple*.

Now, back to the formula. This formula is just a shorthand way of reminding us of what we've already observed: to calculate simple interest, multiply the principal times the rate as a decimal times the time. The formula summarizes that idea and also gives us a useful framework to help organize our thoughts when solving these types of problems. An example will illustrate this well.

Example 1.1.11 ***Heather borrows $18,500 at 5⅞% simple interest for 2 years. How much interest will she pay?***

The principal P = $18,500, the interest rate R = 0.05875, and the time T = 2 years. So we begin with our formula:

$$I = PRT$$

Replace each letter whose value we know:

$$I = (\$18{,}500)(0.05875)(2)$$

And then follow the formula's instruction and multiply:

$$I = \$2{,}173.75$$

The formula now tells us that I, the amount of interest, is $2,173.75.

Loans in Disguise

Sometimes interest is paid in situations we might not normally think of as loans. If you deposit money in a bank account, you probably expect to be paid interest, even though

Even though we don't usually think of it this way, a deposit is a loan. © Keith Brofsky/Getty Images/DIL

you probably don't think of your deposit as a loan. In reality, though, it actually *is* a loan. When you are depositing money to a bank account you are actually loaning that money to the bank. While it is in your account the bank has the use of it, and in fact does use it (to make loans to other people.)

There are many different types of bank deposits. Checking and savings accounts are familiar examples of ways in which we loan money to banks. These are sometimes referred to as ***demand accounts***, because you can withdraw your money any time that you want (i.e., "on demand.") Another common type of account is a ***certificate of deposit***, or ***CD***. When you deposit money into a CD, you agree to keep it on deposit at the bank for a fixed period of time. For this reason, CDs are often also referred to as ***term deposits*** or other similar names. CDs often offer better interest rates than checking or savings accounts, since with a CD the bank knows how long it will have the money, giving it more opportunity to take advantage of longer term loans on which it can collect higher interest rates.

Example 1.1.12 ***Jake deposited $2,318.29 into a 2-year CD paying 5.17% simple interest per annum. How much will his account be worth at the end of the term?***

Recall that the phrase per annum simply means per year.

$$I = PRT$$
$$I = (\$2,318.29)(0.0517)(2)$$
$$I = \$239.71$$

At the end of the term, his account will contain both the principal and interest, so the total value of the account will be $2,318.29 + $239.71 = $2,558.00.

There are many different types of financial institutions that offer checking and savings accounts, CDs, and other types of deposit accounts. Jake might have opened his CD at a ***savings and loan*** or ***credit union*** just as well as at a commercial bank. While there are differences in the range of services offered, eligibility to open accounts, and government regulation among these different types of institutions, the basic principles we are working with apply equally well to any of them. As is common practice in business, when we use the term *bank* in this book, it should be understood that we are not necessarily referring only to commercial banks, but to any sort of financial institution that offers loans and deposit accounts.

EXERCISES 1.1

A. Interest as Difference

1. Adrian borrowed $2,000 and paid back a total of $2,125. How much interest did he pay?

2. Sarah loaned Andrew $12,375 for 6 months. Andrew paid back $12,500. How much interest did he pay?

3. Kelli loaned Kerri $785.82, and 2 years later Kerri will pay back $854.29. How much total interest will Kelli receive?

4. Logan borrowed $24,318.79 and will have to repay a total of $27,174.25. How much interest will he pay?

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B. Adding Interest to Determine Repayment Amounts

5. Tony loaned Josh $2,000. Josh agreed to pay Tony $300 interest for this loan. How much will Josh pay back?

6. Hannah borrowed $4,200 from Fifth National Bank, agreeing to pay $400 in interest for this loan. How much will she pay in total?

7. Jonas is borrowing $249.76 from Katrina for 1 year, and has agreed to pay $35.50 in interest. How much will Katrina receive when he pays her back?

8. Haley has agreed to loan Taylor $85,529.68 and Taylor has agreed to pay $7,261.13 in interest. How much in total will Taylor have to give Haley when she repays the loan?

C. Terminology

In each of the following situations, identify (a) the principal, (b) the term, (c) the creditor, and (d) the debtor.

9. Jin's parents loaned her $2,500. She promised to pay them back $2,750 in 2 years.

10. Promethean Combustion Products borrowed $800,000 from Venture Capital Funding Corp. Three years from now Promethean will be required to pay back a total of $965,000.

D. Rewriting Percents as Decimals

11. Rewrite each of the following percent interest rates as decimals.

a.	12% ________	g.	9 ½% ________
b.	15.3% ________	h.	19 ¾% ________
c.	8% ________	i.	5 ⅝% ________
d.	4.35% ________	j.	20 7/16% ________
e.	0.75% ________	k.	7/8% ________
f.	125% ________	l.	375 ⅛% ________

E. Interest as a Percent (One Year Loans)

12. Taneisha is loaning Jim $12,000 for 1 year. They have agreed that the simple interest rate for this loan will be 8%. Find the total amount of interest Jim will pay.

13. Alonzo loaned Jeremy $325.18 for 1 year at a simple interest rate of 12 ⅝%. How much interest will Jeremy have to pay?

14. Samir borrowed $7,829.14 for 1 year at a simple interest rate of 9 ¾% per annum. How much will he need to repay the loan?

15. Terri has borrowed $8,200 for 1 year at a simple interest rate of 11.5% per annum. What is the total amount she will need to repay the loan?

F. Interest as a Percent (Multiple-Year Loans)

16. Westerman Capital Corp. loaned Milford Financial Inc. $100,000 for 2 years at 8% simple interest. How much interest will Milford Financial pay?

17. Kyle borrowed $800 from Gavin for 4 years at 5½% simple interest. How much interest will Kyle pay for this loan?

18. Reza borrowed $16,000 from Wiscoy Savings and Loan for 3 years at 9.65% simple interest. How much total interest will he pay?

19. Wendy loaned Tom $2,896.17 for 8 years at 6.74% simple interest per annum. How much total interest will Wendy earn?

20. Yushio is borrowing $3,525 from Houghtonville National Bank for 2 years at 12.6% simple interest. How much will he need to repay the loan?

21. Mary has agreed to loan Karen $1,125.37 for 5 years at 7⅞% simple interest. How much will Karen receive when the loan is repaid?

22. Tris borrowed $25,300 at 9¼% simple interest for 3 years. How much will he need to pay off this loan?

23. Glenys made a loan of $16,425.75 for 3 years at 14.79% simple interest. How much in total will she receive when the loan is repaid?

G. Grab Bag

24. Bob deposited $15,000 in a CD for 3 years paying 4.33% simple interest. How much total interest will he earn?

25. When I went out to lunch with a few coworkers last week I forgot my wallet. One of my coworkers paid my $10.75 check, and I paid her back $12 at the end of the week. How much interest did I pay?

26. June plans to deposit $800 in a certificate of deposit paying 6⅜% simple interest for 2 years. What will her CD be worth at the end of the term?

27. Hassan has decided to deposit $3,257.19 into a bank CD paying 3.25% simple interest for 1 year. What will the CD be worth at the end of the year?

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28. Larissa has opened a CD at Canandaigua Federal Bank by depositing $27,392.04. The term of the CD is 4 years, and it pays 5.44% simple interest. How much will she have in this account at the end of the term?

29. The Village of West Rochester made a short-term deposit of $476,903 in a local bank. When the village withdrew its funds, the account had grown to $479,147. How much interest did it earn on the deposit?

30. Express $9\frac{12}{25}$% as a decimal.

31. If you invested $1,825 at $5\frac{7}{8}$% simple interest, how much would your money have grown to after 2 years?

32. Find the amount of interest that would be paid on a $5,255.52 deposit at 5.25% simple interest for 1 year.

33. Levar's Landscaping borrowed $79,500 to finance the purchase of new equipment. The simple interest rate was $8\frac{3}{8}$%, and the term of the loan was 1 year. Calculate the total interest that the business will pay for this loan.

34. Martina deposited $4,257.09 in a certificate of deposit. Two years later, the value of her account had grown to $4,503.27. How much interest did she earn?

35. Find the total amount that will be required to pay off a 3-year loan of $14,043.43 at 6.09% simple interest.

36. Express the following rates as decimals: (a) 4.37%, (b) 12.5%, and (c) 300%.

H. Additional Exercises

37. Sheldon paid $4,255 to settle a debt. The total interest he paid was $375. How much did he borrow originally?

38. Each of the following decimals represents an interest rate. Rewrite the rate as a percent.

a.	0.03	____________	e.	0.387	____________
b.	0.0475	____________	f.	0.008	____________
c.	0.125	____________	g.	.06569	____________
d.	1.5	____________	h.	.0025	____________

39. a. Tom deposited $5,000 in a 2-year certificate of deposit paying 8% simple interest. What was the value of his account at the end of the 2 years?

    b. Jerry deposited $5,000 in a 1-year certificate deposit paying 8% simple interest. At the end of the first year, he took his money and opened up a new 1-year certificate of deposit, also paying 8% simple interest. How much was Jerry's account worth at the end of the 2 years?

c. Since Tom and Jerry both had the same amount of money, the same amount of time, and the same interest rate, it would seem that they should both have ended up with the same amount of money. Why didn't they?

40. Mireille has been offered the opportunity to own a restaurant franchise. Right now, she makes $60,000 per year as a computer analyst, but she projects that she would be able to earn $85,000 annually by quitting her current job and working full time managing the restaurant. However, she would need to invest $500,000 in the business up front. If she were to invest this money elsewhere, she believes she could earn 7% simple interest per year on her money. Would she *really* be making more money from the franchise? Explain.

41. Determine the simple interest for a $2,000 loan at 5.25% for 6 months.

42. Determine the simple interest for a loan of $5,250 for 1 year if the simple interest rate is 1.25% *per month*.

1.2 The Term of a Loan

So far, we've considered only loans whose terms are measured in whole years. While such terms are not uncommon, they are certainly not mandatory. A loan can extend for any period of time at all. When the interest rate is per year, and the term is also in years, we hardly even need to think about the units of time at all. When dealing with loans whose terms are not whole years, though, we have to take a bit more care with the units of time.

Loans with Terms in Months

It stands to reason that the units of time used for the interest rate must be consistent with the units used for the term. Since interest rates are normally given per year, this usually means that we must convert the term into years to be consistent. The following example will illustrate how we have to handle a loan when the term is not a whole number of years.

Example 1.2.1 If Sarai borrows $5,000 for 6 months at 9% simple interest, how much will she need to pay back?

As before, we can calculate her interest by using I = PRT, plugging in P = $5,000 and R = 0.09. T is a bit more complicated. We must be consistent with our units of time.

The term is 6 months, but we can't just plug in T = 6, since the 9% interest rate is assumed to be a rate ***per year****. Since the interest rate is per year, when we use it we must measure time in years. Plugging in T = 6 would mean 6* ***years,*** *not 6 months.*

T should give the term of the loan in years. Since a year contains 12 months, 6 months is equal to 6⁄12 of a year, and so we plug in T = 6/12. (Another way of looking at this would be to say that since 6 months is half of a year, T = 1/2. Either way, we get the same result since 6/12 = 1/2 = 0.5.)

Thus

$$I = PRT$$
$$I = (\$5{,}000)(0.09)(6/12)$$

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I = ($5,000)(.09)(0.5)
I = $225

So Sarai will pay back $5,000 + $225 = $5,225.

So as we've just seen, when the term is given in months, we need to divide the number of months by 12 to convert the term to years. In this first example, 6 divides into 12 nicely, but of course, the same principle can be applied even when the numbers do not divide so neatly.

Example 1.2.2 ***Zachary deposited $3,412.59 in a bank account paying $5\frac{1}{4}$% simple interest for 7 months. How much interest did he earn?***

We can use the simple interest formula once again, plugging in P = $3,412.59 and R = 0.0525. Since the term is expressed in months, we divide 7/12 to get T = 0.583333333.

I = PRT
I = ($3,412.59)(0.0525)(0.583333333)
I = $104.51

So Zachary earned $104.51 in interest.

This example raises an issue. Since 7/12 does not come out evenly, can it be rounded? In general, the answer is no. In business it is accepted that a certain amount of rounding is necessary, but a reasonable degree of accuracy is obviously expected. Too much rounding, especially midway though a calculation, can cause results that are unacceptably far off of the correct answer. In this text, rather than getting bogged down in determining how much rounding is too much, we will follow the general rule that up until the final answer numbers should be carried out to the full number of decimal places given by your calculator. In the example above, the value was shown out to nine decimal places. Your calculator may have more or fewer, but this will not be a problem. As long as you use the full precision of your calculator, any differences will be small enough to be lost in the final rounding.

Fortunately, we can avoid the nuisance of having to write out or type in the entire unrounded decimal. On most calculators, you can simply enter the whole expression into the calculator at the same time:

Operation	Result
3412.59*.0525*7/12=	104.51056875

which rounds to the expected answer of $104.51.

We will use that approach in the next example.

Example 1.2.3 ***Yvonne deposited $2,719.00 in an account paying 4.6% simple interest for 20 months. Find the total interest she earned.***

I = PRT
I = ($2,719.00)(0.046)(20/12)

We evaluate the right side of this equation on the calculator:

Operation	Result
2719*.046*20/12=	208.4566667

Rounding this result we get:

I = $208.46

So Yvonne earned $208.46.

Depending on your prior math background, you may be uncomfortable with the fraction $^{20}/_{12}$. You may have been told at some point that the numerator (the top) of a fraction must be smaller than the denominator (the bottom). Fractions whose numerators are larger are called ***improper*** but there really is nothing mathematically improper about them at all. There are cultural reasons why people may prefer to avoid such fractions—a recipe that called for $\frac{3}{2}$ cups flour would seem strange, while a recipe calling for $1\frac{1}{2}$ cups wouldn't—but these reasons are a matter of tradition and style, not mathematical necessity. While we could rewrite $\frac{20}{12}$ as $1\frac{8}{12}$, simplify that to $1\frac{2}{3}$, and then convert it to a decimal, this would accomplish nothing except needlessly adding steps. We will freely use "improper" fractions whenever they show up.

Loans with Terms in Days—The Exact Method

After we have dealt with loans whose terms are measured in months, it's not surprising that our next step is to consider loans with terms in days. The idea is the same, except that instead of dividing by 12 months, we divide by the number of days in the year.

Example 1.2.4 Nick deposited $1,600 in a credit union CD with a term of 90 days and a simple interest rate of 4.72%. Find the value of his account at the end of its term.

Since there are 365 days in a year, we divided by 365 instead of 12, since 90 days is $\frac{90}{365}$ of a year.

$$I = PRT$$
$$I = (\$1,600)(0.0472)(90/365)$$
$$I = \$18.62$$

And so Nick's ending account value will be $1,600 + $18.62 = $1,618.62.

Since we divided by 12 when the term was in months (since there are 12 months in the year), it only makes sense that we should divide by 365 when the term is in days (since there are 365 days in the year) as we did in this example.

Unfortunately, this is not quite as clear cut as it might seem. While there are exactly 12 months in each and every year, not every year has exactly 365 days. Leap years, which occur whenever the year is evenly divisible by four[2] (such as 1996, 2000, 2004, 2008, . . .) have an extra day, and if the year is a leap year we really should use 366.

This example didn't state whether or not it occurred in a leap year, so we don't know for certain whether to use 365 or 366. And heaven help us if the term of the loan crosses over two calendar years, one of which is a leap year and the other isn't! Calculating interest based on days can clearly become quite complicated. But even that is not the end of the story; we can carry things even further if we really want to be precise. It actually takes the earth $365\frac{1}{4}$ days to circle the sun (the extra $\frac{1}{4}$ is why leap years occur one out of every 4 years). In some cases interest may be calculated by dividing by 365.25 regardless of whether or not the year is a leap year. Taking that approach might be a little bit extreme, and it is unusual but not completely unheard of to see it used in financial calculations.[3]

Some businesses always use the correct calendar number of days in the year (365 in an ordinary year, 366 in a leap year). Others simply assume that all years have 365 days, while still others use 365.25. Having this many different approaches can be confusing, but it is an unfortunate fact of life that any one of them could be used in a given situation. The good news is that the difference among these methods is very small, as the next example will illustrate.

[2] Actually, the rule is a bit more complicated: A year is a leap year if it is divisible by 4, except in cases where it is also divisible by 100. But even this exception has an exception: if the year is also divisible by 400, it is a leap year after all! Since the last time a year divisible by 4 was not a leap year was 1900, and the next time it will happen is 2100, for all practical purposes we can ignore the exceptions.

[3] For the truly obsessive, an even more exact value for the time required to circle the sun is 365.256363051 days, called a **sidereal year**. The pointlessness of carrying things this far should be obvious.

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Example 1.2.5 ***Calculate the simple interest due on a 120-day loan of $1,000 at 8.6% simple interest in three different ways: assuming there are 365, 366, or 365.25 days in the year.***

365 days: $I = PRT = (\$1{,}000)(0.086)(120/365) = \$28.27$

366 days: $I = PRT = (\$1{,}000)(0.086)(120/366) = \$28.20$

365.25 days: $I = PRT = (\$1{,}000)(0.086)(120/365.25) = \$28.26$

This example shows that, while the number of days used does indeed make a difference, the difference is quite small—a few pennies on a $1,000 loan. The differences are not large but they can be annoying, causing discrepancies that are small enough to not matter much but still large enough to be frustrating.

Interest that is calculated on the basis of the actual number of days in the year is called ***exact interest***; calculating interest in this way is known as the ***exact method***. For the sake of simplicity (and sanity), it is not uncommon to adopt the rule of always assuming that a year has 365 days, since that is the more common number of days for a year to have, and using 365 or 366 makes very little difference. Always using a 365-day year may be referred to as the ***simplified exact method***. In this text we will adopt the rule that *unless otherwise specified, interest is to be calculated using the simplified exact method (i.e. 365 days per year)*.

Example 1.2.6 ***Calculate the simple interest due on a 150-day loan of $120,000 at 9.45% simple interest.***

Following the rules stated above, we assume that interest should be calculated using 365 days in the year.

$$I = PRT$$
$$I = (\$120{,}000)(0.0945)(150/365)$$
$$I = \$4{,}660.27$$

Loans with Terms in Days—Bankers' Rule

There is another commonly used approach to calculating interest that, while not as true to the actual calendar, can be much simpler. Under ***bankers' rule*** we assume that the year consists of 12 months having 30 days each, for a total of 360 days in the year.

Bankers' rule was adopted before modern calculators and computers were available. Financial calculations had to be done mainly with pencil-and-paper arithmetic. Bankers' rule offers the desirable advantage that many numbers divide nicely into 360, while very few numbers divide nicely into 365. This simplifies matters and reduces the tediousness of calculations without sacrificing too much accuracy. Five days out of an entire year does not amount to much.

Since financial calculations today are mostly done with calculators and computers, bankers' rule has lost a lot of its appeal. There actually still are some reasons to like bankers' rule (we will run into a few later on) even with technology to do our number crunching, but by and large bankers' rule has been fading away. But it has been widely used for a very long time and, thanks to its longstanding status as a standard method, remains in common use today.

Calculations with bankers' rule really aren't done any differently than with the exact method. The only difference is that you divide the days by 360.

Example 1.2.7 ***Rework Example 1.2.5 using bankers' rule:***

Calculate the simple interest due on a 120-day loan of $10,000 at 8.6% simple interest using bankers rule.

360 days: $I = PRT = (\$1{,}000)(0.086)(120/360) = \$28.67$

Comparing this example to the results of Example 1.2.5, we can see that, while bankers' rule does make a difference, the difference is not enormous.

Because these different methods do give different results, it is important to be clear on which method is being used in any given situation. Even though the differences are not big, it is easy to see how confusion and disputes could arise if the choice of method were left unclear. In practice, if the term of the loan is to be measured in days, the terms of the loan should specify which method will be used in order to prevent misunderstanding.

You might suspect that the differences between bankers' rule and the exact method leave an opportunity for sneaky banks to manipulate interest calculations to their benefit. After all, what prevents a bank from always choosing whichever rule works to its advantage (and thus to the customer's disadvantage)? In practice, the method to be used will be specified either in a bank's general policies, government regulations, or in the paperwork for any deposit or loan, and in any case, as we've seen above, the difference is slight. It is probably true that some banks select one method or the other to nudge things to their favor, but their benefit from doing this would be minimal. A bank that wants to pay less interest on a deposit or charge more on a loan won't get very far playing games with the calculation method, and is far more likely to just charge a higher or pay a lower rate pure and simple. In any case, an informed consumer can (and should) use mathematics to compare different rates and calculation methods.

Loans with Other Terms

It is possible to measure the term of the loan with units other than years, months, or days. While such situations are far less common, they can be handled in much the same way.

Example 1.2.8 ***Bridget borrows $2,000 for 13 weeks at 6% simple interest. Find the total interest she will pay.***

The only difference between this problem and the others is that, since the term is in weeks, we divide by 52 (since there are 52 weeks per year).

$$I = PRT$$
$$I = (\$2000)(0.06)(13/52)$$
$$I = \$30$$

So Bridget's interest will total $30.

There is some ambiguity here, though. A year does not contain exactly 52 weeks; 52 weeks times 7 days per week adds up to only 364 days. Each year thus actually contains $52\frac{1}{7}$ (or, if it is a leap year, $52\frac{2}{7}$) weeks. Since weeks are not often used, there is no single standard accepted way of dealing with the extra fractional weeks. In this text we will follow the reasonable approach used above, and simply assume 52 weeks per year.

EXERCISES 1.2

A. Loans with Terms in Months

1. Find the interest that would be paid for a loan of $1,200 for 6 months at 10% simple interest.

2. If Josh loans Adam $500 for 8 months at 5.4% simple interest, how much interest will Adam pay?

3. Allison loaned Lisa $15,453 for 22 months. The simple interest rate for the loan was $11\frac{5}{8}$%. Find the total amount of interest Allison earned.

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4. How much interest would you have to pay for a 30-month loan of $1,735.53 if the simple interest rate were 7.11%?

5. Zeropoint Energy Systems has just borrowed $800,000 from a private investor for 19 months, at a simple interest rate of 9.53%. Find the total amount Zeropoint will have to repay.

B. Terms in Days: Exact Method

Use the simplified exact method (365 days/year) for the exercises in this section.

6. Toby loaned Jae $500 at a simple interest rate of 7.3%. Find the total interest Toby will earn if the loan's term is 150 days.

7. Bushnell Savings and Loan borrowed $2,500,000 from Fullam Federal Bank for 10 days at a simple interest rate of 2.17%. Find the total interest the savings and loan will pay.

8. If I deposit $1,875 in a CD that pays 3.13% simple interest, what will the value of the account be after 100 days?

9. Peg borrowed $3,715.19 at 15 ⅞% simple interest for 438 days. How much will she need in total to pay the loan back?

10. How much interest will Hanif earn if he makes a loan of $4,280 for 210 days at 10% simple interest?

C. Terms in Days: Bankers' Rule

Use bankers' rule (360 days/year) for the exercises in this section.

11. The Hsang-wha Trading Company borrowed $720,000 for 30 days at 14.4% simple interest. Find the total amount of interest the company paid.

12. Find the total interest owed for a 120-day loan of $815 if the simple interest rate is $8\frac{13}{16}$%.

13. Alan agreed to loan Shane $215.50 for 500 days. Assuming that the simple interest rate is 20%, how much will Alan earn from this loan?

14. One credit union agrees to make a short-term loan to another in the amount of $10,560,350. The loan will be paid back, together with 3.75% simple interest, in 14 days. Find the total amount of the repayment.

15. Summer deposited $2,251.03 in a 264-day bank certificate of deposit paying 0.87% simple interest. What will her account value be at the end of the term?

D. Grab Bag

For exercises where the term is given in days, use the simplified exact method (365 days/year) unless otherwise specified.

16. Fishers Capital loaned Valentown Property Services Corp $40,000 for 7 months at 7 3/16% simple interest. Find the total interest to be paid.

17. Determine the value of a certificate of deposit at the end of its 300-day term if the initial deposit was $5,038.77 and the simple interest rate was 6.35%.

18. B.O.Y. McTastee's Goode-Tyme Burger Emporium temporarily financed a shipment of new fixtures with a 20-day loan at 12% simple interest calculated using bankers' rule. The amount borrowed was $538,926. Find the total interest paid.

19. Elaine loaned Madison $250 for 8 weeks at 7.77% simple interest. How much interest did she earn from this loan?

20. How much interest would you earn if you deposited $808.08 in a certificate of deposit paying 1¾% simple interest for 2 months?

21. In order to cover a temporary funding crunch, the Eastfield Central School District had to borrow $1,700,000 at a simple interest rate of 5.22% for 35 days. How much will the interest on this loan cost the district?

22. Erica borrowed $20,000 for 250 days. The terms of the loan require her to pay 18.99% simple interest calculated using bankers' rule. How much will she need to pay off the loan?

23. Sanjay loaned his brother $25,000 as start-up funds for a new business. They agreed that he would be repaid 21 months later, together with simple interest at a rate of 2.50%. How much interest will Sanjay's brother pay?

24. Ovid National Bank loaned Braeside Corporation $10,983,155.65 for 159 days at 9⅝% simple interest. How much interest will the bank be paid?

25. If you invest $1,935.29 at 6.385% simple interest for 281 days, how much will you earn on the investment?

26. I loaned my brother $250 for 9 months at 5% simple interest. How much interest did he pay?

27. Contrapolar Power Controls borrowed $25,000,000 for 420 days at 6% simple interest. Assuming that bankers' rule is used, what is the total amount the company will need to repay?

28. A roofing contractor estimated that a reroofing job for a retail store would cost $15,700. The store's owner cannot afford to pay cash, but the roof is leaking badly and needs to be replaced right away. The contractor offers to make

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a loan for the cost of the job for 1 year at 10% simple interest. How much interest would the storeowner pay if she accepted this offer?

29. A volunteer ambulance company was conducting a fund drive to buy a new ambulance when the old one broke down entirely and had to be replaced. The fund drive was going well, but the company had not yet reached its goal, and so could only pay for part of the cost of the new ambulance. They financed the remaining $22,453 with a 5-month loan at 8.23% simple interest. Find the amount they will need to raise to pay off this loan.

30. Gustavo borrowed $2,400 for 1 year at 12.253% simple interest. How much will he need to repay the loan?

31. Calculate the simple interest on a $47,539 loan at 14¾% for 211 days.

E. Additional Exercises

32. Three years ago, Andre opened a CD at Hopewell National Bank with a deposit of $3,000. The certificate pays a simple interest rate of 5.58%. The term of the certificate will end 1 year from now. What will the value of his account be at that time?

33. 45 days ago, Liam borrowed $800 from Tammy at 14% simple interest. He will pay her back 120 days from now. How much interest will he owe Tammy at that time?

34. Two months from now Jessaca will repay a loan that she took out 7 months ago. The principal was $450 and the simple interest rate is 10.3%. How much will she need to repay the loan?

35. Suppose that you deposit $500 at 4% simple interest for 20 days. How much interest will you earn?
    a. If interest is calculated using bankers' rule.
    b. If interest is calculated using the simplified exact method.

36. Assuming that the simple interest rate is the same either way, would a borrower prefer bankers' rule or the exact method? Which would a lender prefer?

37. Ralph deposited £2,948.35 in the Bank of Old South Wales for 200 days at 5.77% simple interest. (*Note:* £ is the symbol for British pounds.) How much was his account worth at the end of the term?

38. Suppose that you deposited $2,000 in a 100-day certificate of deposit near the end of 2007. The simple interest rate is 7.22%, and the bank calculates interest using the exact method, using the exact number of days in the year. Thirty-nine days of the certificate's term fell in 2007, which was not a leap year; the rest fell in 2008 which was a leap year. Calculate the interest for this deposit.

1.3 Determining Principal, Interest Rates, and Time

So far, we have developed the ability to calculate the amount of interest due when we know the principal, rate and time. However, situations may arise where we already know the amount of interest, and instead need to calculate one of the other quantities. For example, consider these situations:

- A retiree hopes to be able to generate $1,000 income per month from an investment account that earns 4.8% simple interest. How much money would he need in the account to achieve this goal?
- Jim borrowed $500 from his brother-in-law, and agreed to pay back $525 ninety days later. What rate of simple interest is Jim paying for this loan, assuming that they agreed to calculate the interest with bankers' rule?
- Maria deposited $9,750 in a savings account that pays 5 ¼% simple interest. How long will it take for her account to grow to $10,000?

In this section, we will figure out how to deal with questions of these types.

Finding Principal

Let's begin by considering the situation of the retiree from above. Since this is a situation of simple interest, it seems reasonable to approach the problem by using the simple interest formula we developed in Section 1.2.

We know the amount of interest is $1,000, and so I = $1,000. We know the interest rate is 4.8%, so R = 0.048. Also, since the interest needs to be earned in a month, we know that T = 1/12. Plugging these values into the formula, we get:

$$I = PRT$$
$$\$1{,}000 = (P)(0.048)(1/12)$$

We can at least multiply the (0.048)(1/12) to get:

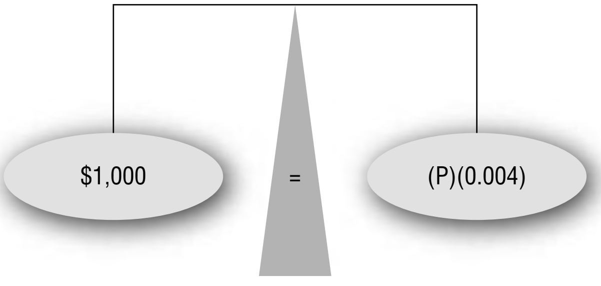
$$\$1{,}000 = (P)(0.004)$$

But now it seems we're stuck. In our earlier work, to find I all we needed to do was multiply the numbers and the formula handed it to us directly. Here, though, P is caught in the middle of the equation. We clearly need some other tools to get it out. We will be able to do this by use of ***the balance principle***.

The Balance Principle

When we write an equation, we are making the claim that the things on the left side of the "=" sign have the exact same value as the things on the other side. We can visualize this by thinking of an equation as a balanced scale. The things on the left side of the equal sign are equal to the things on the right. If we imagine that we placed the contents of each side on a scale, it would balance.

Using this idea with our present situation, $1,000 = (P)(0.004), we'd have:



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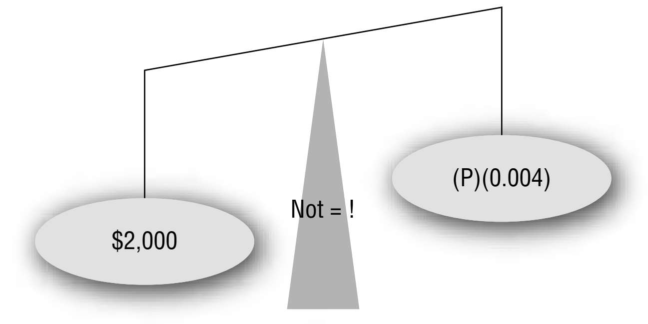
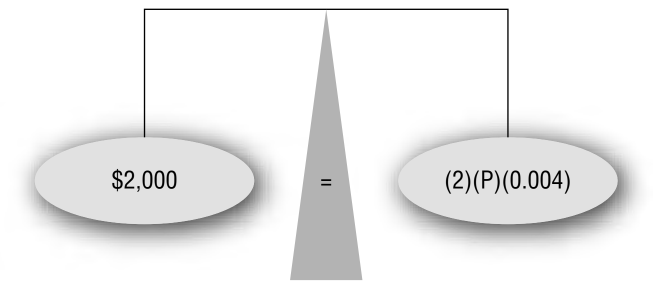
Now, suppose that you double the left side, turning $1,000 into $2,000. This would make the left side "heavier" than the right, and the scale would then tilt:



But, if we also doubled the right side, the scale would be back in balance:



And so, even if we do something that changes the "weight" on one side, we can still have a balanced scale if we make the same change to the other side. In this example we doubled both sides, but it should be clear that it would also have been OK to have divided both sides in half or added $50 to both sides. Or we could have subtracted $35 from both sides had we wanted to. And so on. The basic idea at work here is:

The Balance Principle

You can make any change you like to one side of an equation, as long as you make an equivalent change to the other side.

Finding Principal (Revisited)

So now let's return to the problem we were considering a moment ago. We had reached a dead end with

$$\$1{,}000 = (P)(0.004)$$

but now the balance principle may be able to help us find P. It allows us to add anything to, subtract anything from, multiply anything by, or divide anything into both sides that we like. Above, we used it to double both sides, and while that is certainly allowed, it really wasn't much help. The resulting equation

$$\$2{,}000 = (2)(P)(0.004)$$

is true but it does nothing to get us closer to the value of P.

But now let's ask ourselves if there might be anything that we could add, subtract, multiply or divide that *would* be helpful.

Our problem is that we need to get rid of the (0.004) multiplied by P on the right side. Now, since any number[4] divided by itself is 1, we can eliminate that 0.004 if we divide both sides by 0.004.

$$\frac{\$1000}{0.004} = \frac{(P)(0.004)}{0.004}$$

A calculator can easily handle the division on the left side. On the right we can divide 0.004 by 0.004 and get 1. Thus we get

$$\$250{,}000 = P(1)$$

and since anything multiplied by 1 is just itself

$$\$250{,}000 = P$$

which is the same as

$$P = \$250{,}000.$$

And so, we can conclude that the amount that our retired friend needs in his account is $250,000.

We can check this. Plugging in $250,000 for P into the simple interest formula, we get:

$$I = PRT$$
$$I = (\$250{,}000)(0.048)(1/12)$$
$$I = \$1{,}000$$

which is precisely what the interest was supposed to be. It worked!

Example 1.3.1 ***How much would you need to have in an investment account to earn $2,000 simple interest in 4 months, assuming that the simple interest rate is 5.9%?***

Following the approach we used above, we start with the simple interest formula, plugging in $2,000 for I, 0.059 for R, and 4/12 for T. So:

$$I = PRT$$
$$\$2{,}000 = P(0.059)(4/12)$$
$$\$2{,}000 = P(0.01966667)$$

We then divide both sides of the equation by 0.01966667 to get:

$$\frac{\$2{,}000}{0.01966667} = \frac{P(0.01966667)}{0.01966667}$$
$$P = \$101{,}694.92$$

So we can conclude that you would need $101,694.92. It is a good idea to check this result by using this value for P together with the R = 0.059 and T = 4/12 to make sure that the interest actually is the desired $2,000.

In our first problem of this type, the number we had to divide by was a short decimal. In Example 1.3.1, that was definitely not the case. Once we calculated 0.01966667, we then had to write it down, and then enter "2000/0.01966667=" into the calculator in order to complete the calculation. Having to write this decimal down and then type it into the calculator is a bit tedious, and it also invites the chance of making an error by accidentally leaving out, mistyping, or switching digits.

Most calculators have a memory feature, which allows you to save a number into the calculator memory and then recall it when you need it. Different calculator models will have their memory keys labeled in a variety of different ways, and may require different steps of keystrokes to save and then recall a stored number. You will need to check your calculator's owner's manual, or ask your course instructor for the specific details of how to do this on whichever calculator you are using.

[4] Actually this is not quite true. Zero divided by zero is not one, it is undefined. We can ignore this one exception, though, as it will not come up in any of the problems we will be dealing with.

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Most Texas Instruments calculators have an "Ans" feature that recalls the result of the last calculation. On most models it is located above the "(-)" key in the bottom row; you can pull "Ans" onto your screen by hitting the "2nd" key followed by the "(-)". If you are using one of these models, you can perform the calculations of Example 1.3.1 as follows (the number of decimal places displayed may vary depending on the model):

Operation	*Result*
0.059*4/12 =	0.01966667
2000/Ans =	101694.9153

While not absolutely necessary, using the calculator memory to store and recall values will prove helpful in both this and future sections. If the instructions above are not sufficient to figure out how to do this on the calculator model you are using, again, consult your owner's manual or your instructor for details on how to use your particular calculator.

Finding The Simple Interest Rate

Now let's try our hand at the second scenario from the start of this section, where the interest rate was the unknown. Jim borrowed \$500 and paid \$25 interest for a 90-day loan using bankers' rule. Plugging in the values we already know into the formula, we get:

$$I = PRT$$
$$\$25 = (\$500)(R)(90/360)$$

which is similar to the previous situation, except that now the R lies between the numbers on the right side. Fortunately, though, this doesn't pose any real problem. Order does not matter with multiplication; 2 times 3 times 5 is the same as 3 times 5 times 2, which is the same as 5 times 2 times 3, and so on. Whichever way you multiply 2, 3, and 5, you get 30. This means that in our equation it makes no difference where the R is, and so we can go ahead on the right and multiply (\$500)(90/360) to get \$125, and so:

$$\$25 = \$125(R).$$

We can now exploit the balance principle to get R alone by dividing both sides by \$125:

$$\frac{\$25}{\$125} = \frac{\$125(R)}{\$125}$$

and so

$$0.2 = R$$

or

$$R = 0.2$$

While this is technically correct, it does require some interpretation. Does this mean that the interest rate is 0.2%? That seems awfully low.

Remember that before we plug an interest rate into the formula, we move the decimal point two places to the left. And so when we solve for R, what we are getting is the rate with the decimal already moved. While R = 0.2 is technically a correct answer, the fact is that we normally express interest rates as percents, and so we should do so here. In order to turn this into a percent, we will need to move the decimal point two places *to the right*.

$$R = 0.2 = 20\%.$$

and so *the simple interest rate is 20%*.

Note that, just as we inserted zeros where necessary to hold the place when moving to the left (8% becomes .08 for example), we had to insert a zero here to hold the place on the right.

We can also check this result:

$$I = PRT$$
$$I = (\$500)(0.20)(90/360)$$
$$I = \$25$$

Example 1.3.2 ***Calculate the simple interest rate for a loan of $9,764.55 if the term is 125 days and the total required to repay the loan is $10,000.***

First we need to find the interest by subtracting. The total interest is $10,000 − $9,764.55 = $235.45. Also, recall that since nothing was said otherwise, we assume the exact method with 365 days.

Plugging values into the formula gives:

$$I = PRT$$
$$\$235.45 = (\$9{,}764.55)R(125/365)$$
$$\$235.45 = (\$3{,}344.02397260)R$$

To solve for R, we divide both sides by 3344.02397260. As in Example 1.3.2 it is probably easiest to take advantage of your calculator's memory to do this. In any case, though, dividing through we get:

$$R = 0.0704091842$$

Moving the decimal two places to the right, we can state the rate as 7.04091842%.

It is not usually necessary to carry the final answer out to this many decimal places, though. There is no absolute rule about how many places to use, but in most situations two or at most three decimal places in the final percent is acceptable. For our purposes in this text, two decimal places will be fine, so we conclude that the interest rate is 7.04%.

Finding Time

Lastly, let's consider the third scenario, where time was the unknown. Maria deposited $9,750 in a savings account that pays 5¼% simple interest and wanted to know how long it would take for her account to grow to $10,000.

We proceed as before:

$$I = PRT$$
$$\$250 = (\$9{,}750)(0.0525)(T)$$
$$\$250 = \$511.875(T)$$
$$\frac{\$250}{\$511.875} = \frac{\$511.875(T)}{\$511.875}$$
$$T = 0.4884004884$$

This answer seems like it must be wrong, since it makes no sense. T = 0.4884004884? What does that tell us?

To make sense of this, we need to remind ourselves that as long as the interest rate is a rate per year (which we assume unless explicitly told otherwise), the time in our formula must also be expressed in years. So what this equation is telling us is that the required time would be 0.4884004884 years. That would be a bit less than half a year, which does seem reasonable. Leaving the answer in this form, though, is probably going to be unacceptable to most people. "A bit less than half a year" is a tad too vague.

When talking about periods of time less than a year, instead of using messy decimals we usually just use smaller units of time, such as months or days. So we need to convert this decimal number of years into months or days.

To convert it into months, let's recall how we went about things in the other direction. When we were given a term in months, we divided by 12 to get the term in years. So:

$$\text{years} = \frac{\text{months}}{12}$$

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If we multiply both sides of this equation by 12, we get:

$$12(\text{years}) = \frac{12(\text{months})}{12}$$

and so:

$$12(\text{years}) = \text{months}$$

This also agrees with common sense. There are 12(1) = 12 months in 1 year, 12(2) = 24 months in 2 years, 12(3) = 36 months in 3 years, etc. So to convert our answer to months we multiply by 12 to get:

$$T = 12(0.4884004884) = 5.860805861 \text{ months.}$$

This answer is also unacceptable, though, since we don't normally talk about messy decimal numbers of months either. We could round this to the nearest month, in which case we'd conclude that the term is six months. However, checking this answer we'd get:

$$I = PRT$$
$$I = (\$9,750)(.0525)(6/12)$$
$$I = \$255.94$$

which is close to what it should be, but not quite right. The rounding is the reason for the discrepancy. The difference between our rounded answer of 6 months and the actual value of 5.860805861 months is approximately 0.14 months, which works out to be about 4 days. That isn't a huge amount of time, but it is enough to cause trouble. We don't have to accept this, since we have the option of converting to days to get a more precise measurement of the term.

Using the same logic for days as we did for months, we multiply by 365 to get

$$T = 365(0.4884004884) = 178.2661783 \text{ days}$$

which we would round to 178 days, following usual rounding rules.

This is still a rounded answer, and in fact since we threw out 0.2661783 in the rounding, it may actually appear as though the rounding is at least as serious an issue as it was with the months. However, checking our answer shows that

$$I = PRT$$
$$I = (\$9750)(.0525)(178/365)$$
$$I = \$249.63$$

which is still not exactly $250, but it is quite a bit closer than before. The rounding is less of a problem here since 0.2661783 days is considerably less time than 0.14 months.

Sadly, though, rounding is once again keeping our answer from being exact. We could deal with this by moving to an even more precise measure of time, such as hours, minutes, or even seconds. For obvious reasons, though, this is almost never done in practice. We simply have to accept the fact that, using any reasonable units of time, we can get close to the result we want, but we will not be able to get the interest to come out to *exactly* $250.

It's pretty unlikely that Maria is going to care too much about being 37 cents short of her goal when we are talking about sums of money on the order of $10,000. We conclude that the best answer to this question is *178 days*.[5]

Before moving on, let's sum up our conclusion about how to convert the units of time for these sorts of problems. Assuming that the interest rate is a rate per year, the solved-for value of T will be in years. To convert this to months, we need to multiply by 12. To convert this to days, we multiply by 365 (or 360 if bankers' rule is being used.) Unless we have a

[5] You might object that even though Maria is only 37 cents short, she's still short. The phrasing of the problem could be taken to mean that she wants *at least* $10,000, which does not happen until day 179. This is technically correct, but in most situations following the usual rounding convention is considered close enough. If the situation did demand that she *must have nothing short of $10,000*, we would indeed need to use 179. It is in general understood that insignificantly small differences between desired and actual results must necessarily occur and aren't worth losing sleep over.

reason why we need to state the term in months, we will usually convert to days since they are the most precise time unit that is commonly used.

Example 1.3.3 ***Suppose that you deposit \$3,850 in an account paying 4.65% simple interest. How long will it take to earn \$150 in interest?***

$$I = PRT$$
$$\$150 = (\$3{,}850)(0.0465)T$$
$$\$150 = (\$179.025)T$$
$$T = 0.837871806$$

This answer is in years. To convert it to more user-friendly terms, we can multiply by 365 to get the answer in days:

$$T = (0.837871806)(365) = 306 \text{ days}$$

A Few Additional Examples

Now that we have developed tools to work with the simple interest formula, it is worth working through a few additional examples to get more comfortable.

Example 1.3.4 ***What amount of money must be invested in a 75-day certificate of deposit paying 5.2% simple interest, using bankers' rule, in order to earn \$40?***

$$I = PRT$$
$$\$40 = (P)(0.052)(75/360)$$
$$\$40 = (P)(0.010833333)$$
$$\frac{\$40}{0.010833333} = \frac{(P)(0.010833333)}{(0.010833333)}$$
$$P = \$3{,}692.31$$

The amount that must be invested is \$3,692.31.

Once again, when the numbers run to many decimal places the use of calculator memory is strongly recommended.

Example 1.3.5 ***If Shay borrows \$20,000 for 9 months and pays interest totaling \$1,129.56, find the rate of simple interest for this loan.***

$$I = PRT$$
$$\$1{,}129.56 = (\$20{,}000)(R)(9/12)$$
$$\$1{,}129.56 = \$15{,}000(R)$$
$$\frac{\$1{,}129.56}{\$15{,}000} = \frac{\$15{,}000(R)}{\$15{,}000}$$
$$R = 0.075304$$

Moving the decimal two places to the right, we find that the rate of simple interest is 7.5304%. Rounding to two decimal places gives that the rate is 7.53%.

Example 1.3.6 ***Suppose Michele borrows \$4,850 at 6¼% simple interest. How long will it be before her debt reaches \$5,050?***

$$I = PRT$$
$$\$200 = (\$4{,}850)(0.0625)(T)$$
$$\$200 = \$303.125(T)$$
$$\frac{\$200}{\$303.125} = \frac{\$303.125(T)}{\$303.125}$$
$$T = 0.659793814$$

Once again, T gives us the time in years, and clearly we need to convert this number to a more reasonable unit of time. We could try months, but might as well just move directly to days. Since the problem did not specify bankers' rule, we'll assume a 365-day year to get that the term is (365)(0.659793814) = 241 days.

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EXERCISES 1.3

A. Finding Principal

1. Find the principal for a loan if the term is 2 years, the simple interest rate is 9%, and the interest totals $63.00.

2. I would like to be able to earn $500 in interest from a certificate of deposit over the next 18 months. Wellsburg Savings and Loan is offering a simple interest rate of 3.275% on 18-month CDs. How much would I need to deposit to achieve my goal?

3. Weston Corporation states in its financial report that the company earned $1,275 in interest from a 6-month loan to the company's CEO. It also reports that the simple interest rate for the loan was 7½%. How much did the company loan the CEO?

4. Because of a clerical error, the paperwork for a loan Simon took out at Protection National Bank failed to list the amount he borrowed. It did, however, specify that the term of the loan would be 100 days, simple interest would be paid at a rate of 12.63%, the interest would be calculated using bankers' rule, and the total interest would be $49.12. How much did Simon borrow?

5. The Capital City School District states in its proposed budget that it expects to need to take out a loan for 45 days late in the school year to cover cash flow needs while the district waits to receive its state aid payment. The budget document also states that this figure assumes a simple interest rate of 3.8%, and the amount the district has budgeted for interest on this loan is $3,000. The budget document does not state how much the district will need to borrow, but as a concerned citizen you would like to know this. How much is the district planning to borrow?

6. How much does Na need in her investment account if she wants to be able to receive $450 per month in income from it, assuming a simple interest rate of 7½%?

B. Finding the Simple Interest Rate

Where necessary, round interest rates to two decimal places. When using days, use the simplified exact method (365 days per year) unless otherwise specified.

7. Convert each of the following decimals to a percent:

   a. 0.0453 ____________________
   b. 0.1011 ____________________
   c. 0.1043 ____________________
   d. 1.043 ____________________
   e. 0.1825 ____________________
   f. 0.1 ____________________

8. Find the simple interest rate for a loan if the principal is $5,000, the term is 2 years, and the interest totals $600.

9. Ahmed lent Chris $100. Two months later, Chris gave Ahmed a total of $105 to pay off the loan. What rate of simple interest did Chris pay?

10. I deposited $4,257.09 in a certificate of deposit, and 83 days later the value of my account had grown to $4,313.53. If the interest was determined using bankers' rule, what was the simple interest rate paid on my certificate?

11. Collamer Financial Corp received $26,653.18 in interest for a 200-day loan of $350,000. What was the simple interest rate?

12. Dantae borrowed $4,000 for 15 months. The interest for this loan will come to a total of $806.25. Find the simple interest rate.

C. Finding Time

Answers should be expressed in an appropriate unit of time. Make sure to clearly state the unit of time that you are using in your answer. When using days remember to assume the exact method unless stated otherwise.

13. Convert 0.84159 years into days.

14. Find the term of a loan if the principal is $20,000, the interest rate is 8%, and the total interest is $320.

15. Jennifer borrowed $7,000 at a simple interest rate of 9⅞%. She paid $590.88 simple interest for this loan. What was the term of her loan?

16. NiftyTronic Corporation lent one of its suppliers $52,347.18 at a simple interest rate of 6.09%. When the supplier repaid the loan, it paid a total of $53,675.50. How long did it take for the supplier to repay NiftyTronic?

17. Pat and Jordan put $11,935.42 into a bank account paying 2.18% simple interest. How long will it take for the account value to grow to $12,000?

18. Tabitha earned $310 in interest for a $2,785 loan. The simple interest rate was 18.99%. What was the term of this loan?

D. Grab Bag

19. Second Law Capital Ventures (SLCV) funded a start-up company with a loan of $2,857,000. Thirty months later, the company repaid SLCV the principal together with interest of $612,826.50. What was the simple interest rate on this loan?

20. Trisha earned $20 interest on a 60 day loan of $500. What simple interest rate does this represent?

21. Alexis borrowed $675 from Peter at 5¾% simple interest. They agreed to calculate the interest using bankers' rule, and also agreed that Alexis would repay the loan when her total debt had reached $700. How long will the loan run?

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22. Justine paid $40.36 simple interest on a 45 days loan of $1,200. Interest was calculated using bankers' rule. Find the simple interest rate.

23. Bloomfield Chemical Company paid $40,000 in interest to settle a 6-month simple interest loan whose interest rate was 7.46%. How much did the company borrow?

24. Allegany City College offers a special program for students who have a short-term financial need. The program allows students to take out loans for short terms at a 3.5% simple interest rate. Robb borrowed $125 from this program, and repaid $125.90. How long was the term of this loan?

25. A dentist purchased $14,763.25 in equipment for her office. The supplier agreed to extend her credit, and allowed her to pay 45 days later. At the end of the 45 days, she paid $15,000 even to settle the debt. What rate of simple interest did she pay?

26. Find the simple interest rate if a 7-month loan of $8,000 resulted in interest of $473.02.

27. Determine how long would it take for me to earn $10 simple interest from an investment of $7,354.29 if the simple interest rate were ¾%.

28. Chan deposited $800 in a credit union certificate of deposit paying 4.21% simple interest. How long will it be before her certificate is worth $820?

29. Convert 1.315946 years into days.

30. The Computer Science Department at Aubuchon Community College wants to establish a scholarship that would award $2,500 each year to a deserving student. In order to do this, the department plans to raise enough funds to create an investment account that would earn $2,500 a year in interest, assuming a simple interest rate of 7%. How much does the department need to reach this goal?

E. Additional Exercises

31. *Usury* refers to the crime (or, in religious terms, sin) of charging excessive interest. In some religious traditions, charging any interest at all is considered excessive and thus condemned as usury.[6] This is of course not the usual legal interpretation in the United States. In the United States, usury is a crime, but "excessive" is defined by state law in most cases, and the threshold for usury varies from state to state. The legal situation is further complicated by the fact that the threshold for usury may vary, depending on the type of lender and/or borrower and/or the type of loan.

    Suppose that the transactions below all occur in a state with a very straightforward usury law: the limit is a simple interest rate of 25% regardless of the lender or loan type. In each case, guess whether or not the lender is guilty of usury. Then calculate the simple interest rate to determine whether or not your guess was correct.

[6] Some verses of the Bible (such as Deut 23:19 and Ezekiel 18) and Quran (such as Al-Baqara 2:275-280 and Al-'Imran 3:130) have also been read as supporting this view. The predominant modern Christian and Jewish view has mellowed to allow for interest that is not exploitive, though many Muslims still refuse to pay or accept interest (though other accommodations usually are made to compensate the lender).

a. Bill loaned Ted $95 for a month. Ted paid $5 interest for the loan.

b. Kendra borrowed $8,350 for 5 years and paid $10,000 simple interest.

c. Nansi borrowed $14,000 for 160 days. She paid $1,500 interest for the loan.

d. Triple-Z Mattress Factory borrowed $200,000 for 3 days. It paid $500 simple interest.

32. If you wanted to earn $500 interest from a bank deposit of $12,500:

a. How long would it take at a simple interest rate of 8%?

b. How long would it take at a 4% simple interest rate?

c. How long would it take at 2%?

d. Make an educated guess as to how long it would take at 1%, and then check your guess by doing the calculation.

e. Make an educated guess for 16%, and then check your guess with the calculation.

33. Anitra's telephone bill for the month of August came to $78.59, and was due on September 20. She paid the bill on October 7, and because she paid it 17 days late she was subject to a late payment penalty of $5. This penalty is not really interest, but we can think of this situation as if Anitra effectively "borrowed" the $78.59 on September 20 when she failed to pay the bill and consider the $5 as if it were the interest paid on this "loan." Looking at things in that way, what was the simple interest rate Anitra paid?

34. Trista and Ryan opened CDs at two different banks. Both earned the same amount of interest. At Trista's bank, the simple interest rate was 3.66%, but at Ryan's bank the simple interest rate was 2.75%. Who made the larger deposit?

1.4 Promissory Notes

Sometimes loans are made with an informal agreement. You would probably be willing to spot a friend $10 until next Friday on a handshake and promise to pay you back. In general, though, it benefits both the borrower and the lender to "get it in writing."

Doing so protects both parties by preventing an unscrupulous lender or borrower from trying to take advantage of the other by changing the terms after the fact. But even when the loan is made between completely trustworthy parties, even among friends or family, having the deal spelled out in writing greatly reduces the chances of honest misunderstandings arising and creating conflict down the road.

Definition 1.4.1

*A **promissory note** (or just **note** for short) is a written agreement between a borrower and a lender that sets out the terms of a loan.*

To fulfill its purpose, a promissory note must contain enough information to clearly specify the amount and timing of any payments that the loan requires. Some of the items that a note may specify are defined below:

Definitions 1.4.2

*The **date** (or **loan date**) of a note is the day it is signed and the loan is made.*

*The **face value** of a note is the amount of the loan for which it is written. (In every situation we have encountered so far the face value is the same as the principal.)*

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On the day that the loan is to be repaid, we say the loan ***comes due*** *and its note* ***matures****. The date on which this happens is the* ***maturity date*** *and the total amount that must be repaid is called the* ***maturity value****.*

Even though a note should provide all of this information, it is not necessary for a note to explicitly spell out all of these items. For example, a note might read:

> *Promissory Note*
> *I, Jane Doe, acknowledge a loan of $1,000 made to me today, April 22, 2006, by the Seventh National Bank of Idaho Falls, and agree to repay this loan in 6 months together with simple interest calculated at a rate of 12%.*

This note does not explicitly state the maturity date or maturity value. Yet the bank certainly wouldn't want that left up in the air! Even though these details aren't stated, there is enough information in this note for us to be able to figure out the maturity date by fast-forwarding six months from April 22, 2006, to arrive at October 22, 2006. To find the maturity value we can use $I = PRT$ to find that the interest should be $60, and then add that onto the principal to arrive at a maturity value of $1060. Now, it might be nice if the note actually spelled out these details explicitly, but as it stands there is no ambiguity or potential for misunderstanding. In fact, you might not want to explicitly state the maturity value and date since it would be redundant, and it would be giving the same information more than once too.

On the other hand, a note like this one is deficient:

> *Promissory Note*
> *This note acknowledges that John Smith has loaned Lisa Jones $250 for 270 days and she hereby agrees to pay him back plus interest.*

Like the first, this one does not explicitly spell out all the relevant details of the loan. But in the first case there was enough information to figure out the missing details. The information given in this note is inadequate—there are details missing that cannot be determined from the information given. It is impossible to determine either the maturity date (it's 270 days from the note's date, but the date is not given) or the maturity value (since the neither the interest amount nor the interest rate is known.)

While the specific format and contents of a note can vary, it should be clear that the details that are given should provide enough information to determine all of the relevant details.

Finding a Note's Term from Its Dates

What about a note like this one?

> *Promissory Note*
> *On this date, March 28, 2007, the West Forklift Federal Credit Union has loaned Spencer Van Etten $2800, which he agrees to repay on October 13, 2007 together with simple interest of 9¾% per year.*

This note does not specify the maturity value, even though that amount would be quite important to both Spencer and the credit union. However, we do know the principal and the rate, and we could figure out the term by counting the days between March 28 and

October 13. So the note does contain enough information for us to be able to determine the interest, and then add it to the principal to arrive at the maturity value.

Counting the days takes some work. There are 3 days remaining in March, plus all of April (30 days), May (31), June (30), July (31), August (31), September (30) and then the first 13 days of October. Totaling these gives a term of $3 + 30 + 31 + 30 + 31 + 31 + 30 + 13 = 199$ days.[7] While counting like this is not exactly difficult, it is tedious, and it would not be hard to accidentally miscount the number of days in a month or leave out a month in the counting.

Matters would be much simpler if the note's date and maturity date fell in the same month. If the loan were made on January 4 and came due on January 25, we could just find the difference by subtracting $25 - 4 = 21$ days. But unfortunately just subtracting like this won't work when the two dates fall in different months.

The problem lies in that the day numbers are reset at the end of each month. After January 31 comes February 1, not January 32. So if a loan is made on January 4 and comes due on February 1, we can't just subtract the way we did when both dates fell in the same month. However, if we think of February 1 as January 32, then we could take $32 - 4 = 28$ days. This approach is not as silly as it sounds. February 1 is not really January 32, but it *is* the 32nd day of the year. So if we think of January 4 as day 4 and February 1 as day 32, the subtraction approach would make perfect sense and work just fine.

A date given as the day of the year without using any month name is sometimes called an ***ordinal date***, or a ***Julian date***.[8] Since February 1 is the 32nd day of the year, February 1 would have an ordinal (Julian) date of 32.

Knowing the ordinal (Julian) dates for ordinary calendar dates can solve our problem with finding a note's term. March 28 happens to be the 87th day of the year, and October 13th is the 286th day of the year. Knowing this, we could find the term of Spencer's loan by subtracting: $286 - 87 = 199$ days.

Of course, this raises the question of how we are supposed to know what day of the year each date is. A *day of the year table* like Table 1.3.1 is a helpful tool that can come to our rescue here.

This table doesn't do anything for us that we couldn't have done ourselves by counting, but it saves the time and effort of doing so. To find out what day of the year March 28 is, we need only look in the March column and 28th row to find that it is day 87. To find October 13, we just look in the October column and 13th row to see that October 13 is day 286.

The following example will provide another illustration of how this table can be used.

Example 1.4.1 ***Find the number of days between April 7, 2003, and September 23, 2003.***

Looking in the table, we see that April 7 is day 97, and September 23 is day 266. So there are $266 - 97 = 169$ days between those two dates.

This table can obviously be quite handy when you need to do these kinds of calculations.

If you need the table and don't have a copy of it handy, you can find ordinal (Julian) dates by building and using an abbreviated version of this table. The key is to create a table that lists the cumulative total number of days that have passed at the end of each month. January has 31 days, and so at the end of January a total of 31 days have passed in the year. February has 28 days (assuming the year is not a leap year) and so at the end of February a total of $31 + 28 = 59$ days have passed. Likewise, March adds 31, so at the end of March $59 + 31 = 90$ days have passed. Continuing on in this way you can construct a table like Table 1.3.2.

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[7]One way to keep track of how many days each month has is to make fists with both hands. Recite the names of the months while moving across your knuckles and the spaces between. January rests on the first knuckle, February on the space between first and second knuckles, March on the second knuckle, and so on. The knuckle months have 31 days, the between-knuckle months don't (30 for all except February.)

[8]Some object to calling this the Julian date, insisting that technically speaking a Julian date is the number of days since January 1 of 4713 B.C.E. The Julian date for July 8, 2006, for example, would be 2453925. Nonetheless, the use of the term as we have defined it is widespread, and so while we take no position on whether or not the term *should* be used in this way, we recognize the fact that it commonly *is* used in this way.

TABLE 1.3.1
Day of the Year Table (For Non-Leap Years)

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
1	1	32	60	91	121	152	182	213	244	274	305	335
2	2	33	61	92	122	153	183	214	245	275	306	336
3	3	34	62	93	123	154	184	215	246	276	307	337
4	4	35	63	94	124	155	185	216	247	277	308	338
5	5	36	64	95	125	156	186	217	248	278	309	339
6	6	37	65	96	126	157	187	218	249	279	310	340
7	7	38	66	97	127	158	188	219	250	280	311	341
8	8	39	67	98	128	159	189	220	251	281	312	342
9	9	40	68	99	129	160	190	221	252	282	313	343
10	10	41	69	100	130	161	191	222	253	283	314	344
11	11	42	70	101	131	162	192	223	254	284	315	345
12	12	43	71	102	132	163	193	224	255	285	316	346
13	13	44	72	103	133	164	194	225	256	286	317	347
14	14	45	73	104	134	165	195	226	257	287	318	348
15	15	46	74	105	135	166	196	227	258	288	319	349
16	16	47	75	106	136	167	197	228	259	289	320	350
17	17	48	76	107	137	168	198	229	260	290	321	351
18	18	49	77	108	138	169	199	230	261	291	322	352
19	19	50	78	109	139	170	200	231	262	292	323	353
20	20	51	79	110	140	171	201	232	263	293	324	354
21	21	52	80	111	141	172	202	233	264	294	325	355
22	22	53	81	112	142	173	203	234	265	295	326	356
23	23	54	82	113	143	174	204	235	266	296	327	357
24	24	55	83	114	144	175	205	236	267	297	328	358
25	25	56	84	115	145	176	206	237	268	298	329	359
26	26	57	85	116	146	177	207	238	269	299	330	360
27	27	58	86	117	147	178	208	239	270	300	331	361
28	28	59	87	118	148	179	209	240	271	301	332	362
29	29		88	119	149	180	210	241	272	302	333	363
30	30		89	120	150	181	211	242	273	303	334	364
31	31		90		151		212	243		304		365

TABLE 1.3.2
Abbreviated Day of the Year Table (For Non-Leap Years)

At the end of:	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
The number of days that have passed is:	31	59	90	120	151	181	212	243	273	304	334	365

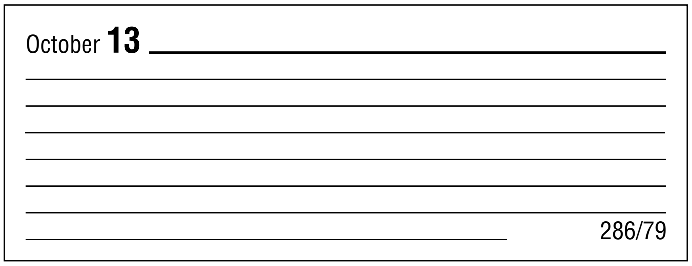
To find the ordinal (Julian) date for any given calendar date, we begin with the total number of days that passed as of the end of the prior month, and then add the number of days we have gone into the current month. For example, for April 7, we would look in this table to find that 90 days passed up through the end of March, and then add the 7 days we have gone into April, $90 + 7 = 97$, so the ordinal date of April 7 is 97. To find the date for

September 23, we would add 243 + 23 = 266. Note that these dates agree with what we found from the longer table in Example 1.4.1.

Example 1.4.2 ***Use the abbreviated day of the year table to find the number of days between August 19, 2006, and October 1, 2006.***

The ordinal date for August 19 is 212 + 19 = 231. The ordinal date for October 1 is 273 + 1 = 274. Subtracting to find the difference, we find there are 274 − 231 = 43 days.

Ordinal dates can also be readily found in other places when a table isn't handy. Many planners or daybooks print the day of the year for each date on the bottom of the page. Often this will give the day of the year followed by the number of days left in the year, as in the illustration below:



The 286/79 indicates that October 13 is the 286th day of the year, and 79 days remain in the year. For our purposes, we are interested only in the 286.

Many computer spreadsheet programs, such as Microsoft Excel, also provide ordinal dates.

Once we can find the term of a note from its dates, we can go on to calculate other values that depend on the term.

Example 1.4.3 ***Find the maturity value for the following promissory note:***

> *Promissory Note*
> *I, Cameron Mills, acknowledge a loan of $2,500 made to me today, January 19, 2005, by William Burgh, and agree to repay this loan on June 30, 2005, together with simple interest calculated at a rate of 10%.*

June 30 is day 181, and January 19 is day 19, so the term of this loan is 181 − 19 = 162 days.

Using this to calculate the interest gives us:

$$I = PRT$$
$$I = (\$2{,}500)(0.10)(162/365)$$
$$I = \$110.96$$

Adding this onto the principal, we arrive at a maturity value of $2,500 + $110.96 = $2,610.96.

Leap Years

Leap years throw a wrench into the problems we have been considering. In leap years, the 60th day of the year is February 29, not March 1, which throws off all dates thereafter. We could deal with this by creating separate tables for leap years. However, a simple way of dealing with the problem is just to add one to the value shown in the table for all dates after February 28 if the year is a leap year.

Recall that for any year between 1900 and 2100, any year divisible by 4 is a leap year. When in doubt, you can check this by dividing the year by 4 on your calculator. If the result is a whole number, then the year is divisible by 4 and is a leap year. If the result is not a whole number, then it isn't divisible by 4 and so it isn't a leap year.

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Example 1.4.4 ***Find the number of days (a) between January 6, 2004, and May 1, 2004, and (b) between September 3, 2004, and November 16, 2004.***

(a) In a non-leap year, January 6 is day 6, and May 1 is day 121. However, 2004 was a leap year. Since January 6 falls before the leap day, it requires no adjustment. Since May 1 falls after the leap day, we add one and conclude that May 1 was day 122 of 2004. Once we've adjusted for the leap year, we proceed as before: $122 - 6 = 116$ days.

(b) In a non-leap year, September 3 is day 246, and November 16 is day 320. We must add one to each of these, though, since both fall after the leap day: $321 - 247 = 74$ days.

If you are getting the day of the year from a day planner or computer program, though, it will have taken the leap year into account already and so you should not add the day in those cases. These would have given January 6, 2004, as day 6, but would have given May 1, 2004, as day 122.

Finding Loan Dates and Maturity Dates

So far, we've found an efficient way to find the term of a note, given its date and maturity date. We can also find the maturity date given the loan date and term, or the loan date given the maturity date and term.

Suppose that you borrow a library book on July 6, and the book will be due in 14 days. It is not hard to see that the due date for the book will be July 20. When we move forward in time, we *add*, and 6 plus 14 makes 20. This applies equally well to longer terms.

Example 1.4.5 ***Find the maturity date of a 135-day note signed on March 7, 2005.***

March 7 is day 66. If we start at day 66 and go forward 135 days, we arrive at day 201, because $66 + 135 = 201$.

Now, to convert this back to the usual calendar. Using the full day of year table, we go looking through the table for day 201 and find that it is July 20, 2005, so that is the maturity date of the note. Using the abbreviated table, we can see that the last day of June is day 181 and the last day of July is day 212, so day 201 must fall in July. Subtract: $201 - 181 = 20$. So it must be the 20th day of July, so we conclude that the maturity date is July 20, 2005.

Suppose on the other hand that you have a movie rental due on October 17. If the rental period is 5 days, it's not hard to see that you must have rented it on October 12. When we move backward in time we *subtract*, and 17 take away 5 makes 12.

Example 1.4.6 ***Find the date of a 200-day note that matures on November 27, 2006.***

November 27 is day 331. If we go backward 200 days, we arrive at day 131, since $331 - 200 = 131$.

If we look through the body of the day of the year table we find that day 131 is May 11. Or, using the abbreviated table, we see that the last day of April is day 120, and $131 - 120 = 11$. Either way, we see the note's date must be May 11, 2006.

Be careful, though, if a leap year is involved.

Example 1.4.7 ***A 100-day note was signed on January 3, 2008. When does the note mature?***

January 3 is day 3, and so 100 days after that is day 103. Remember, though, that 2008 is a leap year. Using the full table, we can see that this must fall after the leap day, so we actually want to find day 102 in the table. The note's maturity date is thus April 12, 2008.

Using the abbreviated table, we remember that every month from February on should have one added to its ending date. The table shows March ending on day 90, which we realize in a leap year should actually be 91. Subtracting, $103 - 91 = 12$, and so we conclude that the maturity date is April 12.

Finding Terms across Two or More Calendar Years

In all of the previous problems the loan date and maturity date have fallen within the same year. But just as date numbers resetting at the start of each month caused us trouble before, day of year numbers resetting at the start of each new year will cause problems when a loan's term crosses from one year to the next. The techniques we have used so far require some adjustment when the dates do not fall in the same year.

For example, suppose a note is dated December 22, 2005, and matures on January 19, 2006. January 22 is day 356, while January 19 is day 19. So far we have found our terms by subtracting the loan date from the maturity date. But $19 - 356$ is a negative number, which does not make any sense. Now it may be tempting to switch the numbers and subtract 19 from 356 to get a positive number. But while this switch will give a positive answer, it won't give a *correct* positive answer. That should be obvious in this case, since subtracting $356 - 19 = 337$ days, which is obviously much longer than the time between December 22 and January 19.

The most straightforward way of handling this is to attack the problem directly at its source. We didn't run into any trouble when everything stayed within the same year, so let's break the term of this note up into pieces: the part in 2005 and the part in 2006. Counting the days in 2005 is no trouble: the year ends on day 365, and so the note exists for $365 - 356 = 9$ days in 2005. Counting the days in 2006 is easy. Since the note runs from the start of the year until day 19, the number of days in 2006 is obviously 19. These two pieces together make up the whole term, and so putting them together, we find that the term of the note is $9 + 19 = 28$ days.

Visualizing this with a time line may be helpful:

A problem with a long term will further illustrate this approach.

Example 1.4.8 ***Find the term of a note dated June 7, 2005, that matured on March 15, 2007.***

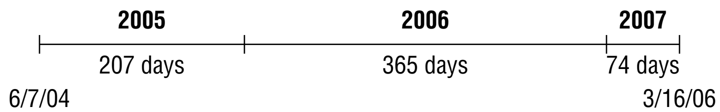
June 7 is day 158, and March 15 is day 74. Since they fall in different years, we split the term of the note by year into the parts that fall in 2005, 2006, and 2007.

In 2005, the note runs from day 158 to the end of the year, day 365. Thus the note lives for $365 - 158 = 207$ days in 2005.

The note's term includes all of 2006, which is 365 days.

In 2007, the note runs from the start of the year until Day 74, so there are 74 days in 2007.

This can be represented by a time line as follows:



Thus, the term of the note is $207 + 365 + 74 = 646$ days.

It is not absolutely essential to draw the time lines, and some people find them unnecessary. However, even if you did not find them necessary in these two examples, as we handle more and more complicated problems you may find that drawing a time line is a very helpful tool to keep track of what is going on in a problem.

Finding Dates across Two or More Calendar Years

What about finding maturity dates and loan dates when the note crosses years? For example, if I sign a 300-day note on November 1, 2006, I know that the maturity date cannot possibly

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still fall in 2006, since there aren't that many days left in the year. It's not surprising, then, that following our original method of adding the days on doesn't work: November 1 is day 305, and day 305 + 300 = day 605, which makes no sense.

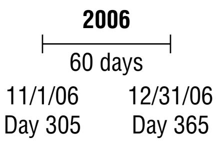
We can adapt the method of Example 1.4.8 to this situation as well though. The note starts in 2006. We can ask ourselves what portion of the note's term will fall in 2006, and then see how much is left over to run into 2007. From day 305 to the end of 2006 would be 365 − 305 = 60 days. So of the total 300 day term, 60 days fall in 2006, which leaves 300 − 60 = 240 days to run into 2007. That would take us to day 240 of 2007, which leads us to the maturity date of 8/28/07.

A time line may once again be helpful to illustrate this. Also, it may be helpful to keep a running tally along the side to keep track of how many days we have left to account for as we fill in the time line.

Running tally

We start with 300 days to account for: 300 days

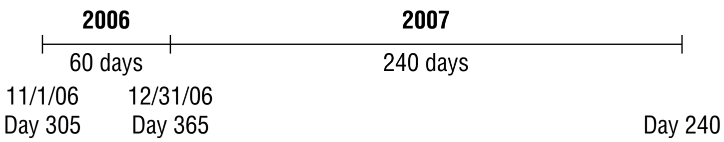
The note lives for 365 − 305 = 60 days in 2006:



−60 days

Which leaves to carry into the next year: 240 days left

Extending 240 days into 2007 takes us to day 240: −240 days



0 days left

Looking to the day of the year table, we see that day 240 is 8/28/07.

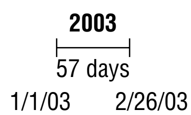
We can follow a similar approach to back up from a maturity date to the loan date, as illustrated in the following example.

Example 1.4.9 ***Find the loan date for a 500-day note that matured on February 26, 2003.***

Running tally

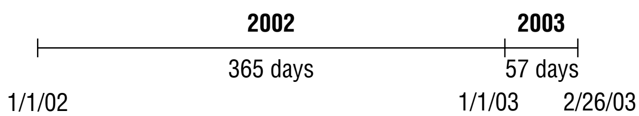
500 days

February 26 is day 57, and obviously we can't go backward 500 days from that date without crossing into a prior year. But we can account for 57 days of the term in 2003:



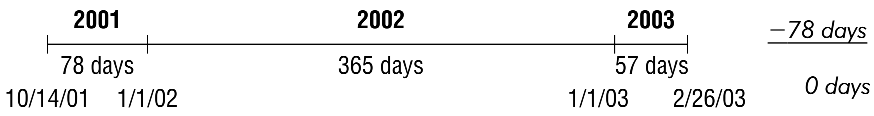
−57 days

Of the remaining 443 days, 2002 takes care of 365: *−365 days*



78 days

We will be able to cover the remaining 78 days in 2001. The year ends on Day 365, so moving 78 days backward takes us to 365 − 78 = Day 287. Seeking out Day 287 from the table gives us a date of October 14.



So we can conclude that the note was signed on October 14, 2001.

EXERCISES 1.4

A. Promissory Notes

For each of the following promissory notes or situations, identify the loan's (a) date, (b) face value, (c) maturity date, (d) maturity value, and (e) term.

1. Tim borrowed $12,000 on January 17, 2004. He signed a note, agreeing to repay a total of $12,800 three months later, on April 17, 2004.

2. The Keuka Park Federal Credit Union loaned Sandy $500 on April 1, 2006. She agreed to repay a total of $510 thirty days later, on May 1, 2006.

3. I, Jerry Wang, acknowledge a loan of $5,350 made to me by Lisa Bryer on July 1, 2006. I agree to repay this loan together with $150 in interest in ninety days, on September 29, 2006.

4. Hottenkold Indoor Comfort Company hereby acknowledges a loan of $75,000 made on January 1, 2006, and to satisfy this debt agrees to repay the Third National Bank of Branchport $80,000 on January 1, 2007.

B. Finding Terms within a Calendar Year

5. Rarefied Air Flight Schools signed a promissory note on June 11, 2005, with a maturity date of November 30, 2005. Find the term of the note.

6. Christopher loaned his friend Evan $750 on March 17, 2006. Evan signed a note, agreeing to pay the loan back (with interest) on December 31, 2006. Find the term of the note.

7. On September 3, 2006 a promissory note that Gilles had signed on February 12 of the same year came due. Find the term of the note.

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8. Serena signed a promissory note on April 1, 2007. The maturity date was June 19, 2007. How many days did she have from when she borrowed the money until she had to repay the loan?

C. Finding Loan and Maturity Dates within a Calendar Year

9. Enrique signed a 200-day promissory note dated May 6, 2005. When did the note mature?

10. Infodynamic Industries International Inc. signed a 75-day note on January 19, 2007. Find the maturity date.

11. To cover the expense of needed roof repairs, Second Congregational Church of Waterport took out a 100-day loan, which matured on August 25, 2005. When was the note signed?

12. Find the date of a 45-day note that matures on October 29, 2009.

D. Finding Terms Crossing Multiple Calendar Years

13. Find the term of a note dated April 7, 2005 that matures on December 1, 2007.

14. Elwood National Bank made a loan to a local business, for which the bank obtained a promissory note dated November 18, 2005, maturing on March 10, 2006. What is the term of this note?

15. Geoff's parents loaned him $2,000 on October 15, 2006. He agreed to pay them back on March 31, 2007. How long does he have to repay the loan?

16. As part of a negotiated relocation package, Garage Management Systems Inc. agreed to make an interest-free loan to one of its executives on April 15, 2006. The loan must be repaid by December 31, 2007. How long does she have to repay the loan?

E. Finding Loan and Maturity Dates Crossing Multiple Calendar Years

17. On May 4, 2001, Karenna took out a 500-day certificate of deposit at her credit union. When did the CD mature?

18. If I borrowed $5,000 for 80 days on November 28, 2006, when would I have had to repay the loan?

19. Nakamura Signs had a 240-day note come due on April 13, 2007. When was the note signed?

20. The records of the Pumpkin Hook Federal Credit Union show that a 525-day note will be coming due on February 14, 2009. When was the note signed?

F. Finding and Using Terms

In each of the following exercises, the term can be found by using the given dates.

21. A certificate of deposit dated June 5, 2009, matures on September 15, 2009. The face value is $10,000 and the simple interest rate is 8%. Find the maturity value of the certificate.

22. If you borrow $1,200 at 9% simple interest on February 28, 2007, and the loan is to be repaid on July 1, 2007, how much interest will you pay?

23. Taylor loaned Miranda $400 on November 7, 2001, which she repaid on April 19, 2002, together with $25 in interest. Find (a) the term and (b) the simple interest rate.

24. Hayward College signed a note on June 30, 2002, with a face value of $125,000. The maturity date of the note was March 15, 2003, and the maturity value was $131,450. Find the simple interest rate paid by the college.

In each of the following exercises, the term can be found by solving $I = PRT$.

25. Bellford Construction signed a note on March 9, 2005. The loan's principal was $12,000, the simple interest rate was 10%, and the note's maturity value was $12,500. What was the note's maturity date?

26. I deposited $1,500 in a certificate of deposit at my local bank on September 30, 2005. The account has a simple interest rate of 4¾%, and the maturity value of the CD is $1,580. When does the CD mature?

G. Finding Terms, Dates, and Maturity Dates with Leap Years

27. Anand signed a note on February 7, 2004. The maturity date was July 5, 2004. Find the term.

28. A note dated January 19, 2008, comes due on November 3, 2008. Find the term.

29. On July 7, 2008, the Arkport Aircompressor Company signed a short-term note with an August 21, 2008, due date. Find the term.

30. A note dated January 3, 2004, came due on February 27, 2004. Find the term.

31. A 150-day note comes due on July 9, 2008. On what date was the note signed?

32. If you take out a 210-day loan on February 6, 2012, when will the note come due?

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33. On December 21, 2007, I signed up for a 100-day trial membership to a wholesale buying club. On what date did my trial membership expire?

34. Find the maturity date for a 110-day note signed on March 23, 2008.

H. Grab Bag

35. Find the maturity value for this promissory note:

> *Promissory Note*
> *Dated October 10, 2006*
> *Truly Trustworthy Property Management Corp hereby acknowledges a loan of $145,000 from Fairhaven First Financial Funding. The loan, together with simple interest calculated at a rate of 8.95%, will be repaid on March 15, 2007.*

36. A note dated May 19, 2007, has a maturity date of July 11, 2008. Find the term of this note.

37. On April 16 Todd loaned Beth $580 for 100 days at 6.85% simple interest. Find the maturity value and maturity date of this loan.

38. Find the maturity date for a 250-day note signed on February 27, 2008.

39. Portable Potables Beverages signed a 200-day note that comes due on May 16, 2010. On what date was the note signed?

40. If you sign a promissory note on March 15 that comes due on October 5, what is the term of the note?

41. Gene and Dean's Clean Green and Pristine Landscaping Service borrowed $2,500 by signing a 13.5% simple interest note on February 8, 2007. The note's maturity date was August 25, 2007. What is the total interest for the loan?

42. A certificate of deposit was opened on November 30, 2005, and comes due on April 18, 2007. Find its term.

43. A note signed on October 15, 2009, has a term of 700 days. What is this note's maturity date?

44. Determine the maturity value for a $3,000 note signed on July 5, 2006, that comes due on April 17, 2007, assuming a 9½% simple interest rate.

45. Find the simple interest rate for this promissory note:

> *IOU. Ted has loaned me $575 on January 18 and I hereby promise to repay him $625 on August 30.*
>
> *Signed,*
> *Jenni*

46. On what date was this promissory note signed?

> *Promissory Note*
> *Sempronius Savings and Loan has made a loan of $12,500 to Jack Miller as follows:*
> *Maturity Value: $13,575*
> *Maturity Date: September 28, 2007*
> *Simple Interest Rate: 18%*

47. Brad didn't pay the property taxes for his house on time. On December 17, 2009, the town sent him a letter warning him that he had to pay his taxes within the next 75 days. When is the deadline?

I. Additional Exercises

48. Suppose that you borrow $2,000 on January 1, 2009. The maturity date of the note that you sign is December 29, 2009, and the simple interest rate is 8% calculated using bankers' rule.
    a. Find the term of the note in days.
    b. Use your answer from part a to find the total amount of interest you will have to pay.
    c. If the term of the loan had instead been 1 year, how much interest would you have had to pay?
    d. Why is part c a trickier question than it might seem to be at first?

49. Find the maturity date for a 200-day note signed on January 25, 2100.

50. Determine whether or not each of the following years is a leap year:
    a. 2013 ____________
    b. 2018 ____________
    c. 2044 ____________
    d. 2058 ____________
    e. 2082 ____________
    f. 2092 ____________
    g. 2380 ____________

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1.5 Nonannual Interest Rates (Optional)

Back in Section 1.1, we noted that, unless it is clearly stated otherwise, an interest rate is always assumed to be a rate *per year*. While this is the accepted convention, there is absolutely no reason why an interest rate could not be stated as a rate per month, week, day, or any other unit of time we want, and in fact this does happen sometimes. We assume annual rates simply because it is customary to do so, and because having a common standard makes it easier to compare different rates.

For example, take a look at this excerpt from a credit card's monthly statement:

	Avg. Daily Balance	Daily Periodic Rate	Nominal Annual Rate	FINANCE CHARGE
Purchases	$882.71	0.05203%	18.99%	$14.23
Cash Advances	$0.00	0.06025%	21.99%	$0.00

Notice the column that gives the daily periodic rate. Because the balance on a credit card can change from one day to the next, it is reasonable to think of the interest as being calculated on a daily basis. And so, it may make sense to talk about a rate per day rather than per year. (Credit cards will be covered in much greater detail in Chapter 10.) This doesn't really pose any new difficulties, except that the units of time used in the interest rate must agree with the units of time used when you are plugging in for T. The following examples will illustrate:

Example 1.5.1 ***Suppose that my credit card charges a daily simple interest rate of 0.05%. How much interest would I owe for 1 day, on which my balance was $1,800?***

*The simple interest formula still applies. However, since the rate is **per day** it stands to reason that the time, T, should be expressed in terms of days as well. Thus*

$$I = PRT$$
$$I = (\$1{,}800)(0.0005)(1)$$
$$I = \$0.90 \text{ or } 90 \text{ cents}$$

Example 1.5.2 ***For the same credit card used in Example 1.5.1, how much interest would I owe for two weeks during which my balance was $2,000?***

Since our interest rate is per day, the term must also be in days. Since 1 week is equal to 7 days, 2 weeks is 2(7) = 14 days. Thus

$$I = PRT$$
$$I = (\$2{,}000)(0.0005)(14)$$
$$I = \$14.00$$

The interest rate being used in these two examples, 0.05%, is much lower than the rates you are probably accustomed to seeing. It seems to run counter to the conventional wisdom about credit card rates—most people think of credit cards as having *high* rates. But the number is actually a little deceptive; it is so small because it covers only a short period of time. This points to another reason why someone might want to use a nonannual interest rate: to make the interest look like less than it would if it were expressed as an annual rate. What would the interest rate look like if it were expressed as a rate per year?

Converting *to* an Annual Simple Interest Rate

Looking back at the credit card statement excerpt, right next to the daily rate there is a column for the nominal annual percentage rate. (For our purposes at the moment, the word *nominal* can be ignored; we'll discuss what that means in Chapter 3.) On the statement, the issuer has stated the interest rate charged both as an annual rate and as a monthly rate. The fact is, even though there may be reasons to use a daily rate, an annual rate actually could have been used just as well. The rate of 0.05203% per day is equivalent to an annual rate of 18.99%.

Example 1.5.3 ***Convert the rate 0.05% per day into an equivalent simple interest rate per year.***

For the (simplified) exact method, there are 365 days in a year. Thus, the interest for a full year would be 365 times the interest for a single day. Thus, since 365(0.05%) = 18.25%, we conclude that 0.05% per day is equivalent to 18.25% per year.

That 0.05% doesn't seem so small now!

We can verify this claimed equivalence by calculating interest using the daily rate and then again using the annual rate.

Example 1.5.4 ***Calculate the interest due for 10 days on a principal of $1,200 using (a) a simple interest rate of 0.05% per day and (b) a simple interest rate of 18.25% annually.***

(a) Since the rate is daily, time should be measured in days, and so $T = 10$.

$$I = PRT$$
$$I = (\$1{,}200)(0.0005)(10)$$
$$I = \$6.00$$

(b) Since the rate is annual, time should be measured in years, and so $T = 10/365$.

$$I = PRT$$
$$I = (\$1{,}200)(0.1825)(10/365)$$
$$I = \$6.00$$

The answers to (a) and (b) agree, as they should. Notice that we had to be careful to make sure that the time units for the value of T agreed with the time units of the interest rate.

Converting *from* an Annual Simple Interest Rate

In Example 1.5.3 we saw how to convert from a given daily rate to an equivalent annual simple interest rate. What if we want to go the other direction?

Example 1.5.5 ***Convert an 18% annual simple interest rate to a rate per month.***

There are 12 months in a year, or, in other words, 1 month is $\frac{1}{12}$ *of a year. Thus, the interest for a month would be just* $\frac{1}{12}$ *of the interest for a full year. Since 18%/12 = 1.5% we conclude that a simple interest rate of 18% per year is equivalent to a simple interest rate of* $1\frac{1}{2}$*% per month.*

We can follow the principles of examples 1.5.3 and 1.5.5 whenever we need to change the time units of an interest rate. If the new time unit is larger (as in Example 1.5.3), we will end up multiplying; if it is smaller, we end up dividing (as in Example 1.5.5). Common sense will also help avoid errors here. If you multiply when you should divide (or vice versa) you will usually wind up with a final answer that is either far too large (or too small) to make sense. It is always a good idea, in these or any other type of problems, to give your answers a quick reality check. A nonsensically large or small answer is an obvious tip off that a mistake has been made and that you should double-check your work.

Note that when an interest rate is given per day, or even per month, the rounding rule of two decimal places that we have been using for annual rates will not provide enough precision. Daily rates should be carried out to at least five decimal places to avoid unacceptably large rounding errors.

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Example 1.5.6 ***Convert an annual simple interest rate of 9.75% into a daily rate.***

Following our default assumption of a 365-day year, we divide by 365: 9.75%/365 = 0.02671%.

Be careful not to move the decimal place. In finding interest rates, we get accustomed to moving the decimal place to convert the decimals used in the formulas to percents. Seeing 0.02671 on the calculator is especially tempting—this result *looks* like the sort of answer whose decimal place needs to be moved. But in this case, there is no reason to do so. The rate is small because a day is a short amount of time.

Converting between Other Units of Time

Suppose that we have a rate per month and want the rate per day, or want to convert from a rate per month to a rate per week, or from per day to a rate per month, and so on. How do we handle these conversions? The idea is the same, except that the number we multiply or divide by will be different.

Example 1.5.7 ***Convert a simple interest rate of 0.045% per day into a rate per month. Assume that bankers' rule is being used.***

Under bankers' rule, we assume that each month has 30 days. Since one month is 30 days, we multiply 0.045% by 30 to get a rate of 1.35% per month.

Unfortunately, as we've seen previously, the calendar poses some problems. Conversions between monthly and yearly rates are never a problem, because there are always exactly 12 months in a year. But there are not always exactly 30 days in a month. If we were not using bankers' rule, we would not have known what to multiply by in Example 1.5.6, since not all months have the same number of days.

Since annual rates are the norm, though, we seldom have much reason to make a conversion where neither time unit is years. Therefore, we will not address this matter any further in this text.

EXERCISES 1.5

A. Using Nonannual Rates

1. Find the interest on a debt of $850.42 for 30 days if the simple interest rate is 0.035% per day.

2. How much interest would you owe on a loan of $4,000 for three weeks if the simple interest rate is 0.0475% per day?

3. If a daily simple interest rate of 0.0225% is charged on a loan of $35,750, how much interest would be paid on a 1-year loan?

4. Find the interest on a 7-month loan of $5,200 if the simple interest rate is ¾% per month.

5. Jens borrowed $875 for 3 months at a simple interest rate of 0.625% per month. Find the total amount he will need to repay the loan.

B. Converting Nonannual to Annual Rates

6. Express each of the following simple interest rates as a simple interest rate per year.
   a. 0.075% per day
   b. 0.075% per week
   c. 0.075% per month

7. Express each of the following simple interest rates as a simple interest rate per year.
   a. 0.01375% per day
   b. 0.25% per month
   c. 0.0825% per day
   d. ½% per day
   e. ½% per month

C. Converting Annual to Nonannual Rates

8. Express each of the following annual simple interest rates as a daily simple interest rate.
   a. 3.65% ____________________
   b. 15% ____________________
   c. 182.5% ____________________

9. Express each of the following annual simple interest rate as (a) a daily simple interest rate and (b) a monthly simple interest rate.
   a. 9% ____________________
   b. 13½% ____________________
   c. 24% ____________________

D. Grab Bag

10. Find the total interest due for a 45-day loan of $835.72 if the simple interest rate is 0.065% per day.

11. Suppose that the annual simple interest rate for a credit card is 19.99%. (a) What is the simple interest rate per day? (b) How much interest would be owed for one week in which the balance on the card was $1,555.93?

12. The monthly simple interest rate on a loan is 0.945%. How much interest would be payable for a 6-month loan for which the principal is $13,546.75? What is the annual simple interest rate for this loan?

13. The Iowa Anvil Company charges a simple interest rate of 0.0575% per day for all late payments. A payment of $4,500 for a shipment was due on March 17. If payment is made on March 30, how much interest would be owed for this late payment?

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14. Find the total interest due for an 8-month loan of $10,857 if the simple interest rate is 1¼% per month.

15. My credit card statement says that the periodic simple interest rate for cash advances is 0.06575% per day. What is this as an annual simple interest rate?

16. Express an annual simple interest rate of 19¾% as a rate per day.

E. Additional Exercises

17. Property taxes in Camden County are due on October 15 of each year. Trent's taxes on his body shop were $5,037.25. If taxes are paid late, interest is charged at a simple interest rate of 1½% per month "or portion thereof." How much would Trent owe if he pays his taxes in full on December 17?

18. A loan of $475.19 was made on April 17, 2007, at a simple interest rate of 0.0475% per day. When the loan was repaid, the total amount required to repay the loan was $499.57. Find the date on which this loan was repaid.

Topic	Key Ideas, Formulas, and Techniques	Examples
The Concept of Interest, p. 3	• Interest is added to the principal of a loan to compensate the lender for the temporary use of the lender's money.	Sam loans Danielle $500. Danielle agrees to pay $80 interest. How much will Danielle pay in total? (Example 1.1.1)
Simple Interest as a Percent, p. 6	• Convert percents to decimals by moving the decimal place • If necessary, convert mixed numbers to decimal rates by dividing the fractional part • Multiply the result by the principal	Bruce loans Jamal $5,314.57 for 1 year at 8.72% simple interest. How much will Bruce repay? (Example 1.1.8)
Calculating Simple Interest for a Loan, p. 8	• The simple interest formula: $I = PRT$ • Substitute principal, interest rate (as a decimal), and time into the formula and then multiply.	Heather borrows $18,500 at $5\frac{7}{8}$% simple interest for 2 years. How much interest will she pay? (Example 1.1.11)
Loans with Terms in Months, p. 14	• Convert months to years by dividing by 12 • Then, use the simple interest formula	Zachary deposited $3,412.59 at $5\frac{1}{4}$% for 7 months. How much interest did he earn? (Example 1.2.2)
The Exact Method, p. 16	• Convert days to years by dividing by the number of days in the year. • The simplified exact method always uses 365 days per year	Calculate the simple interest due on a 150-day loan of $120,000 at 9.45% simple interest. (Example 1.2.5)
Bankers' Rule, p. 16	• Convert days to years by dividing by 360	Calculate the simple interest due on a 120-day loan of $10,000 at 8.6% simple interest using bankers' rule. (Example 1.2.6)
Loans with Terms in Weeks, p. 17	• Convert weeks to years by dividing by 52	Bridget borrows $2,000 for 13 weeks at 6% simple interest. Find the total interest she will pay. (Example 1.2.8)
Finding Principal, p. 23	• Substitute the values of I, R, and T into the simple interest formula • Use the balance principle to find P; divide both sides of the equation by whatever is multiplied by P	How much principal is needed to earn $2,000 simple interest in 4 months at a 5.9% rate? (Example 1.3.1)
Finding the Interest Rate, p. 25	• Substitute into the simple interest formula and use the balance principle just as when finding principal • Convert to a percent by moving the decimal two places to the right • Round appropriately (usually two decimal places)	Calculate the simple interest rate for a loan of $9,764.55 if the term is 125 days and the total to repay the loan is $10,000. (Example 1.3.2)
Finding Time, p. 27	• Use the simple interest formula and balance principle just as for finding principal or rate • Convert the answer to reasonable time units (usually days) by multiplying by 365 (using the simplified exact method) or 360 (using bankers' rule)	If Michele borrows $4,800 at $6\frac{1}{4}$% simple interest, how long will it take before her debt reaches $5,000? (Example 1.3.6)

(Continued)

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Topic	Key Ideas, Formulas, and Techniques	Examples
Finding the Term of a Note from Its Dates (within a Calendar Year), p. 33	• Convert calendar dates to Julian dates using the day of the year table (or the abbreviated table) • If the year is a leap year, add 1 to the Julian date if the date falls after February 29. • Subtract the loan date from the maturity date	Find the number of days between April 7, 2003, and September 23, 2003. (Example 1.4.1)
Finding Maturity Dates (within a Calendar Year), p. 36	• Convert the loan date to a Julian date • Add the days in the term • Convert the result to a calendar date by finding it in the day of the year table	Find the maturity date of a 135 day note signed on March 7, 2005. (Example 1.4.5)
Finding Loan Dates (within a Calendar Year), p. 36	• Convert the maturity date to a Julian date • Subtract the days in the term • Convert the result to a calendar date by finding it in the day of the year table	Find the date of a 200-day note that matures on November 27, 2006. (Example 1.4.6)
Finding Terms Across Multiple Years, p. 37	• Draw a time line, dividing the term up by calendar years • Find the number of days of the note's term that fall within each calendar year • Add up the total	Find the term of a note dated June 7, 2004, that matures on March 15, 2006. (Example 1.4.8)
Finding Dates Across Multiple Years, p. 38	• Draw a time line • Work through the portion of the term that falls in each calendar year separately • Keep a running tally of how much of the term has been accounted for in each calendar year until the full term is used	Find the loan date for a 500-day note that matured on February 26, 2003.
Using Nonannual Interest Rates (Optional), p. 44	• Convert the term into the same time units used by the interest rate • Use the same techniques as with annual interest rates	Find the simple interest on $2,000 for 2 weeks if the rate is 0.05% per day. (Example 1.5.2)
Converting Between Nonannual and Annual Rates (Optional), p. 45	• To convert to an annual rate, multiply by the number of time units (days, months, etc.) per year • To convert from an annual rate, divide by the number of time units (days, months, etc.) per year	Convert 0.05% per day into an annual simple interest rate. (Example 1.5.3)

The following exercises are a mixture of problems from the topics covered in Chapter 1. One of the objectives of doing these exercises is to be able to correctly identify which topics and tools from this chapter are needed for each problem. All of the material covered in this chapter is fair game, except for topics that were identified as optional.

1. Find the amount that you would need to deposit at a simple interest rate of 8.45% in order to earn $200 in 3 months.

2. Express 5 7/8% as a decimal.

3. Mike loaned Jim $2325.17 for 80 days at 7 ½% simple interest. Find the total amount Jim will have to repay.

4. How long would it take for $1,835 to grow to $2,000 at 9.31% simple interest?

5. On January 7, 2005, Ming-zhu borrowed $186,547 at 11.35% simple interest. The note that he signed came due on October 29, 2005. Find the note's maturity value.

6. I borrowed $80 and paid $100 when the loan came due. How much interest did I pay?

7. If a note is signed on December 17, 2005, and matures on April 13, 2007, find its term.

8. Ingrid invested $12,500 at 4.35% simple interest for 300 days. If bankers' rule is used to calculate interest, how much interest did she earn?

9. Lockwell Lockers Company borrowed $182,600 at 6 5/16% simple interest on July 15, 2004. The note that the company signed matured on February 3, 2005. How much interest did the company pay?

10. Johanna borrowed $350 for 100 days. She paid $375 in total to settle the loan when it came due on September 25. Find (a) the amount of interest she paid and (b) the date the loan was taken out.

11. Find the maturity date and maturity value of the following promissory note.

> *I hereby acknowledge a loan of $5,235.17 made to me on October 11, 2005, and agree to pay this loan back in 300 days, together with simple interest calculated at a rate of 12½%.*

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12. Suppose that I loan someone $5,000 for 90 days. I will be repaid $5,125 when the loan comes due. Find the simple interest rate.

13. Apercu Industries borrowed $92,350 for 200 days at 15% simple interest. Find the total interest they will pay (a) using bankers' rule and (b) using the exact method.

14. On March 1, Serge deposited $1,200 in a certificate of deposit paying 3.91% simple interest. The certificate's maturity date was November 21. Find the value of the certificate at maturity.

15. What date is 890 days from March 29, 2005?

16. Calvin loaned Erin $8,145 on August 10, 2005. On April 17, 2006, Erin paid off the loan with $8,525.47. Find the simple interest rate for this loan.

17. Axel put $939.39 in a bank account paying 5.35% simple interest. How long will it take before his account has grown to $1,000?

18. Axel put $939.39 in a bank account paying 5.35% simple interest on June 7, 2004. When will his account be worth $1,000?

19. Find the interest due on a loan of $89,000 for 7 months at 7.38% simple interest.

20. Express each of the following percent interest rates as decimals:
    a. 8%
    b. $4\frac{1}{8}$%
    c. 15.3%
    d. 125%
    e. 0.75%

21. On July 7, a real estate developer offered Danny $275,000 for a vacant building that he owns. The developer states that this offer will expire in 35 days. On what date does this offer expire?

22. How much would you need to have in an investment account in order to generate monthly income of $850, assuming a 7.5% simple interest rate?

23. Convert each of the following percent interest rates to decimals:
    a. $4\frac{3}{4}$%
    b. $10\frac{7}{8}$%
    c. $13\frac{9}{16}$%
    d. $5\frac{11}{32}$%

24. Carl loaned Jerry $500 on February 12, 2008, and Jerry repaid a total of $515.93 on July 15, 2008. Find the simple interest rate for this loan.

25. Find the maturity value of this promissory note:

> *Promissory Note*
> *Today, July 30, 2007, I, Erwin Addison, acknowledge a loan of $25,000 made to me by the Fourth National Bank of Portville, and agree to repay this loan on September 19, 2007, together with simple interest calculated at a rate of 9% per annum.*

26. According to doomsayers, the ancient Mayan calendar terminates on December 21, 2012, and therefore the world will end on that date. How many more days do we have left?

27. Andriy deposited $4,500 in a 1-year CD paying 6.15% simple interest. How much interest will he earn?

28. Calculate the simple interest on an $80,000 loan for 210 days at a simple interest rate of 12.285%. Assume that bankers' rule is being used.

29. A small business owner borrowed $19,275.13 and paid $21,045.61 to settle the debt. How much total interest did she pay?

30. The Tilsonboro Savings Bank pays 5.38% simple interest on 6-month certificates of deposit. If you deposit $12,500 in such a CD, how much will it be worth when it matures?

31. A roofing contractor took out a short-term loan for $15,000 to pay for materials. The simple interest rate was 8% per annum, and the term was 21 days. Find the total interest paid.

32. Your mobile phone statement shows a balance of $178.59 and the payment is due on April 20. The statement says that payments received more than 15 days late will be charged a $20 late payment fee. By what date must you pay your balance to avoid the fee?

33. Ralyd Pharmaceuticals' patent on its top-selling drug expires on May 15, 2009. How many days of this patent were left on November 8, 2006?

34. Determine the total amount required to repay a $5,000, 3-year loan if the simple interest rate is 10.3% per annum.

35. Find the total interest for a 15-month loan of $98,359.25 if the simple interest rate is 5.05%.

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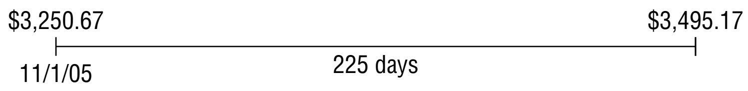
36. Qisun borrowed $4,818 at a 5% simple interest rate on September 5, 2005. When did the total amount she owed reach $5,000?

37. St. Stephen's Church is considering borrowing $850,000 to fund a building expansion. At the annual financial meeting, the chair of the finance committee states that with the financing the committee is considering, the church would not be required to repay any principal for 5 years, but would be required to make an interest payment of $20,000 per quarter. What is the simple interest rate for this loan?

38. For the loan represented by the time line shown below, determine:
   a. The principal
   b. The amount of interest
   c. The term
   d. The maturity date
   e. The simple interest rate



39. Find the interest for a 1-year loan of $400,000 at a 12 1/8% simple interest rate.

40. Dick made a loan to George. Who is the creditor and who is the debtor?

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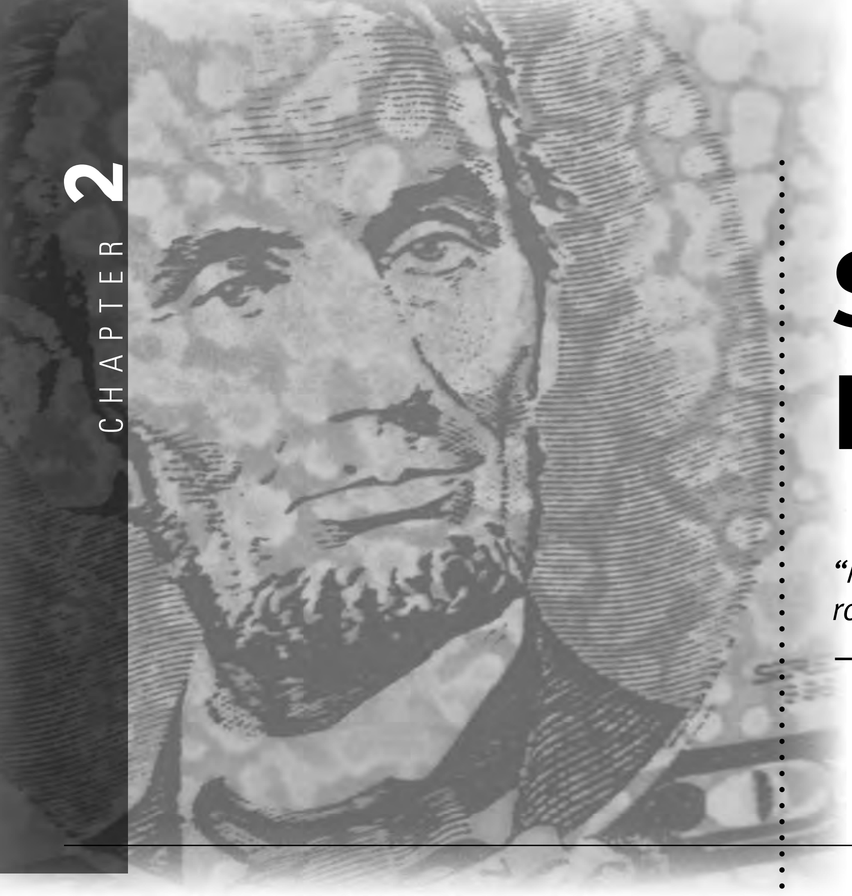

CHAPTER 2

Simple Discount

"Money is power, freedom, a cushion, the root of all evil, the sum of blessings."

—Carl Sandburg, "The People, Yes"

Learning Objectives

LO 1 Understand the concept of simple discount, and recognize situations where it may be used.

LO 2 Calculate the amount of simple discount for, and the proceeds of, a simple discount note.

LO 3 Use the simple discount formula and basic algebra techniques to find the maturity value, simple discount rate, or term, given the other details of a discount note.

LO 4 Compare simple interest rates to simple discount rates and calculate the equivalent simple interest rate for a discounted note.

LO 5 Analyze secondary sales of promissory notes and perform calculations necessary to evaluate the simple interest rates earned by different parties to a given financial transaction.

Chapter Outline

2.1 Simple Discount

The size of a loan usually depends on the amount the borrower needs (or wants) to borrow. That was certainly how we looked at things in Chapter 1, and it is probably the most natural point of view in most situations.

However, there are times when a loan's size depends more on the amount to be repaid than on the amount to be borrowed. Consider the following few examples:

- Jennifer knows she will be getting a paycheck for $500 at the end of the week, but she needs money now. She takes out a loan against this paycheck, borrowing as much as she can pay off with her check when it arrives.
- When you file your federal income taxes, you find that you are owed a refund of $737.15. Your tax preparer offers the option of getting your money right away, instead of waiting for the IRS to process your return and send your refund check. In exchange for agreeing to sign over your refund check when it arrives, the preparer agrees to lend you $707.15 today.
- The Cedar Junction Central School District is scheduled to receive a state aid check for $72,500 on April 1. The district needs the funds by mid-February to meet expenses, but unfortunately the state is unwilling to make the payment early. In order to avoid a cash crunch, the district borrows $71,342 from First Terrapin National Bank, to be repaid on April 1 when the state aid arrives. The amount borrowed is based on the $72,500 that the district will have available for repayment when the aid check arrives.

In each of these cases, a loan is being made, and so it is reasonable for the lenders to expect to be paid some "rent" on their money. In the previous chapter, we have described that rent as interest, and thought of it as being added on to the principal. Looking at things as *principal* + *interest* = *maturity value*, we might describe the tax refund loan example as:

$$\begin{array}{r} \$707.15 \\ +\ \$30.00 \\ \hline \$737.15 \end{array}$$

While there is nothing wrong with this description, it doesn't really represent things quite the way they are happening. The figures are correct, but this mathematical description suggests that you decided to borrow $707.15, added on $30 interest, and then arrived at the $737.15 maturity value as a consequence. But that isn't what happened at all. A truer representation of the situation might be to turn the description around:

$$\begin{array}{r} \$737.15 \\ -\ \$30.00 \\ \hline \$707.15 \end{array}$$

This second representation is financially and mathematically equivalent to the first, but it presents a different way of looking at things. It suggests that you knew you could repay $737.15, gave up $30.00 "interest," and arrived at the $707.15 "principal" as a consequence. This is much truer to what actually happened.

These examples demonstrate an alternative way of looking at the time value of money. With this new way of looking at things, we will need some new terminology:

Definitions 2.1.1

A loan that is made on the basis of a fixed maturity value is called a ***discount loan***. *The lender subtracts an amount, called the* ***discount***, *from the maturity value, and pays the result, called the* ***proceeds***, *to the borrower.*

If we were looking at the tax refund example as interest, we would call the $707.15 the principal and the $30 the interest, and add them together to obtain the $737.15 maturity value. Looking at it as discount, we start with the $737.15 maturity value, subtract the $30 discount, and arrive a $707.15 in proceeds.

Most of the rest of our terminology remains the same, with one notable exception. *Term, maturity date,* and *maturity value* all still mean the same thing. Recall though that the ***face value*** of a simple interest note has heretofore always been the same as its principal. The "size" of the loan is the amount most likely to be shown prominently on the note's face, hence the term. When a promissory note is based on discount, however, the size of the

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U.S. Savings Bonds are sold at a discount to their face value.
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loan is more apt to be thought of as the maturity value. And so:

Definition 2.1.2

*The **face value** of a discount note is its maturity value.*

Thus, the meaning of the term *face value* depends on whether we are talking about interest or discount.

One common example of a discount note is U.S. Savings Bonds (since 2001 sometimes referred to as "Patriot Bonds"). Savings bonds can be purchased at most banks, and many employers offer a program that allows their workers to buy them by payroll deduction. A savings bond is actually a promissory note issued by the U.S. government, so when you buy a savings bond you are loaning money to Uncle Sam. Savings bonds used to be a much more popular investment than they are today, but many people still do use them as a way to save, and they are often given as gifts for occasions like the birth or adoption of a child.

There are several types of savings bonds, but the type most familiar to most people (Series EE bonds) is sold for half of its face value. This is one reason why savings bonds are often used as contest prizes or promotional giveaways since a "$50 U.S. savings bond" actually only costs $25, allowing the prize to be advertised as being twice its actual cost and current value.[1] The $50 face value reflects the amount the government guarantees that it will pay for the bond on maturity, which is usually many years away. (As of early 2005, the maturity date for a savings bond was 20 years from the date purchased, though the time to maturity changes depending on prevailing interest rates when the note is issued.) Since the $50 face value encourages us to think of $50 as the size of the note, it is natural to consider a savings bond as a type of discount note.[2]

Treasury bills (also known as T bills for short) are another common example of a discount note. While they may be less familiar than savings bonds, they are actually less complicated and in many ways better example of a discount loan. Treasury bills are short-term loans to the U.S. federal government, carrying terms ranging from a few days to 6 months, though the most common terms are 4, 13, or 26 weeks. They are sold at a discount to their maturity values. For example, you might buy a $1,000 face value T bill for $985, a $15 discount from the maturity value.

T bills cannot normally be purchased through banks or payroll deductions the way that savings bonds can, and while savings bonds are issued with face values as low as $50, T bills are sold in $1,000 increments, with $1,000 being the minimum face value. T bills can be purchased through investment brokers, or directly from the government through its Treasury Direct program (www.treasurydirect.gov).

Also, unlike savings bonds, the price of a T bill is not set in advance. When a T bill is issued, the government offers it for sale on the market. The prices are set ***at auction***, meaning that the bonds are sold to whoever is willing to pay the highest prices for them. Like anything else sold on the open market, the selling price of a $1,000 T bill is the best offer that the seller (in this case, the federal government) can get for the thing being sold. Whatever the price turns out to be, though, the note will clearly sell at a discount to its maturity value.

[1]Let's be honest: that's also a reason why they make great gifts.

[2]Savings bonds are actually quite complicated animals, with all sorts of features that ordinary discount notes lack. They can, for example, be cashed in before the maturity date or left to grow beyond the maturity date. The rates paid on savings bonds may vary, depending on the specific type of savings bond, date of purchase, and prevailing interest rates. Since the rates vary, they actually usually reach their maturity value long before their supposed maturity date, which is based on a guaranteed minimum interest rate. Furthermore, though they resemble discount notes in many ways, their values actually are not calculated by simple discount. Nonetheless, their basic premise makes an accessible and familiar illustration of a discount note. Because of all the extra bells and whistles, though, savings bonds will not be further discussed in this chapter.

The Simple Discount Formula

We can arrive at a formula for simple discount by thinking back to how we developed the simple interest formula. We observed that the amount of simple interest should be proportional to the amount borrowed, and that it should also be increased or decreased in proportion to the loan's term. From this, we arrived at the simple interest formula:

$$I = PRT$$

The same logic applies to discount. If a $500 note is discounted by $20, it stands to reason that a $5,000 note should be discounted by $200. If a 6-month discount note is discounted by $80, it stands to reason that a 12-month note would be discounted by $160. Thus, modeling from what we did for interest, we can arrive at:

FORMULA 2.1
The Simple Discount Formula

$$D = MdT$$

where
D represents the amount of simple DISCOUNT for a loan,
M represents the MATURITY VALUE
d represents the interest DISCOUNT RATE (expressed as a decimal)
and
T represents the TERM for the loan

The simple discount formula closely mirrors the simple interest formula. The differences lie in the letters used (D rather than I and d in place of R, so that we do not confuse discount with interest) and in the fact that the discount is based on maturity value rather than on principal. Despite these differences, the resemblance between simple interest and simple discount should be apparent, and it should not be surprising that the mathematical techniques we used with simple interest can be equally well employed with simple discount.

Solving Simple Discount Problems

The following examples illustrate the use of the simple discount formula.

Example 2.1.1 ***A $10,000 face value discount note has a term of 4 months. The simple discount rate is 6%. Find the amount of the discount.***

$$D = MdT$$
$$D = (\$10{,}000)(0.06)(4/12)$$
$$D = \$200.00$$

The note would be discounted by $200.

Example 2.1.2 ***A $5,000 face value note has a term of 219 days. The simple discount rate is 9⅜%. Find the proceeds of the note.***

$$D = MdT$$
$$D = (\$5{,}000)(0.09375)(219/365)$$
$$D = \$281.25$$

The proceeds can be found by subtracting the discount from the maturity value: $5,000 − $281.25 = $4,718.75. Thus the proceeds of the note would be $4,718.75.

With simple interest, we saw that we could use algebra on the formula to find the principal, the interest rate, or the term. It comes as no surprise that we can do the same sort of thing with simple discount. The next several examples will demonstrate how the techniques we developed for simple interest can be applied for simple discount.

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Example 2.1.3 ***A 3-month note is discounted by \$28.75. The simple discount rate is 5¾%. Determine the maturity value and the proceeds of the note.***

$$\begin{aligned} D &= MdT \\ \$28.75 &= M(0.0575)(3/12) \\ \$28.75 &= M(0.014375) \\ \frac{\$28.75}{0.014375} &= M \\ M &= \$2{,}000 \end{aligned}$$

This gives us the maturity value. To find the proceeds, remember that discount is subtracted from the maturity value. Since \$2,000 − \$28.75 = \$1971.25, we conclude that the maturity value is \$2,000 and the proceeds are \$1,971.25.

Example 2.1.4 ***When Nestor filed his federal income taxes, he was happy to learn that he was due a refund of \$799.45. He was less happy to learn that it would take 45 days for his refund check to arrive. His tax preparer offered to give him \$775.00 on the spot, in exchange for Nestor's tax refund check when it arrives. What simple discount rate does this offer equate to?***

The amount of discount is the difference between the \$799.45 and the \$775.00, so D = 799.45 − 775.00 = \$24.45. Using this in the formula to find the rate, we get:

$$\begin{aligned} D &= MdT \\ \$24.45 &= (\$799.45)d(45/365) \\ \$24.45 &= \$98.56232877d \\ d &= 0.248066379 \end{aligned}$$

Moving the decimal place and rounding, we get that the simple discount rate is 24.81%.

Example 2.1.5 ***A \$10,000 T bill with 182 days to maturity sold at auction for \$9,753.16. What is the simple discount rate?***

The amount of discount is \$10,000 − \$9,753.16 = \$246.84.

$$\begin{aligned} D &= MdT \\ \$246.84 &= (\$10{,}000)d(182/365) \\ \$246.84 &= (\$4{,}986.3013699)d \\ d &= 0.04950363 \end{aligned}$$

The simple discount rate is 4.95%.

Example 2.1.6 ***Killawog Financial Corp invested \$49,200 in discount bonds with face values totaling \$50,000. The discount rate was 4%. How long will it be until the notes mature?***

The amount of discount is \$50,000 − \$49,200 = \$800.

$$\begin{aligned} D &= MdT \\ \$800 &= \$50{,}000\,(0.04)(T) \\ \$800 &= \$2{,}000\,(T) \\ T &= 0.4 \text{ years} \end{aligned}$$

Multiplying by 365 to convert this term to days, we find that the notes will mature in 146 days.

Other types of problems that we solved with simple interest can also be solved if the loan is made using simple discount instead. The date calculation techniques of Chapter 1 can be readily adapted to simple discount problems as well.

Example 2.1.7 ***The Reeds Corners Central School District borrowed \$4,959,247 on March 7, 2005, in anticipation of receiving a state aid payment of \$5,000,000. The loan was based on a simple discount rate of 4¼%. On what date will the district receive its state aid?***

$$\begin{aligned} D &= MdT \\ \$40{,}753 &= \$5{,}000{,}000(0.0425)(T) \\ T &= 0.1917788235 \text{ years or 70 days} \end{aligned}$$

March 7 is the 65th day of the year, so the note matures on the 66 + 70 = 136th day of the year. The state aid will be received on May 16, 2005.

The exercises include a number of other problems that make use of many of the skills developed in Chapter 1.

EXERCISES 2.1

A. Terminology

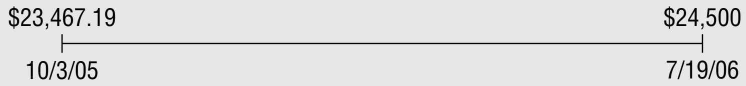
In each of the following situations, find (a) the maturity value, (b) the proceeds, and (c) the amount of discount.

1. A lender loaned a borrower $750 on May 8, and the borrower repaid the loan by giving the lender $800 on June 10.

2. A note will mature for its face value of $20,000 in 100 days. You buy the note for $1,000 less than its face value.

3. Stewart's New Arts Gallery received $18,000 from a lender on March 1, and paid off the loan on October 15 with a check for $20,000.

4. The payments between a borrower and lender are illustrated in the following time line:



5. Kyle will be getting a paycheck of $1,308.55 at the end of the month. Sammy agreed to give him $1,275 today in exchange for the check when it comes.

6. If you give me $20 today, I will pay you back $25 next Tuesday.

B. Calculating Simple Discount and Proceeds

7. A note has a maturity value of $10,000. If the term is 200 days and the simple discount rate is 9%, find (a) the amount of discount and (b) the proceeds.

8. To finance the building of a new fire station, the Farmview Fire District issued discount notes, each having a face value of $1,000 and a term of 5 years. The notes were sold at a simple discount rate of 3%. Find the proceeds of each note sold.

9. Derron knows that he will be receiving $100,000 to settle an insurance claim, but he will not actually be paid for another 2 months. Needing cash now, he is considering an offer from a finance company to buy this payment from him by paying him cash today. The simple discount rate they are offering him is 35%. How much would he receive if he takes this deal?

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10. Brittany bought a discount bond issued by the State of California with a maturity value of $20,000. The discount rate was 3.19% and the term was 175 days. How much did Brittany pay for the bond?

C. Finding Maturity Value, Discount Rate, and Term

11. Find the maturity value of a 6-month discount note if the discount is $250 and the discount rate is 12.5%.

12. Hall's Homestyle Horseradish Farms signed a $25,000 discount note with a term of 90 days. The company received proceeds of $24,325. Find the simple discount rate for this note.

13. Suppose that an investment manager buys a $10,000 U.S. government discount note for $9,759.16. If note matures in 217 days, find the simple discount rate.

14. A note with a maturity value of $5,000 is sold at a simple discount rate of 4%. If the proceeds are $4,975, what is the term of the note?

15. Handy Dandy Discount Loans offers Jason $785.21 in exchange for a paycheck of $800, which he will be receiving in a few days. If the simple discount rate is 75%, in how many days is Jason expecting his paycheck?

D. Grab Bag

16. A $2,000 face value discount note matures on June 17. Marco bought this note on April 8 at a simple discount rate of 5.72%. How much did he pay for the note?

17. You agree to pay me $180 one month from now if I will give you $160 today. What is the simple discount rate for this deal?

18. Find the amount of discount if an $8000 maturity value T bill is sold at auction for $7,856.02.

19. A $5,000 T bill is sold at auction for $4,876.52. The term is 182 days. Find the simple discount rate.

20. Find the proceeds for a discount loan if the maturity value is $23,507 and the amount of discount is $1,851.03.

21. Ed's Worldwide Lawn Furniture Emporium has received an order for $875,000 worth of picnic tables. The buyer will pay for the tables on delivery, 45 days from now. In need of cash right away, Ed considers selling this payment to a finance company at a simple discount rate of 24%. What proceeds would he receive if he took this deal?

22. On June 28, 2005, I decided to invest some money by buying a $1,000 face value discount note, with a maturity date of December 31, 2005 and a simple discount rate of 5.89%. How much of a discount will I get off the note's face value?

23. If you are due a tax refund of $875.39 in 50 days and your tax preparer offers to give you $850.00 now in exchange for your refund when it arrives, what is the simple discount rate?

24. What would be the proceeds of the following promissory note if it is sold on April 1, 2007, at a simple discount rate of 8.21%?

The County of Conesus will pay the owner of this note $10,000 on November 15, 2007.

25. A $10,000 T bill with 28 days to maturity is sold with a simple discount rate of 5.32%. Find the selling price.

26. A $1,000 T bill is sold for $982.56. The term is 91 days. Find the simple discount rate.

27. The City of Chester Rock, experiencing dire financial problems, has been promised a $17.5 million aid payment from the state. The city desperately needs cash to pay its bills no later than May 1. Unfortunately, the state aid will not be paid to the city until May 15. A group of investors agrees to provide the city with a cash payment on May 1 in exchange for the May 15 state aid check. The simple discount rate for this loan will be 3.98%. How much profit will the investors make on this deal? How much will the city receive on May 1?

28. At a 4% simple discount rate, a $5,000 face value note sold for $4,905.75 on March 12. Find (a) the term and (b) the maturity date.

29. On April 3, 2007 Burana Financial Investment Corp. bought a $100,000 T bill for $99,353.29. The simple discount rate was 4.44%. When is the maturity date?

E. Additional Exercises

30. Life insurance policies pay their death benefit on the death of the person insured. A **viatical settlement** is a business agreement in which someone with a terminal illness may "sell" his or her insurance death benefit to an investor at a discount. The investor becomes the beneficiary of the insurance policy, and receives the death benefit when the insured person passes away.

    Suppose that Ellwood has a $150,000 life insurance policy, and has been diagnosed with a terminal illness. His doctors have told him that he can expect to live only another 9 months. A viatical company offers to buy his death benefit for $117,300. Assuming that the doctors are correct about his life expectancy, what is the simple discount rate being offered?

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2.2 Simple Discount vs. Simple Interest

Suppose that Lysander Office Supply borrowed $38,000 for 1 year from Van Buren Capital Funding Corp. The maturity value of the note was $40,000. Was this loan based on simple discount or simple interest?

There is no way of knowing this for sure from the information given. We could look at this as simple interest. In that case, we would say that:

\$38,000	(principal)
+ \$2,000	(interest)
\$40,000	(maturity value)

But we could equally well look at this as simple discount, in which case we would say:

\$40,000	(maturity value)
− \$2,000	(discount)
\$38,000	(proceeds)

In fact, we could do this for any of the loans we've looked at so far, whether they were actually simple interest or simple discount. While any given loan may be set up with either simple interest or simple discount, the reality is that interest and discount are actually just two different ways of looking at the same thing. In many cases, interest is the more natural way of looking at things. In others, such as the examples of Section 2.1, discount may be more natural. In some cases there may be legal, regulatory, or accounting reasons why a loan must be treated as interest or discount. But whichever viewpoint is more natural, the fact remains that any simple interest loan can be looked at as discount if we want to, and vice versa.

Without knowing any of the details surrounding this particular loan, we just can't tell whether it was made with interest or discount. We can, though, take a look at the deal both ways, and use it as an example to compare simple interest and simple discount rates.

Example 2.2.1 ***For the transaction described above find (a) the simple interest rate and (b) the simple discount rate.***

(a)
$$I = PRT$$
$$\$2{,}000 = (\$38{,}000(R)(1)$$
$$R = 0.0526 = 5.26\%$$

(b)
$$D = MdT$$
$$\$2{,}000 = \$40{,}000(d)(1)$$
$$d = 0.05 = 5.00\%$$

The results of Example 2.2.1 may be surprising. You might have expected that the "rate" would be the same either way. After all, in both cases it is a percentage, and we've already seem that the *amounts* of simple interest and simple discount are the same (in this case, it's \$2,000 either way.) In fact, though, the simple interest and simple discount *rates* are not the same thing.

A rate is a percent, and a percent must be *of something.* For simple interest, that something is the principal, but in the case of simple discount, that something is the maturity value. When we calculated the simple interest rate, we looked at \$2,000 as a percent of \$38,000. With discount, the rate was found by looking at that same \$2,000 as a percent of \$40,000. Since these "of" amounts are different, it actually stands to reason that the percents will end up being different.

In fact, we can take this observation a step further. Since the principal and maturity are always different, for a given loan *the simple interest rate and simple discount rate will never be the same!* (The only exception would be where no interest/discount is being paid. If the interest and discount are both zero, then simple interest and simple discount rates are both 0%.)

Notice also that in Example 2.2.1 the simple interest rate turned out to be larger than the simple discount rate. This happened because \$2,000 is a larger percentage of \$38,000 than it is of \$40,000. Since the principal is smaller than the maturity value, the interest rate must be higher than the discount rate to arrive at the same \$2,000. Ignoring the possible but unlikely case of negative interest, the principal will always be less than the maturity value for any loan. Thus, *the simple interest rate will always be larger than the simple discount rate.*

Even though the simple interest rate will always be higher, it does seem reasonable to expect that, as we have seen in the examples above, the simple interest rate should be in the same ballpark as the simple discount rate. The principal/proceeds in a given transaction (on which the interest rate is based) are usually not far from the maturity value (on which the discount rate is based). And so it makes sense that the rates based on them will not be too terribly far from each other. This is not always true, though—when there is a big difference between the principal/proceeds and maturity value, the rates can be widely different. This will happen when the amount of interest is large in relation to the size of the loan, which can result from very high rates and/or long terms. Some examples of these situations are given in the Additional Exercises at the end of this section.

Determining an Equivalent Simple Interest Rate

From ordinary experience, we are much more accustomed to thinking about loans from the point of view of interest rather than discount. Thus, when we hear a discount rate, we are likely to want to interpret it as an interest rate, even though we have just seen that simple interest and simple discount rates are not in fact the same thing. For example, back in Example 2.1.6 a finance company bought bonds with a 4% discount rate. You probably thought of that as basically the same thing as earning 4% simple interest. In fact, though, we now know that the interest rate would be a bit higher.

Example 2.2.2 (Example 2.1.6 revisited) ***Killawog Financial invested $49,200 in bonds whose maturity values totaled $50,000. The remaining term of the bonds was 146 days. The simple discount rate was 4%, but what would the equivalent simple interest rate be?***

To answer this question, we look at the same transaction as before, but now we interpret it as though it were simple interest. Thus:

$$I = PRT$$
$$\$800 = (\$49{,}200)(R)(146/365)$$
$$\$800 = \$19{,}680(R)$$
$$R = 4.07\%$$

As expected, the simple interest rate is higher than the equivalent simple discount rate.

Even when simple discount is the logical and natural way of looking at things, we may want to know the equivalent simple interest rate. Since most of us are more accustomed to thinking in terms of interest, discount rates can be deceiving. It is easy to mistakenly read a discount rate as an interest rate. The following example will illustrate:

Example 2.2.3 ***An investment manager is weighing a choice between two possible investments for a fund that she manages. She originally had planned to invest in a $10,000 face value, 9-month simple discount note issued by the Levy Pants Company, which she was offered at a simple discount rate of 8%. On the other hand, the company has offered to borrow the same amount of money from her fund by signing a note carrying a simple interest rate of $8\frac{1}{4}\%$. Which is the better deal for the investment fund?***

On the face of it, this looks like a pretty simple question; $8\frac{1}{4}\%$ is higher than 8%, and so obviously a lender would prefer the higher rate.

However, despite appearances, this really is not so simple, because one rate is interest, while the other is discount, and so the comparison is not really "apples to apples."

Suppose that she invested in the 8% discount note. Then:

$$D = MdT$$
$$D = (\$10{,}000)(0.08)(9/12)$$
$$D = \$600$$

And so the fund would pay $10,000 − $600 = $9,400 for the note.

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On the other hand, if we look at this note as simple interest:

$$I = PRT$$
$$\$600 = (\$9{,}400)(R)(9/12)$$
$$\$600 = \$7{,}050(R)$$
$$R = 8.51\%$$

*The 8% simple **discount** rate is actually equivalent to earning an 8.51% simple **interest** rate. Despite initial appearances, the fund would actually earn a better return from the 8% discount note.*

When rates quoted for investments are based on discount, often both the interest and discount rate will be given. For example, when the rates for T bills, which are discount notes, are reported in newspapers or financial publications, it is not unusual to see both interest and discount rates given. For example, rates for T bills might be shown in a table that would look something like this one:

U.S. TREASURY BILL CURRENT MARKET RATES

Maturity Date	*Discount Rate*	*Interest Rate*
8/31/06	4.95%	4.97%
11/30/06	5.13%	5.20%

Note that the rates differ, depending on the maturity date for the note. This will usually be the case. The amount of time for which an investor/lender is tying up her money often will affect the rate of interest or discount she expects to receive. The rates for longer term notes are usually higher than for shorter term ones, though there is no reason why this must always be the case.

Rates in Disguise

Suppose that you are due a $750 paycheck at the end of the week, but want or need to get your hands on the cash on Monday. A ***payday lender*** is an individual or business that will offer to give you immediate cash in exchange for your agreement to sign over your check to the lender when you receive it.[3] Suppose that such a lender offers to make this deal with you for a fee of 2% of the paycheck. This is, for all intents and purposes, a loan. Your $750 is the maturity value, and the 2% will be calculated off of it, and subtracted from it, to determine your proceeds.

Even though this scenario provides a classic example of discount, we need to be careful in interpreting that 2% "rate." It would be quite easy to think of that 2% as the simple discount rate in the usual sense, and thus to treat it as a rate of 2% per year. But read that description over once again—that is not the deal at all! The fee is "2% *of the paycheck.*" No one said that the 2% was a *rate* of simple discount, annual or otherwise. The lender's cut is 2%, not 2% "per" anything. So the lender will take a discount of (0.02)($750.00) = $15.00, leaving you with $735.00.

What is the equivalent simple discount *rate*? The term of the loan is 4 days, and so using this in the formula gives:

$$D = MdT$$
$$\$15.00 = \$750(d)(4/365)$$
$$d = 1.825 = 182.50\%$$

A 2% fee doesn't sound like all that much to pay, but putting it in terms of an annual simple discount rate puts a new perspective on it. The equivalent simple interest rate is just as astounding:

$$I = PRT$$
$$\$15.00 = \$735(R)(4/365)$$
$$I = 1.8622 = 186.22\%$$

[3]This may be done by having you write a post-dated check from your own checking account.

Whether or not it is worth paying $15.00 out of $750.00 to get your cash sooner is of course a personal decision, depending entirely on your preferences and circumstances. The point of this illustration is only that in evaluating the cost of a loan it is important to be very clear about just what the "percent" really means. We all tend to automatically think of any percent as an annual interest rate. That assumption can be very misleading, as this example, hopefully, has made clear.

The fee deducted may not always be technically considered interest; some or all of it may be considered a "service fee" or some other terminology. The technical legal treatment of these fees really does not concern us here all that much though. An amount being subtracted from the eventual payment amount is in concept discount or interest, regardless of how it is treated for accounting or legal purposes.

Example 2.2.4 ***Ginny is expecting a $795 paycheck in 8 days. A payday lender offers to give her cash now for this check. The lender's fee is 1.5% of the amount, plus a $10 service fee. Find the equivalent simple interest and simple discount rates.***

First, we need to determine how much Ginny will be giving up: 1.5% of her paycheck is (1.5%)($795) = $11.93. Adding in the $10 service fee, the total is $21.93. Ginny would receive $795 − $21.93 = $773.07.

As simple discount:

$$\begin{aligned} D &= MdT \\ \$21.93 &= (\$795)d(8/365) \\ d &= 1.2586 = 125.86\% \end{aligned}$$

As simple interest:

$$\begin{aligned} I &= PRT \\ \$21.93 &= (\$773.07)R(8/365) \\ R &= 1.2943 = 129.43\% \end{aligned}$$

Businesses also sometimes do something similar to raise cash. An amount that a business is owed to be paid in the near future is called a ***receivable***, and if a business needs cash before the payment is due it may ***discount the receivable*** by "selling" this payment at a discount. The amount of the discount might be expressed as a straight percent, as in our examples above, or may be expressed as a rate. Since the difference between 2% discount and 2% simple discount rate is so enormous, it is important to pay careful attention to how things are worded.

EXERCISES 2.2

A. Terminology

Each of the following situations could be thought of either as simple interest or as simple discount. First describe the loan in terms of simple interest, identifying (a) the principal, (b) the maturity value, (c) the amount of interest, and (d) the face value. Then describe the loan in terms of simple discount, identifying (e) the proceeds, (f) the maturity value, (g) the amount of discount, and (h) the face value.

1. I signed a note to borrow $875 and paid back $900.

2. A state government issues a bond to a group of investors. The investors pay the state $4,753,259 for the bond, and will be paid back $5,000,000 two years later.

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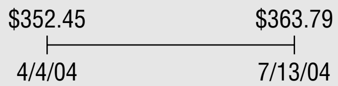
3. The payments exchanged by a borrower and lender are as illustrated in the timeline below:



4. The following promissory note:

> *On February 14, 2002, I borrowed $18,355.17 from The National Bank of Northern South Dakota, and agree to pay back $18,759.15 on February 14, 2003.*
>
> *Walter W. Walters*

B. Interest Rates and Discount Rates

For each of the following situations, find (a) the simple interest rate and (b) the simple discount rate.

5. A $10,000 face value discount note is sold for $9,393.93. The term is 300 days.

6. Port Gibson Mutual Insurance Company lends $62,500 to a real estate developer in exchange for a $70,000 payment 9 months later.

7. Mara loans her brother $15 and he pays her back $20 fourteen days later.

8. Stannards Capital Investment Corp invested $576,300 to buy a note that matures in 2 years for $725,000.

C. Finding Equivalent Interest Rates

In each of the following exercises, find the simple interest rate that would be equivalent to the stated simple discount rate.

9. The simple discount rate is 12%, the maturity value is $7,500, and the term is 1 year.

10. A $5,000 maturity value note with a 150-day term is sold at a simple discount rate of 8.35%.

11. A real estate agent knows that he will receive a commission of $4,250 from the sale of a property when the deal is completed 37 days from now. Needing cash to meet his expenses today, though, he signs a discount note at a credit union, using his expected commission as the maturity value. The discount rate is 9.55%.

12. A $10,000 T bill with 73 days till maturity is sold at a 5.48% simple discount rate.

13. Algernon has just learned that his long-lost uncle has passed away, leaving him $120,000 in his will. He will not receive the money, though, until his uncle's estate is fully settled, which will take 1 year. Unfortunately, Algie could really use some cash today to settle his gambling debts. He is able to arrange a discount note from a moderately disreputable lender, using the money from his uncle as the maturity value, at a discount rate of 40%.

D. Rates in Disguise

14. Denarea will receive a $1,043.59 paycheck in 7 days. A payday lender offers to give her cash today for this check for a fee of 1½% of the check amount. Find (a) the amount of the fee, (b) the amount she will receive, (c) the equivalent simple discount rate, and (d) the equivalent simple interest rate.

15. Tom will be paid $789.95 in 10 days. A payday lender offers to give him cash today for his paycheck, taking as its fee a discount of 1.75% plus an additional service fee of $5. Find (a) the amount of the lender's fee, (b) the equivalent simple discount rate, and (c) the equivalent simple interest rate.

16. An insurance agent has just sold a large policy and will be paid a commission of $3,279.46 in 30 days. His brother-in-law offers to give him cash right now in exchange for his commission check for a fee of 4%. Thinking that 4% sounds like a pretty terrific rate for a loan, the agent agrees. What is the actual simple discount rate he is paying in this deal? What is the equivalent simple interest rate?

17. Benny runs a machine tool business. A large customer owes him a payment of $750,000 due in 16 days, but Benny needs cash now. A finance company offers to give him cash now for this debt; the fee would be 1%. What is the actual simple discount rate Benny is being offered? What is the equivalent simple interest rate?

E. Grab Bag

18. The Titan Siren Company is due to receive a payment of $475,000 in 27 days. A finance company offers to buy this receivable today for $471,500. Find the simple discount rate being offered. Find the simple interest rate being offered.

19. A $10,000 T bill with 135 days to maturity is sold at a 4.44% simple discount rate. What is the equivalent simple interest rate?

20. Suppose you buy a note with a $500 maturity value for $482.36. First describe the loan in terms of simple interest, identifying (a) the principal, (b) the maturity value, (c) the amount of interest, and (d) the face value. Then describe the loan in terms of simple discount, identifying (e) the proceeds, (f) the maturity value, (g) the amount of discount, and (h) the face value.

21. An annual subscription to *Unpopular Sports Weekly* costs $79.95. My subscription renewal isn't due for another 3 months, but the company offers to give me a 10% discount off the price if I renew now. If we looked at this early payment as a loan, what would the simple interest rate be?

22. A $25,000 T bill with 18 days to maturity is sold with a 5.09% simple discount rate. Find the equivalent simple interest rate.

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23. The Gorham Widget Corporation will receive a payment of $1.8 million (i.e. $1,800,000) for a large order 3 months from now. Unfortunately, in order to cover the costs of producing the items ordered, Gorham needs to get its hands on the money sooner. If a finance company offers make a loan to them at a simple discount rate of 9.5%, how much could Gorham borrow against this payment? If another finance company offers them a loan at a simple interest rate of 9.63%, would Gorham be better off taking that?

    Justify your answer.

24. My tax preparer offered to give me $1,249.35 up front in exchange for my tax refund of $1,295, which I expect to receive in 45 days. What is the amount of discount for this offer?

25. My tax preparer offered to give me $1,249.35 up front in exchange for my tax refund of $1,295, which I expect to receive in 45 days. I want to use this money to buy a new refrigerator. If I buy the refrigerator on my credit card, I will pay a simple interest rate of 21.99%. How does the simple interest rate my tax preparer is offering me compare to the rate on my credit card?

F. Additional Exercises

26. Wilson borrowed $14,357 for 100 days at 15% simple interest. If we looked at this loan as simple discount instead, what would the simple discount rate be?

27. There is an error in this table of T bill rates. For one of the maturities, an editor accidentally switched the interest and discount rates. Which maturity had its rates switched?

U.S. TREASURY BILL CURRENT MARKET RATES

Maturity Date	*Discount Rate*	*Interest Rate*
4/30/07	5.55%	5.60%
5/31/07	5.57%	5.65%
6/30/07	5.62%	5.72%
7/31/07	5.80%	5.66%
8/30/07	5.82%	5.94%

28. Determine the equivalent simple interest rate for a 2-year simple discount loan, assuming a maturity value of $1,000 and a simple discount rate of 25%.

29. Determine the equivalent simple interest rate for a 5-year simple discount loan, assuming a maturity value of $5,000 and a simple discount rate of 18%.

30. Suppose that a $10,000 maturity value note is discounted with a simple discount rate of 6%. Determine the equivalent simple interest rate assuming that the term is (a) 1 month, (b) 3 months, (c) 6 months, (d) 1 year. Is there anything that someone might find surprising about these answers?

2.3 Secondary Sales of Promissory Notes

Suppose that you borrow $1,000 from Friendly Neighborhood National Bank (FNNB). You sign a 1-year note with a simple interest rate of 8%. By now, it is a simple matter for us to calculate that this means that you will be required to pay back the original $1,000 plus $80 simple interest for a total of $1,080 one year from the loan date.

But now suppose that 1 month after you sign the note, the bank decides that it wants to be repaid early. Maybe it realized that it made more loans than it should have, maybe it needs the cash for some other purpose, or maybe it just wants to take advantage of an opportunity to loan that $1,000 to someone else at a better rate. Whatever the reason, the bank wants its cash back 11 months sooner than originally agreed. Does it have the right to demand that you repay the loan before the maturity date?

In most cases, the answer to this question would be no.[4] Unless the terms of the loan specifically allow the lender to demand early payment, you would not normally be obligated to pay up so much as a single day sooner than the maturity date. (Of course, there is nothing to prevent a lender from *asking* for early payment, maybe even offering some incentives to entice you to pay early.) But in the end, if the note says you have to pay up 3 weeks from next Tuesday, then you are under no obligation to pay until 3 weeks from next Tuesday.

The lender does have other options, though. A key fact about notes is that they are ***negotiable***. This term can be misleading—it does not mean that the terms of the note can be negotiated (that may well be true, but that is not what is meant by this term). The term *negotiable* means that notes can be bought and sold.[5] Friendly Neighborhood National Bank may not be able to collect from you before the maturity date, but it probably can sell your note to someone else. Such a transaction is often referred to as a ***secondary sale*** of the note. (Notes can be sold again and again and again, but no matter how many times a note has changed hands any such sale is still called secondary.)

Here is how it works. Suppose FNNB sells your note to Cheery Community Savings and Trust (CCST). On the maturity date your $1,080 will go to Cheery Community instead of Friendly National. This really makes no difference to you. The sale does not affect the maturity value or date, so you will still cough up the same $1,080 on the same maturity date as before. At worst, this may require you to make your payment check out to CCST instead of FNNB, and send it to a different address than you originally would have. And actually it may not even require that, since sometimes the original lender simply agrees to pass on your payment to the note's new owner. In that case, you might not even be aware that the note has been sold at all.

If FNNB does sell the note to CCST, how much should CCST pay? FNNB would certainly be quite happy if it could get the full $1,080 maturity value, but then CCST would make no profit on the deal. There is nothing to prevent CCST from paying $1,080 (or, for that matter, even more), but it is hard to imagine why it would willingly do that. In all likelihood, CCST will pay something less than the $1,080 that it will receive at maturity. For example, the two banks might agree to take $60 off the maturity value. The note would then change hands for $1,080 − $60 = $1,020.

Notice that when FFNB and CCST decide on a selling price for this note, they are likely to think of that price in terms of subtracting something from the maturity value. In other words, they are likely to think in terms of discount. When you first borrowed the money, the situation looked like simple interest, but matters look different when a note that already

[4]Sometimes a loan may include the provision that the lender ***can*** demand early repayment. Bank certificates of deposit are an example; though the term of the CD is fixed, you (as the lender) typically can withdraw your money sooner, though doing so usually will involve a significant financial penalty, such as forfeiting some of the interest earned. Loans that can require early payment are not common for most situations, but such loans do exist. You should always read the fine print of any agreement. Since you are the type of person who reads footnotes in math books, you probably knew that already.

[5]It is possible for a note to have a provision that the lender can not sell it, or can sell it only under certain circumstances, but that would be unusual.

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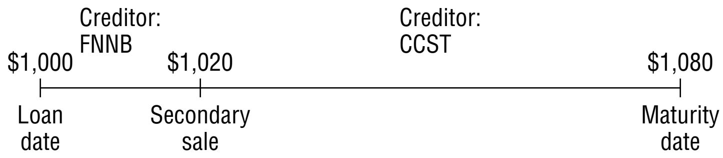
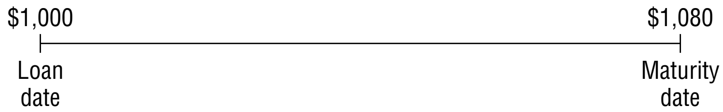
exists is being sold. The simple interest loan that created this note is water under the bridge. It is now natural to think of the note in terms of discount rather than interest.

This secondary sale divides the creditor's role between the two banks. This can be visualized with a time line as



However, from your point of view as the borrower, the loan still looks like this:



Even if you are aware that a sale has taken place, you could still look at the note in this way since the sale does not affect you financially.

Example 2.3.1 ***Suppose that John loans Paul $20,000 for 1 year at 8% simple interest. Three months later, John sells the note to Ringo at a simple discount rate of 7¾%. How much does Ringo pay for the note?***

There are two different transactions to consider here: the original loan and the secondary sale. While the question deals with the secondary sale, we first have to work out the details of the original loan. We can't figure out the secondary sale price until we know the maturity value of the note being sold.

Original loan: *The original loan was made using simple interest.*

$$I = PRT$$
$$I = (\$20{,}000)(0.08)(1)$$
$$I = \$1{,}600$$

The maturity value of the note would then be $20,000 + $1,600 = $21,600.

Secondary sale: *The secondary sale is made using simple discount. Since 3 months of the original 1-year term have already expired, the remaining term is 1 year − 3 months = 9 months.*

$$D = MdT$$
$$D = (\$21{,}600)(0.0775)(9/12)$$
$$D = \$1{,}255.50$$

Subtracting this from the maturity value gives a price of $21,600 − $1,255.50 = $20,344.50.

The discount rate used in a secondary sale does not necessarily have any connection to the original interest rate. Under ordinary circumstances it is reasonable to expect that the rates would be comparable. If the interest rate for the original loan is, say, 10%, then it stands to reason that the rate that the secondary buyer would expect to be able to earn on the note would be in the same ballpark. This does not necessarily have to be the case, as the next example illustrates.

Example 2.3.2 ***Tinker loaned Evers $997.52 for 235 days at a simple interest rate of 6.54%. But 74 days later, he sold the note to Chance using a simple discount rate of 9.95%. How much did Chance pay for the note?***

Original loan:

$$I = PRT$$
$$I = (\$997.52)(0.0654)(235/365)$$
$$I = \$42.00$$

Maturity Value = $997.52 + $42.00 = $1,039.52

Secondary sale: *The remaining term is 235 − 74 = 161 days.*

$$D = MdT$$
$$D = (\$1{,}039.52)(0.0995)(161/365)$$
$$D = \$45.62$$

Chance then would have paid \$1,039.52 − \$45.62 = \$993.90

In this example, Tinker sold the note for less than the original principal, so he actually lost money on the deal! This can, and does, happen. There is nothing to guarantee that a given note can be sold for more than its original principal. If the best deal that a note's owner can find is to sell at a loss, so be it. If that is unacceptable, then the owner of the note has the option of holding on until maturity, or at least until a more attractive deal comes along. It is not the buyer's responsibility to guarantee the seller a profit.

It is very common for a lender to sell a note prior to maturity. Government bonds (such as the T bills discussed in the prior section, as well as other, longer term bonds) as well as bonds of corporations are frequently bought and sold by investors and financial institutions. Also, loans such as car loans and mortgages are very widely bought and sold. Since those types of loans involve payments made during the term of the loan, though, they are more mathematically complex and will not be dealt with in this section. But we will work with them later in the text.

Measuring Actual Interest Rate Earned

As we discussed in Section 2.2, when simple discount is used there is often good reason to calculate the equivalent simple interest rate as well. Similarly, when notes are bought and sold, it is often worthwhile to calculate the actual simple interest rate earned by each party, so that each party can evaluate the deal in terms of the interest rate earned or paid. The following example will illustrate.

Example 2.3.3 ***For the scenario described in Example 2.3.1, calculate the simple interest rate that:***

- ***(a) John actually earns***
- ***(b) Ringo actually earns (assuming he does not sell the note prior to maturity)***
- ***(c) Paul actually pays***

The overall scenario has many different parts, yet in each of these cases we are asked to look at the problem only from one party's perspective. So to find John's interest rate, we need to look at only what happened from John's point of view; we can ignore everything else that happened in the life of this note. In each case we will begin by putting ourselves in one person's shoes and then look at the deal only from his point of view.

Drawing time lines, while not essential, is often helpful.

(a) John made the loan at an 8% simple interest rate, and if he had held the note until maturity that is the rate he would have earned. However, he didn't do that, and since he sold early the actual return on his investment may have been more or less than an 8% rate.

We can draw a time line and look at the story from John's point of view:

From John's viewpoint, the principal is \$20,000, the interest is \$344.50, and the term is 3 months.

$$I = PRT$$
$$\$344.50 = (\$20{,}000)(R)(3/12)$$
$$\$344.50 = \$5{,}000(R)$$
$$R = 6.89\%$$

So the rate John actually earned was 6.89%.

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(b) Though we already know that the price Ringo paid for the note was found with a 7¾% simple ***discount*** *rate, we also know from Section 2.2 that the simple* ***interest*** *rate will be different. Look at the story from Ringo's perspective:*

$20,344.50 ——— 9 months ——— $21,600

From Ringo's point of view, the principal is $20,344.50, the interest is $1,255.50 and the term is 9 months.

$$I = PRT$$
$$\$1255.50 = (\$20{,}344.50)(R)(9/12)$$
$$\$1{,}255.50 = \$15{,}258.375(R)$$
$$R = 8.23\%$$

So we can conclude that Ringo earned a simple interest rate of 8.23%.

(c) Following the approach used in (a) and (b), we could draw a time line from Paul's point of view and then calculate the interest from there. That would give the correct answer, but there is no need to go to that trouble. Paul agreed to pay 8% simple interest, and the fact that his note was sold does not change that, no matter how many times it is sold. So the simple interest rate that Paul pays is still 8%.

Even if a note is sold hundreds of times, we can always draw a time line from any individual's point of view and use that to calculate the simple interest rate that the individual actually earned (or paid).

In Example 2.3.2, Tinker actually lost money. It is interesting to ask what sort of simple interest rate that translates into.

Example 2.3.4 ***For the situation described in Example 2.3.2, calculate the simple interest rate that Tinker actually earned.***

$997.52 ——— 74 days ——— $993.90

In this case, instead of gaining interest, Tinker lost money. In other problems, we found the amount of interest by subtracting, and following that approach we find that his interest was $993.90 − $997.52 = −$3.62. The result is negative because Tinker lost money on the deal. Proceeding to calculate his simple interest rate, we get:

$$I = PRT$$
$$-\$3.62 = (\$997.52)(R)(74/365)$$
$$-\$3.62 = \$202.232876712(R)$$
$$R = -1.79\%$$

While a negative interest rate may sound ridiculous, it actually describes this situation quite well. It stands to reason that losing money on a loan could be expressed as "earning" a negative interest rate.

Negative rates seem strange because we aren't used to seeing them. You can imagine a bank's marketing nightmares trying to attract deposits by offering negative interest rates on deposits! ("Come watch your account value drop!") Of course charging negative interest rates for loans might make the marketing department's job easy ("Pay back less than you borrowed!"), but it wouldn't do much to help keep the bank in business. For these reasons, if no others, negative interest rates are almost never actually used directly.[6] But when a note or other investment is sold for less than was paid for it, negative rates can arise and do make sense.

Since Tinker did so poorly on this deal, it is worth asking how Chance made out.

[6]Amazingly, this does happen sometimes. In the 1990s some Japanese savings accounts actually offered negative interest rates. At the time the inflation rate in Japan was essentially zero, and huge losses in the stock and real estate markets left people willing to put up with "negative interest" in accounts where they felt their money would at least be kept safe. While it might be psychologically difficult to watch your savings balance erode thanks to the interest it "earns," this really isn't any sillier than depositing your money in a "safe" account paying 3% when prices are rising by 5%. In either case, the real value of your money is less at the end than at the start, and so you are losing either way.

Example 2.3.5 ***Find the simple interest rate that Chance actually earned in Example 2.3.2.***

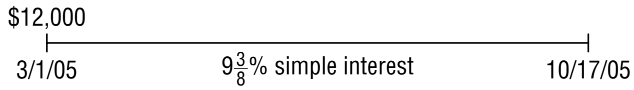
$$I = PRT$$
$$\$45.62 = (\$993.90)(R)(161/365)$$
$$\$45.62 = \$438.405205479(R)$$
$$R = 10.41\%$$

Since Chance bought the note for less than its principal, he actually earned more than Tinker would have. Moreover, since he made the purchase closer to maturity, it took him less time to earn this larger profit. So it stands to reason that Chance's interest rate would turn out to be quite high compared to the original simple interest rate.

One word of caution and advice: situations such as these that involve multiple parties are more complicated that the simpler borrower and lender problems we've seen in prior sections. It is important to read carefully. Drawing time lines often helps approach these types of problems in an organized way. An example may serve to illustrate:

Example 2.3.6 ***On March 1, 2005, Simon loaned Paula $12,000 at 9⅜% simple interest. The maturity date of the note was October 17, 2005. Fifty days prior to maturity, Simon sold the note to Randy at a simple discount rate of 8¼%. What simple interest rate did Simon actually earn?***

Original loan:

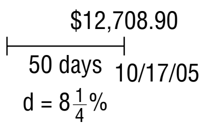


October 17 is day 290 of the year; March 1 is day 60. The term of the note is thus 290 − 60 = 230 days.

$$I = PRT$$
$$I = (\$12{,}000)(0.09375)(230/365)$$
$$I = \$708.90$$

The maturity value of the note is thus $12,000 + $708.90 = $12,708.90.

Secondary sale: *We can represent the note at the time of its sale by means of a time line.*



$$D = MdT$$
$$D = (\$12{,}708.90)(0.0825)(50/365)$$
$$D = \$143.63$$

So the selling price of the note was $12,708.90 − $143.63 = $12,565.27.

Simon's P.O.V.: *Since the original term was 230 days and he sold with 50 days left, Simon was in on the deal for 180 days.*

$$I = PRT$$
$$\$565.27 = (\$12{,}000)(R)(180/365)$$
$$\$565.27 = 5917.808219178(R)$$
$$R = 9.55\%.$$

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When dealing with problems with this much complexity, never try to rush through things. It is tempting to try to just jump into plugging in and then solving with one of our formulas to get to the answer as quickly as possible. However, as the above examples hopefully have illustrated, the values that you need to plug into the formulas really depend entirely on whose perspective a given question asks you to take. Take your time and think things through carefully.

Secondary Sales with Interest Rates (Optional)

When a note is sold, the maturity value is already fixed, and, as we have seen, discount is a natural way of handling secondary sales. Nonetheless, such sales can be determined by using simple interest instead, though the calculation will require a bit more effort.

Consider this situation, for example. A note has a maturity value of $10,000 and a remaining term of 5 months. For what price would this note be sold if the simple *interest* rate of the sale were 6%?

We cannot use the simple discount formula—this isn't simple discount. However, we have a problem with the simple interest formula. While we can plug in $R = 6\%$ and $T = 5/12$, we don't know the principal! It is tempting to plug in $P = \$10{,}000$, but this would be incorrect; $10,000 is the maturity value, not the principal, and in fact we know that principal cannot possibly be $10,000. It must be something less, though how much less is an open question.

We do know that maturity value is equal to principal plus interest. So:

$$P + I = \$10{,}000$$

From the simple interest formula we know that $I = PRT$, and so we can use this in the equation above, replacing I with PRT to get:

$$P + PRT = \$10{,}000$$

Substituting the known values, we get:

$$P + P(0.06)(5/12) = \$10{,}000$$
$$P + P(0.025) = \$10{,}000$$

The rules of algebra allow us to combine the P with the P(0.025) on the left side of this equation through the process of ***combining like terms***. If this is familiar to you, fine—if not, the next step can be thought of in the following way. If you had seven P's and put them together with 2 P's, it stands to reason that you would have 9 P's altogether. In this case you have 1 P, and when you put it together with 0.025 P's you end up with 1.025P.

However we choose to think about it, we simplify the left side to get:

$$1.025(P) = \$10{,}000$$

and then divide through to get:

$$P = \$9756.10$$

EXERCISES 2.3

A. Basic Secondary Sales

Make sure to read the wording of each exercise carefully, especially when finding the remaining term of the note when it is sold.

1. Jasper Savings Bank loaned Colline $3,000 for 125 days at 8.45% simple interest then 45 days later sold the note to Troupsburg Trust at a simple discount rate of 7.68%. Find (a) the maturity value of the note and (b) the amount Troupsburg Trust paid for it.

2. Harry borrowed $5,255 from Ron for 200 days at a simple interest rate of 12.25%. Then, 80 days later, Ron sold the note to Hermione, using a simple discount rate of 9.28%. Find (a) the maturity value of the note and (b) the amount Hermione paid.

3. Audra borrowed $2,750 from Thierry for 100 days at a simple interest rate of 16%. When 30 days were left until maturity, Shawn offered to buy the note, using a simple discount rate of 12%. How much did Shawn offer for the note?

4. Estimable Comestibles Catering borrowed $8,000 from Allegany Federal Credit Union at a simple interest rate of 10.3%. The term of the note was 220 days; 90 days later, the note was sold to Limestone Capital Venture Corp. at a simple discount rate of 11.4%. What proceeds did Allegany Federal receive from the sale?

B. Secondary Sales with Dates

5. On January 16, Dan borrowed $10,000 to upgrade the ovens at his pizzeria. The note matured on November 15 of the same year, and the simple interest rate was 9.92%. The note was sold on March 25, at a simple discount rate of 8¼%. Find the proceeds from this sale.

6. Tien borrowed $2,500 from Kevin on February 11, 2007. The note matured on July 5, 2007, and the simple interest rate was 15.02%. On February 26, Kevin sold the note to Byron at a simple discount rate of 9.31%. How much did Byron pay?

7. Neela loaned Davis $6,000 on October 18, 2005. The term of the note was 200 days, and the simple interest rate was 6.75%. On November 23, 2005, she sold the note to Vic at a simple discount rate of 12.81%. How much did Neela get from Vic?

8. On April 30, 2006, Wayneport General Hospital borrowed $538,000 from Macedon Funding Corp. The note's maturity date was February 7, 2007, and the simple interest rate was 4.59%. On October 16, 2006, the note was sold to Palmyra Mutual Investment Company with a simple discount rate of 6.00%. Find the proceeds of the sale.

C. Measuring Actual Interest Earned

In each exercise, assume that no additional sales of the notes took place. That is, assume that the secondary buyer in each exercise held onto the note until maturity.

9. In Exercise 1, find the actual simple interest rate (a) earned by Troupsburg Trust, (b) earned by Jasper Savings Bank, and (c) paid by Colline.

10. In Exercise 2, find the actual simple interest rate (a) earned by Ron, (b) paid by Harry, and (c) earned by Hermione.

11. In Exercise 3, find the actual simple interest rate (a) earned by Shawn, (b) earned by Thierry, and (c) paid by Audra.

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12. In Exercise 4, find the actual simple interest rate (a) paid by Estimable Comestibles, (b) earned by Allegany Federal Credit Union, and (c) earned by Limestone Capital Ventures.

13. In Exercise 5, find the actual simple interest rate that (a) Dan paid, (b) the original lender earned, and (c) the secondary buyer earned.

14. In Exercise 6, find the actual simple interest rate (a) earned by Kevin, (b) earned by Byron, and (c) paid by Tien.

15. In Exercise 7, find the actual simple interest rate (a) that Neela earned, (b) that Vic earned, (c) that Davis paid.

16. In Exercise 8, find the actual simple interest rate (a) paid by the hospital, (b) earned by Macedon Funding, and (c) earned by Palmyra Mutual.

D. Secondary Sales with Interest Rates (Optional)

17. A note with a maturity value of $13,576.25 is sold 75 days before maturity with a simple *interest* rate of 6.75%. Find the selling price of the note.

18. A note with a maturity value of $7,645.14 was sold 20 days before maturity. The simple interest rate used to determine the price was 9.36%. How much was paid for the note?

19. Howard loaned Jenni $5,000 for 60 days at 7½% simple interest. 45 days later he sold the note to Timo, using a simple interest rate of 8%. How much did Timo pay for the note?

20. A $10,000 T bill with 38 days till maturity is sold at a simple interest rate of 5.21%. Find the selling price.

E. Grab Bag

21. Maria signed a $2,000 face value promissory note at Montour Falls Savings and Trust, with a term of 100 days and a simple interest rate of 9.37%. Thirty days later, Watkins Glen National Bank bought the note with a simple discount rate of 8.95%. Find the price paid for this note.

22. To finance a shipment of new cars, Cantor's Auto World borrowed $547,813 from Alephone Financial Funding Corp. by signing a 60-day promissory note with a simple interest rate of 6.44%. Then, 25 days later, the note was sold with a simple discount rate of 6.75%. What rate of simple interest did Alephone Financial earn on this loan?

23. Andy loaned Bryan $800 on May 8. Bryan signed a note agreeing to pay the loan back on October 15 together with 5½% simple interest. On July 17, Bryan sold the note to Curt. The discount rate of the sale was 3.88%. What simple interest rate did Andy earn? Assuming Curt held the note to maturity, what simple interest rate did he earn?

24. Emerson loaned Lake $29.375 for 180 days at a 8¼% simple interest rate. Thirty days later he sold the note to Palmer at a 7⅛% simple discount rate. What simple interest rate did Emerson earn from this deal?

25. To be able to buy seed and fertilizer, a farmer borrowed $35,000 from a local bank, signing a 120-day note carrying a simple interest rate of 7.5%. Thirty days before maturity, the bank sold the note to a private investor, with a discount rate of 5.2%. Find (a) the maturity value of the note, (b) the price for which the note was sold, (c) the rate of simple interest the bank earned, and (d) the rate of simple interest the private investor earned.

26. On May 6, 2007, Ronda bought a $10,000 T bill maturing on July 7, 2007, at a 6.53% simple discount rate. She sold the note for $9,984.50 on June 19. What rate of simple interest did she earn?

27. Groucho loaned Chico $293,547.17 for 257 days at a 13.29% simple interest rate. Then, 118 days before the note matured, he sold the note to Karl at an 11¾% simple discount rate. What simple interest rate did Chico actually pay?

28. On April 1, 2008, the Cattarauqua Ginseng Company borrowed $40,000 for 100 days at 11.63% simple interest. On May 12, 2008, the note was sold at a simple discount rate of 24.39%. What simple interest rate did the original lender earn?

29. To encourage The Superwonderful Stuff Company to expand its warehouse operation in the community, the Town of Localville agreed to lend the company $2,500,000 for 3 years at a simple interest rate of 3.25%. One year later, facing a budget crunch, the town was forced to sell the note to raise cash. Municipal County agreed to buy the note, with a simple discount rate of 4.33%. Find both the amount of interest and the simple interest rate Localville earned.

30. What financial impact does Localville's sale of the note (from Exercise 29) have on the Superwonderful Stuff Company?

F. Additional Exercises

31. Sean borrowed $480 from Shawn for 300 days at a simple interest rate of 7%. Fifty days later, Shawn sold the note to Siann at a discount rate of 11.5%, and 120 days after that, Siann sold the note to Shaun at a discount rate of 9.25%. What simple interest rate did Sean actually pay?

32. AAA Enterprises made a loan to the BBB Company at a simple interest rate of 5%. A while later, AAA Enterprises sold the note to CCC Inc. at a simple discount rate of 5%. Was the actual interest rate earned by AAA Enterprises less than 5%, equal to 5%, or more than 5%?

33. An investment manager purchases a $10,000 face value simple discount government bond. The simple discount rate was 3.57%, and the remaining term of the note was 147 days; 93 days later, he sold the note at a simple discount rate of 3.41%. Find the simple interest rate he actually earned.

34. On July 5, Crassus loaned Cesar $25,000 for 155 days at a simple interest rate of 10%. Crassus sold the note to Pompey for $25,800. The simple discount rate used was 8%. On what date did Pompey buy the note?

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CHAPTER 2
SUMMARY

Topic	Key Ideas, Formulas, and Techniques	Examples
The Concept of Discount, pp. 56–57	• Discount is subtracted from the maturity value of a note • Discount is useful when the maturity value must be a set amount	Discussion at beginning of Chapter 2.1
Calculating Simple Discount and Proceeds, p. 59	• Use the simple discount formula D = MdT to calculate discount • Subtract discount from maturity value to find the proceeds	A $5,000 face value discount note has a term of 219 days. The simple discount rate is $9\frac{3}{8}$%. Find the proceeds. (Example 2.1.2)
Finding Maturity Value, Simple Discount Rate, and Term, p. 60	• Plug the known quantities into the simple discount formula, and use the principle of balance to find the unknown quantity • Adjust the result using appropriate rounding and/or units • Approach is the same as for similar simple interest problems	A $10,000 T bill with 182 days to maturity is sold at auction for $9753.16. What was the simple discount rate? (Example 2.1.5)
Simple Interest Versus Simple Discount, p. 64	• Regardless of how it was originally set up, a loan can be regarded as using simple interest or simple discount. • To find the simple interest rate, use the simple interest formula to solve for R • To find the simple discount rate, use the simple discount formula to solve for d	Lysander Office Supply borrowed $38,000 for 1 year. The note's maturity value was $40,000. Find the simple discount rate, and find the simple interest rate. (Example 2.2.1)
The Equivalent Simple Interest Rate for a Discount Note, p. 65	• Find the proceeds of the note using the simple discount rate and formula • Use the proceeds as the principal in the simple interest formula and solve for R	A $10,000 face value, 9-month simple discount note is offered with an 8% simple discount rate. What is the equivalent simple interest rate? (Part of Example 2.2.3)
Rates in Disguise, p. 66	• Discount may be expressed as a percent of the maturity value without regard to time • To find the proceeds, multiply the percent (as a decimal) by the maturity value, and subtract. An additional flat dollar amount may sometimes also be subtracted. • An equivalent simple interest or discount rate can be found using the simple interest or simple discount formula	Ginny is expecting a $795 paycheck in 8 days. A payday lender offers cash now, charging a fee of 1.5% plus $10. Find the equivalent simple interest and simple discount rates. (Example 2.2.4)
Secondary Sales of Promissory Notes, p. 71	• The owner of a promissory note may sell the note to someone else. • This sale does not affect the maturity value or maturity date of the note, so these should be calculated first. • The selling price is based on the previously determined maturity value and date.	John loans Paul $20,000 for 1 year at 8% simple interest; 3 months later, John sells the note to Ringo at a $7\frac{3}{4}$% simple discount rate. How much does Ringo pay? (Example 2.3.1)

(Continued)

Topic	Key Ideas, Formulas, and Techniques	Examples
Measuring Actual Interest Earned, pp. 72–73	• The simple interest rate earned by each party to a promissory note transaction can be determined. • Draw a time line that reflects what happened from a given party's point of view. • Substitute the information from that time line into the simple interest formula to solve for the rate that party actually earned.	For the scenario of Example 2.3.1 (see above), calculate the simple interest rate that John earns, the rate that Ringo earns, and the rate that Paul pays. (Example 2.3.2)
Negative Interest Rates, p. 74	• When someone loses money on a deal, we can consider this as earning negative interest, and express the investment result with a negative interest rate.	Tinker loaned Evers $997.52 and 74 days later he sold the note to Chance for $993.90. What simple interest rate did Tinker earn? (Example 2.3.4)
Secondary Sales with Interest Rates (Optional), p. 76	• A secondary sale may be determined by using simple interest rather than simple discount • Substitute the interest rate, term, and maturity value into $P + PRT = M$ • Use algebraic steps to solve for P	A $10,000 maturity value note with 5 months remaining term is sold at a 6% simple interest rate. Find the selling price. (See page 000.)

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CHAPTER 2 EXERCISES

The following exercises are a mixture of problems primarily from the topics covered Chapter 2. One of the objectives of these exercises is to be able to correctly identify which topics and tools are needed for each problem. While the emphasis is on material covered in Chapter 2, some problems covering Chapter 1 material are also included. All of the material covered in this chapter is fair game, except for optional topics, which are not included in these exercises.

1. A note with a maturity value of $18,340 is due in 90 days, and is discounted at a rate of 11.3%. Find the amount of the discount and the proceeds of the loan.

2. A note with a maturity value of $25,000 is discounted at a rate of 14%. The maturity date of the note is December 15, and the loan date was February 28. (a) Find the loan proceeds. (b) Find the equivalent rate of simple interest for this loan.

3. Nancy loaned Lisa $12,700 for 200 days at a simple interest rate of 5.2%. Thirty days later, she sold the note to Lynn at a simple discount rate of $4\frac{3}{4}$%. Find the amount Lynn paid for the note.

4. Find the term of a discount note if the maturity value is $10,000, the proceeds are $9,715, and the simple discount rate is 5.09%.

5. Topical Tropical Fruit Company is borrowing money by offering to sell discount notes that mature for $20,000 on March 30, 2007. On September 25, 2006, Bestinvest Financial Management bought one of the notes at a discount rate of 6.27%. Find the amount paid for the note, and the simple interest rate for this loan.

6. If a 6-month note has a simple discount of $200 and a simple interest rate of 8%, what is its maturity value?

7. A note that matures for $700 is sold for $680. If the term is 2 months, find the simple discount rate and simple interest rate.

8. Tom bought a 180-day note from Jerry for $11,346. The note's maturity value is $11,857. Find the rate of simple interest for the note. Find the rate of simple discount for this note.

9. Hugh borrows $23,000 from Derron, signing a 150-day note at a simple interest rate of 9.3%. Sixty days later, Derron sells the note to Louise at a discount of 8.26%. How much did Louise pay for the note?

10. Larry borrows $20,000 from Bob, signing at 215-day note at a simple interest rate of 11.5%. Ninety days later, Andy buys the note from Bob at a simple discount of 11.8%. (a) How much did Andy pay? (b) What simple interest rate does Bob actually earn on the transaction? (c) What simple interest rate does Larry actually pay?

11. Find the term of a discount note with a maturity value of $100,000 if the proceeds are $92,984.17 and the discount rate is 8.73%.

12. On June 5, 1998, Dudley purchased a 1-year $10,000 Treasury bill at a simple discount rate of 4.2%. On January 14, 1999, he sold it on the secondary market at a simple discount rate of 4.1%. (a) How much did he pay for the note? (b) How much did he sell the note for? (c) What simple interest rate did he earn on the transaction?

13. A $13,575 maturity value note is discounted at an 8%. The term is 100 days. Find the proceeds.

14. Calculate the amount of discount if the maturity value is $4,000 and the proceeds are $3,750.

15. Josie is expecting a paycheck of $1,435.19 in 10 days. If I pay her $1,400 today in exchange for this check when it is received, what is the simple discount rate for the loan? What is the simple interest rate?

16. Alvin borrowed $8,912.35 from Theodore for 125 days at a simple interest rate of 8½%. Find the amount of simple interest and the total amount to repay the loan.

17. On April 17 Lucy made a loan to Linus, for which he signed a 100-day note at 7.85% simple interest. The total interest he agreed to pay is $31.58. What is the maturity date of the note, and how much will he have to pay Lucy on that date?

18. Find the amount of discount on a note with a maturity value of $5,000 if the proceeds are $4,848.59.

19. Janelle loaned Nicole $2,569 on January 18, 2005. The simple interest rate for the loan was 7% and the term was 300 days. On August 1 she sold the note to Parvati, using a simple discount rate of 8%. What rate of simple interest did Nicole pay?

20. On April 1, 1999 Josie signed a 300-day note for $20,000 at a simple interest rate of 12¾%. Find (a) the maturity date of the note and (b) the maturity value.

21. A note with a $3,000 maturity value is sold for $2,857.16. What is the amount of discount?

22. Samuel bought a discount note with a $5,000 maturity value at 6% simple discount for 45 days, using bankers' rule. Find the proceeds.

23. On April 1, 2009, Presho Potato Products Company borrowed $15,000 at 8¾% simple interest for 200 days. On July 8, 2009, the lender sold the note with a 7.97% simple discount rate. Find the selling price of the note.

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24. Find the simple interest and simple discount rates for the loan illustrated by the following time line:

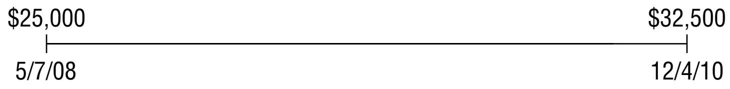
25. A $10,000 T bill with 130 days until maturity is sold with a 5.01% simple discount rate. Find the proceeds.

26. Faryal sold a note with a $20,000 maturity value, with a $1575 discount. Find her proceeds.

27. Alfie loaned Betty $3,000 for 200 days at 11% simple interest. Thirty days later he sold the note to Gemma with a 15% simple discount rate. How much did Gemma pay for the note?

28. Virginia borrowed $3,500 for 100 days from Matta National Bank at 8.43% simple interest. Twenty days later, the bank sold the note to Po Financial at a $7\frac{3}{4}\%$ simple discount rate; 25 days after that the note was sold to NiVestment Corp with a 9.26% simple discount rate. What simple interest rate did Virginia actually pay?

29. For the loan illustrated by the time line shown below, determine the (a) proceeds, (b) maturity value, (c) simple interest, and (d) simple discount.



30. A $50,000 maturity value note with 100 days till maturity is sold at a 4.44% simple discount rate. Find the proceeds.

31. A $10,000 maturity value note is sold for $9,899.43 with a 6.57% simple discount rate. What is the note's remaining term?

32. A $10,000 maturity value note is sold for $9,899.43 with a 6.57% simple discount rate. What is the equivalent simple interest rate? (*Hint:* Use the answer to Exercise 31.)

33. Jessamyn is due an $824.15 tax refund, which she expects to receive on April 30. On March 17 her tax preparer offered to buy this refund by giving her $800 on the spot. What is the simple discount rate for this offer? What is the simple interest rate?

34. Austin is due to receive a commission check for $1,845.17 in 9 days. He needs cash now. A business associate offers to give him cash now in exchange for the check when it is received. Austin would have to give up a fee of 1% of his check. What is the simple interest rate equivalent to this offer?

35. Jersey Shore Financial Funding made a $10,000 loan to a small business for 300 days. The maturity value of the note was $10,850. Fifty days later, Jersey Shore sold the note to the Bank of Shamokin Dam for $9,795. What rate of simple discount was used for this secondary sale? What simple interest rate did the original lender earn?

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CHAPTER 3

Compound Interest

"The most powerful force in the universe is compound interest."

—Quote often attributed to Albert Einstein

Learning Objectives

LO 1 Recognize the difference between simple and compound interest, and understand the reasoning behind compound interest.

LO 2 Find the future value or present value of a sum of money invested at compound interest.

LO 3 Calculate compound interest using different compounding frequencies.

LO 4 Evaluate the impact of compounding frequency on the total amount of interest earned.

LO 5 Find and interpret the effective interest rate for a given nominal rate and compounding frequency.

LO 6 Apply the mathematical tools for compound interest to other types of compound growth.

Chapter Outline

3.1 Compound Interest: The Basics

So far our discussion of interest (and discount) has dealt only with *simple* interest and discount. Saying "simple interest" as opposed to just plain "interest" suggests that there is some other kind. Though we have not yet used the term in this text, you have nonetheless probably heard the term *compound* interest at some point or other. In this chapter we will investigate compound interest and see how it differs from its simple cousin.

To begin with, let's consider a $5,000 loan for 5 years at 8% simple interest. From our previous work we can easily calculate the interest:

$$I = PRT$$
$$I = (\$5{,}000)(0.08)(5)$$
$$I = \$2{,}000$$

and so the maturity value would be $5,000 + $2,000 = $7,000.

What we have not yet looked at is the way the loan progresses toward this goal as time passes through the 5-year term. This calculation takes us from principal to maturity value without any thought about *how* the amount of interest grows over the loan's term. Suppose, then, that instead of just jumping from the $5,000 principal to the $7,000 maturity value 5 years later we take a look at the loan year by year along the way.

In the first year, the interest would be

$$I = PRT$$
$$I = (\$5{,}000)(0.08)(1)$$
$$I = \$400$$

In the second year, the interest would be the same. The principal is still $5,000, the interest rate is still 8%, and the second year is 1 year long, just like the first one. Likewise, the interest in the third, fourth, and fifth years would also be the same. Thus under simple interest the loan is growing at a constant rate of $400 per year:

Year	*Interest*	*Balance*
Start	N/A	**$5,000.00**
Year 1	$400.00	**$5,400.00**
Year 2	$400.00	**$5,800.00**
Year 3	$400.00	**$6,200.00**
Year 4	$400.00	**$6,600.00**
Year 5	$400.00	**$7,000.00**

We see that at this pace we arrive at an ending balance of $7,000.00 at maturity, just as expected.

This is all well and good, but now imagine that this is a deposit that you have made in a bank certificate. At the end of the first year, you would have $5,400.00 in your account. That money belongs to you. By leaving it on deposit at the bank for the second year, you are in effect loaning the bank your $5,400.00. Yet, according to our simple interest calculation, you are being paid interest only on your original $5,000.00. Even though your account balance grows and grows because of the accumulation of interest, you continue to be paid interest only on that original $5,000.00.

That is how simple interest works. No matter how long the loan continues, under simple interest the borrower pays (and lender receives) interest only on the original principal, not on any interest that accumulates along the way. Thus, in year 2 you get interest on the $5,000, since that was original principal, but you are not entitled to any interest on the $400. Even though that money is yours, and even though you are letting your bank have use of it, it is not considered principal and it does not earn interest. The same thing happens in years 3, 4, and 5. It doesn't matter how big your balance gets; only the original $5,000 earns interest. Even if you left the account open at the same rate for 10,000 years (in which time your balance would grow to more than $4 million), you would still continue to earn interest at the same plodding rate of $400 per year.

This doesn't seem quite fair. It seems reasonable that you should receive interest on the entire amount of your account balance. If the bank has the use of $5,400 of your money in year 2, then you have every reason to expect that it should pay you

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interest on the $5,400. In other words, you want to receive interest on your accumulated interest!

That is precisely the point of ***compound interest***. With compound interest, interest is paid on both the original principal and on any interest that accumulates along the way. When interest is paid on interest, we say that it ***compounds***. To see how this works, let's revisit the $5,000 bank account, this time calculating compound interest.

Compound Interest

Consistent with our first view of this account, suppose that we look at the account balance year by year, assuming that interest is credited to your account annually. Then in the first year you would earn interest on your original $5,000, just as before:

First Year

I = PRT
I = ($5,000)(0.08)(1)
I = $400.00

Things change though in the second year. Since we are now using compound interest, when we calculate interest for the second year we will treat the entire $5,400.00 as the principal:

Second Year

I = PRT
I = ($5,400.00)(0.08)(1)
I = $432.00

Not surprisingly, in the second year you earn more interest. The extra $32 is due entirely to "interest on interest." You can verify for yourself that $32 is 8% interest on $400 for 1 year.

In the third year, the interest grows even larger, since interest will now be paid on the accumulated balance of $5,400 + $432 = $5,832.

Third Year

I = PRT
I = ($5,832.00)(0.08)(1)
I = $466.56

As the balance grows year by year, so does the interest. It is worthwhile to compare the growth of the account year by year under compound interest versus its growth under simple interest.

	SIMPLE INTEREST			COMPOUND INTEREST			
Year	*Start*	*Interest*	*End*	*Start*	*Interest*	*End*	*Difference in End Balance*
1	$5,000.00	$400.00	**$5,400.00**	$5,000.00	$400.00	**$5,400.00**	**$0.00**
2	$5,400.00	$400.00	**$5,800.00**	$5,400.00	$432.00	**$5,832.00**	**$32.00**
3	$5,800.00	$400.00	**$6,200.00**	$5,832.00	$466.56	**$6,298.56**	**$98.56**
4	$6,200.00	$400.00	**$6,600.00**	$6,298.00	$503.88	**$6,802.44**	**$202.44**
5	$6,600.00	$400.00	**$7,000.00**	$6,802.44	$544.20	**$7,346.64**	**$346.64**

Note that, as time goes by and the account balance grows, so does the impact of compounding. Over the first couple of years, the difference between simple and compound interest is not all that significant. But as the principal used for compound interest grows larger and larger, the interest earned on that principal also grows and grows. Even more interestingly, while the pace of growth under simple interest remains constant, the account balance under compound interest is not only growing, but it is growing at an accelerating rate!

Looking back to the problems from Chapters 1 and 2, it is worth noting that, with few exceptions, the terms have generally been fairly short, with most under 1 year and very few extending further than 2 or 3 years. The reason for this should now be apparent. A lender would have to be quite foolish to pass up the opportunity to profit from the power of compounding over the long term. Even on this 5-year example, the advantage (to the lender) of compounding is obvious.

What about even longer terms? There is certainly every reason to expect that the gap between simple and compound interest will continue to widen. What if, instead of 5 years, we consider our $5,000 at 8% for 50 years? It is no trouble to see what the account would grow to under simple interest:

$$I = PRT$$
$$I = (\$5{,}000)(0.08)(50)$$
$$I = \$20{,}000$$

and so the account would grow to a total of $25,000, which doesn't sound too bad for a $5,000 investment.

Yet we know that under compound interest you would fare even better. How much better? Before we actually try to calculate the difference, it's worth venturing a guess. Could the difference be as much as double the balance, or $50,000? $100,000 would be 4 times as much, which would be a huge difference. But, in fact, these guesses are both far too low, as can be seen from Table 3.1:

TABLE 3.1 Comparison of Simple and Compound Interest
Assumptions: $5,000 deposited at 8% interest; compound interest is compounded annually.

	SIMPLE INTEREST			COMPOUND INTEREST			
Year	*Start*	*Interest*	*End*	*Start*	*Interest*	*End*	*Difference in End Balance*
1	$5,000.00	$400.00	**$5,400.00**	$5,000.00	$400.00	**$5,400.00**	**$0.00**
2	$5,400.00	$400.00	**$5,800.00**	$5,400.00	$432.00	**$5,832.00**	**$32.00**
3	$5,800.00	$400.00	**$6,200.00**	$5,832.00	$466.56	**$6,298.56**	**$98.56**
4	$6,200.00	$400.00	**$6,600.00**	$6,298.56	$503.88	**$6,802.44**	**$202.44**
5	$6,600.00	$400.00	**$7,000.00**	$6,802.44	$544.20	**$7,346.64**	**$346.64**
6	$7,000.00	$400.00	**$7,400.00**	$7,346.64	$587.73	**$7,934.37**	**$534.37**
7	$7,400.00	$400.00	**$7,800.00**	$7,934.37	$634.75	**$8,569.12**	**$769.12**
8	$7,800.00	$400.00	**$8,200.00**	$8,569.12	$685.53	**$9,254.65**	**$1,054.65**
9	$8,200.00	$400.00	**$8,600.00**	$9,254.65	$740.37	**$9,995.02**	**$1,395.02**
10	$8,600.00	$400.00	**$9,000.00**	$9,995.02	$799.60	**$10,794.62**	**$1,794.62**
11	$9,000.00	$400.00	**$9,400.00**	$10,794.62	$863.57	**$11,658.19**	**$2,258.19**
12	$9,400.00	$400.00	**$9,800.00**	$11,658.19	$932.66	**$12,590.85**	**$2,790.85**
13	$9,800.00	$400.00	**$10,200.00**	$12,590.85	$1,007.27	**$13,598.12**	**$3,398.12**
14	$10,200.00	$400.00	**$10,600.00**	$13,598.12	$1,087.85	**$14,685.97**	**$4,085.97**
15	$10,600.00	$400.00	**$11,000.00**	$14,685.97	$1,174.88	**$15,860.85**	**$4,860.85**
16	$11,000.00	$400.00	**$11,400.00**	$15,860.85	$1,268.87	**$17,129.71**	**$5,729.71**
17	$11,400.00	$400.00	**$11,800.00**	$17,129.71	$1,370.38	**$18,500.09**	**$6,700.09**
18	$11,800.00	$400.00	**$12,200.00**	$18,500.09	$1,480.01	**$19,980.10**	**$7,780.10**
19	$12,200.00	$400.00	**$12,600.00**	$19,980.10	$1,598.41	**$21,578.51**	**$8,978.51**

(Continued)

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TABLE 3.1 Comparison of Simple and Compound Interest *(Continued)*

	SIMPLE INTEREST			COMPOUND INTEREST			
Year	***Start***	***Interest***	***End***	***Start***	***Interest***	***End***	***Difference in End Balance***
20	$12,600.00	$400.00	**$13,000.00**	$21,578.51	$1,726.28	**$23,304.79**	**$10,304.79**
21	$13,000.00	$400.00	**$13,400.00**	$23,304.79	$1,864.38	**$25,169.17**	**$11,769.17**
22	$13,400.00	$400.00	**$13,800.00**	$25,169.17	$2,013.53	**$27,182.70**	**$13,382.70**
23	$13,800.00	$400.00	**$14,200.00**	$27,182.70	$2,174.62	**$29,357.32**	**$15,157.32**
24	$14,200.00	$400.00	**$14,600.00**	$29,357.32	$2,348.59	**$31,705.90**	**$17,105.90**
25	$14,600.00	$400.00	**$15,000.00**	$31,705.90	$2,536.47	**$34,242.38**	**$19,242.38**
26	$15,000.00	$400.00	**$15,400.00**	$34,242.38	$2,739.39	**$36,981.77**	**$21,581.77**
27	$15,400.00	$400.00	**$15,800.00**	$36,981.77	$2,958.54	**$39,940.31**	**$24,140.31**
28	$15,800.00	$400.00	**$16,200.00**	$39,940.31	$3,195.22	**$43,135.53**	**$26,935.53**
29	$16,200.00	$400.00	**$16,600.00**	$43,135.53	$3,450.84	**$46,586.37**	**$29,986.37**
30	$16,600.00	$400.00	**$17,000.00**	$46,586.37	$3,726.91	**$50,313.28**	**$33,313.28**
31	$17,000.00	$400.00	**$17,400.00**	$50,313.28	$4,025.06	**$54,338.35**	**$36,938.35**
32	$17,400.00	$400.00	**$17,800.00**	$54,338.35	$4,347.07	**$58,685.41**	**$40,885.41**
33	$17,800.00	$400.00	**$18,200.00**	$58,685.41	$4,694.83	**$63,380.25**	**$45,180.25**
34	$18,200.00	$400.00	**$18,600.00**	$63,380.25	$5,070.42	**$68,450.67**	**$49,850.67**
35	$18,600.00	$400.00	**$19,000.00**	$68,450.67	$5,476.05	**$73,926.72**	**$54,926.72**
36	$19,000.00	$400.00	**$19,400.00**	$73,926.72	$5,914.14	**$79,840.86**	**$60,440.86**
37	$19,400.00	$400.00	**$19,800.00**	$79,840.86	$6,387.27	**$86,228.13**	**$66,428.13**
38	$19,800.00	$400.00	**$20,200.00**	$86,228.13	$6,898.25	**$93,126.38**	**$72,926.38**
39	$20,200.00	$400.00	**$20,600.00**	$93,126.38	$7,450.11	**$100,576.49**	**$79,976.49**
40	$20,600.00	$400.00	**$21,000.00**	$100,576.49	$8,046.12	**$108,622.61**	**$87,622.61**
41	$21,000.00	$400.00	**$21,400.00**	$108,622.61	$8,689.81	**$117,312.42**	**$95,912.42**
42	$21,400.00	$400.00	**$21,800.00**	$117,312.42	$9,384.99	**$126,697.41**	**$104,897.41**
43	$21,800.00	$400.00	**$22,200.00**	$126,697.41	$10,135.79	**$136,833.20**	**$114,633.20**
44	$22,200.00	$400.00	**$22,600.00**	$136,833.20	$10,946.66	**$147,779.86**	**$125,179.86**
45	$22,600.00	$400.00	**$23,000.00**	$147,779.86	$11,822.39	**$159,602.25**	**$136,602.25**
46	$23,000.00	$400.00	**$23,400.00**	$159,602.25	$12,768.18	**$172,370.43**	**$148,970.43**
47	$23,400.00	$400.00	**$23,800.00**	$172,370.43	$13,789.63	**$186,160.06**	**$162,360.06**
48	$23,800.00	$400.00	**$24,200.00**	$186,160.06	$14,892.80	**$201,052.87**	**$176,852.87**
49	$24,200.00	$400.00	**$24,600.00**	$201,052.87	$16,084.23	**$217,137.09**	**$192,537.09**
50	$24,600.00	$400.00	**$25,000.00**	$217,137.09	$17,370.97	**$234,508.06**	**$209,508.06**

With time, compound interest roars ahead, leaving simple interest in the dust. The compound interest earned in the last year alone is almost as much as the total simple interest for the entire 50 years! Clearly, compounding is a powerful force.

A Formula for Compound Interest

Looking at things year by year is a good way to get a sense of how compound interest works, but a tedious and impractical way of doing an actual calculation. Hopefully we can find a more efficient way to calculate compound interest.

Let's begin by noting that 8% interest for a single year on 1 dollar amounts to exactly 8 cents. Another way of saying this would be to note that in 1 year $1.00 turns into $1.08. What happens to $5,000.00? Well, it stands to reason that a starting balance

5,000 times as large should grow to an ending balance 5,000 times as large. Following this idea suggests:

$$\text{End of year 1 balance} = \$5{,}000(1.08) = \$5{,}400.00$$

which agrees with the result in the table.

What about the next year? Since crediting one year's interest is equivalent to multiplying by 1.08, to credit the next year's interest, we multiply by 1.08 again. Or, in other words, to get to the second year's balance we multiply the original balance by 1.08 twice:

$$\text{End of year 2 balance} = \$5{,}000(1.08)(1.08) = \$5{,}832.00$$

which once again agrees with the result in the table.

The rationale behind this approach is logical, and the results so far have agreed with the table. So we have every reason to expect continued success. Using it for 5 years, we find:

$$\text{End of year 5 balance} = \$5{,}000(1.08)(1.08)(1.08)(1.08)(1.08) = \$7{,}346.64$$

which agrees with the table once again. We seem to be on to something here. In fact, however many years are involved, we can find the amount the account balance will grow to by repeatedly multiplying by 1.08.

This approach is far more efficient than building an entire table, yet the repeated multiplications are still tedious. We can accomplish the same thing more efficiently by using *exponents*.

An ***exponent*** is a way of denoting repeated multiplication of the same number. In general, x^y means y-many x's multiplied together. For example, 2^3 means three 2s multiplied together, or (2)(2)(2), which equals 8. Likewise:

$$(1.08)(1.08)(1.08)(1.08)(1.08) = (1.08)^5$$

Most calculators have a key for exponents; the calculator that you are using with this text should have one. Different calculator models, though, may mark their exponent keys differently. Often this key will be labeled "^", or "x^y", or "y^x". (Be careful, though; many calculators also have keys labeled "e^x"—this is not the key you are looking for.) Whichever symbol is used on your calculator, locate this key and let's try it out. To multiply five 1.08s together:

1. Enter 1.08
2. Press the exponent key
3. Enter 5
4. Press "=" (or "Enter")

The result should be 1.469328. This was what we multiplied by the original principal, and so, taking things one step further, if we multiply this result by $5,000, we come up with $7,346.64, the same as when we went to the trouble of actually doing the multiplication five times.

Using exponents with a calculator frees us to compound interest over longer periods of time. For example, we can now find an account balance after 50 years with no more effort than it took for 5.

$$(1.08)^{50} = 46.901612513$$

(Your calculator may give more or fewer decimal places, but that is nothing to worry about; the difference will be insignificant in the final answer.)

Multiplying this result by $5,000, we arrive at $234,508.06, which once again agrees with the answer we know from the table.

If the interest rate were different, it should be clear that we could use the same logic, just using the new rate the same way we used the 8%. Likewise, a different principal

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would simply take the place of the $5,000. We can sum up all of these observations in a formula:

FORMULA 3.1.1
The Compound Interest Formula

$$FV = PV(1 + i)^n$$

where
FV represents the FUTURE VALUE (the ending amount)
PV represents the PRESENT VALUE (the starting amount)
i represents the INTEREST RATE (per time period)
and
n represents the NUMBER OF TIME PERIODS

This formula introduces two new terms: *present value* and *future value*. As defined above, ***present value*** refers to the amount at the beginning of the term. In this context, the present value is essentially the same idea as the principal. Likewise, since the ***future value*** is the amount at the end of the time period in question, it can be thought of as maturity value. (However, future value is not always the same as maturity value. We might use the formula to figure out an account balance at some point prior to a note's actual maturity.)

To see how the formula works, let's revisit a problem we've already worked out.

Example 3.1.1 ***Use the compound interest formula to find how much $5,000 will grow to in 50 years at 8% annual compound interest.***

$$FV = PV(1 + i)^n$$
$$FV = \$5{,}000(1 + 0.08)^{50}$$
$$FV = \$5{,}000(1.08)^{50}$$
$$FV = \$5{,}000(46.9016125132)$$
$$FV = \$234{,}508.06$$

Order of Operations

Notice that in the above calculation, we used the exponent before multiplying, even though reading from left to right it might appear as though the multiplication should have come first. For that matter, we added what was inside the parentheses before multiplying, also defying left to right order. While in English we read from left to right, there is a different set of "rules of the road" in mathematics known as ***order of operations***, which determines the order in which calculations are performed. These rules are summarized below:

Order	Operation
1st	Parentheses
2nd	Exponents
3rd	Multiplication/division
4th	Addition/subtraction

Following order of operations, we must first do whatever is inside parentheses, and so we first added $1 + 0.08$ to get 1.08. Next come exponents, ahead of multiplication, so this requires that we evaluate 1.08^{50} before multiplying by $5,000.

Aside from needing to play by the rules, there is another very good reason for doing things in this order as well. If we had worked from left to right and multiplied $5,000(1.08) and then raised the result to the 50th power, the number the exponent applies to would have included the $5,000 in it. So instead of just multiplying fifty 1.08s (which we know we want), we also would have been throwing in fifty $5,000s (which we don't want). It stands to reason that we would have to apply the exponent before multiplying to avoid including things to which it should not apply.

Many calculators are programmed to follow order of operations automatically. This is true of most (but not all) newer models. On those calculators, if you enter the entire expression 5000*1.08^50 or 5000*(1 + .08)^50, the result will be correct. These calculators understand the order in which the operations should be done. On other models, though, the calculator performs operations in the order you enter them, and so with those calculators you must first add 1 + .08, then enter 1.08^50, obtain the result, and then multiply by 5000. Using your calculator, you should at this point try working through Example 3.1.1 (where you know what the answer should be) to determine how the model you are using works. If your calculator does not recognize order of operations, it may be worth investing in one that does, since the work we have ahead will be much easier with calculator we can trust to observe order of operations correctly.

Before moving on, let's work one more example to get the hang of the formula:

Example 3.1.2 ***Suppose you invest \$14,075 at 7.5% annually compounded interest. How much will this grow to over 20 years?***

In this case, \$14,075, the amount you start with, is the present value, so PV = \$14,075. The interest rate is 7.5%, so i = 0.075. And since the term is 20 years, n = 20. Plugging those into the formula gives:

$$FV = PV(1 + i)^n$$
$$FV = \$14{,}075(1 + .075)^{20}$$
$$FV = \$14{,}075(1.075)^{20}$$
$$FV = \$14{,}075(4.247851100)$$
$$FV = \$59{,}788.50$$

Of course, your calculator may allow you to get this result entering the expression to be calculated all at once. If so, it is not necessary to work it through line by line.

Calculating Compound Interest

The simple interest formula calculates the amount of interest directly. If you want to know maturity value, you need to take the extra step of adding the interest onto the principal. The compound interest formula, though, works a bit differently. This formula calculates the future value directly. No extra adding step is needed. Because it eliminates the need for an extra step, the compound interest formula is a bit more convenient when the final balance is our goal.

But what if we instead want to know the total interest earned? Then the situation is reversed. While the simple interest formula answers that question directly, the compound interest formula does not. A few moments thought, though, reveals the solution.

With a high rate of compound interest, even small investments can grow to large future values over time. © S Meltzer/PhotoLink/Getty Images/DIL

Example 3.1.3 ***Suppose you invest \$14,075 at 7.5% annually compounded interest. How much total interest will you earn over 20 years?***

In Example 3.1.2 we found that the future value for this account would be \$59,788.50. Of that account value, \$14,075 comes from your original principal, so the rest must be the interest, \$59,788.50 – \$14,075 = \$45,713.50. So \$45,713.50 is the total interest you will earn.

Be careful when working problems (and also of course when using these formulas in the real world) to be clear about whether it is total interest or future value that you are after. The extra step of subtracting isn't too great

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an inconvenience, but it is easy to overlook. Read questions carefully to make sure that, whether you are finding future value or total interest, you actually are answering the question asked.

Example 3.1.4 ***Suppose that $2,000 is deposited at a compound interest rate of 6% annually. Find (a) the total account value after 12 years and (b) the total interest earned in those 12 years.***

(a) Finding the future account value is just a matter of using the formula. To wit:

$$FV = PV(1 + i)^n$$
$$FV = \$2,000(1 + 0.06)^{12}$$
$$FV = \$4,024.39$$

(b) The difference between the future value and present value is the total interest. Thus:

$$\text{Total interest} = \$4,024.39 - \$2,000 = \$2,024.39$$

In all of our examples in this section, we are assuming that the interest is credited each year, just as it did in the example we used to start this chapter. Because the interest compounds each year, we say that it is ***compounded annually***. It is possible to have interest that compounds more often than once a year, though we will not see any examples of this until Section 3.2.

Finding Present Value

The algebraic tools we have developed in prior sections are still useful with the compound interest formula as well.

Example 3.1.5 ***How much money should I deposit today into an account earning 7⅜% annually compounded interest in order to have $2,000 in the account 5 years from now?***

Before beginning with the formula, it is important to notice that in this case the $2,000 is the FV, not the PV.

$$FV = PV(1 + i)^n$$
$$\$2,000 = PV(1 + 0.07375)^5$$
$$\$2,000 = PV(1.42730203237)$$

It is the PV that we are after. Using the same reasoning that we used in Chapters 1 and 2, we can divide both sides by the 1.42730203237 to get PV by itself. (Remember that it may be helpful to use your calculator's memory to avoid having to type in the long decimal when you divide.)

Doing this gives us a final answer of:

$$PV = \$1,401.25$$

I should therefore deposit $1,401.25.

In Chapter 1 we used algebra on the I = PRT formula to obtain each of the quantities included in that formula. We solved for P, then solved for R, then solved for T. In Chapter 2, we did the same thing with the simple discount formula. So it is natural to expect that our next move would be to look at problems where we need to find the interest rate i or the time n in the compound interest formula. Unfortunately, solving for these values requires a much more significant algebraic investment, and so we will dodge those questions for now. We can, though, find approximate answers using a handy tool known as the Rule of 72.

The Rule of 72

The Rule of 72 is a useful rule of thumb for estimating how quickly money will grow at a given compound interest rate.

FORMULA 3.1.2
The Rule of 72[1]

The time required for a sum of money to double at a compound interest rate of x% is *approximately* 72/x years.
(The interest rate should *not* be converted to a decimal.)

Look back at Example 3.1.4, where we had $2,000 invested at 6% compound interest for 12 years. The rule says that the time required to double your money at 6% compound interest should be approximately 72/6 = 12 years. And in fact this was pretty much what happened; the original $2,000 a bit more than doubled in 12 years.

The Rule of 72 is strictly an approximation; it is no substitute for the compound interest formula, but it does have its uses. It is a useful tool for coming up with quick, ballpark estimates, and can also be useful as a quick way of validating the reasonableness of a compound interest calculation. The following examples will illustrate this.

Example 3.1.6 ***Jarron deposited $3,200 into a retirement account, which he expects to earn 7% annually compounded interest. If his expectation about the interest rate is correct, how much will his deposit grow to between now and when he retires 40 years from now? Use the Rule of 72 to obtain an approximate answer, then use the compound interest formula to find the exact value.***

Using the Rule of 72, we know that his money should double approximately every 10 years, since 72/7 ≈ 10.2857, a bit more than 10. So in 40 years, his account should experience approximately 40/10 = 4 doublings. Thus:

Time	*Approximate Account Value*
Start	$3,200
After 10 years	$6,400
After 20 years	$12,800
After 30 years	$25,600
After 40 years	$51,200

We could also have arrived that this by doubling his account balance four times using an exponent:

$$FV \approx \$3{,}200(2)(2)(2)(2) = \$3{,}200(2^4) = \$3{,}200(16) = \$51{,}200$$

So we expect Jarron's future value to be "in the neighborhood of $50,000." Using the compound interest formula, we have

$$FV = \$3{,}200(1.07)^{40} = \$47{,}918.27$$

The actual result is not particularly close to the $51,200 given by the Rule of 72. The difference between the two is over $3,000. However, we shouldn't kid ourselves about what Rule of 72 is—it is strictly a means of getting a quick and rough estimate. While there is a gap between the two answers, they are in roughly the same ballpark. If Jarron needs a precise future value, the Rule of 72 is not all that helpful, but then no one ever claimed that it would be. However, if he just wants to get a rough idea of how large the account might grow, the Rule of 72 would suffice. It is also useful as a way of validating the more precise calculation; $47,918.27 seems like an awfully large amount for $3,200 to grow to, but the

[1]The Rule of 72 is sometimes instead called the Rule of 70, in which case, as the name suggests we use 70 in place of 72. This actually gives a slightly better approximation in some cases. However, tradition sides with 72, which has the advantage of being evenly divisible by more numbers than 70 is. In any case, either rule gives only a rough approximation, so we could just as well use the "Rule of Whatever Number in the Low 70s That You Like." Be careful, though: there is also a Rule of 78, which, despite the similar sounding name, has nothing at all to do with what we are talking about here.

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fact that the two calculations are consistent should give him some comfort that he did the calculation correctly.

Using the Rule of 72 to Find Rates

The Rule of 72 can also be used to estimate the rate needed to achieve a desired future value. Suppose you have $30,000 in an investment account, and hope that the account will grow to $1,000,000 by the time you retire 30 years from now. It would be helpful to know what sort of rate that would require, so that you can figure out whether or not your hope is realistic.

We can adapt the rule to deal with problems like these. With a little algebraic effort, we can set up the Rule of 72 to answer this question.

$$\text{doubling time} = \frac{72}{\text{rate}}$$

Multiplying both sides by the rate gives:

$$(\text{doubling time})(\text{rate}) = 72$$

And then dividing by doubling time we get:

$$\text{rate} = \frac{72}{\text{doubling time}}$$

This leads us to:

FORMULA 3.1.3
The Rule of 72 (Alternate Form)

The compound interest rate required for a sum of money to double in x years is *approximately* 72/x percent.

Example 3.1.7 ***What compound interest rate is required to double $50,000 in 5 years?***

72/5 = 14.4, and so the interest rate would need to be approximately 14.4%. (The fact that the amount was $50,000 is irrelevant.)

Returning to the question we posed a bit earlier, consider this example:

Example 3.1.8 ***What compound interest rate is required for $30,000 to grow to $1,000,000 in 30 years?***

This does not ask for just a single doubling, and so we first must determine how many doublings are required. One way to do this would be to take the initial amount and repeatedly double it until it reaches the target FV. Doing so gives:

Number of Doublings	Account Value
Start	$30,000
1	$60,000
2	$120,000
3	$240,000
4	$480,000
5	$960,000
6	$1,920,000

From this table we can see that we reach $1,000,000 just a bit after 5 doublings.

Alternatively, we could note that $1,000,000/$30,000 ≈ 33.33. Now, no number of doublings will give you exactly 33.33 times your original money, but we will try to get as close as possible. One doubling gives you twice your original money, two doublings gives you (2)(2) = 2^2 = 4 times your money, and so on. Keeping this up, we soon see that five doublings would give 2^5 = (2)(2)(2)(2)(2) = 32 times your original money, and so we see that we need a bit more than 5 doublings.

Whichever way we arrive at the number of doublings, we can finish the problem by noting that since the time allowed is 30 years, the money would need to double approximately every 30/5 = 6 years. Then 72/6 = 12, and so we conclude that the interest rate would need to be approximately 12%.

It is important to remember though that this is only a rough approximation. You can see for yourself just how close it is by using this interest rate with the $30,000 for 30 years to see how close the future value is to the desired $1,000,000. You can improve on the approximation by finding the future value and tweaking our 12% estimate. If 12% gives a value that is too high, try a rate that is a bit lower. If 12% gives a too low future value, try using a rate that is a bit higher. Improving on the Rule of 72 estimate in this way requires a bit of trial and error, but it does allow us to come up with a more precise value if we need it.

EXERCISES 3.1

A. Basics of Compound Interest

1. Suppose that you deposit $2,500 in an account paying 7% interest that compounds annually for 4 years. Fill in the missing values in the table below, which shows how your account value would grow:

Year	Start of Year	Interest	End of Year
1	$2,500.00		
2			
3			
4			

2. If the account from Exercise 1 had paid simple interest, how much less would you have had at the end of the 4 years?

3. Suppose that you deposit $4,250 in an account that pays 6½% annually compounded interest for 5 years. Set up a table similar to the one used in Exercise 1 and use it to show the growth in this account's value over the course of its 5-year term.

B. Using the Compound Interest Formula for FV

Be sure to read each question carefully to determine whether it is asking for the future value or the total interest.

4. If I invest $25,112 at 7.24% annually compounded interest for 25 years, how much will I end up with?

5. Tris deposited $3,275.14 into a 5-year credit union certificate of deposit paying 5.13% interest compounded annually. How much will his CD be worth at maturity?

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6. How much interest would you earn if you invested $1,000 at 8% annually compounded interest for 5 years?

7. Suppose you put $5,000 into a retirement account that earned 9% annually compounded interest. How much would your account be worth in 10 years? In 20 years? In 40 years? (Assume that you make no additional deposits into the account.)

8. Ken took out a personal loan for $2,500. Interest on this loan compounds annually at a rate of 12¾%. If he pays the loan off in its entirety 5 years later, how much total interest will he have paid?

9. If your great-great-great-great-great-great-great-great grandfather had invested $5 at 8% annually compounded interest 200 years ago, and his account had been passed down from generation to generation without any additional deposits or withdrawals, how much would the account be worth today?

C. Using the Compound Interest Formula for PV

10. How much would you need to deposit today into an account paying 6.52% annually compounded interest in order to have $5,000 in 3 years?

11. Jake's bank statement shows that he has $4,268.97 in a CD that he opened 4 years ago. The account's interest rate has been 5.04% compounded annually. How much did he originally deposit?

12. Marisol wants to set up an account for her newly born granddaughter, in the hopes that with compound interest it will grow to $1,000,000 by the time her granddaughter is 70 years old. Assuming that this account will earn 5% annually compounded interest, how much should Marisol deposit? How much total interest would the account earn? What if the account instead earns 10% compounded annually?

13. Find the PV if the FV = $4,250, the rate is 5.67% compounded annually, and the term is 6 years.

14. Determine the present value if the future value is $300, the interest rate is 11⅝ % compounded annually, and the term is 10 years.

D. The Rule of 72

In each of the following problems, use the Rule of 72 to obtain an approximate answer.

15. How long will it take to double your money at an annually compounded interest rate of (a) 1%, (b) 2%, (c) 4%, (d) 6%, (e) 8%, (f) 12%?

16. I have $1,000 in an account which pays 5.45% annually compounded interest. Assuming this interest rate does not change, how long will it take for my account to grow to $2,000?

17. What interest rate would you need to earn in order to double your money in (a) 1 year, (b) 2 years, (c) 4 years, (d) 6 years, (e) 8 years, (f) 12 years?

18. Tanya plans on retiring in 7 years. She has an investment account that she is using to save for retirement, and right now she has about half of the amount she wants to have in the account on retirement. Assuming she makes no additional deposits to this account, what interest rate will she need to earn in order to achieve her goal?

19. Laurie has $2,500 in an investment account. What annually compounded interest rate would she need to earn in order for the account to grow to $10,000 in 20 years?

20. Rafael has just invested $1,000 at 8% annually compounded interest. How long will it take for this to grow to $8,000?

21. Use the interest rate that you found in Exercise 19 and find the future value of $2,500 at that rate in 20 years. How good an approximation was your answer for Exercise 19?

22. Use the time you found in Exercise 20 and find the future value of $1,000 at 8% compounded annually. How good an approximation was your answer in Exercise 20?

E. Grab Bag

23. Common sense says that a high interest rate will result in more interest than a low one, and a long period of time will result in more interest than a short one. This problem will demonstrate just how significant the size of the rate and length of time can be with compound interest.

    Fill in the following table by calculating the future value of $1,000 at the interest rate given by the row and the period of time given by the column. For example, the first entry, which is filled in for you, gives the future value of $1000 in 5 years at 3% compound interest.

	5 years	*10 years*	*20 years*	*40 years*
3%				
6%				
9%				
12%				

24. An alternative way of looking at the effect of time and rate would be to look at things from the point of view of accumulating a specific target future value. The higher the rate and the longer the time, the less money you need to deposit in order to achieve the same result.

    Fill in the table below by calculating the present value needed to grow to $100,000 at the interest rate given by the row and the period of time given by the column. For example, the first entry is filled in for you, showing the amount needed to grow to $100,000 in 5 years at 3% compound interest.

	5 years	*10 years*	*20 years*	*40 years*
3%	$86,260.88			
6%				
9%				
12%				

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25. At a town board meeting, the financial manager states that the town's financial situation has improved. However, while it is no longer borrowing money to fund its operations, interest on prior borrowings continues to compound, and if nothing is done the town's debt will double in the next 10 years. Approximately what interest rate is the town paying on its debt? At this rate, how long would it take for the debt to quadruple?

26. Gita plans to open a CD with a goal of having $2,500 in the account after 2 years. If the best rate she can find is 5.97% compounded annually, how much should she deposit to reach her goal?

27. Five years ago, Jill deposited $1,172.39 in an account that has consistently paid 4.11% compounded annually. How much interest has she earned?

28. Wes put $2,000 into an investment account that pays 7.5% annually compounded interest. Approximately how long will it take for his deposit to grow to $32,000? Using this period of time, find the future value of $2,000 at 7.5% compounded annually to see how good your approximation was.

29. Moshe invested $2,500 at 6% compounded annually for 8 years. How much did he end up with?

30. Find the future value of $3,255.09 at 6.17% annually compounded interest for 1 year.

31. I invested $1,011.35 in a 6-year CD that pays 7.15% compounded annually. How much more interest will I earn than if I had invested my money at 7.15% simple interest?

32. Ryann borrowed $3,031.95 at 9.34% annually compounded interest for 2 years. How much total interest will she pay?

33. As of July 26, 2005, the total debt of the U.S. federal government was $7,870,499,539,830, which works out to $26,534 per person. Assume that the only growth in the debt per person comes from interest on the existing debt, and not from any changes in the U.S. population or from any additional borrowing.[2] Assuming an interest rate of 4.5% compounded annually, how much will your share of the debt have grown to in 2035?

F. Additional Exercises

34. Oliver and Olivia both opened 4-year CDs on the same day. Oliver deposited $3,500, and his account paid 6.25% compounded annually. Olivia's account paid only 5.55% compounded annually, but she ended up with exactly the same account value as Oliver. How much did Olivia deposit?

35. I invested $1,011.35 in a 1-year CD that pays 7.15% compounded annually. How much more interest would I earn than if I had invested my money at 7.15% simple interest? Compare your result to Exercise 31.

[2]This problem is obviously completely hypothetical.

36. Ernie deposited $1,000 with annually compounded interest for 10 years, and wound up with an account worth $4,000. If his interest rate had been twice as large, would he have had more than $8,000, less than $8,000, or exactly $8,000?

37. In some situations it may be illegal for a lender to charge compound interest to a borrower. Suppose that you agree to lend a business associate $10,000 for 5 years at 8% annually compounded interest. Later, though, you learn that it is illegal to charge compound interest for this loan. You realize, though, that you can still end up with the same future value using simple interest by using a different rate. What *simple* interest rate would you need to charge to wind up with the same future value?

38. The Rule of 70 is a variation on the Rule of 72. It works in the same way, but uses 70 instead of 72. Use the Rule of 72 to approximate the interest rate needed to double your money in 4 years. Do the same thing using the Rule of 70. Then, find the future value of $1,000 in 4 years using each approximation.

3.2 Compounding Frequencies

The comparisons between simple and compound interest make it plain that the "interest on interest" that compounding provides makes an enormous difference over time. Yet the difference only becomes significant after several compoundings have taken place.

In the examples we looked at in Section 3.1, there was no difference between simple and compound interest at the end of the first year. Since no interest was credited to those accounts until the end of the first year, there was no opportunity for interest on interest until after that first crediting took place. In the second year, interest on interest began to make a difference, and the difference became more pronounced the following year, as interest on interest combined with interest on interest on interest. As the years went by the cumulative effect of compounding on compounding accounted for the dramatic end results we saw.

Of course, none of this can happen until interest is first credited to the account. Yet there is no reason why we must wait an entire year for this to happen. Interest could be credited, say, at the end of each month. Then, rather than waiting for an entire year for compounding to begin, we would have to wait only one month for compound interest to start working its magic. What's more, over the course of each year compounding would take place a total of 12 times, rather than just once. The advantage of this (to the lender at least) is obvious. Putting the power of compounding to work sooner and more often can't help but add up to more interest and bigger account balances.

But how much more interest would this monthly compounding mean? If we invested $5,000 at 8% compounded *annually*, when the first interest is credited at the end of the first year we would have $5,400. Now suppose that we look at the same $5,000 at the same 8%, but this time we compound the interest *every month*.

The interest earned for the first month would be:

$$I = PRT$$
$$I = (\$5{,}000.00)(0.08)(1/12)$$
$$I = \$33.33$$

So at the end of the first month the account balance would be $5,033.33. Following the same approach we find that the next month's interest works out to $33.56—a bit more—and continuing on through the rest of the year we get:

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Month	Beginning Balance	Interest Earned	End of Month Balance
1	$5,000.00	$33.33	$5,033.33
2	$5,033.33	$33.56	$5,066.89
3	$5,066.89	$33.78	$5,100.67
4	$5,100.67	$34.01	$5,134.68
5	$5,134.68	$34.23	$5,168.91
6	$5,168.91	$34.46	$5,203.37
7	$5,203.37	$34.69	$5,238.06
8	$5,238.06	$34.92	$5,272.98
9	$5,272.98	$35.15	$5,308.13
10	$5,308.13	$35.39	$5,343.52
11	$5,343.52	$35.62	$5,379.14
12	$5,379.14	$35.86	$5,415.00

Notice that this is $15.00 more than we had at the end of the first year with annual compounding. Not an enormous amount more, but more nonetheless. As we expected would happen, more compounding results in more interest.

The Compound Interest Formula for Nonannual Compounding

The previous example shows that monthly compounding doesn't work all that differently from annual compounding. Using the same reasoning as in Section 3.1, we can observe that crediting the first month's interest is the same as multiplying the principal by $(1 + 0.08/12)$, and that crediting 12 months' interest is the same as multiplying by $(1 + 0.08/12)^{12}$.

Still, it might come as a bit of a surprise that we can use the same compound interest formula as before. But in fact, even though we didn't take note of it in Section 3.1, this was already built into the compound interest formula. Recall that i is the interest rate *per time period*, and n is the number of *time periods*. While in Section 3.1 the time periods were always years, there is no reason that the time periods couldn't be months, or days, or whatever.

We repeat it here:

FORMULA 3.1 (AGAIN)
The Compound Interest Formula

$$FV = PV(1 + i)^n$$

where
FV represents the FUTURE VALUE (the ending amount)
PV represents the PRESENT VALUE (the starting amount)
i represents the INTEREST RATE (per time period)
and
n represents the NUMBER OF TIME PERIODS

Since interest rates are usually given as annual rates, when using other compounding periods we will have to divide. If the term is stated in years, as it often is, we will usually have to multiply. The following examples will illustrate.

Example 3.2.1 ***Find the future value of $2,500 at 6% interest compounded monthly for 7 years.***

The PV is $2,500, but the values of i and n require some work.

Since the interest is compounded monthly, the 6% (per year) needs to be divided by 12 (since each month is 1/12 of a year) to make it monthly. By the same token, the term of 7 years *must be expressed in months, and so we must multiply it by 12 (since there are 12 months per year.) So*

$$i = \frac{0.06}{12} = 0.005$$

and

$$n = 7(12) = 84$$

Substituting these into the formula we get:

$$FV = PV(1 + i)^n$$
$$FV = (\$2{,}500)(1 + 0.005)^{84}$$
$$FV = \$3{,}800.92$$

Let's work through a second example to get comfortable with how this works.

Example 3.2.2 **Find the future value of \$3,250 at 4.75% simple interest compounded daily for 4 years.**

Since there are 365 days in a year, we must divide the annual rate by 365 to get a daily rate, and multiply the term in years by 365 to get the term in days.

$$i = \frac{0.0475}{365} = 0.000130137$$

and

$$n = 4(365) = 1{,}460$$

Substituting these values into the formula we get:

$$FV = PV(1 + i)^n$$
$$FV = \$3{,}250(1.000130137)^{1460}$$
$$FV = \$3{,}930.01$$

In Example 3.2.2 the value of i unfortunately did not come out to a nice neat value the way it did in Example 3.2.1. This will often happen, especially with daily compounding, since few numbers divide by 365 evenly. It is important to be careful with rounding in these cases. Since i is so small, it must be carried out to many decimal places to avoid significant rounding errors. (For example, if we had rounded to four decimal places and used $i = 0.0001$, our answer would have come out to be \$3,760.86, which is pretty far off the mark.) The value of i should always be taken out to the full number of decimal places given by your calculator. (The number of decimal places on your calculator may be more or fewer than those shown in this text, but as long as you take the value to your calculator's full precision there will be no problems.)

Tediousness (and potential for errors due to typos) can be avoided by finding the value of n first. Then, when we find i, we can just leave that number stored in the calculator, add 1, raise the result to the n power, and multiply by the PV. If your calculator does not have order of operations built-in, the keystrokes (after finding n) would then be:

Operation	Result
.0475/365=	0.000130137
+1=	1.000130137
^1460=	1.20923465
*3250=	3930.01

If your calculator recognizes order of operations, the entire formula can be entered all at once.

Operation	Result
3250*(1+.0475/365)^1460=	3930.01

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Assuming that this all works out all right with your calculator, you are likely to find that entering everything at once is easier. The danger is the potential for a keystroke error when entering such a lengthy expression. It is especially easy to misplace or forget the parentheses, a seemingly minor error with disastrous results. You may want to try working through a few problems with each approach to see which you like best. Remember too that whenever you get an answer, you should ask yourself whether the numbers you end up with are reasonable. A quick reality check will often, though not always, catch mistakes.

Let's work through one more example. You may want to try evaluating the value both ways to see which you like best. Of course, make sure that you choose an approach that will give the correct answer with whatever model of calculator you are using.

Example 3.2.3 ***Find the future value of $85.75 at 8.37% compounded monthly for 15 years,***

Since there are 12 months in the year:

$$i = \frac{0.0837}{12} \quad \text{and} \quad n = (15)(12) = 180$$

Plugging this into the compound interest formula we get:

$$FV = PV(1 + i)^n$$
$$FV = \$85.75\left(1 + \frac{0.0837}{12}\right)^{180}$$

We can evaluate this on the calculator as:

Operation	*Result*
.0837/12=	0.006975
+1=	1.006975
^180=	3.494330151
*85.75=	299.64

Or:

Operation	*Result*
85.75*(1+.0837/12)^180=	299.64

Either way, the future value comes out to be $299.64.

Just as we saw in Section 3.1, we can use the formula to find the present value needed to grow at compound interest to a desired future value.

Example 3.2.4 ***How much do I need to deposit today into a CD paying 6.06% compounded monthly in order to have $10,000 in the account in 3 years?***

Since interest compounds monthly, $i = 0.0606/12$ and $n = 3(12) = 36$

$$FV = PV(1 + i)^n$$
$$\$10{,}000 = PV\left(1 + \frac{0.0606}{12}\right)^{36}$$
$$\$10{,}000 = PV(1.19882570)$$

As in the past, we divide both sides through to get the PV. Also as in the past, using calculator memory may make things easier.

$$PV = \$8{,}341.50$$

So I need to deposit $8,341.50.

Comparing Compounding Frequencies

From everything we've seen so far, it seems reasonable to expect that the more often interest compounds, the larger the total. This is in fact correct. The following example will illustrate this by finding a future value with a number of commonly used compounding frequencies.

Example 3.2.5 ***Find the future value of $5,000 in 5 years at 8% interest compounded annually, semiannually, quarterly, monthly, biweekly, weekly, and daily.***

The results are displayed in the table below. (You may want to work through these calculations yourself to get some more practice in using your calculator.)

Frequency	**Times/Yr**	***i***	***n***	**Formula** $FV = PV(1+i)^n$	**Future Value**
Annual	1	0.08	5	$FV = \$5{,}000(1+0.08)^5$	$7,346.64
Semiannual	2	0.08/2	5(2) = 10	$FV = \$5{,}000\left(1+\frac{0.08}{2}\right)^{10}$	$7,401.22
Quarterly	4	0.08/4	5(4) = 20	$FV = \$5{,}000\left(1+\frac{0.08}{4}\right)^{20}$	$7,429.74
Monthly	12	0.08/12	5(12) = 60	$FV = \$5{,}000\left(1+\frac{0.08}{12}\right)^{60}$	$7,449.23
Biweekly (Fortnightly)	26	0.08/26	5(26) = 130	$FV = \$5{,}000\left(1+\frac{0.08}{26}\right)^{130}$	$7,454.54
Weekly	52	0.08/52	5(52) = 260	$FV = \$5{,}000\left(1+\frac{0.08}{52}\right)^{260}$	$7,456.83
Daily (bankers' rule)	360	0.08/360	5(360) = 1,800	$FV = \$5{,}000\left(1+\frac{0.08}{360}\right)^{1{,}800}$	$7,458.79
Daily (exact method)	365	0.08/365	5(365) = 1,825	$FV = \$5{,}000\left(1+\frac{0.08}{365}\right)^{1{,}825}$	$7,458.80

As we would have expected, the table shows that more frequent compounding does indeed result in more interest. This is true in general, though it may be a bit disappointing that the gain in interest as the compounding frequency increases is not all that impressive beyond a certain point. The gain in interest between annual and, say, monthly is far greater than the gain between monthly and daily. This effect is due to the fact that, while a small a time interval means plenty of compoundings, it also means that the interest rate is divided among so many compoundings that each time interval brings only a miniscule amount of interest. This is illustrated by the following very silly example.

Example 3.2.6 ***Find the future value of $5,000 at 8% interest for 5 years, assuming that the interest compounds every minute.***

This problem is basically the same as the previous example, except that we first need to determine how many minutes there are in a year. Since each day has 24 hours, and each hour has 60 minutes:

(365 days/year)(24 hours/day)(60 minutes/hour) = 525,600 minutes/year And so we add a new line to the table from Example 3.2.3

Frequency	**Times/Year**	***i***	***n***	**Formula** $FV = PV(1+i)^n$	**Future Value**
Every minute	525,600	0.08/525,600	5(525,600) =2,628,000	$FV = \$5{,}000\left(1+\frac{0.08}{525{,}600}\right)^{2{,}62}$	$7,459.12

While more than 2½ million compoundings sounds astounding, in fact each compounding contributes such a miniscule amount of interest that the end result produces just a whopping *32 cents* more than plodding along with daily compounding! Ridiculously frequent compounding actually produces rather dull results, and thus (aside from the occasional

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marketing gimmick) it is seldom considered seriously. For any practical purposes, there is seldom any point in compounding more frequently than daily.

Continuous Compounding (Optional)

But what if we *really* push the envelope? While compounding every minute didn't produce much of a gain, it still did earn more than just daily compounding. What if we compound interest every second? Or every thousandth of a second? How about compounding every nanosecond?[3] Compounding interest a billion times each second *must* result in some pretty staggering gains! Even if the quantity of interest earned each nanosecond is vanishingly small, surely the power of interest building upon itself so mind-bogglingly often can be expected to result in a staggering future value!

Yet the astounding answer to this question is that even incomprehensibly fast compounding produces little extra beyond dowdy, boring old daily compounding. In fact, mathematicians discovered hundreds of years ago that there is a limit to just how much the interest can grow to, *no matter how fast it compounds*. Assuming that the interest compounds every fraction of a fraction of a fraction of a second is known as ***continuous compounding***. Continuous compounding can be a tricky idea to wrap your mind around. How fast is it? Continuous compounding is so fast that no matter how fast you want your interest compounded, continuous compounding is faster. Continuous compounding is the "limiting case," the result of assuming that interest compounds the fastest it even theoretically could.

If we go ahead and pretend that our interest is compounding infinitely often, our compound interest formula becomes useless. We would have to divide our interest rate by infinity (whatever that means) and then use an infinite exponent (whatever that means). However, the following remarkable formula gets the job done:

FORMULA 3.2.1
The Continuous Compound Interest Formula

$$FV = PVe^{rt}$$

where
FV represents the FUTURE VALUE (the ending amount)
PV represents the PRESENT VALUE (the starting amount)
e is a mathematical constant (approximately 2.71828)
r represents the ANNUAL INTEREST RATE
and
t represents the NUMBER OF YEARS

The e used in the formula is a mathematical constant. Like its better known cousin π, e is an ***irrational number***, meaning that when we try to represent it with a decimal, the decimal never stops or forms any repeating pattern. This of course makes the precise value of e impossible to work with. However, for all practical purposes, we only need to use an approximate value for e that is accurate to the first few decimal places (just as the value of π is often approximated by 3.14 or 22/7.) For us, the approximation $e \approx 2.71828$ will be close enough for all practical purposes.

Example 3.2.7 ***Find the future value of $5,000 at 8% interest for 5 years, assuming that the interest compounds continuously.***

This problem is yet another variation to add to the table from Example 3.2.5. This time, we use the continuous compounding formula to find the result:

$$FV = PV(e^{rt})$$
$$FV = \$5,000(2.71828)^{(0.08)(5)}$$

[3] A nanosecond is one billionth of a second.

$FV = \$5{,}000(2.71828)^{0.40}$
$FV = \$5{,}000(1.491824296)$
$FV = \$7{,}459.12$

This is the same answer we had when interest compounded every minute. The answers aren't really the same; continuous compounding does provide slightly more. However, that "more" is very slight indeed—less than a penny—and so the miniscule difference is lost in the rounding.

Frequency	**Times/Year**	**I**	**N**	**Formula**	**Future Value**
Continuous	"Infinitely many"	n/a	n/a	$FV = \$5{,}000e^{0.40}$	$7,459.12

You may find it strange that the exponent in the above problem was not a whole number. The most familiar explanation of what an exponent is, and the one we have used in developing the compound interest formula, defines exponents in terms of repeated multiplication. Thus e^5 would be understood as five e's multiplied together, or (e)(e)(e)(e)(e). But $e^{0.40}$ makes no sense with this view; how do you multiply together 0.40 e's? The answer to this puzzle is that our usual understanding of an exponent is fine when the exponent is a whole number, but there is much more to the story. Exponents have a deeper meaning that allows them to be negative numbers, fractions, decimals, irrational numbers, and worse. A proper explanation of this lies well beyond the scope of this book. The curious reader can find more information in a college algebra or precalculus textbook; the less curious reader can be content with the knowledge that any calculator will be able to handle these exponents, and that will suffice for practical business purposes.

The astute (and/or paranoid) reader may also notice that in evaluating this formula we appear to have violated order of operations. Order of operations says that exponents should be done before multiplication, yet we multiplied (5)(0.08) before we used the result as an exponent. The reason for this apparent violation of order of operations is that when we write the "rt" together up in the exponent, it is understood that they make up the exponent together, and so we must evaluate the result of multiplying them together first. It is as though there were parentheses around them. In mathematical notation, it is common practice to not bother to write in parentheses when it is clear from the way things are written that things are meant to be grouped together in this way. This is known as an ***implied grouping***.

This does become an issue, though, when working out compound interest with a calculator. If you are entering the entire formula at once and trusting the calculator to follow order of operations, we have to insert parentheses around the things in the exponent. When typing an expression into a calculator, there is no "up there" making it clear what is, or is not, in the exponent. So, to perform the calculation from Example 3.2.6, you would need to enter:

Correct Operations	*Result*
5000*2.71828^(.08*5)=	7459.12

If you leave out the parentheses, the result is far off the mark

INCORRECT Operations	*Result*
5000*2.71828^.08*5=	27082.18

If you are concerned about forgetting this, you may want to actually write the implied parentheses into your formula. While this is not the standard way of writing it, it is a correct alternative form.

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FORMULA 3.2.1
The Continuous Compound Interest Formula
(Alternative Version)

$$FV = PVe^{(rt)}$$

where
FV represents the FUTURE VALUE (the ending amount)
PV represents the PRESENT VALUE (the starting amount)
e is a mathematical constant (approximately 2.71828)
r represents the ANNUAL INTEREST RATE
and
t represents the NUMBER OF YEARS

Compound Interest with "Messy" Terms

So far, all of the problems we have looked at in this section have had terms measured in whole years. This is of course not required in the real world, and so we will need to consider other terms as well—terms that do not happen to be a whole number of years. Those sorts of terms don't require any new formulas, but require a bit more effort in determining the value of n.

The key fact to remember is that n must give the term in the time units of the compounding. The following examples will illustrate:

Example 3.2.8 ***Luisa deposited $2,850 in a credit union certificate of deposit, paying 4.8% compounded monthly for 2½ years. What will the maturity value of her certificate be?***

We must convert years to months. In previous problems, we have multiplied years by 12 to get months; that doesn't change here. So n = (2½ years)(12 months/year) = (2.5 years)(12 months/year) = 30 months.

$$FV = PV(1 + i)^n$$
$$FV = \$2,850\left(1 + \frac{0.048}{12}\right)^{30}$$
$$FV = \$3,212.60$$

For some students, a non–whole number in the term is sometimes distracting enough to cause confusion about whether to multiply or divide when they are finding n. Remember that however we calculate it, n is supposed to represent the term *in months.* If you had gotten confused and divided here instead of multiplying, you would have come up with 2.5/12 = 0.208333333333. It should be obvious that 2½ years is not the same as 0.2083333333 months, and so thinking about whether the value you are using for n makes sense will spot any mistakes of this type.

There is never any ambiguity in converting between months and years. There are always 12 months in every year, so the number we use in our conversion will always be 12. Converting between months and days, though, presents a bit of a problem, since a month may have 28, 29, 30, or 31 days, and unless we know the specific months in question we can't know the correct number of days. One way of dealing with this is to extend bankers' rule. We can pretend that the year has 360 days, and it is divided into 12 months of 30 days each.

Example 3.2.9 ***Nigel deposited $4,265.97 in a savings account paying 3.6% compounded daily using bankers rule. He closed the account 3 years, 7 months, and 17 days later. How much did he have in his account when he closed it?***

Here we have a term expressed in a variety of units. The simplest approach is to convert each to days and then find the total. Under bankers' rule we pretend that the year has 360 days, which we divide equally into months of 30 days each. So:

(3 years)(360 days/year) = 1,080 days
(7 months)(30 days/month) = 210 days
17 days = 17 days
And so in total n = 1,080 + 210 + 17 = 1,307 days

$$FV = PV(1 + i)^n$$
$$FV = \$4{,}265.97\left(1 + \frac{0.036}{360}\right)^{1{,}307}$$
$$FV = \$4{,}861.58$$

If we are using the exact method, though, matters are a bit more complicated.

Example 3.2.10 ***Nigel deposited \$4,265.97 in a savings account paying 3.6% compounded daily using the exact method. He closed the account 3 years, 7 months, and 17 days later. How much did he have in his account when he closed it?***

This problem looks identical to the previous one. Unfortunately, this switch away from bankers' rule presents a big problem in the conversion of months to days. Since not every month contains the same number of days, it is actually impossible to know for sure how many days were in the term, which makes it impossible to answer the question exactly unless we know which specific months were included. (If we knew, for example, the dates on which he opened and closed his account we could find the exact number of days it was open.) For that matter, to get an exact count of the days we would also need to know whether or not the term included a leap year.

There is no definitive way to deal with this. One typical approach would be to use 365 days per year but assume 30 days per month. In most (but not all) cases this will slightly understate the term, but it will never be too far from the exact value. Using this approach we would have

$$n = (3)(365) + 7(30) + 17 = 1{,}322 \text{ days}$$

In which case:

$$FV = PV(1 + i)^n$$
$$FV = \$4{,}265.97\left(1 + \frac{0.036}{360}\right)^{1{,}322}$$
$$FV = \$4{,}860.07$$

Another not so commonly used but more accurate approach is to assume that each month has 30.5 days (sort of "averaging" the 30 and 31 day months), discarding any half days in the end (to recognize that February has fewer.) Using this approach we would have

$$n = (3)(365) + 7(30.5) + 17 = 1{,}325.5 \text{ days, which we drop to } 1{,}325 \text{ days}$$

In which case:

$$FV = PV(1 + i)^n$$
$$FV = \$4{,}265.97\left(1 + \frac{0.036}{360}\right)^{1{,}325}$$
$$FV = \$4{,}861.51$$

Overall, though, this is all much ado about not much. The difference is quite small, and in either case the answer can never be any more than an approximation. Whichever approach is used, the most important thing is to be clear about the fact that the answer is an approximation, and not necessarily the actual exact future value. Neil's actual future value will depend on the actual number of days; while we don't know this, the bank knows the actual days and months involved, and so they will. There is nothing wrong with approximating, so long as we realize that that is what we are doing, and Nigel is not likely to be much bothered by the small difference between the approximation and actual values. Approximations can lead to conflicts and misunderstandings, though, when they are mistaken for exact numbers and so we should make sure to be crystal clear that our answer is only an approximation.

We will close with one more example, this time illustrating the need to not overthink things!

Example 3.2.11 ***Suppose I deposit \$1,200 in a certificate of deposit paying 6% compounded monthly for 9 months. How much interest will I earn?***

Since interest is compounded monthly, $i = 0.06/12$. For n, we need to know the term in months. Don't overthink this—there is no conversion to do! The term is already given in months, so $n = 9$.

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$$FV = PV(1 + i)^n$$
$$FV = \$1{,}200\left(1 + \frac{0.06}{12}\right)^9$$
$$FV = \$1{,}200(1.04591058)$$
$$FV = \$1{,}255.09$$

Since the question asked for the amount of interest to be earned, we subtract to get $1,255.09 − $1,200 = $55.09.

Nonannual Compounding and the Rule of 72

In Section 3.1 we discussed the Rule of 72 when interest compounds annually. Can we still use this rule if the interest compounds more often? The answer to this question is yes. While, as we have seen, compounding frequency does make a difference, the difference is modest, and the Rule of 72 is an approximation anyway. The frequency of the compounding can thus be ignored when you are using this rule. Compounding frequency makes a difference, of course, but the difference is not large enough to seriously affect the Rule of 72.

EXERCISES 3.2

A. The Concept of Nonannual Compounding

1. A bank account with an initial balance of $1,450 pays 6% interest compounded quarterly. Using the simple interest formula to find the interest earned each quarter, fill in the missing values in the following table, showing the growth in the account's value over the first year.

Quarter	*Beginning Balance*	*Interest Earned*	*End of Quarter Balance*
1	$1,450.00		
2			
3			
4			

B. Using the Compound Interest Formula

2. For each of the following combinations of interest rates, compounding frequencies and terms, find the value of i (the interest rate per period) and n (the number of periods) that would be used in the compound interest formula. The value of i may be either left as a fraction or divided out to its decimal form.

   a. 8%, quarterly compounding, 10 years

   b. 9%, compounded monthly, 7 years

   c. 15%, compounded semiannually, 15 years

   d. 5%, daily compounding using bankers rule, 8 years

3. For each of the following combinations of interest rates, compounding frequencies and terms, find the value of i (the interest rate per period) and n (the number of periods) which would be used in the compound interest formula. The value of i may be either left as a fraction or divided out to its decimal form.

   a. 5%, daily compounding using the simplified exact method, 8 years

   b. 7.19%, compounded quarterly, 2 years

   c. 3.03%, compounded daily, 1 year

   d. 4¾%, compounded monthly, 29 years

4. For each of the following problems, find the appropriate values of i and n to use in the compound interest formula. Then find the future value of the given amount of money, assuming interest compounds at the stated interest rate and compounding frequency for the given period of time.

   a. $7,250 at 5% compounded quarterly for 3 years

   b. $1,175.09 at 9.27% compounded semiannually for 9 years

   c. $2,025.50 at 7.3% compounded daily for 5 years

   d. $500 at 3⅞% compounded monthly for 25 years

5. For each of the following problems, find the appropriate values of i and n to use in the compound interest formula. Then find the future value of the given amount of money, assuming interest compounds at the stated interest rate and compounding frequency for the given period of time.

   a. $31,805.23 at 5.47% compounded daily for 10 years

   b. $3,000 at 10% compounded quarterly for 1 year

   c. $97,500 at 7% compounded daily (bankers rule) for 2 years

   d. $2,000 at 9¼% compounded daily for 38 years

6. Jeremy deposited $2,500 in a certificate of deposit paying 4.49% compounded daily for 3 years. How much will he end up with in his account?

7. Kim invested $13,700, which earned 5.25% compounded monthly for 5 years. How much interest did she earn?

8. The interest rate offered for a 7-year certificate by Kossuth Savings Bank is 6.11% compounded daily. How much total interest would the bank pay on a $3,200 deposit into one of these certificates?

9. How much would I need to deposit into an account paying 4.33% compounded quarterly in order to have $1,000 in 4 years?

10. What present value is needed to grow to $2,800 in 10 years if the interest rate is 11.5% compounded daily (bankers' rule)?

11. Ramesh wants to have $50,000 in an investment account when his son starts college 17 years from now. Assuming that the account pays 5.61% compounded daily, how much should he have in the account now? (Assume that he will not make any additional deposits.)

12. The interest rate offered on a 5-year certificate of deposit at the National Bank of Emporium is 8.42% compounded daily. If Art deposits $72,500 in one of these CDs, how much will it be worth at maturity?

13. Olivier invested $362,000. He expects to earn 7¼% compounded quarterly. Assuming he is right about the rate he will earn, how much will his investment be worth in 20 years?

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C. Comparing Compounding Frequencies

14. Complete the following table comparing the future value of $1,600 at 9% interest in 20 years, using different compounding frequencies.

Frequency	***Times/Year***	***i***	***n***	***Formula***	***Future Value***
Annual					
Semiannually					
Quarterly					
Monthly					
Daily (bankers' rule)					
Daily (exact method)					

D. Continuous Compounding (Optional)

15. Xiaoyi has deposited $2,199.50 in a 2-year CD paying 4.84% compounded continuously. How much will this account be worth at the end of its term?

16. How much should I invest in order to have $5,000 at the end of 3 years if my account will earn 5.4% compounded continuously?

17. Tessa deposited $1,056.25 in a 4-year CD paying 6.01% compounded daily. How much more interest would she earn if the CD paid the same rate compounded continuously?

18. Suppose that you borrowed $850 at 11.3% compound interest for 5 years, but don't remember how often the interest compounds. What is the largest amount you could possibly need to pay off the loan at maturity?

E. Compound Interest With "Messy" Terms

19. Find the future value of the given amount of money, assuming interest compounds at the stated interest rate and compounding frequency for the given period of time.
    a. $1,300 at 6.6% compounded monthly for 3½ years
    b. $2,125 at 9.25% compounded quarterly for 5¾ years
    c. $913.75 at 4% compounded daily (bankers' rule) for 4½ years
    d. $4,000 at 10.09% compounded monthly for 7 years and 5 months
    e. $6,925.35 at 5½% compounded monthly for 63 months
    f. $1,115.79 at 2⅜% compounded daily (bankers' rule) for 4 years, 7 months, and 15 days.
    g. $75,050.95 at 3.85% compounded daily (bankers' rule) for 19 years, 2 months, and 28 days
    h. $3,754.77 at 4.31% compounded daily (exact method) for 535 days

20. How much interest would you earn from a $4,200 deposit in 30 months if the interest rate is 5.64% compounded monthly?

21. Heather bought a new dining room set from a furniture store that was offering a special "make no payments and pay no interest until 2010!" promotion. Even though she could pay for the furniture up front, she figured that she might as well take advantage of the offer, put the money in a CD, and earn some interest until she has to pay up. The last day that she can pay off the furniture without being hit with interest is 2 years, 7 months, and 3 days away. The furniture cost $3,007.79, and her bank will pay an interest rate of 4.76% compounded daily (bankers' rule) for the CD. How much does she need to deposit in order to have the right amount when the time comes?

F. Grab Bag

22. Find the total interest earned in 5 years on a deposit of $83,000 at 7.5% compounded daily.

23. How much should I deposit in an account today in order to have the value grow to $18,000 in 5 years, assuming the account earns 5.37% compounded quarterly.

24. Interest compounds on my savings account daily. What interest rate would I need to earn in order to double my account's value in 10 years?

25. Find the future value of $1,735.12 at 3.75% compounded monthly for 20 years.

26. Trevor invested $32,500 in a 30 month certificate of deposit which paid 4.63% compounded monthly. How much interest did he earn?

27. Keitha invested $24,915 in a 30-month certificate of deposit that paid 5.07% compounded daily (bankers' rule). How much interest did she earn?

28. Cherise deposited $1,955.19 at 6% interest compounded daily for 7 years. How much more did she earn than if the interest had compounded quarterly?

29. Use the Rule of 72 to approximate how long it would take me to double my money if I can earn 5.85% compounded daily.

30. Shimura-Taniyama Corp. invested $475,000, which earned 5⅝% interest compounded daily for 4 years, 8 months, and 17 days. Can you determine the exact amount of interest the company earned? If so, find it. If not, calculate a reasonable approximation of the amount.

31. Find the future value of $13,355.09 at 7.07% compounded monthly for 5 years.

32. Find the future value of $13,355.09 at 7.07% compounded monthly for 60 months.

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G. Additional Exercises

33. In all of the problems we have looked at so far, we have assumed that the interest rate remains constant. For some situations (such as most CDs) this is a reasonable assumption. However, in many other situations the interest rate may well change.

    Suppose that Sandra deposits $1,350 in a bank account for 5 years. For the first 2 years, the account pays 6% compounded monthly, then for the last 3 years the account pays 5.4% compounded quarterly.

    a. Find the value of Sandra's account at the end of the first 2 years.

    b. Find the value of Sandra's account at the end of the next 3 years. (*Hint:* Your answer to (a) will be the PV for the last 3 years.)

34. In Example 3.2.9 we said that using a 365-day year and 30 days per month will usually, but not always, underestimate the number of days. Give an example where this rule does not underestimate the number of days.

35. On April 3, 2006, I deposited $4,302.59 in a certificate of deposit paying 5.73% compounded daily. The certificate matured on July 9, 2007. What was the maturity value?

36. Repeat Exercise 38 from Section 3.1, this time assuming that the interest is compounded daily. Does daily compounding affect your assessment of Rule of 70 versus Rule of 72?

37. Find the amount of interest earned on a $3,000 investment at 5.5% for 3 years, assuming interest compounds every nanosecond.

3.3 Effective Interest Rates

Comparing Interest Rates

Whether managing a business or our own personal finances, we are often presented with some choice in options for loans, deposits, and other investments. As borrowers, we want to pay as little interest as possible, and so want to be able to find a low interest rate; as lenders, the shoe is on the other foot and we want to seek out a high interest rate.

Of course, rates are not the only thing to consider. You probably would not choose to open a savings account at a bank with inconvenient branches and rude customer service just to earn a slightly higher rate on your account. But all other things being equal you want to find the best rate you can.

Now, it would seem that comparing interest rates would be a fairly easy thing to do: if you are borrowing go for the smallest percent, and if you are lending go for the highest. And in some cases, it really is that simple. For example:

Example 3.3.1 Which of the following banks is offering the best rate for a certificate of deposit?

Bank	*Rate*	*Compounding*
Maplehurst Savings and Loan	3.79%	Annual
Killbuck First Financial	$3\frac{3}{4}\%$	Annual

Both rates are compounded annually, so for comparison we need only look at the rates. Since $3\frac{3}{4}\% = 3.75\%$*, it is not hard to see that Maplehurst S&L is offering a slightly higher rate.*

Here is another easy choice:

Example 3.3.2 ***Which of these two banks is offering the best CD rate?***

Bank	**Rate**	**Compounding**
Cato National Bank and Trust	4.35%	Daily
Meridian Mutual Building and Loan	4.35%	Quarterly

Both rates are the same, but Cato National compounds interest daily. Since more frequent compounding means more interest overall, we know that Cato National will end up paying more interest, and so their rate is the better one.

Example 3.3.2 illustrates the fact that in making comparisons it is not enough to look at the rate alone. Compounding also must be taken into account. This makes matters a bit more complicated. What are we to make of a choice such as this one?

Bank	**Rate**	**Compounding**
Bank of Bolivar	4.04%	Annual
Richburg Savings Bank	3.98%	Daily

On the one hand, Bank of Bolivar's rate is higher. Yet on the other, Richburg Savings is offering daily compounding. It is possible that Richburg's daily compounding will more than make up for its lower rate, and that despite initial appearances Richburg may in the end be offering more interest. Yet it is also possible that Richburg's daily compounding is not enough to catch up with Bolivar's higher rate.

Whether or not a higher rate is better than more frequent compounding really depends on the rates involved. If the annual rate were *much* higher, we would know which to pick. We know that even though daily compounding results in more interest, it isn't *that* much more. Given a choice between 8% compounded quarterly and 2% compounded daily and we can easily tell which is the better rate. Likewise, if the rates were very, very close to each other we would also know which to choose. Given a choice between 5.51% compounded daily versus 5.52% compounded annually and we know that daily compounding will more than make up for a mere 0.01% difference in the rates. But in cases like this one, where the rates are far, but not too far, apart, there is no easy way to tell simply by looking at them which is the winner.

We can resolve the question by actually trying both rates out. We can determine the future value of the deposit at each bank and see which one is higher. Unfortunately, we don't know how much money is involved, nor do we know how long the account will be open. Fortunately, though, this doesn't matter. For comparison purposes, we can use any amount of money as a present value. Interest is calculated as a percent, and so whichever rate is higher for a $100 deposit will also be higher for a deposit of $100,000, or any other amount for that matter. Likewise, we can use any period of time we like for comparison purposes. Whichever rate is faster for 1 year will also be faster for 3½ years; if you always run faster than I do, you will win a race with me regardless of how long the race is.

Since we can choose whatever PV and time we like, we might as well choose some nice round numbers. Suppose we use PV = $100 and a term of 1 year. Then:

$$\textbf{Bolivar:}\quad FV = \$100(1.0404)^1 = \$104.04$$

$$\textbf{Richburg:}\quad FV = \$100\left(1 + \frac{0.0398}{365}\right)^{365} = \$104.06$$

From this, we can see that in this case daily compounding does indeed make up for the lower rate. Richburg wins by a small margin. Of course, it would be possible for Bolivar to raise its rate enough to match, or even beat, Richburg. A worthwhile question to consider is this: How much would Bolivar have to raise its rate to match Richburg's? Readers are encouraged to attempt to answer this question for themselves, though we will hold off answering this for a bit.

Let's consider another comparison of different rates and compounding frequencies.

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Example 3.3.3 ***Leo has a life insurance policy with Trustworthy Mutual Life of Nebraska. The company credits interest to his policy's cash value[4] and offers Leo the choice of two different options:***

Option	***Rate***	***Compounding***
Daily Dividends	8.00%	Daily (bankers' rule)
Annual Advancement	8.33%	Annual

Which option would give Leo the most interest?

Using the same approach as before, we find the future value using any amount and any term that we like. As before, we will make "nice" choices of PV = $100 and time = 1 year.

Daily Dividends: $FV = \$100\left(1 + \frac{0.08}{360}\right)^{360} = \$108.33$

Annual Advancement: $FV = \$100(1 + 0.0833)^1 = \$108.33$

This one ends in a draw. Both options pay the same result, and so it makes no difference ratewise which option Leo chooses.[5]

When two rates and compoundings give the same result, we say that they are ***equivalent***. Since both options result in the same amount of interest being paid, we can use the two interchangeably. Even if Leo chose the Daily Dividends option paying 8% compounded daily, the company could just as well credit his interest using the 8.33% with annual compounding.

For any given interest rate and compounding frequency, there is always an equivalent annually compounded rate. Before taking this idea further, let's introduce some terminology to keep things straight.

Definitions 3.3.1

The annually compounded rate which produces the same results as a given interest rate and compounding is called the ***equivalent annual rate*** *(EAR) or the* ***effective interest rate****. The original interest rate is called the* ***nominal rate****.*

So, for example, we could say that if the *nominal rate* is 8.00% compounded daily (bankers' rule), then the *equivalent annual rate* (or the *effective rate*) is 8.33%.

How to Find the Effective Interest Rate for a Nominal Rate

Now let's return to the question posed immediately before Example 3.3.3. Recall that the Bank of Bolivar was offering an annually compounded interest rate that did not quite match the daily compounded rate offered by Richburg Savings Bank. What annually compounded interest rate should Bolivar offer to match 3.98% compounded daily? Or in other words, what effective rate is equivalent to 3.98% compounded daily?

Suppose we call the effective rate R. Then, if we were to use R to find the future value of $100 in 1 year, we would need to end up with $104.06, the same FV we got using 3.98% compounded daily. So, plugging R into the future value formula, we would get:

$$FV = \$100(1 + R)^1$$
$$\$104.06 = \$100(1 + R)^1$$

Since an exponent of 1 doesn't really do anything, it can be ignored, and so we then get:

$$\$104.06 = \$100(1 + R)$$

[4] The cash value of a life insurance policy is an amount of money for which the policy can be "cashed in." Not all types of policies have cash values, but for those that do the cash value usually grows with time.

[5] Actually, there is a very slight difference between the two. The future value using the annual rate is exactly $108.33, while the future value with the daily rate actually comes out to be $1.0832774,399, which rounds to $108.33. The difference is slight, but it is still there, and might not be lost in rounding if we use a higher PV. However, since it amounts to less than a penny on a $100 deposit over the course of an entire year, in just about any situation we would consider the difference far too small to matter.

Then, dividing both sides by 100 we get:

$$1.0406 = 1 + R$$

And then subtracting 1 from both sides gives us:

$$R = 0.0406 \quad \text{or} \quad 4.06\%.$$

This should come as no surprise, and in fact you may well have figured this out for yourself without bothering with the algebra. The 3.98% compounded daily earned $4.06 interest above and beyond the original $100, and figuring out what percent that represents of $100 is simple. In fact, this was actually the reason why we chose to use $100 and 1 year originally—because with these choices the effective interest rate is easy to see.

We can use this approach whenever we need to find the effective rate for a given nominal rate.

"FORMULA" 3.3.1
Finding the Effective Interest Rate

To find the effective rate for a given nominal rate and compounding frequency, simply find the FV of $100 in 1 year using the nominal rate and compounding. The effective interest rate (rounded to two decimal places) will be the same number as the amount of interest earned.

The following example will illustrate:

Example 3.3.4 ***Find the equivalent annual rate for 7.35% compounded quarterly.***

Following the procedure given above:

$$FV = \$100\left(1 + \frac{0.0735}{4}\right)^4 = \$107.56.$$

The interest earned is $7.56, and so we conclude that the equivalent annual rate is 7.56%. We can verify this by using this rate to find the same future value:

$$FV = \$100(1 + 0.0756)^1 = \$107.56.$$

Of course, the equivalent annual rate in Example 3.3.4 is not *exactly* 7.56%. The future value was rounded to two decimal places, as is customary, and so the rate based on this answer is also rounded to two decimal places. In many situations two decimal places is good enough; however, if more precision is needed we can get it by carrying the future value out to as many decimal places as we need.

Example 3.3.5 ***Rework Example 3.3.4, this time finding the rate to three decimal places.***

$$FV = \$100\left(1 + \frac{0.0735}{4}\right)^4 = \$107.555$$

As strange as it may seem to carry a dollar value out to three decimal places, this allows us to conclude that the equivalent annual rate is 7.555% (to three decimal places.)

A Formula for Effective Rates (Optional)

The procedure given above is perfectly adequate to find effective rates in any circumstance. For those who desire a more traditional style of formula, though, we can develop one. Suppose we let:

r = the nominal rate
c = the number of compoundings per year
R = the effective rate

Then using the fact that the nominal and effective rates will give the same future value, and once again using $100 and 1 year, we get the equation:

$$\$100\left(1 + \frac{r}{c}\right)^c = \$100(1 + R)^1$$

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Writing off the exponent of 1 and dividing both sides by \$100 we get:

$$\left(1 + \frac{r}{c}\right)^c = (1 + R)$$

Then, subtracting 1 from both sides, we obtain the formula:

FORMULA 3.3.2
The Effective Rate Formula

$$\textbf{Effective rate} = \left(1 + \frac{r}{c}\right)^c - 1$$

where
r = the NOMINAL INTEREST RATE
and
c = the number of COMPOUNDINGS PER YEAR

Example 3.3.6 ***Rework Example 3.3.5, this time using the formula***

$$\text{Effective rate} = \left(1 + \frac{.0735}{4}\right)^4 - 1 = 1.07555 - 1 = .07555 = 7.555\%$$

Note that, since the formula gives the effective rate as a decimal, in order to have three decimal places in the percent we needed to round to 5 decimal places, so that we would have the correct number when we moved the decimal to the left.

Using Effective Rates for Comparisons

One of the main advantages of effective rates is that they allow us to readily compare nominal rates with different compounding frequencies. By using equivalent annual rates, we can make an "apples to apples" comparison, making any comparison of rates as simple as the one in Example 3.3.1.

The following example will illustrate:

Example 3.3.7 ***Which of these interest rates is actually the highest?***

Bank	*Rate*	*Compounding*
Tully Savings Bank	5.95%	Annual
Bank of Manlius	5.85%	Monthly
Cincinnatus Trust	5.75%	Daily

To compare these nominal rates, we convert each to its equivalent annual (effective) rate.[6]

For Tully Savings, no work is required since the rate is already annual.

$$\text{For Bank of Manlius: } FV = \$100\left(1 + \frac{0.0585}{12}\right)^{12} = \$106.01$$

$$\text{For Cincinnatus Trust: } FV = \$100\left(1 + \frac{0.0575}{365}\right)^{365} = \$105.92$$

This allows us to complete a new table using the effective rates:

Bank	*Nominal Rate*	*Effective Rate*
Tully Savings Bank	5.95% annually	5.95%
Bank of Manlius	5.85% monthly	6.01%
Cincinnatus Trust	5.75% daily	5.92%

By comparing the effective rates, we see that Bank of Manlius is actually offering the highest interest rate, followed by Tully Savings Bank, with Cincinnatus bringing up the rear.

[6] While in this example we use the \$100 and 1-year procedure to determine the effective rates, the optional formula can equally well be used.

Effective Rates and The Truth in Lending Act

Under most circumstances, financial institutions must disclose the effective rate for deposit accounts. The nominal rate may or may not be given, but the effective rate usually must be. The point of requiring this is to enable consumers to readily compare different rates without being confused by the effects of different compounding frequencies.

The Truth in Lending Act (also known as Regulation Z) is the main federal law in the United States that requires financial institutions to make these disclosures. The regulations that require this, however, are complex, and the terms *effective rate* or *equivalent annual rate* will not generally be used. The most common term for the effective rates when dealing with deposit accounts is APY, or *annual(ized) percent yield.* Depending on the type of financial institution and type of deposit, investment, or loan, various different terms may be used, including:

- Annual percentage yield (APY)
- Effective rate
- Effective yield
- Effective annual rate
- Annualized yield

and so on.

Matters are made worse by the fact that in some cases the rate labeled as the APY may not actually match what we would have calculated as the effective rate, if the financial product requires payments of certain types of fees, or if the interest rate can vary in the future. The full details fall far outside the scope of this book. However, it is important to recognize that by whatever name it is given, this rate is intended as a means of "apples to apples" comparison. In many cases it will simply be the effective rate that we have been using. But even when it is not, the purpose of this disclosure is to allow the consumer to compare the overall interest rates being offered from different financial institutions and/or products.

It should also be noted that similar rules apply to the rates for loans as well as deposits. With loans, the term "APR" (annual percentage rate) is often used. However, since most loans require payments to be made during their terms, compounding is not quite so straightforward as with deposits. The APR for a loan is not the same as the APY for a deposit. However, the "apples to apples" purpose is pretty much the same.

Normally, though, when we are talking about deposit accounts, the term APY will be used, and unless the type of account has some unusual fees required or "bonuses" offered, the APY will correspond to the effective rate as we have been discussing it. This example is typical of how nominal and effective rates may be presented side by side:

Account Type	***Interest Rate***	***APY***
Interest Checking	0.88%	0.88%
Premium Checking	1.25%	1.26%
Premium Select Savings	1.85%	1.88%
CD 3 months–1 year	2.45%	2.48%
CD 1 year–3 years	3.10%	3.15%
CD 3 years–10 years	3.65%	3.72%

Again, the purpose of this disclosure is to make comparisons easier. If another bank is offering a *nominal* rate of 3.66% for a 5-year CD, we can't really tell from this whether or not that rate beats the one offered in the table above, since we don't know the compounding frequencies, and even if we did we cannot directly compare nominal rates with different frequencies. However, if we know that the *APY* for a 5-year CD at that other bank is 3.69%, we can directly compare that to the APY shown above and see that the competing bank's interest rates are in fact a bit lower.

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Using Effective Rates

Though interest may be actually credited to an account by using the nominal rate and compounding, we can also use effective rates, since they are by definition equivalent to the nominal rate and compounding. The important thing to remember is that the effective rate is treated as though the compounding is annual regardless of the actual compounding frequency used with the nominal rate.

***Example 3.3.8** Twelve Corners Federal Credit Union compounds interest on all of its accounts daily. The credit union is offering an effective rate of 7.33% on its 5-year certificates of deposit. If someone put $20,000 into one of these CDs, how much would the certificate be worth at maturity?*

Even though we know that the interest will compound daily, we do not know the nominal rate being used with that daily compounding. We do have the effective rate, though, which we can use just as well since it is equivalent to the nominal rate. The effective rate uses annual compounding though. So:

$$FV = PV(1 + i)^n$$
$$FV = \$20{,}000(1.0733)^5$$
$$FV = \$28{,}486.27$$

It is critically important to be clear about whether a given rate is nominal or effective. Even though we know that the interest in the previous example will actually be compounded daily, it will not be compounded daily with a 7.33% rate. While it would be an easy and entirely understandable error to use that 7.33% with daily compounding, this would be incorrect. The nominal compounding frequency is used with the nominal rate; the effective rate is always treated as annual compounding regardless of how frequently interest actually compounds.

Using Effective Rate with "Messy" Terms

In the previous example, annual compounding made matters quite a bit simpler, since it meant we didn't have to divide the rate by anything to get i, or do anything to the term to get n. This will not always work out so well, though.

***Example 3.3.9** Twelve Corners FCU also allows its customers to set up a CD with whatever term suits them. For CDs with terms between 2 and 5 years, the effective rate currently being offered is 6.25%. If a customer deposits $20,000 to a CD with a term of 1,000 days, how much will it be worth a maturity?*

Here, since the effective rate is annual, n must be in terms of years. In the past, whenever we have had to calculate n, we have multiplied. To convert years into days, for example, we have multiplied by 365 (or 360 if bankers' rule is being used.) Here we want to go the other direction, and so we instead divide by 365 to get $n = 1{,}000/365 = 2.739726027$. While it may seem strange to have a fraction or decimal in an exponent, this is actually perfectly permissible mathematically. Marching forward:

$$FV = PV(1 + i)^n$$
$$FV = PV(1.0625)^{2.739726027}$$
$$FV = \$23{,}613.70$$

Typing the entire decimal into the calculator is annoying, and can be avoided just by entering the fraction as the exponent. In that case, though, parentheses need to be put around the fraction. Using this with the previous example, we would enter:

Operation	Result
1+.0625=	1.0625
^(1000/365)=	1.180685089
*20000=	23613.70

Or, if we enter the expression all at once:

Operation	*Result*
20000*1.0625^(1000/365)=	23613.70

The reason the ( ) are required is order of operations. Without them, the calculator will take the exponent prior to division, and so 1.0625 will be raised to the 1,000 power, and then the result will be divided by 365. That is not the same as using 1,000/365 as the exponent. The good news is that it is *so* different that if you accidentally forget to put in the parentheses in most cases the result will be so absurdly far from a reasonable answer that it should be immediately obvious that it is not correct.

Alternatively, we could divide "1000/365" and store the result in the calculator's memory, reclaiming that value to use in the exponent in the next step.

When "Interest" Isn't Really Interest

In all of our discussions so far, we have considered only situations where the growth from present value to future value was due to the action of interest. There are, though, many similar situations which don't involve actual *interest*—"rent"paid for the temporary use of money—but do look quite similar. Consider the following examples:

- According to an article in an industry publication, the suggested retail price of a pair of Treadworthy Sneaker Company's TWS-42 shoes was $75.49 seven years ago, and has increased at an annual rate of 4.8% since then. How much would the suggested retail price of the shoes be today?
- Amazingly Awesome Internet Stuff Corp.'s stock has increased in value over the last 5 years at a rate of 28.4% annually. If you had invested $2,000 in the company's stock 5 years ago, how much would it be worth today?
- A union contract sets a worker's hourly wage at $16.35 an hour this year, and also states that this hourly rate will be increased by 3.5% in each of the next 4 years. What will the hourly rate be at the end of the 4 years?
- The population of the Republic of Freedonia was 1,534,670 according to the 2005 national census. If the population grows at an annual rate of 2.4%, what will the population be in 2015?

Let's look at the first example. Clearly, this is not a situation where interest is involved. When you buy a pair of sneakers, you are not lending the manufacturer or retailer any money. You are buying their product. This is a purchase, not a loan.[7] However, the situation seems entirely analogous to the one where the increases were due to interest. The price grows by a percentage each year, just as an account value grows by earning interest. Each year's increase is a percentage on top of the prior year's price, just as each year's interest is earned on top of the previous year's. Even though the price increases really have nothing to do with any actual *interest*, it seems as though the mathematics could be worked out in much the same way.

The other examples are similar. If you buy stock in a company, you are putting your money to use, but you are not *lending* it to the company or to anyone else for that matter. Stock represents ownership of a company, and so when you buy stock, you are buying a piece of the company. Likewise, hourly wage rates are not in any sense a loan, and their contractual growth most certainly is not interest. It would be even more absurd to think of a country's population as a loan, or its growth rate as interest on one.

However, each of these cases does bear a strong similarity to compound interest. In each case something is growing, the rate of growth is a set percent, and each year's growth builds upon the previous years. These were the essential attributes of compound interest

[7] Even if you borrow the money to buy the shoes, that is a separate issue, and has nothing to do with the suggested retail price that the company sets.

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used to develop our formulas, and we could use exactly the same reasoning in each of these cases. Thus, it is mathematically legitimate to use the compound interest formula in these cases. The following example will illustrate:

Example 3.3.10 ***Using the information given above, find the current suggested retail price of TWS-42 sneakers.***

$$FV = PV(1 + i)^n$$
$$FV = \$75.49(1.048)^7$$
$$FV = \$104.81$$

The "future value" here represents the current price, so we conclude the current retail price is $104.81.

It is left as an exercise (Exercises 29 to 31) for the reader to work out the other three examples.

In this book, we will usually distinguish actual interest from situations such these which are analogous to, but not really the same as, interest. We will use the more generic terms *growth rate* and *compound growth* when the growth may not actually be interest. You should be aware, though, that in practice people are often not so precise, and the terms "interest" and "interest rate" will often be used for things that really aren't. While this is an abuse of terminology, it is a mostly harmless abuse, and in business it is just an unspoken assumption that everyone understands the real deal.

EXERCISES 3.3

A. Comparing Interest Rates

Don't be afraid to do calculations to answer these questions, but try to avoid doing calculations that aren't really necessary.

1. Luis is shopping around for the best rate on a deposit account. Which of the following is offering the best rate?

Bank	Rate	Compounding
Lockport National	5.07%	Monthly
Brockport Mutual	5.15%	Monthly
Rockport Savings and Loan	5.09%	Monthly

2. K.T. is shopping around for the best rate on an auto loan. Which of the following is offering the best rate?

Bank	Rate	Compounding
Whitney Point National	8.37%	Daily
McGraw Mutual	8.11%	Daily
Homer Savings and Loan	8.08%	Daily

3. Robinson plans on opening a new savings account. Which of the following is offering the highest rate?

Bank	Rate	Compounding
Bank A	2.44%	Quarterly
Bank B	2.44%	Monthly
Bank C	2.44%	Daily (Bankers Rule)

4. Melissa plans on opening a 5-year CD. She has gotten rate quotes from three different financial institutions:

Bank	***Rate***	***Compounding***
Ellicott Savings Bank	4.97%	Daily
Little Valley S&L	5.19%	Annually
Seneca Junction Savings	5.16%	Monthly

Which would pay her the most interest?

5. Mohan needs to take out a personal loan. He has gotten rate quotes from three different places. Which one is offering the best rate?

Bank	***Rate***	***Compounding***
First Cattaraugus Bank	11.25%	Annually
Chautauqua Mutual Trust	11.01%	Monthly
Allegany Federal Credit Union	10.95%	Daily

B. Finding Effective Rates

Interest rates should be rounded to two decimal places.

6. Find the effective rate equivalent to each of the following nominal rates:
   a. 4.35% compounded daily
   b. 12.99% compounded monthly
   c. 7.05% compounded quarterly
   d. 19¾% compounded daily
   e. 5% compounded annually

7. Find the effective rate equivalent to each of the following nominal rates:
   a. 9% compounded annually
   b. 9% compounded semiannually
   c. 9% compounded quarterly
   d. 9% compounded monthly
   e. 9% compounded daily

8. Find the effective rate equivalent to 9.78% compounded daily with bankers' rule.

9. Find the effective rate equivalent to 13.29% compounded monthly.

10. Find the effective rate equivalent to 4⅝% compounded quarterly.

11. Find the effective rate equivalent to 7½% compounded semiannually.

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C. Using Effective Rates for Comparisons

12. Find the effective rate equivalent to each of the three rates in question 5. Which of the effective rates is lowest? Does this agree with your answer to question 5?

13. Find the effective rate equivalent to each of the rates given in the table below, and use them to decide which rate is actually the highest and which is actually the lowest.

Bank	***Rate***	***Compounding***
First National Bank	8.81%	Daily
Second National Bank	8.99%	Quarterly
Third National Bank	9.12%	Annually
Fourth National Bank	8.84%	Monthly
Fifth National Bank	9.04%	Annually

14. Which of the following financial institutions is offering the most attractive rate for personal loans?

Bank	***Rate***	***APR***
Bank of Newark	10.95%	11.57%
Port Gibson Financial	11.15%	11.74%
Seneca Castle Savings	10.99%	11.56%

15. Which of the following financial institutions is offering the most attractive rate for certificates of deposit?

Bank	***Rate***	***APY***
Perinton Trust	3.97% compounded daily	4.05%
Penfield Mutual	4.19% compounded annually	4.19%
Pittsford Savings Bank	4.06% compounded monthly	4.14%

D. Using Effective Rates in Calculations

16. Emma put $1,209.35 into a 4-year CD. The effective interest rate for her account was 3.99%. How much will her CD be worth when it matures?

17. Javier invested $2,881 at an effective interest rate of 7.25% for 2 years. How much interest did he earn?

18. If you can earn an effective rate of 7.29% on an investment, how much would you need to invest in order to have $1,000 in 3 years?

19. Toby's bank compounds interest daily in his account. If the effective interest rate is a constant 6.02%, how much will $750 grow to in 5 years?

20. The Bank of Machias compounds interest on all of its CDs monthly. Ciara opened a 5-year CD there by depositing \$20,000. The effective rate is 5.85%. How much total interest will she earn?

21. When Keo opened a CD at Shinglehouse National Bank, he was told that the APY was 6.59%. Interest on his CD compounds daily. He deposited \$40,000 and the CD's term was 3 years. What will the CD's value be at maturity?

22. Adam hopes to have \$250,000 in his retirement savings account when he retires in 35 years. How much would he need to have in his account today to reach this goal if he earns an effective interest rate of 7.2%.

E. Using Effective Rates with "Messy" Terms

23. Irma invested \$50,000 for 2½ years at an effective interest rate of 5.9%. How much was her investment worth at the end of its term? How much total interest did she earn?

24. How much interest would a credit union pay on a \$475 deposit for 215 days if the effective interest rate was 2.83%?

25. Find the future value of \$372,000 at a 9.92% effective rate for 9 months.

26. Mike has a line of credit at his bank, a flexible type of loan which allows him to borrow money and pay it back as he wishes. The effective rate is 17.45%. How much interest would he pay if he borrowed \$835.00 and then paid it back with interest 37 days later?

27. How much interest would be paid on a 10-month loan of \$3,000 if the effective interest rate were 9.00%?

28. Determine the present value required to produce a \$25,000 future value in 500 days assuming a 12¾% effective rate.

F. When "Interest" Isn't Really Interest

29. Amazingly Awesome Internet Stuff Corp.'s stock has increased in value over the last 5 years at a rate of 28.4% annually. If you had invested \$2,000 in the company's stock 5 years ago, how much would it be worth today?

30. A union contract sets a worker's hourly wage at \$16.35 an hour this year, and also states that this hourly rate will be increased by 3.5% in each of the next 4 years. What will the hourly rate be at the end of the 4 years?

31. The population of the Republic of Freedonia was 1,534,670 according to the 2007 national census. If the population grows at an annual rate of 2.4%, what will the population be in 2017?

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32. In 2005 it was estimated that there were 100 million Internet users in China. If this number grows at an annual rate of 8% for the next 5 years, how many internet users will China have in 2010?

33. Over the course of the twentieth century, the inflation rate in the United States averaged somewhere between 3 and 4% (depending on how the rate is calculated). Suppose that for the next 50 years the rate of inflation runs at 3½%, and that candy prices rise at this same rate. How much will a candy bar that now costs 59¢ cost 50 years from now?

G. Grab Bag

34. Find the effective interest rate equivalent to 4.92% compounded daily.

35. Would you be better off with an investment that pays 6.55% compounded daily or one that pays 6.68% compounded quarterly? Justify your answer.

36. Vince has just opened an 18-month certificate of deposit on which interest compounds daily. He deposited $4,635 and the effective rate of interest is 4.73%. How much total interest will he earn?

37. According to a mutual fund company, their Core Equity mutual fund has on average grown at an effective rate of 10.75%. If you invest $10,000 in this fund, and it grows at this claimed rate for the next 10 years, what will your account be worth?

38. In 2005, total U.S. petroleum demand was estimated to be 21.5 million barrels per day, and was projected to grow at a 2% effective rate in the future. Assuming these figures are correct, what will the daily demand be in 2020?

39. Jacinda invested $1,250 in an account that pays 5% compounded quarterly. Would she have been better off with an account that pays 4.93% compounded daily? Justify your answer.

40. Holland and Wales Federal Credit Union states that it is offering an interest rate of 11.99% compounded daily for personal loans. What is the effective rate?

41. Which of these financial institutions is offering the best rate for a 3-year CD?

Bank	***Rate***
Duke Center Trust	5.33% compounded daily
Bradford Financial	5.51% compounded annually
Knapp Creek National Bank	5.45% compounded quarterly

42. Interest on my 2-year certificate of deposit compounds quarterly. The APY is 7.11%. If I originally deposited $5,432.10, find this CD's maturity value.

43. Find the effective rate equivalent to 6.45% compounded annually.

H. Additional Exercises

44. Tamara walked into the lobby of her bank and saw the following sign:

> *New Account Spectacular!*
> *Open a new 6-month CD and enjoy a 5.25% interest rate!*
> *(Effective rate 5.18%)*

She immediately knew that whoever wrote the sign had made a mistake. How did she know?

45. Marta opened a CD for 2 years that paid an effective rate of 8%. At the end of its term, she took the proceeds and invested them in a new 2-year CD that paid a 12% effective rate. At the same time as Marta opened her original account, Dylan opened up a 4-year CD paying an effective rate of 10%. Both Marta and Dylan started by depositing the same amount of money. Who had more at the end of 4 years?

46. Portageville Savings and Loan offers a 3.25% nominal rate on its savings accounts, compounded monthly. What interest rate **compounded quarterly** would be equivalent to this rate?

47. Find the effective rate equivalent to 6% compounded continuously.

48. The population of Waldburg is presently 58,273 and is predicted to decline at a 1.4% annual rate in the future. According to this prediction, what will the population be in 20 years?

3.4 Comparing Effective and Nominal Rates

In the previous section we developed the idea of an annually compounded rate equivalent to a given nominal rate. In this section, we will explore this equivalence, and take note of some issues in using effective rates in place of nominal ones.

If we know the effective rate for a given nominal rate, it would seem that we could use the two rates interchangeably. This is true, by and large, as the following examples will illustrate.

Example 3.4.1 ***Melvin deposited $1,896 in a savings account for 4 years at 3.98% compounded daily. In the previous chapter, we found that this nominal rate is equivalent to 4.06% compounded annually. Find his future value using both rates and compare the results.***

Nominal rate: $FV = \$1{,}896\left(1 + \frac{0.0398}{365}\right)^{1{,}460} = \$2{,}223.18$

Effective rate: $FV = \$1{,}896(1.0406)^4 = \$2{,}223.17$

The answers are not exactly the same, but they are very close.

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Example 3.4.2 ***In Example 3.3.7 we found that a nominal rate of 5.75% compounded daily is equivalent to a rate of 5.92% compounded annually. Suppose you deposit $3,475 for 2 years at this nominal rate. Find the future value using the nominal rate, find the future value using the effective rate, and compare the two results.***

Nominal rate: $n = 2(365) = 730 \qquad FV = \$3{,}475\left(1 + \frac{0.0575}{365}\right)^{730} = \$3{,}898.47$

Effective rate: $FV = \$3{,}475(1 + 0.0592)^2 = \$3{,}898.62$

Despite the fact that the two rates are supposed to be equivalent, the future values are not the same. The difference is small, but it is still a difference.

What gives? The claim that the rates are equivalent implies that they should give the same results. In fact, that was the basis of our very definition of equivalent rates! While the difference between the two results in both examples was quite small, there were differences nonetheless.

The reason for these differences lies in the fact that both of these effective rates were *rounded to two decimal places.* Since these effective rates are rounded, there is a bit of inaccuracy in them, and so it is actually not surprising that they do not give perfectly accurate results. Any time rounding is involved, there will exist at least the potential for these sorts of small discrepancies. In Example 3.4.1 the discrepancy was extremely small because, even though the rate was rounded, the amount of rounding involved was very small. In Example 3.4.2, the rounding was more significant, and so the resulting discrepancy was also larger.

One way to get around this problem is to demand more decimal places from our effective rates. If we went out to, say, five decimal places, the effective rate in Example 3.4.2 would have been 5.91805%. Using this as our effective rate, the future value comes out to be $3,898.47, and the discrepancy vanishes. Actually, though, the two future values are still different, but the difference is less than a penny and so is lost when we round the final answers. With a much larger present value and/or a much longer term, the difference might once again become large enough to show up in the final answer.

Another way around the problem is simply to just decide to live with it. Suppose that you calculated your future value with the 5.92% effective rate, and so expected a future value of $3,898.62. Your bank, however, used the nominal rate, and so when you arrive to claim your future value on the maturity date you find that the account is worth 15 cents less than you expected. While you might wonder why the two numbers differ, you are hardly likely to be too concerned about such a trivial difference. Is a 15-cent discrepancy on a nearly $4,000 account balance, over 2 years, really worth worrying about or making an issue over?[8]

Rounding is a necessary evil, and as an unavoidable consequence there will always be some minor discrepancies in financial calculations. So long as the rounding is not excessive, the discrepancies will not be large enough to be a cause for concern. We have been using two decimal places for effective rates, since in most cases that will give results that are close enough; in cases where greater precision is required more decimal places can be used.

Accountants sum up this fact of financial life as the ***principle of materiality***. Unless we have some reason why we need to demand a very high degree of precision, we accept small discrepancies as the unavoidable price of the convenience of being able to do a reasonable amount of rounding. Rather than lose sleep over these minor discrepancies, we instead simply decide to live with them.

[8] I would bet that if you did insist on making an issue of this, the branch manager would gladly fork over the extra 15 cents from his own pocket just to make you go away. I would also bet that he would be telling stories about you and your precious 15 cents at cocktail parties for many years to come!

In an ideal financial world we might all like all of our answers to be infinitely precise, but it is actually just common sense to temper that expectation. Small errors, while undesirable, are not worth the cost and effort of chasing down. If you find a $500 discrepancy when balancing your checkbook, it is well worth investing a couple of hours to chase down the problem. But wasting an evening to chase down a nickel is a waste of time. It would be nice to have everything match to the penny, but it is a far better decision to write off the nickel difference than to waste valuable time and effort to chase it down.

This is not license for sloppiness, nor is it a statement that we don't really care all that much about getting the right answer. Taking a materiality point of view simply means that we agree to accept some small inaccuracies as a regrettable fact of life. It does not mean that we throw accuracy to the wind and stop caring about having the right answers. We still will make every *reasonable* attempt to see everything work out exactly. We just decline to exert *unreasonable* effort for to-the-penny exactness.

For most purposes, effective rates rounded to two, or at most three, decimal places result in discrepancies that are small enough to be ignored for all practical purposes.

Example 3.4.3 ***Tris deposited $5,000 at 6.38% compounded daily for 16 years. Find his account's future value in two ways: (a) using the nominal rate and (b) by finding the effective rate and using it to find the future value.***

(a) n = (16 years)(365 days/year) = 5,840 days. Then:

$$FV = \$5{,}000\left(1 + \frac{0.0638}{365}\right)^{5{,}840}$$

$$FV = \$13{,}875.83$$

(b) We first need to find the effective rate. Even though we know that Tris actually deposited $5,000 and the account was actually open for 16 years, we do not need (or want) these details when finding the effective rate.

$$FV = PV(1 + i)^n$$

$$FV = \$100\left(1 + \frac{0.0638}{365}\right)^{365}$$

$$FV = \$106.59$$

From which we conclude that the effective rate is 6.59%.

Now to use the effective rate. Remember that even though we know that Tris's interest actually compounds daily, the effective rate is always annually compounded.

$$FV = PV(1 + i)^n$$

$$FV = \$5{,}000(1 + 0.0659)^{16}$$

$$FV = \$13{,}881.40$$

The actual balance in Tris's account would be $13,875.83, since the nominal rate would be the one really used. The disagreement between these two answers is once again due to the rounding of the effective rate.

Example 3.4.3 illustrates the fact that "close enough" depends on context. An error of more than $5 would be quite serious if the Tris's account's size were $20. However, with an account balance of many thousands of dollars over a term of 16 years, this isn't much. This is only common sense, but it does demand that we think about the situation and use sound judgment when asking whether or not a difference is big enough to matter. Considering the size of the account and time period, the discrepancy between the two answers is unlikely to be much of an issue. If, however, a situation did demand greater precision, we could get it by taking the effective rate out to more decimal places.

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EXERCISES 3.4

A. Comparing Effective and Nominal Rates

1. An bank is offering to pay an interest rate of 6.3% compounded monthly. The APY is given as 6.49%. Find the future value of $3,560.75 in 2 years, using the nominal rate and then using the APY. Do your answers match? Should they?

2. A credit union is offering to pay an interest rate of 4.92% compounded daily. The APY is given as 5.04%. Find the present value required to produce $20,000 in 1 year, first using the nominal rate, and then using the APY. Do your answers match? Should they?

3. Melina invested $4,000 for 7 years at a nominal rate of 5.43% compounded quarterly.
   a. Find the effective interest rate which is equivalent to 5.43% compounded quarterly.
   b. Find Melina's future value using the nominal rate.
   c. Find Melina's future value using the effective rate.
   d. Do your answers to (b) and (c) agree? Should they?

4. Bob deposited $17,345.92 at a nominal rate of 4¾% compounded daily for 1 year.
   a. Find the effective rate of interest.
   b. Find the future value using the nominal rate.
   c. Find the future value using the effective rate.
   d. Do your answers to (b) and (c) agree? Should they?

5. Kelsey deposited $39,559.01 in a 5-year certificate of deposit paying 5.98% compounded monthly.
   a. Find Kelsey's future value using the nominal rate.
   b. Find the effective interest rate.
   c. Find the future value using the effective rate.
   d. Your answers to (a) and (c) will not be the same. Which answer gives the actual value of her account at the end of 5 years?

6. Paolo borrowed $3,945 for 6 months at an interest rate of 12.99% compounded monthly.
   a. Find the total interest he will pay using the nominal rate.
   b. Find the effective interest rate.
   c. Find the total interest he will pay using the effective rate.
   d. Which is the actual amount of interest he will pay, (a) or (c)?

7. Meryl invested $2,500 for 2 years at 9.75% compounded daily.
   a. Find the total interest he will earn using the nominal rate.
   b. Find the effective rate, and use it to find the total interest he will earn.

c. Which answer, (a) or (b), gives the actual amount of interest he will earn?

8. Oswayobank offers a 7.52% rate compounded monthly for a 7-year deposit account.
   a. Find the APY for this account.
   b. Use the APY to determine how much someone would need to deposit in order to end up with a $5,000 account value at maturity.
   c. Rework (b) using the nominal rate.

B. Additional Exercises

9. Parul deposited $4,000 in a CD paying 6.29% compounded continuously. Find the effective interest rate and use it to find the future value after 4 years. Also, find the future value after 4 years by using the nominal rate.

10. Howard invested $67,945.16 for 3 years, 8 months, and 17 days in an account paying 8.19% compounded daily using bankers' rule. Find the total interest he will earn, both directly by using the nominal rate, and then by finding the effective rate and using it.

3.5 Solving for Rates and Times (Optional)

So far, we have dodged the question of how to use the compound interest formula to find interest rates and terms. While the Rule of 72 provides a way to approximate these answers, in this section we will see how to find exact answers. This will, however, require more advanced algebraic techniques than we have been using so far, and this section assumes that the reader has more than basic algebra skills. Solution methods are presented in the following by means of examples.

Solving for the Interest Rate (Annual Compounding)

Example 3.5.1 ***Suppose that an investment grows from $2,500 to $3,000 in 5 years. What is the effective interest rate earned by this investment?***

We begin by substituting the values we have into the compound interest formula.

$$FV = PV(1+i)^n$$
$$\$3{,}000 = \$2{,}500(1+i)^5$$

We need to whittle away at this equation to isolate the i. A first step toward this goal would be to eliminate the $2,500 by dividing both sides.

$$1.2 = (1+i)^5$$

This helps, but it leaves the difficulty of getting rid of the exponent. To do this, we will use the fact that when something raised to a power is itself raised to a power, you can multiply the exponents. We've previously noted that exponents do not have to be whole numbers, and so if we raise both sides of this equation to the 1/5: power we get:

$$(1.2)^{1/5} = \left((1+i)^5\right)^{1/5}$$
$$(1.2)^{1/5} = (1+i)^1$$
$$1.0371372893 = 1+i$$

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We then subtract 1 from both sides to get $i = 0.0371372893$. Moving the decimal, we conclude that $i \approx 3.71\%$.

Remember that there is a particular need for caution when using fractional exponents with your calculator due to a side effect of order of operations. If you enter 1.2^1/5 = on most calculators, the result will be 0.24, not the correct 1.0371372893. The reason is that under order of operations, exponents take precedence over division, and so the calculator interprets this entry as "first raise 1.2 to the power 1, then divide the result by 5." To obtain the correct answer, you must use parentheses around the 1/5, and so instead enter 1.2^(1/5) = which will give the correct result.

Solving for the Interest Rate (Nonannual Compounding)

Example 3.5.2 ***Kelli has a certificate of deposit on which the interest compounds monthly. The balance in the account is $4,350.17 today, and the certificate will mature 2 years from today with a value of $4,715.50. What is the interest rate on her certificate?***

We follow the same basic steps here as in the previous problem (being careful to note that n must be in months):

$$\begin{aligned} FV &= PV(1+i)^n \\ \$4{,}715.50 &= \$4{,}350.17(1+i)^{24} \\ 1.083980626 &= (1+i)^{24} \\ 1.083980626^{1/24} &= 1+i \\ 1.003365652 &= 1+i \\ i &= 0.003365652 \end{aligned}$$

This gives i as a rate per month, however. Since i is the annual rate divided by 12, we must multiply by 12 to get the annual interest rate: $(0.003365652)(12) = 0.040387829$, and so the interest rate to two decimal places is 4.04%.

Converting from Effective Rates to Nominal Rates

In Section 3.3 we discussed finding the effective rate for a given nominal rate. We did not discuss going the other direction. We can, however, use a similar approach to that in Example 3.5.2 to accomplish this.

Example 3.5.3 ***What quarterly compounded nominal interest rate is equivalent to an effective rate of 17.4%?***

We know that compounding at the unknown nominal rate for one year provides the same result as using the effective rate for one year. Therefore:

$$\begin{aligned} 1.174^1 &= (1+i)^4 \\ 1.174^{1/4} &= 1+i \\ 1.040919212 &= 1+i \\ i &= 0.040919212 \end{aligned}$$

Multiplying this by 4 gives a nominal quarterly rate of 16.37%.

Solving for Time

Example 3.5.4 ***How long will it take for $100 to grow to $500 at 6% interest compounded quarterly?***

We begin once again by substituting into the formula, remembering to divide the rate by 4 because it is compounded quarterly:

$$\begin{aligned} FV &= PV(1+i)^n \\ \$500 &= \$100(1+.015)^n \\ 5 &= (1.015)^n \end{aligned}$$

The difficulty now is to get the n out of the exponent and by itself. This can be accomplished by using a logarithm:

$$\log(5) = n\log(1.015)$$
$$n = \frac{\log(5)}{\log(1.015)}$$
$$n = 108.09858293$$

So the term is just over 108 quarters. To convert to years, we must divide by 4 to get a term of just over 27 years.

EXERCISES 3.5

A. Solving for Interest Rate (Annual Compounding)

1. In Example 3.1.5 we used the Rule of 72 to estimate the rate required for \$30,000 to grow to \$1,000,000 in 30 years. Using the techniques of this section, find the effective rate. How good was our estimate?

2. If Max has \$40,057.29 in his investment account, what effective interest rate would he need to earn for this account to grow to \$500,000 in 25 years?

3. What effective rate does David need to earn on his investments for their value to grow from \$594,895 to \$1,000,000 in 8 years?

4. The population of Oakview Junction is now 23,500, and an urban planner has predicted that it will grow to 30,000 in 10 years. What growth rate is assumed in this projection?

5. What effective rate would you need to earn for \$20 to grow to \$100,000 in 20 years? In 50 years? In 100 years?

B. Solving for Interest Rate (Nonannual Compounding)

6. If Silvestre's account balance grew from \$1,595 to \$1,703.27 in 3 years, and interest compounded daily, what nominal rate did the account earn?

7. What nominal rate, compounded quarterly, would I need to be able to have \$21,500 grow to \$30,000 in 5 years' time?

8. Find the nominal rate compounded monthly that is equivalent to an effective rate of 8.25%.

9. A bank advertises an effective rate is 5.35% on a certificate of deposit. The interest actually compounds monthly. Find the nominal rate.

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C. Solving for Time

10. According to his financial planner, Rodney's investment portfolio can be expected to earn an 8½% effective rate. Assuming this is correct, how long will it take for his portfolio's value to grow from $259,000 to $500,000?

11. If my savings account pays a constant 3.65% compounded daily, how long will it take for my $1,893.25 account balance to grow to $2,000?

12. Use the Rule of 72 to approximate the time needed to double a sum of money at a 6.5% effective rate. Then use the techniques of this section to find the actual time required. How good was the estimate?

D. Grab Bag

13. Find the nominal rate compounded daily that would be equivalent to an effective rate of 8.77%.

14. Chey's retirement account has a balance of $797,503. Assuming that she makes no deposits or withdrawals, and that her account earns 6¼% compounded monthly, how long will it take for her account balance to reach $1,000,000?

15. A wind turbine manufacturer's sales were $18,576,950 in 2005. An investment analyst predicts that the company's sales will grow at an effective rate of 18% per year. Assuming this prediction is correct, how long will it be before the annual sales reach $100,000,000?

16. Jack's investment portfolio was worth $37,500 at the start of 2004, and had grown to $65,937 by the start of 2007. What was the effective growth rate of his account during that time period?

E. Additional Exercises

17. Suppose that Kimberly invests $10,000, which grows at an effective rate of 8% for 3 years. For the next 2 years the investment grows at an effective rate of 11%, and then for the last 5 years it grows at only 5%. Taking the entire 10 years as a whole, what was the overall effective rate that she earned?

18. Irakli invested $30,000 in a stock. Eight years later it was worth $75,000. In the last 3 years, the stock grew at a steady 4% rate. What was its effective growth rate for the first 5 years?

CHAPTER 3 SUMMARY

Topic	Key Ideas, Formulas, and Techniques	Example(s)
The Concept of Compound Interest, pp. 86–88	• Simple interest is only paid on the original principal, not on accumulated interest • Compound interest is paid both on principal and on accumulated interest	See discussion at the beginning of Chapter 3.1.
Calculating Future Value with Compound Interest, p. 93	• The compound interest formula: $FV = PV(1 + i)^n$ • Substitute present value, interest rate per period, and number of time periods into the formula • Be careful to follow order of operations when evaluating the formula	Suppose you invest $14,075 at 7.5% annually compounded interest. How much will this grow to in 20 years? (Example 3.1.2)
Calculating Compound Interest, p. 93	• Find the future value, using the compound interest formula • The amount of compound interest is the difference between the PV and the FV	Suppose you invest $14,075 at 7.5% annually compounded interest. How much interest will you earn over 20 years? (Example 3.1.3)
Calculating Present Value with Compound Interest, p. 94	• Substitute future value, interest rate per period, and number of time periods into the compound interest formula • Use the principle of balance to solve for PV	How much money should I invest today into an account paying $7\frac{3}{8}$% annually compounded interest to have $2,000 five years from now. (Example 3.1.5)
The Rule of 72, p. 96	• The doubling time for money growing at x% compound interest is approximately 72/x • The compound interest rate required to double a sum of money in x years is approximately 72/x%	What compound interest rate is required for $30,000 to grow to $1,000,000 in 30 years? (Example 3.1.8)
Compounding Frequencies, p. 103	• If interest compounds more frequently than once a year, the same compound interest formula is used • i must be the interest rate *per period.* The annual rate must be divided by the number of time periods per year • n must be the number of time periods. The term in years must be multiplied by the number of time periods per year.	Find the future value of $3,250 at 4.75% interest compounded daily for 4 years. (Example 3.2.2)
Comparing Compounding Frequencies, pp. 104–105	• The more frequently a given interest rate compounds, the more interest.	Compare the future value of $5,000 in 5 years at 8% compounded annually, semiannually, quarterly, monthly, biweekly, and daily. (Example 3.2.5)
Continuous Compounding (Optional), p. 106	• Continuous compounding assumes that interest compounds "infinitely many" times per year • The continuous compounding formula: $FV = PV\,e^{rt}$ • e is a mathematical constant, approximately 2.71827 • Continuous compounding only provides slightly more interest than daily compounding	Find the future value of $5,000 at 8% for 5 years assuming continuous compounding. (Example 3.2.7)

(Continued)

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Topic	Key Ideas, Formulas, and Techniques	Example(s)
Compound Interest with "Messy" Terms, pp. 108–109	• n in the compound interest formula must be the number of time periods • The term must be converted to the same time units as the compounding.	Nigel deposited $4,265.97 in a savings account paying 3.6% compounded daily. He closed the account 3 years, 7 months, and 17 days later. How much did he have when he closed it? (Example 3.2.10)
Effective Interest Rates, p. 117	• The effective interest rate for a given nominal interest rate is the annually compounded rate which would give the same results as the given rate. • To find the effective rate, find the future value of $100 for one year at that rate. The effective rate is the same number as the amount of interest earned. • Or, as an alternative, Formula 3.3.2 can be used.	Find the equivalent annual rate for 7.35% compounded quarterly. (Example 3.3.4)
Using Effective Rates for Comparisons, p. 118	• Nominal rates cannot be directly compared if their compounding frequencies are not the same • To compare nominal rates fairly, compare their effective rates.	Which rate is highest: 5.95% compounded annually, 5.85% monthly, or 5.75% daily? (Example 3.3.7)
Using Effective Rates with "Messy" Terms, p. 120	• Effective rates compound annually, so n in the compound interest formula must be in years. • If the term is not a whole number of years, decimals and fractions can be used in an exponent.	Find the future value of a $20,000 CD for 1000 days at a 6.25% effective rate. (Example 3.3.9)
Compound Growth Other than Interest, pp. 121–122	• The compound interest formula can be used any time something is growing at a constant percent rate, even if the growth is not interest.	A pair of sneakers cost $75.49 seven years ago. The price has been rising at a 4.8% growth rate. What is the price today? (Example 3.3.10)
Effective versus Nominal Rates, p. 129	• Calculations done with the effective rate should match calculations done with the nominal rate • Results will not match exactly because the effective rate is rounded.	Tris deposited $5,000 at 6.38% compounded daily for 16 years. Find his account's future value two ways: using the nominal rate and by finding the effective rate and using it. (Example 3.4.3)
Solving for Rates and Times (Optional), pp. 131–132	• The compound interest rate can be used to solve for times and interest rates • The algebra to do this requires the use of fractional exponents and/or logarithms • The Rule of 72 can provide approximate answers without requiring as much algebra	How long will it take for $100 to grow to $500 at 6% compounded quarterly? (Example 3.5.4)

CHAPTER 3
EXERCISES

The following exercises are a mixture of problems primarily from the topics covered Chapter 3. One of the objectives of these exercises is to be able to correctly identify which topics and tools are needed for each problem. While the emphasis is on material covered in Chapter 3, some problems covering material from Chapters 1 and 2 may also be included. All of the material covered in this chapter is "fair game", except for optional topics, which are not included in these exercises.

1. Abigail deposits $4,000.00 in a certificate of deposit at a bank for 3 years at an interest rate of 8.36% compounded quarterly. Find (a) the value of the certificate at the end of the 3 years and (b) the total amount of interest earned.

2. Bruce needs to borrow $7,300 for 2 years. Colden National Bank offers him the loan at 7.26% compounded daily, and Angola National Bank offers him the loan at 7.35% compounded monthly. (a) How much would he need to pay back at each bank? (b) Which bank is offering the better deal?

3. Sylvia is shopping around for a bank to open a savings account. She considers three: NetnetBank, which offers a rate of 4.35% compounded annually; SNB Bank, which offers 4.29% compounded weekly; and TJBank, which offers 4.17% compounded daily. Which of the three banks offers the best deal?

4. Jayson is 35 years old and has set the goal of retiring at age 60 with $750,000. (a) How much money does he need to set aside today to reach his goal, assuming his investments earn an effective annual interest rate of 11.4%? (b) How much would he have had to set aside if he had started 15 years earlier? (c) Rework both (a) and (b), assuming that instead his investments earn only 5%.

5. Eden Financial is offering an interest rate of 6.25% compounded daily on 1-year certificates of deposit. Derby Savings Bank is offering a higher rate, 6.35%, but that rate compounds quarterly. Which would be the better choice for your deposit?

6. Cheyenne has invested $3,000 in a 5-year bank certificate on which the interest rate is 6¼% compounded daily. Find the value of the certificate at the end of five years in two ways:
   a) Calculate the FV using the 6¼% compounded daily rate.
   b) Find the effective rate and then use it to find the FV.
   c) Do your answers to (a) and (b) match? If not, explain why not.

7. Find the effective interest rate equivalent to 12.7% compounded daily (bankers' rule).

8. Frank has borrowed $12,349.16 at an interest rate of 14% compounded monthly for 17 months.
   a) Calculate the amount needed to repay the loan using the given interest rate.
   b) Find the equivalent annual rate for the given interest rate.
   c) Calculate the amount needed to repay the loan using the equivalent annual rate.

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9. Deidre borrowed $1,236.95 for 14 months at an interest rate of 9.2% compounded daily. Find the payoff amount for the loan.

10. The State of Wyoming issued a $50,000 face value discount bond that matures on July 1, 2007. Pike Capital Corp bought this note for $48,359 on November 18, 2006. Find the simple interest rate and the simple discount rate.

11. Harry has a store charge account on which the effective interest rate is 21.6%. Suppose he charges a new stereo for $750, and makes no payments for 10 months. (a) How much will he owe at the end of the 10 months? (b) Suppose instead that he forgets about the account and moves to a monastery in Nepal (where the store's collection department can't find him) and returns 12 years later. How much will he owe when he returns?

12. Vladimir opened a savings account with a deposit of $3,279.14 at an interest rate of 7.2% compounded daily. The account remained open for 2 years, 3 months, and 17 days, during which time he made no deposits or withdrawals. How much was the account worth at the end when it was closed?

13. Jacqui borrowed $575.00 for 1 year and 7 months at an interest rate of 6.3% compounded monthly. How much did she have to repay?

14. Find the effective interest rate equivalent to 3.59% compounded (a) daily, (b) monthly, (c) quarterly, (d) semiannually, and (e) annually.

15. Which of the following financial institutions is offering the best rate for a certificate of deposit?

Bank	*Rate*	*Compounding*
Naples Savings and Trust	9.55%	Annually
National Bank of Cohocton	9.35%	Quarterly
Springwater Savings Bank	9¼%	Daily

16. Find the effective rate equivalent to 7.17% compounded annually.

17. How much would I need to deposit at 5.22% compounded daily for an investment to grow to $35,000 in 30 years?

18. Find the future value of $478,923 in 5 years, given an APY of 9.12%.

19. Lois deposited $1,375.29 in a savings account that paid 4.79% compounded daily for 300 days. Would she have been better off if she had earned a 4.87% effective rate? Justify your answer.

20. The population of Allegany City is 27,540. An urban planning consultant predicts that the population will grow at a 1.95% effective rate for the next 10 years. What is she predicting the population will be in 10 years?

21. Find the effective rate equivalent to 8.01% compounded monthly.

22. Approximately how long would it take to double your money at a rate of 4.59% compounded quarterly?

23. Burali-Forti Logic Processors Inc. had sales of $375,000,000 in the last year. The company predicts that its sales will grow at an 8% annual rate for the foreseeable future. According to this prediction, what will the company's sales be in 5 years?

24. Gabriela invested $3,500 at $4\frac{13}{16}$% compounded monthly for $4\frac{1}{2}$ years. How much interest did she earn?

25. Approximately what effective interest rate would you need to earn in order for a $2,500 investment to grow to $10,000 in 12 years?

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CHAPTER 4

Annuities

"No man acquires property without acquiring with it a little arithmetic also."

—Ralph Waldo Emerson

Learning Objectives

LO 1 Recognize financial situations that can be described as annuities, and distinguish between present and future values and between ordinary annuities and annuities due.

LO 2 Calculate the future value of an annuity, using annuity factors obtained from formulas, tables, and/or calculator or computer programs.

LO 3 Find the payments necessary to accumulate a desired future value goal, given a period of time and interest rate.

LO 4 Calculate the present value of an annuity, using annuity factors obtained from formulas, tables, and/or calculator or computer programs.

LO 5 Find the annuity payments that are equivalent to a specified present value, given a period of time and interest rate.

LO 6 Apply present and future value annuity calculations to real-world financial situations, such as long-term investment programs or loans.

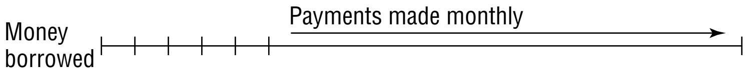
LO 7 Construct amortization tables for loans, and use them as a tool to illustrate and draw conclusions about the workings of loan interest.

Chapter Outline

4.1 What Is an Annuity?

All of the situations we have considered so far, whether simple interest, simple discount, or compound interest, have had something in common. In every case, a sum of money is lent and then the loan is repaid in full all at once at the maturity date. While the terminology and formulas have varied, timelines for any of these loans have looked like:

This is all well and good, but there are many financial situations where this sort of timeline is far too simple. Take a car loan for example. Suppose you borrow \$14,500 at 8% compounded monthly for 5 years to buy a car. The deal is unlikely to be that you get the \$14,500 up front, do nothing for 5 years, and then repay the entire loan at the end, all in a single lump sum. Such an arrangement would not make sense either to you or to the lender. Instead, the usual arrangement requires you pay off the loan by making payments every month. A timeline for your car loan would instead look more like this:



This loan's consistent and regular payments are an example of an ***annuity***. Clearly, we will need to go beyond the mathematics we have so far to deal with such situations. Before we do that, though, we'll need to first define some new terminology.

Definition 4.1.1

*An **annuity** is any collection of equal payments made at regular time intervals.*

Annuities are very common in business and personal finance. A number of examples are given below; with a few moments' thought you can probably come up with many others.

Some Examples of Annuities

- Monthly payments on a car loan, student loan, or mortgage
- Your paycheck (if salaried or if hourly with hours that are always the same)

The payments on a car loan are an example of an annuity. © Jack Star/PhotoLink/Getty Images/DIL

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- Rent payments (until the rent changes)
- Pension or Social Security payments (as long as payments do not change)
- "Budget plan" utility bills (same amount paid each month)
- Regularly scheduled deposits to a savings program, such as an Individual Retirement Account (an IRA)

Some Examples That Are Not Annuities

- Monthly credit card payments (unless you pay the same amount each month)
- Your paycheck (if hours vary, or if paid on commission)
- Daily receipts of a business
- "Pay as you go" utility bills

A few words of caution about terminology are in order. The term *annuity* can be used for a type of insurance contract (including such variations as fixed annuities, variable annuities, and tax-sheltered annuities). While these products often have features that involve a series of payments that fits our definition, it is important not to confuse the annuities we are talking about with these types of insurance contracts. For clarity's sake, we will use the term ***insurance annuity*** when referring to these contracts, though you should be aware that in the real world people are not always so clear and the plain "annuity" is generally used for them. Fortunately, it is usually clear from context which sort of annuity one is talking about, and the terminology doesn't create any more confusion than the fact that Washington is the name of both a city and a state.

Present and Future Values of Annuities

In some situations, such as IRA deposits, the annuity payments are made for the purpose of accumulating a sum of money at a future date. In others, such as a mortgage or car loan, the payments are being made in exchange for a sum of money received at the start of the annuity. In both cases, the sum of money involved is important enough to have a name attached to it.

Definitions 4.1.2

A sum of money to which an annuity's payments and interest accumulate in the end is called the annuity's ***future value***.

A sum of money paid at the beginning of an annuity, to which the annuity's payments are accepted as equivalent, is called the annuity's ***present value***. *(The present value is also sometimes called the* ***amount*** *of the annuity.)*

Note that we already have used the terms *present value* and *future value* with compound interest. Their meanings here are consistent with what these terms meant in the compound interest sections: present value comes at the beginning, future value comes at the end.

Some examples will help illustrate these terms:

Example 4.1.1 ***Distinguishing between present value and future value of an annuity.***

(a) Dylan deposits $25 from each paycheck into a 401(k) savings plan at work. He will keep this up for the next 40 years, at which time he plans to retire, hopefully having accumulated a large balance in his account. Since equal payments are being made into the account at regular intervals, this is an annuity. Is the value of the account when Dylan retires a present value or a future value?

The value of the account when Dylan reaches retirement would be the future value.

(b) Thalissa borrowed $160,000 to buy a house. To pay off this mortgage loan, she agreed to make payments of $1,735.52 per month for 30 years. Since her mortgage

payments are all equal, and are made at regular intervals, they constitute an annuity. Was the amount she borrowed this annuity's present value or future value?

The $160,000 was received at the start of the annuity payments. Therefore it would be the present value of the annuity.

Just as with compound interest, the terms "present" and "future" here do not necessarily correspond to the actual present or future. When Dylan reaches retirement, the value of his account will still be considered a future value, even though his retirement will no longer be in the future. Likewise, regardless of how long ago Thalissa took out her loan and bought her house, we still call the $160,000 the annuity's present value.

The Timing of the Payments

Our definition of an annuity demands that payments be made at regular intervals (annually, monthly, weekly, daily, etc.), but it is silent about the timing of the payments within each interval. If I agree to pay $245 monthly for 60 months for a car loan, this clearly fits the definition of an annuity. But do my payments begin right away (the payment due at the start of each month) or can I wait 30 days before making the first payment (the payment is due at the end of each month)? Our annuity definition says nothing about this detail.

An annuity actually can be set up either way; either timing of the payments would meet the definition. However, since the timing of the payments does matter, we do want to distinguish annuities whose payments come at the start of each period from those whose payments come at the end.

Definitions 4.1.3

*An **ordinary annuity** is an annuity whose payments are made at the end of each time period.*

*An **annuity due** is an annuity whose payment are made at the beginning of each time period.*

As the name *ordinary* suggests, annuities are assumed to be ordinary unless otherwise specified.

Example 4.1.2** **Distinguishing between ordinary annuities and annuities due.

(a) John took out a car loan on May 7. Payments will be made monthly. The first payment is not due until June 7 (the second will be due on July 7, etc.). Is this an ordinary annuity or an annuity due?

*Because his first payment is not made until the **end** of the first month, the second at the end of the second month, and so on, his car payments are an ordinary annuity.*

(b) Jenna won a lottery jackpot which will pay her $35,000 per year for the next 26 years. She does not have to wait an entire year to get her first check—she will be paid the first $35,000 right away. Her second payment will come a year from now, at the start of the second year, and so on. Is this an ordinary annuity or an annuity due?

*Since the payments come at the **start** of each year, her prize payout is an annuity due.*

Don't misunderstand the terms *start* and *end* here. In John's example, we said the payments were made at the end of the month, even though they were in fact due on the seventh day—nowhere near the time to turn the page of the calendar. *End of the month* refers to a month counted from the date the annuity begins. It does *not* mean end of the *calendar* month. Likewise, Jenna does *not* necessarily receive her lottery payment at the start of each calendar year (on January 1). Rather, the years in question are measured from the date her payout annuity begins. She would only receive her payments on January 1 if by coincidence that happened to be the starting date of the annuity.

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EXERCISES 4.1

A. The Definition of an Annuity

Determine whether or not each of the following situations describes an annuity. If the situation is not an annuity, explain why it is not.

1. A car lease requires monthly payments of $235.94 for 5 years.

2. Your cell phone bill.

3. The money Adam pays for groceries each week.

4. Ashok bought a guitar from his brother for $350. Since he didn't have the money to pay for it up front, his brother agreed that he could pay him $25 a week until his payments add up to $350.

5. Caries' Candy Counter pays $1,400 a month in rent for its retail store.

6. The rent for the Tastee Lard Donut Shoppe is $850 a month plus 2% of the monthly sales.

7. Cheryl pays for her son's day care at the beginning of every month. Her provider charges $35 for each day her son is scheduled to be there during the month.

8. Every single morning, rain or shine, Cieran walks to his favorite coffee shop and buys a double redeye latte.

9. According to their divorce decree, Terry is required to pay his ex-wife $590 a month in child support until their daughter turns 21.

10. In response to her church's annual stewardship campaign, Peggy pledged to make an offering of $20 each week.

B. Present and Future Values

Each of the following problems describes an annuity. Determine whether the amount indicated is the annuity's present value or future value.

11. Artie bought a policy from an insurance company that will pay him $950 a month guaranteed for the next 20 years. Is the amount he paid a present value or future value?

12. The Belcoda Municipal Electric Company expects that in 5 years' time it will need to make significant upgrades to its equipment. In order to set aside enough money to pay these expenses, the utility has begun depositing $98,000 each quarter into an investment account. Is the amount they are trying to accumulate a present or future value?

13. Wyatt just attended a sales presentation held by the investment company that administers his employer's 401(k) savings plan. According to the sales representative, if he puts just $125 into the plan with each paycheck he could be a millionaire when he hits retirement age. Is the $1,000,000 projection a present or future value?

14. Otisco County won a judgment against a former contractor that will require the contractor to pay the county $52,000 per year for the next 20 years. Instead of having to make payments for the next 20 years, the contractor offers to pay the county $835,000 in one lump sum today. Would this be a present value or future value?

15. Fifteen years ago, Ken quit smoking, and instead of spending $30 a week on cigarettes he put that money in a special savings account each week. Is the value of his account today a present or future value?

16. Three years ago, Len bought a new car and borrowed the entire cost. Is the cost of the car the present value of his monthly payments or the future value?

17. During his college years, Teddy took out student loans to pay his tuition and living expenses. Now he is making equal payments each month on those loans. Is the amount he borrowed the present value or the future value of his loans?

18. Tessa was laid off from her job and was out of work for 6 months. Every month, she borrowed $1,000 from her father to help cover her living expenses. Was the total she owed him at the end of the 6 months the present value or the future value of the amounts she borrowed?

C. Ordinary Annuities and Annuities Due

Each of the following problems describes an annuity. Determine whether it is an ordinary annuity or an annuity due. Justify your answer.

19. John won a lottery jackpot, which will pay him $100,000 every year for 20 years, starting right away.

20. At the end of each quarter, the TiJiBi Acoustic Electric Corporation pays each of its shareholders a dividend of 25 cents for each share owned.

21. Because he is self-employed, Dylan does not have income taxes withheld from paychecks. Instead, he is required to make estimated tax payments to the IRS. This year, he has to pay $1,845 each quarter.

22. Every month I pay my insurance company $78.59 for my car insurance premium.

23. The rent for Lourdes' apartment is $845 a month and is due at the beginning of each month.

24. A customer of the Mendon Falls Gas and Electric Company pays his utility bills on a budget plan. Instead of having his bill fluctuate from month to month, he pays the utility $165 at the end of each month.

25. Jenny and Rob's monthly mortgage payment is $1,257.09.

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D. Grab Bag

For each of the following scenarios, determine whether or not the payments describe an annuity. If they do, determine whether they are an annuity due or an ordinary annuity. Also, determine whether the amount indicated is a present value or a future value.

26. For the past 20 years, Steve has put $100 in a special savings account at the beginning of each month. He now has $31,575.16 in that account.

27. Ever since their daughter was born, Jim and Lara have been setting aside money to pay for her college. At the end of each month, they have always deposited at least $200 into a college savings account for her. They presently have $18,735.92 in the account.

28. Delsimar Devices Corp. sets aside $275,000 at the end of each quarter. The money goes into a special fund the company plans to use for a long-term expansion plan. The company expects that in 5 years it will have accumulated $5,800,000 in this account.

29. Two years ago, I took out a $25,000 business loan. The payments are $500 per month.

30. Sonja paid an insurance company $400,000 for a policy that will pay her $3,000 a month in income for the rest of her life.

4.2 Future Values of Annuities

In the previous section, we saw several examples of present and future values of annuities, but did not discuss how to actually find the present or future value of any given annuity. This will require more sophisticated mathematical tools than the ones we have been using so far. In every calculation we have done so far, there has always been a single starting amount (the principal, proceeds, or present value) and a single ending amount (the maturity value or future value), with no payments made along the way. Yet with annuities, there are lots of payments being made along the way.

In this section, we'll begin to develop the mathematical tools we'll need to work with annuities. At first, we will focus just on finding future values, though later on in the chapter, we will also address present values. At first we will concern ourselves only with ordinary annuities. Later in this section we will see how to handle annuities due as well.

Looking ahead, you will no doubt notice that this is a very long section. It takes quite a bit of space to develop the formulas and techniques used to handle annuity future values. Depending on the level and objectives of the course that you are taking, your instructor may or may not be covering all of the material in this chapter. *If you have any doubts about which material is being covered in your class, make sure to check with your instructor*.

The Future Value of an Ordinary Annuity

Suppose that you deposit $1,200 at the end of each year into an investment account that earns 7.2% compounded annually. Assuming you keep the payments up, how much would your account be worth in 5 years? Your annual payments constitute an ordinary annuity, and since we are asking about their accumulated value with interest at the end, we are looking for the future value.

It turns out that we actually can find this future value by using the tools we already have. We will consider this example problem from several different lines of attack, in the hope of developing approaches that we can use with annuities in general.

Approach 1: The Chronological Approach

A natural way to approach this problem is to build up the account value year by year, crediting interest as it comes due and adding new payments as they are made.

At the end of the first year, $1,200 is deposited. No interest would be paid, since there was no money in the account prior to this deposit.

At the end of the second year, 1 year's worth of interest would be paid on the $1,200, raising the account value to $1,200(1.072) = $1,286.40. In addition, $1,200 comes in from the second-year deposit, bringing to total to $2,486.40.

We could continue along this way until the end of the 5 years, the results of which can be summarized in the table below:

Year	*Starting Balance*	*Interest Earned*	*Deposit*	*Ending Balance*
1	$0.00	$0.00	$1,200.00	$1,200.00
2	$1,200.00	$86.40	$1,200.00	$2,486.40
3	$2,486.40	$179.02	$1,200.00	$3,865.42
4	$3,865.42	$278.31	$1,200.00	$5,343.73
5	$5,343.73	$384.75	$1,200.00	**$6,928.48**

This approach is logical, does not involve any new formulas, and it nicely reflects what goes on in the account as time goes by. But it is obvious that we wouldn't want to use it to find the future value of 40 years of weekly payments. This chronological approach may be fine in cases where the total number of payments is small, but clearly we have reason to find an alternative.

Approach 2: The Bucket Approach

Suppose now that you opened up a new account for each year's deposit, instead of making them all to the same account. Assuming that all of your accounts paid the same interest rate, and assuming you didn't mind the inconvenience of keeping track of multiple accounts, would this have made any difference to the end result?

The bucket approach arises from the observation that keeping separate accounts would make no difference whatever in the total end result. It doesn't make any difference whether you have a dollar or 20 nickels, and it doesn't make any difference whether you have one large account or 20 smaller ones. Since everything would come out the same if each deposit (and its interest) were kept in a separate "bucket," we *pretend* that this is actually what happens. That's probably not what actually does happen, but since the result would be the same if it did, pretending this happens won't change our final answer.

The first payment of $1,200 is placed in the first bucket. This money is kept on deposit from the end of year 1 until the end of year 5, a total of 4 years. So, at the end of the fifth year, its bucket will contain a total of $1,200(1.072)^4 = $1,584.75.

We do the same for each of the other five payments, and then put them all back together again at the end for the total:

Payment from Year	*Payment Amount*	*Years of Interest*	*Future Value*
1	$1,200	4	$1,584.75
2	$1,200	3	$1,478.31
3	$1,200	2	$1,379.02
4	$1,200	1	$1,286.40
5	$1,200	0	$1,200.00
	Grand total		***$6,928.48***

Notice that we arrive at the same total as with the chronological approach, as we would have hoped. Unfortunately, the bucket approach would still be very tedious to use if there were a lot of payments instead of just five. As things stand, we still have not found our goal

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of a final way to handle these situations, though, as we will see before long, the bucket approach can be adapted to efficiently handle those other situations.

Approach 3: The Annuity Factor Approach

There is a third alternative. Suppose that instead of payments of $1,200, the payment were instead $3,600, three times as much. It makes sense that the future value would then be three times as much. So to find the future value of the $3,600 annuity, we wouldn't need to start from scratch, we could just multiply the $1,200 annuity's future value by 3 to get FV = 3($6,928.48) = $20,785.44. If you have doubts about this, you can verify it for yourself by working through the future value of a $3,600 annuity using either of the approaches we used above.

In general, a larger or smaller payment changes the future value proportionately. Exploiting this, we can define the *future value annuity factor*.

Definition 4.2.1

For a given interest rate, payment frequency, and number of payments, the ***future value annuity factor*** *is the future value that would accumulate if each payment were $1. We denote this factor with the symbol* $s_{\overline{n}|i}$*, where n is the number of payments and i is the interest rate per payment period. [For convenience, this symbol can be pronounced "snigh" (rhymes with "sigh").]*

When we find annuity factors, we carry the results out to more than the usual two decimal places, to avoid losing too much in the rounding. It is conventional not to write a dollar sign in front of the annuity factor, since, among other reasons, using the dollar sign will look strange with numbers carried out to more than two decimal places. Now, let's revisit our problem by using an annuity factor. Since there are 5 annual payments, and since the interest rate is 7.2% per year, the annuity factor we are looking for is $s_{\overline{5}|.072}$. Using the bucket approach, this time with payments of $1, we get

Payment from Year	Payment Amount	Years of Interest	Future Value
1	1.00	4	1.32062387
2	1.00	3	1.23192525
3	1.00	2	1.14918400
4	1.00	1	1.07200000
5	1.00	0	1.00000000
	Grand total		***5.77373311***

Since our payments were actually $1,200, or 1,200 times as much, it is logical that the future value we want is:

$$FV = \$1{,}200(5.77373311)$$
$$FV = \$6{,}928.48$$

The Future Value Annuity Formula

It should be apparent that if the amount of the payments were something other than $1,200, if the time period were something other than 5 years, and if the interest rate were something other than 7.2%, the same principles would apply. This allows us to generalize a future value formula:

FORMULA 4.2.1
The Future Value of an Ordinary Annuity

$$FV = PMTs_{\overline{n}|i}$$

where
FV represents the FUTURE VALUE of the annuity
PMT represents the amount of each PAYMENT
and
$s_{\overline{n}|i}$ is the FUTURE VALUE ANNUITY FACTOR (as defined in Definition 4.2.1)

It's nice to have a formula, but at this point alarms should be going off in your head. Sure, we do have a general formula now, but in order to use it we need to calculate the annuity factor, and the way we did that was not much of an improvement over just using the bucket approach in the first place! At least this formula would nicely allow us to change the payment amount if we wanted to without having to recalculate everything from scratch. But a far greater improvement would be some more efficient way of getting the annuity factors. Fortunately, we can do this.

Finding Annuity Factors Efficiently—Tables

One way to find annuity factors is simply to have a table listing the factors for certain interest rates and numbers of payments. An example of such a table is given below:

SAMPLE TABLE OF ANNUITY FACTORS

Number of Payments (n)	RATE PER PERIOD (i)					
	0.50%	**1.00%**	**2.00%**	**4.00%**	**6.00%**	**8.00%**
1	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
2	2.00500000	2.01000000	2.02000000	2.04000000	2.06000000	2.08000000
3	3.01502500	3.03010000	3.06040000	3.12160000	3.18360000	3.24640000
4	4.03010012	4.06040100	4.12160800	4.24646400	4.37461600	4.50611200
5	5.05025063	5.10100501	5.20404016	5.41632256	5.63709296	5.86660096
8	8.14140879	8.28567056	8.58296905	9.21422626	9.89746791	10.63662763
10	10.22802641	10.46221254	10.94972100	12.00610712	13.18079494	14.48656247
15	15.53654752	16.09689554	17.29341692	20.02358764	23.27596988	27.15211393
20	20.97911544	22.01900399	24.29736980	29.77807858	36.78559120	45.76196430
25	26.55911502	28.24319950	32.03029972	41.64590829	54.86451200	73.10593995
30	32.28001658	34.78489153	40.56807921	56.08493775	79.05818622	113.28321111
40	44.15884730	48.88637336	60.40198318	95.02551570	154.76196562	259.05651871
50	56.64516299	64.46318218	84.57940145	152.66708366	290.33590458	573.77015642

This table allows us to look up the value of the appropriate annuity factor for any of the interest rates and values of n that the table includes. For example, if we need the annuity factor for a 15-year annual annuity at 8%, we would look in the n = 15 row and 8% column and see that the value is 27.15211393. Using our annuity factor notation, we would write $s_{\overline{15}|.08} = 27.15211393$.

The following example will illustrate the use of a table factor to find the future value of an annuity.

Example 4.2.1 ***How much will I have as a future value if I deposit $3,000 at the end of each year into an account paying 6% compounded annually for 30 years?***

The payments are equal and at regular intervals, and their timing is at the end of each period, so we have an ordinary annuity. Using Formula 4.2.1 with the annuity factor from the table above:

$$FV = PMTs_{\overline{n}|i}$$
$$FV = (\$3,000)s_{\overline{30}|.06}$$

We can find the value of the needed annuity factor from the table. We look in the n = 30 row and the 6% column and find that the value of the annuity factor is 79.05818622. We can then plug this into our formula and finish the calculation:

$$FV = (\$3,000)(79.05818622)$$
$$FV = \$237,174.56$$

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The obvious shortcoming of this approach is that a reasonably sized table can list only a limited number of interest rates and number of payments. If the interest rate was 6½%, or the time period 35 years, we'd be out of luck. The obvious solution is to get a bigger table, but a table that gave every possible rate together with every possible number of payments would be cumbersome. Still, this has historically been a commonly used approach, and many libraries still have in their collections books containing lengthy tables of annuity and other financially useful factors. Fortunately technology has largely eliminated the need for such monstrous tables.

That being said, such tables do exist. At most large bookstores you can find books for sale that include such annuity factor tables, and they are available on some Internet sites as well.

Finding Annuity Factors Efficiently—Calculators and Computers

Some calculators and software packages have features to automatically calculate annuity factors. This is not a standard feature on most calculators on the market today, but it is not an unusual feature either, particularly on models sold specifically as "business" or "financial" calculators. For the course that you are taking, you may or may not be using a calculator with this feature. If you are, your instructor or the calculator's owner's manual can provide the details of how to obtain these factors from your calculator.

It is also possible to quickly obtain annuity factors by using many common types of business software, such as Microsoft Excel.

It is not necessary to buy a special calculator or have access to a computer to find annuity factors, however. We will derive a formula for $s_{\overline{n}|i}$ that can be worked out on almost any calculator.

A Formula for $s_{\overline{n}|i}$

Deriving the $s_{\overline{n}|i}$ formula requires a bit of algebra and a bit of cleverness. Readers without a solid algebra background can skip over this derivation can go directly to Formula 4.2.2 (though doing so will deprive you of all the fun . . . .)

Let's look back at the bucket approach's calculation of $s_{\overline{n}|i}$. The values that we added up to get $s_{\overline{n}|i}$ were:

$$s_{\overline{n}|i} = 1(1.072)^4 + 1(1.072)^3 + 1(1.072)^2 + 1(1.072) + 1$$

Now (for entirely nonobvious reasons) let's multiply both sides of this equation by 1.072 to get:

$$(1.072)s_{\overline{n}|i} = 1.072[1(1.072)^4 + 1(1.072)^3 + 1(1.072)^2 + 1(1.072) + 1]$$

Distributing on the right side, and neatening things up a bit, we get:

$$(1.072)s_{\overline{n}|i} = (1.072)^5 + (1.072)^4 + (1.072)^3 + (1.072)^2 + (1.072)$$

Now (once again mysteriously) suppose we add 1 to both sides to get:

$$(1.072)s_{\overline{n}|i} + 1 = (1.072)^5 + (1.072)^4 + (1.072)^3 + (1.072)^2 + (1.072) + 1$$

Now it looks like we've have really made a mess—that is, until we notice that most of the right-hand side is something we've seen before:

$$(1.072)s_{\overline{n}|i} + 1 = (1.072)^5 + \underbrace{(1.072)^4 + (1.072)^3 + (1.072)^2 + (1.072) + 1}_{\text{Hey, that's } s_{\overline{n}|i}!}$$

So:

$$(1.072)s_{\overline{n}|i} + 1 = (1.072)^5 + s_{\overline{n}|i}$$

Now, if we subtract 1 from both sides, and also subtract $s_{\overline{n}|i}$ from both sides, this becomes:

$$(1.072)s_{\overline{n}|i} - s_{\overline{n}|i} = (1.072)^5 - 1$$

which leads to:

$$(0.072)s_{\overline{n}|i} = (1.072)^5 - 1$$

Finally, dividing both sides by 0.072, we get:

$$s_{\overline{5}|.072} = \frac{1.072^5 - 1}{0.072}$$

Evaluating this gives:

$$s_{\overline{5}|.072} = 5.77373311$$

which is the same annuity factor we calculated earlier. It worked!

Now this may seem like an awfully complicated way of getting at $s_{\overline{n}|i}$, far more effort than just the direct bucket approach. However, notice that we could repeat this process for any interest rate and number of payments. There was nothing special about n = 5 and I = 0.072. If we have n = 40 and i = 0.565, the same process would work. Rather than slog through the details a second time, though, we can just skip to the end and get:

$$s_{\overline{40}|.0565} = \frac{1.0565^{40} - 1}{0.0565} = 141.79020544$$

Since this reasoning would hold up for any n and i, it turns out that what we have found here is what we were seeking all along; a formula that will enable us to efficiently find annuity factors, and thus to find annuity future values. We can now generalize this as a formula:

FORMULA 4.2.2
The Future Value Annuity Factor

$$s_{\overline{n}|i} = \frac{(1 + i)^n - 1}{i}$$

where
i represents the INTEREST RATE per payment period
and
n represents the NUMBER OF PAYMENTS

The following example will illustrate the use of this formula:

Example 4.2.2 ***Find the future value annuity factor for a term of 20 years with an interest rate of 7.9% compounded annually.***

Here, i = 20 and n = 0.079. Plugging these values into Formula 4.2.2 we get:

$$s_{\overline{n}|i} = \frac{(1 + i)^n - 1}{i} = \frac{(1 + 0.079)^{20} - 1}{0.079}$$

We need to be careful evaluating this. This formula is far more complicated than others we have used so far. Order of operations demands that we must first add the 1 + 0.079 because it is in ( )'s, and next raise the result to the 20th power, to get 4.575398165. Order of operations next says to do multiplication and division, but clearly it is impossible to divide 4.575398165 − 1 by 0.079 without first doing the subtraction. In this case, we are allowed to subtract first. The fraction line is considered an ***implied grouping symbol****, which means that we treat it as though there were ( )'s around both top and bottom even though we don't bother to write them—it is simply assumed that everyone knows that they are there. And so we next do the subtraction, getting 3.575398165, which we divide by 0.079 to get a final answer of*

$$s_{\overline{20}|.079} = 45.258204619$$

That description was probably a bit hard to follow. It will probably be much easier to follow if we summarize the calculator steps in a table. If your calculator does not follow order of

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operations, or if you do not want to try to enter a formula this large all at once, the steps would be:

Operation	Result
1+.079=	1.079
^20=	4.575398165
−1=	3.575398165
/.079=	45.25820462

If in previous work you have been entering the entire formulas into the calculator in one step, you can do that here also. You must, though, be careful about the implied grouping. Since you cannot enter things on top of one another, the calculator does not know about the implied grouping. You have to make the implied grouping explicit to the calculator, and so you must make sure to put in parentheses around the top yourself.

Operation	Result
((1+.079)^20−1)/(.079)=	45.25820462

Make sure not to overlook these implied parentheses! The result of leaving them out will usually be an answer that is miles away from the correct one. Because of the length of the formula and the many opportunities it presents to make a mistake typing it in, the all-at-once approach is not recommended.

A third alternative would be to enter by combining some, but not all of the steps. For example, we could calculate this annuity factor as:

Operation	Result
(1+.079)^20=	4.575398165
−1=	3.575398165
/.079=	45.25820462

You may want to try calculating annuity factors each way for yourself to see which approach you like best.

Now that we've figured out how to calculate the annuity factors, we can put them to work to find actual future values.

Example 4.2.3 ***Suppose that $750 is deposited each year into an account paying 7.9% interest compounded annually. What will the future value of the account be?***

Since the payments are equal and made at regular intervals, this is an annuity; since the timing of the payments is unspecified, we assume it to be an ordinary annuity and use Formula 4.2.1. The annuity factor we need is the one calculated in Example 4.2.2.

$$FV = PMT s_{\overline{n}|i}$$
$$FV = \$750\ s_{\overline{20}|.079}$$
$$FV = (\$750)(45.258204619)$$
$$FV = \$33{,}943.65$$

Getting comfortable with these calculations will no doubt take additional practice. There will be numerous opportunities to get that practice from the remaining examples and exercises in this section.

Nonannual Annuities

Even though the annuities we've worked with so far in this section have all had annual payments, there is no reason why we can't use the same techniques with an annuity whose payments are quarterly, monthly, weekly, or any other frequency. The transition is basically the same as the one we made with compound interest when we moved from annual compounding to other compounding frequencies.

Example 4.2.4 ***Find the future value of quarterly payments of $750 for 5 years, assuming an 8% interest rate.***

We first determine the values of i and n. This is done exactly the same we did it with compound interest in Chapter 3:

$$i = \frac{0.08}{4} = 0.02$$

and

$$n = 5(4) = 20$$

Before we can proceed, we need to find the annuity factor. To obtain the factor by using the formula, we plug in these values of n and i to get:

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} = \frac{(1+0.02)^{20} - 1}{0.02}$$

Evaluating this on the calculator step by step we get:

Operation	Result
1+.02=	1.02
^20=	1.48594740
−1=	0.48594740
/.079=	24.2973698

Alternatively:

Operation	Result
(1+.02)^20=	1.48594740
−1=	0.48594740
/.079=	24.2973698

(You should find the same value for the annuity factor if you are using a table, or if you are using a preprogrammed calculator feature as well.)

Now that we have the annuity factor, we can complete the calculation. Plugging into our annuity formula, we get:

$$FV = PMT\ s_{\overline{n}|i}$$
$$FV = (\$750)\ s_{\overline{20}|0.02}$$
$$FV = (\$750)(24.2973698)$$
$$FV = \$18{,}223.03$$

While lengthy, this last problem was not much different from a problem with annual payments. Calculating annuity factors for nonannual annuities can become a bit stickier, though, since nonannual compounding means that we will usually have to divide to get i. If the result does not come out evenly, this can require an awful lot of tedious typing with rich potential for typos. The following example will illustrate ways to accurately complete this calculation.

Example 4.2.5 ***Find the future value annuity factor for a monthly annuity, assuming the term is 15 years and the interest rate is 7.1% compounded monthly.***

Here,

$$i = \frac{0.071}{12} = 0.005916667$$

and

$$n = (15)(12) = 180$$

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We can work this out exactly as before:

Operation	*Result*
1+.005916667=	1.005916667
^180=	2.891749861
−1=	1.891749861
/.005916667=	319.732352909

Though the decimal makes this a bit tedious, this will work fine, and this is the preferred method if you happen to be using a calculator that does not follow order of operations.

It may be more efficient to avoid the decimal and enter i as a fraction, combining steps as before. Make sure to put ( )'s around the denominator!

Operation	*Result*
(1+.071/12)^180=	2.891749861
−1=	1.891749861
/(.071/12)=	319.732352909

If you have been entering the formula all at once, you can do that here as well, but between the fractions and the added percents it gets awfully long, which invites typos.

Either of the approaches demonstrated in Example 4.2.5 will work out fine. The important thing is to make sure that you can consistently and correctly use whichever approach suits you and your calculator best.

Example 4.2.6 ***Each month, Carrie deposits $250 into a savings account that pays 4.5% interest (compounded monthly). Assuming that she keeps this up, and that the interest rate does not change, how much will her deposits have grown to after 5 years?***

Carrie's payments are equal, and made at regular intervals, and so her deposits make up an annuity. Since the timing of the payments is not specified, we assume it is an ordinary annuity, and so we can use Formulas 4.2.1 and 4.2.2 to find its future value. We first must find the annuity factor, and to do that we must first find:

$$i = \frac{0.045}{12}$$

and

$$n = 5(12) = 60$$

Note that we have decided here to leave i as a fraction. Substituting into Formula 4.2.2, we get:

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} = \frac{\left(1 + \frac{0.045}{12}\right)^{60} - 1}{\frac{0.045}{12}} = 67.14555214$$

Using this in Formula 4.2.1, we get

$$FV = PMT s_{\overline{n}|i}$$
$$FV = \$250(67.14555214)$$
$$FV = \$16{,}786.39$$

Notice that in all of our examples the payment and compounding frequencies are the same. We quietly assumed this in developing our formula for $s_{\overline{n}|i}$. It is of course possible for the annuity payments and compounding frequencies to differ, but this difference considerably complicates matters. A way of dealing with this situation is outlined at the end of this section, but for most purposes we would normally just assume that the frequencies are identical.

Finding the Total Interest Earned

Our annuity formulas do not provide any easy, direct way to find the total amount of interest earned. However, we can easily find this out with a slightly indirect approach:

Example 4.2.7 ***How much total interest did Carrie (from Example 4.2.6) earn?***

To solve this, we first observe that the money in her account comes from two sources: the money she deposits, and the interest she earns. It is not hard to determine the total of her deposits. She made monthly deposits for 5 years, for a total of $n = 60$ deposits. Each one was for \$250, so in total she deposited (60)(\$250) = \$15,000.

If \$15,000 of her future value came from her deposits, the rest must have come from interest. So

$$\text{Carrie's total interest} = \$16{,}786.39 - \$15{,}000 = \$1{,}786.39.$$

The Future Value of an Annuity Due

With an annuity due, payments are made at the beginning of each period rather than the end. Each payment is made earlier, so it stands to reason that an annuity due would have a larger future value than an ordinary annuity, since the payments have longer to earn interest.

To see how much more, let's revisit the 5 year, \$1,200 per year annuity with 7.2% interest, this time, though, as an annuity due. The first payment would earn interest from the start of the first year until the end of the fifth year, for a total of 5 years. The second payment would earn interest for 4 years, the third payment for 3 years, and so on. Thus, following the bucket approach just as we did before, we get that the future value would be:

Payment from Year	*Payment Amount*	*Years of Interest*	*Future Value*
1	\$1,200	5	\$1,698.85
2	\$1,200	4	\$1,584.75
3	\$1,200	3	\$1,478.31
4	\$1,200	2	\$1,379.02
5	\$1,200	1	\$1,286.40
	Grand total		***\$7,427.33***

Notice that this is the same table we used for the ordinary annuity, except that every payment is being credited with 1 additional year of interest. Since crediting 1 year's extra interest is equivalent to multiplying everything through by 1.072, we get our future value by multiplying the ordinary annuity's future value by 1.072. Trying it out, we see that \$6,928.48(1.072) does indeed equal \$7,427.33.

This leads us to a formula for annuities due:

FORMULA 4.2.3
Future Value for an Annuity Due

$$\mathbf{FV = PMT\ s_{\overline{n}|i}(1 + i)}$$

where
FV represents the FUTURE VALUE of the annuity
PMT is the amount of each PAYMENT
i is the INTEREST RATE per period
and
$s_{\overline{n}|i}$ is the FUTURE VALUE ANNUITY FACTOR (as defined in Definition 4.2.1)

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Example 4.2.8 ***On New Year's Day 2004, Mariano resolved to deposit $2,500 at the start of each year into a retirement savings account. Assuming that he sticks to this resolution, and that his account earns 8¼% compounded annually, how much will he have after 40 years?***

The payments are equal and made at the start of each year, so this is an annuity due. Thus:

$$FV = PMTs_{\overline{n}|i}(1 + i)$$
$$FV = \$3{,}000\, s_{\overline{40}|.0825}\,(1 + 0.0825)$$

Before going any further, we must find the annuity factor. Using Formula 4.2.2 we get:

$$s_{\overline{n}|i} = \frac{(1 + i)^n - 1}{i} = \frac{(1 + 0.0825)^{40} - 1}{0.0825} = 276.72205752$$

Returning to the FV formula by plugging in this annuity factor, we get:

$$FV = \$3{,}000(276.72205752)(1.0825)$$
$$FV = \$898{,}654.88$$

So if Mariano does stick with the program, he will have $898,654.88, or nearly $900,000 after 40 years.

Summing Up

We've covered a lot of ground in this section. Before wrapping up with discussion of some optional topics, it's worthwhile to briefly summarize the formulas of this section together in one place.

Future Value of Annuity Formulas Summary

Ordinary annuities: $FV = PMT\, s_{\overline{n}|i}$

Annuities due: $FV = PMT\, s_{\overline{n}|i}\,(1 + i)$

For both formulas, the annuity factor is given by the formula:

$$s_{\overline{n}|i} = \frac{(1 + i)^n - 1}{i}$$

When Compounding and Payment Frequencies Differ (Optional)

As noted previously, we normally assume that interest compounds at the same frequency as payments are made to an annuity. When interest compounds more frequently than payments are made, the future value can be found by first finding an equivalent interest rate that compounds at the same frequency as the payments. The following example will illustrate.

Example 4.2.9 ***Suppose that $300 is deposited each quarter into an account paying 6% interest compounded monthly. Find the future value of the account in 5 years.***

There are 3 months in each quarter, and so the interest compounded each quarter will amount to multiplying the balance by $(1.005)^3$*. Thus, we use this factor in place of the* $(1 + i)$*, and in place of i we use* $(1.005)^3 - 1$ *in the* $s_{\overline{n}|i}$ *formula to get:*

$$s_{\overline{n}|i} = \frac{(1 + i)^n - 1}{i} = \frac{\left[(1.005)^3\right]^{20} - 1}{(1.005)^3 - 1} = 23.1407801$$

Thus the future value would be $FV = \$300(23.1407801) = \$6{,}942.23$.

Note that if we had simply ignored the monthly compounding and just used 6% compounded quarterly, the future value would have been $2,312.37. While the monthly compounding does make a difference, that difference is not large.

In cases where the payments are made more frequently than interest compounds, everything depends on how the between-compoundings payments are treated. It is likely the case (though not necessarily so) that payments made between compoundings will earn interest for the portion of the period for which they are in the account. Assuming this, the approach is basically the same as in Example 4.2.8, though the exponents are uglier.

Example 4.2.10 ***Suppose $100 per month is deposited in an account for which the interest is 6% compounded quarterly. Find the future value after 5 years.***

Each month is one-third of a quarter, and so the interest compounding for a month would be $(1 + 0.06/4)^{1/3}$. Proceeding as in the previous example, we get

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} = \frac{\left[\left(1 + \frac{0.06}{4}\right)^{1/3}\right]^{60} - 1}{\left(1 + \frac{0.06}{4}\right)^{1/3} - 1} = 69.7167087$$

And so the future value would be ($100)(69.7167087) = $6,971.67

If we have instead just used 6% compounded monthly, the future value would have been $6,977.00. Once again, there is a difference, but it is not large.

If, however, the between-compounding payments earn no between-compounding interest, then the situation is the same as if all of the payments for a given compounding period were made all at once at the end of the compounding period.

Example 4.2.11 ***Suppose that $100 per month is deposited in an account for which the interest is 6% compounded quarterly. Find the future value after 5 years, assuming that each quarter's interest is paid only on the funds that were in the account at the start of the quarter.***

It makes no difference here whether you make $100 payments each month or instead just keep the payments in a coffee can until the end of the quarter and then make a single $300 deposit then. So the future value can be found by assuming payments of $300 per quarter, in which case the future value will be $6,937.10.

Recall from Chapter 3 that, while the compounding frequency does matter, the difference between monthly and quarterly compounding is not huge. In all of the previous examples, had we just assumed that interest compounded at the same frequency as the payments, the difference would not have been all that great. If the situation demands an exact value, this difference would of course matter, and so we would not be able to ignore it. However, if the future value is a projection or illustration, where it is understood that the figure given is not meant to be taken as exact, using quarterly compounding would probably be close enough. It is often understood that the future value in question is not to be taken too literally, and so it is usually reasonable to stick with the same frequency assumption. With the exception of a few clearly marked exercises in this section, we will assume matching frequencies for the remainder of this book.

EXERCISES 4.2

In all of the exercises in this section, assume that there are no additional deposits or withdrawals from the accounts other than those described.

A. The Chronological and Bucket Approaches

1. Suppose that you deposit $1,359.55 at the end of each year into an investment account that earns 5.7% interest compounded annually for 4 years. Determine the future value of your payments using (a) the chronological approach and (b) the bucket approach by completing the tables below:

   a. Chronological

Year	Starting Balance	Interest Earned	Deposit	Ending Balance
1	$0.00	$0.00	$1,359.55	
2			$1,359.55	
3			$1,359.55	
4			$1,359.55	

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b. Bucket

Payment from Year	Payment Amount	Years of Interest	Future Value
1	$1,359.55		
2	$1,359.55		
3	$1,359.55		
4	$1,359.55		
	Grand total		

B. Finding Future Values by Using Annuity Factors (Ordinary Annuities, Annual Payments)

Exercises 2 and 3 are intended only for those evaluating annuity factors by using the formulas.

2. Calculate $s_{\overline{48}|0.0042}$ as follows.
   a. Write the annuity factor formula, plugging in the appropriate values for n and i.

   b. Evaluate the formula by following the calculator steps shown below and filling in the values in the Result column of the table below.

Operation	*Result*
(1+.0042)^48=	
−1=	
/.0042=	

3. Calculate $s_{\overline{54}|.065}$ as follows
   a. Write the annuity factor formula, plugging in the appropriate values for n and i.

   b. Evaluate the formula by following the calculator steps shown below and filling in the values in the Result column of the table below.

Operation	*Result*
(1+.065)^54=	
−1=	
/.065=	

To find the annuity factors for these problems, use whatever method your instructor is requiring you to use in the course you are taking. Depending on the precision of your calculator (or table) your answers may have more or fewer decimal places than the "back of the book" answers.

4. Calculate $s_{\overline{n}|i}$ if n = 24 and i = .015.

5. a. Calculate $s_{\overline{n}|i}$ if n = 20 and i = .095.
   b. Use your answer from (a) to find the future value of $1,000 invested at the end of each year for 20 years, assuming a 9.5% interest rate.

6. a. Calculate $s_{\overline{4}|0.057}$.
   b. Use your answer from (a) to find the future value of the annuity from Problem 1.

7. You deposit \$3,000 each year into an investment account that earns 7.5% interest for 25 years.
   a) Find n.
   b) Find i (expressed as either a fraction or a decimal).
   c) Find the value of $s_{\overline{n}|i}$.
   d) Find the future value of the annuity.

8. a. Calculate $s_{\overline{n}|i}$ if n = 60 and i = .03.
   b. Use your answer from (a) to find the future value of \$500 invested at the end of each year for 60 years, assuming a 3% interest rate.
   c. Use your answer from (a) to find the future value of \$500 invested at the end of each quarter for 15 years, assuming a 12% interest rate.

9. Donni has decided to invest \$2,000 each year in an account that she expects to earn 8.25%. If she keeps this up for 35 years, how much would she end up with in her account?

10. Find the future value of \$857.35 per year for 20 years at 6½%.

11. Saint Viateur College does not have much of an endowment,[1] and the college's administration has developed a multiyear plan to build it up by appealing to their alumni for contributions. The administration believes that it can raise \$800,000 in each of the next 7 years, and can invest this money at 7.3%. If it is successful, how much will the plan have raised at the end of the 7 years?

12. Suppose I deposit \$5,000 each year into a retirement account. (a) How much will I have in 40 years if my account earns 5%? (b) How much will I have in 40 years if my account earns 10%?

C. Future Values for Nonannual Ordinary Annuities

13. Calculate $s_{\overline{72}|.082/12}$ as follows.
   a. Write the annuity factor formula, plugging in the appropriate values for n and i.
   b. Evaluate the formula by following the calculator steps shown below and filling in the values in the Result column of the table below.

Operation	*Result*
(1+.082/12)^72	
−1=	
/(.082/12)=	

[1]An endowment is a fund owned by an organization such as a college, charity, church, hospital, or similar organization that is invested and used to support the organization's mission. Colleges, for example, might use their endowments to provide scholarships, keep tuitions lower, build better facilities, or fund research.

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For each of Problems 14 to 17:

- a) *Find n.*
- b) *Find i (expressed as either a fraction or a decimal).*
- c) *Find the value of* $s_{\overline{n}|i}$.
- d) *Find the future value of the annuity.*

14. You deposit $175 each month into an account that earns 6% interest for 10 years.

15. You deposit $1,200 quarterly into an account that earns 5.94% for 18 years.

16. You deposit $47.26 each week into an account that earns 11.33% for 37 years.

17. You deposit $135.29 monthly into an account paying 8¾% for 27 years.

18. At the end of every month, Adam puts $100 into his savings account, which pays 3% interest. Assuming this keeps up for 5 years, how much will his deposits grow to?

19. Kulbir has set up an "automatic account builder" plan with an investment company. At the end of each month, the company automatically deducts $125 from his checking account and deposits it into an account he has set up to save for his kids' college costs. If this continues for 17 years, and the account earns 6.36%, how much will his account grow to?

20. The Fillmore Fan Company sets aside $28,500 from each quarter's profits and invests the money in a special account to help pay for future planned improvements to its facilities. If this account earns 4¾%, how much will it have grown to in 3 years?

21. Chun has a Holiday Club account at her local savings bank. At the end of each week, she deposits $20 into the account, which pays 2.8% interest. How much will the account have grown to at the end of the year?

D. Finding Total Interest Earned

22. If you put $40 each month into an account that earned 7.8% interest for 30 years, how much total interest would you earn?

23. Find the total interest earned on an ordinary annuity of $65.49 per month for 8 years, assuming an 8.43% interest rate.

E. Finding Future Values Using Annuity Factors: Annuities Due

24. Suppose that Donni (from Problem 9) made her deposits at the beginning of each year, but that everything else remained the same. How much would she end up with?

25. Find the future value of an annuity due of $502.37 per year for 18 years at 5.2%.

26. Suppose that you deposit $3,250 into a retirement account today, and vow to do the same on this date every year. Suppose that your account earns 7.45%. How much will your deposits have grown to in 30 years?

27. a. Lisa put $84.03 each month into an account that earned 10.47% for 29 years. How much did the account end up being worth?
    b. If Lisa had made her deposits at the beginning of each month instead of the end of the month, how much more would she have wound up with?

F. Differing Payment and Compounding Frequencies (Optional)

28. Find the future value of an ordinary annuity of $375 per month for 20 years assuming an interest rate of 7.11% compounded daily.

29. Find the future value of an ordinary annuity of $777.25 per quarter for 20 years, assuming an interest rate of 9% compounded annually, and assuming interest is paid on payments made between compoundings.

30. Repeat Problem 29, assuming instead that no interest is paid on between-compounding payments.

G. Grab Bag

31. Anders put $103.45 each month in a long-term investment account that earned 8.39% for 32 years. How much total interest did he earn?

32. J.J. deposits $125 at the start of each month into an investment account paying 7¼%. Assuming he keeps this up, how much will he have at the end of 30 years?

33. A financial planner is making a presentation to a community group. She wants to make the point that small amounts saved on a regular basis over time can grow into surprisingly large amounts. She is thinking of using the following example:

    Suppose you spend $3.25 every morning on a cup of gourmet coffee, but instead decide to put that $3.25 into an investment account that earns 9%, which falls well within the average long-term growth rate of the investments my firm offers. How much do you think that account could grow to in 40 years?

    Calculate the answer to her question.

34. Find the future value of a 25-year annuity due if the payments are $500 semiannually and the interest rate is 3.78%.

35. How much interest will I earn if I deposit $45.95 each month into an account that pays 6.02% for 10 years? For 20 years? For 40 years?

36. Find the future value annuity factor for an ordinary annuity with monthly payments for 22 years and an 8⅝% interest rate.

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37. Suppose that Ron deposits $125 per month into an account paying 8%. His brother Don deposits $250 per month into an account paying 4%. How much will each brother have in his account after 40 years?

38. Suppose that Holly deposits $125 per month into an account paying 8%. Her sister Molly deposits $250 per month into an account paying 4%. How much will each sister have in her account after 16 years?

39. The members of a community church, which presently has no endowment fund, have pledged to donate a total of $18,250 each year above their usual offerings in order to help the church build an endowment. If the money is invested at a 5.39% rate, how much will they endowment have grown to in 10 years?

40. Jack's financial advisor has encouraged him to start putting money into a retirement account. Suppose that Jack deposits $750 at the end of each year into an account earning 8¾% for 25 years. How much will he end up with? How much would he end up with if he instead made his deposits at the start of each year?

H. Additional Exercises

41. A group of ambitious developers has begun planning a new community. They project that each year a net gain of 850 new residents will move into the community. They also project that, aside from new residents, the community's population will grow at a rate of 3% per year (due to normal population changes resulting from births and deaths). If these projections are correct, what will the community's population be in 15 years?

42. a. Find the future value of $1,200 per year at 9% for 5 years, first as an ordinary annuity and then as an annuity due. Compare the two results.

    b. Find the future value of $100 per month at 9% for 5 years, first as an ordinary annuity and then as an annuity due. Compare the two results.

    c. In both (a) and (b) the total payments per year were the same, the interest rate was the same, and the terms were the same. Why was the difference between the ordinary annuity and the annuity due smaller for the monthly annuity than for the annual one?

43. Suppose that Tommy has decided that he can save $3,000 each year in his retirement account. He has not decided yet whether to make the deposit all at once each year, or to split it up into semiannual deposits (of $1,500 each), quarterly deposits (of $750 each), monthly, weekly, or even daily. Suppose that, however the deposits are made, his account earns 7.3%. Find his future value after 10 years for each of these deposit frequencies. What can you conclude?

44. (Optional.) As discussed in this chapter, we normally assume that interest compounds with the same frequency as the annuity's payments. So, one of the reasons Tommy wound up with more money with daily deposits than with, say, monthly deposits, was that daily compounding results in a higher effective rate than monthly compounding. Realistically speaking, the interest rate of his account probably would compound at the same frequency regardless of how often Tommy makes his deposits. Rework Problem 43, this time assuming that, regardless of how often he makes his deposits, his account will pay 7.3% *compounded daily.*

4.3 Sinking Funds

So far we have looked at annuities from the point of view that the payments determine the future value. If instead we set up an annuity with a future value goal in mind, we would need to look at things in the opposite direction. We call such annuities *sinking funds*.

Definition

*A **sinking fund** is an annuity for which the amount of the payments is determined by the future value desired.*

The difference between a sinking fund and a garden variety annuity is one of attitude, not substance. Mathematically, there is no difference between a sinking fund and any other annuity. The same terms and formulas we developed for annuities in general apply equally well whether the annuity is a sinking fund or not.

Example 4.3.1 ***Suppose Calvin has set for himself a goal of having \$10,000 in a savings account in 5 years. He plans to make equal deposits to the account at the end of each month, and expects the account to earn 3.6% interest. How much should each of his deposits be?***

Using the future value formula, we get:

$$FV = PMTs_{\overline{n}|i}$$
$$\$10{,}000 = PMTs_{\overline{60}|.003}$$
$$\$10{,}000 = PMT(65.63160098)$$

Dividing through both sides to solve for PMT gives:

$$PMT = \$152.37$$

Note that in this example, we did not show the work done to calculate $s_{\overline{60}|.003}$. In Section 4.2 we got quite a bit of practice with these, and so we are assuming at this point that everyone is able to work out the value of any future value annuity factor needed. From this point forward in the text, when an annuity factor is required we will give its value, but not show the steps done to obtain it. If you are having difficulty calculating these factors yourself, you should go back to Section 4.2 for additional practice before continuing on with the remainder of this chapter.

Sinking funds for annuities due work in much the same way:

Example 4.3.2 ***Shauna owns a software development company, and as part of a new product she has licensed the right to include in it some code owned by her friend Elena. To allow Shauna time to develop and market the product, Elena has agreed to wait 2 years before getting the \$10,000 Shauna has agreed to pay. If, in anticipation of paying Elena, she decides to make equal deposits at the start of each quarter into an account paying 4.8% how much should each deposit be?***

This represents a sinking fund, since her deposits are being made for the purpose of accumulating a specific future value—the \$10,000 she will need to pay Elena. Since she is making these deposits at the start of each quarter, though, this is an annuity due and so we need to begin with the annuity due formula.

$$FV = PMTs_{\overline{n}|i}(1 + i)$$
$$\$10{,}000 = PMTs_{\overline{8}|.012}(1.012)$$
$$\$10{,}000 = PMT(8.344186128)(1.012)$$
$$\$10{,}000 = PMT(8.444316362)$$
$$PMT = \$1{,}184.23$$

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Sinking Funds with Loans

Knowing that her company had a large expense ahead on the horizon, it was prudent of Shauna to set up a fund to be able to meet it, rather than just wait and then get hit with a major expense all at once. However, Example 4.3.2 didn't say that she was under any *obligation* to do this. Shauna could just as well have chosen to not worry about the obligation until the end of the 2 years, hoping that she would be able to come up with the money when needed. Of course, this risks the unpleasant possibility of being unable to come up with the required amount to meet the obligation, or being able to come up with it but financially jeopardizing her company's finances if its financial position is not strong enough to comfortably weather a $10,000 payment all at once. Shauna was smart to not want to take that chance.

But it is not only Shauna who should be concerned about this. Elena must realize that if Shauna can't come up with the $10,000 when it is due, she may not get paid. If funds have not been set aside along the way, and if Shauna's business isn't flush enough with cash to pay the $10,000 all at once, Elena may find that trying to collect her money is trying to get blood from a stone. Elena may have legal contracts to show that Shauna must pay, and Shauna might be a decent and honest person with no intention of cheating her friend—but if the money isn't there, the money isn't there. Elena might have to wait longer than she agreed to before she can collect, she might have to hound Shauna or even sue her to get the funds, or she may not even be able to collect at all.

Given these possibilities, Elena has good reason to want Shauna to be building up that balance along the way. To help ensure that she will in fact be able to pay her when the time comes, it is possible that Elena might insist that as part of their agreement Shauna be required to set up and pay into a sinking fund. This would not be an unusual provision to include in their deal.

Loans provide another situation where one party might require another to have a sinking fund. The following example will illustrate.

Example 4.3.3 ***The Shelbyville Water and Sewer District has borrowed $1.5 million from a group of investors. The note carries an interest rate of 5.52% compounded annually, and matures in 7 years. No payments will be made to the investors during the term of the loan, but the deal requires the district to establish a sinking fund, and make semiannual deposits into this fund in order to accumulate the full maturity value. As required, the district sets up an account at a local bank, which offers an interest rate of 3.8%. How much should each of the deposits be?***

Before determining the sinking fund payment, we must first determine the future value to be accumulated. The future value will be the $1.5 million borrowed, together with the interest that it accumulates. This is not an annuity, but rather is a case of regular compound interest:

$$FV = PV(1 + i)^n$$
$$FV = \$1{,}500{,}000(1.0552)^7$$
$$FV = \$2{,}184{,}915.96$$

On an amount this large, anything less than a dollar is insignificant, so we can reasonably round this future value to $2,184,916.

Now that we know the desired future value, we can readily find the sinking fund payment.

$$FV = PMTs_{\overline{n}|i}$$
$$\$2{,}184{,}916 = PMTs_{\overline{14}|.019}$$
$$\$2{,}184{,}916 = PMT(15.86753836)$$
$$PMT = \$137{,}697.23$$

While we have generally tried to avoid having to write down and retype long decimals, in these cases trying to combine everything into a single calculator entry is impractical.

Of course, given the size of the deposits involved, the district would most likely round the payments to $137,697, or even $137,700. On a payment of this size, rounding to the nearest dollar does not seem unreasonable. From this point forward in the text, we will feel free to round large values to the nearest dollar in our final answers. This of course raises the question of how large is "large"? This is a judgment call, but from here on in we will use the following rule of thumb: *when a dollar amount in a final answer is larger than $10,000, we will feel free to round to the nearest dollar.* Of course, it is also still correct to keep dollar and cent accuracy if you prefer.

Notice that the interest rate that the district pays to the investors is completely independent from the rate that it receives from the bank on its deposits. While the sinking fund was required as a term of the loan, financially it is a separate account and there is no necessary connection between the rates.

Situations such as these are actually the source of the term *sinking fund*. The purpose of the deposits is to take care of that looming future value, and so, with each deposit, more and more of the future obligation is taken care of, and the amount of the obligation that is not covered declines, or "sinks." Hence the name.

Sinking Funds and Retirement Planning

One particularly useful application of sinking funds is in retirement planning. It is hard to avoid being bombarded with news reports and commentaries about the pressing need for all of us to "save more for retirement." Many companies offer 401(k) or similar retirement savings plans and encourage workers to use them to save, and the perpetual debate over the future of the U.S. Social Security system includes plenty of discussion about ways to require or provide incentives for people to save for their long-term financial needs.

While the benefits of saving money over time are obvious, it is less obvious just how much saving is enough. On general principle, the answer should probably be "as much as possible," but that is hopelessly vague and not especially helpful. Treating a retirement goal as a sinking fund can help to provide a clearer answer.

Suppose that Joe is now 25 years old, and hopes to be able to retire 45 years from now, at age 70. To ensure that he can do so comfortably, he has decided to start making deposits into an investment account which he assumes can earn an average of 9%. The deposits will be automatically deducted from his paycheck when he receives it on the first and fifteenth of each month. To get some sense of how large his deposits should be, he has set a goal of having $1,000,000 in the account. These deposits will form an annuity, and since he is letting the goal future value determine their size, this is a prime example of a sinking fund. We can thus work out the size of each payment:

Example 4.3.4 ***Using the scenario described above, determine how much Joe should have deducted from each paycheck to reach his goal.***

First, to determine n and i. Joe is paid twice a month (semimonthly), which would be 24 times per year (12 months times 2 paychecks/month). Over the course of 45 years, that is $n = 45(24) = 1{,}080$ deposits. For i, we assume as usual that interest compounds at the same frequency as the payments, or semimonthly, so $i = 0.09/24 = 0.00375$.

$$FV = PMTs_{\overline{n}|i}$$
$$\$1{,}000{,}000 = PMTs_{\overline{1080}|.00375}$$
$$\$1{,}000{,}000 = PMT(14923.819794054)$$
$$PMT = \$67.01$$

To reach his goal on the basis of these assumptions, Joe needs to be deducting $67.01 from each paycheck.

This answer may seem surprising; $67.01 out of every paycheck is not exactly peanuts, but it seems awfully small compared to future value that sounds like a lottery jackpot. We can get some insight into how this happens by finding the total interest Joe will earn.

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Example 4.3.5 ***For the scenario of the previous example, determine how much total interest Joe would be earning.***

Since he is making 1,080 deposits of $67.01 each, his total deposits add up to (1,080)($67.01) = $72,370.80. So the total interest earned is:

$$\text{Total interest} = \$1{,}000{,}000 - \$72{,}370.80 = \$927{,}629.20$$

So most of Joe's million comes from the result of compound growth over time. He is planning on earning a fairly high rate, and while his account balance probably won't be much in the early years, what there is will be earning compound interest for a very long period of time, and, as we have repeatedly seen, compound interest over a long period of time is powerful stuff indeed.

Whether or not Joe's assumptions (that he should have $1,000,000 and that he can earn 9%) are reasonable is a matter that will be addressed in other parts of this book, in Chapters 6 and 7. However, whether they are too high, too low, or a mix, the inescapable mathematical conclusion here is that compound interest working over long periods of time can accumulate a large sum from surprisingly small deposits. The following example will illustrate the importance of time.

Example 4.3.6 ***How much would Joe's semimonthly deposits need to be to accumulate $1,000,000 at age 70, again assuming a 9% growth rate, assuming that he starts at (a) age 65, (b) age 50, (c) age 35, (d) age 25 (already done), (e) age 18, and (f) at age 2 (obviously assuming someone started the deposits for his benefit)? For each starting age, determine how much of the $1,000,000 comes from his deposits, and how much comes from interest on the deposits.***

Each of the calculations will be essentially the same as the one we did above, so we will not show them here. For practice, though, you should verify the calculations for at least one or two of the rows. The results are displayed in the table below:

Starting Age	Number of Deposits	Annuity Factor	Each Deposit	Total Deposits	Total Interest
2	1,632	119,650.08513797	**$8.36**	$13,643.52	**$986,356.48**
18	1,248	28,221.56380297	**$35.43**	$44,216.64	**$955,783.36**
25	1,080	14,923.81979405	**$67.01**	$72,370.80	**$927,629.20**
35	840	5,919.72900240	**$168.93**	$141,901.20	**$858,098.80**
50	480	1,341.15067989	**$745.63**	$357,902.40	**$642,097.60**
65	120	151.19807368	**$6,613.84**	$793,660.80	**$206,339.20**

It should come as no surprise that the sooner the deposits start, the smaller they can be, but it may still be surprising just how much of an impact this has.

EXERCISES 4.3

A. Sinking Funds

1. A community college is planning to add a new academic building 5 years from now. The college's administration wants to set aside some money from its annual budget in each of the next 5 years in order to accumulate a fund of $1,200,000 to use toward the project. If the college can earn 4.55% interest, how much should it set aside each year to meet this goal?

2. Brad has nothing in his savings account right now, but realizes it would be a very good idea to build up a savings balance. His account pays 4% interest. How much should he deposit each month if he wants to have $10,000 in this account in 2 years.

3. Find the quarterly payment needed to accumulate $30,000 in 8 years, assuming an interest rate of 5.17%.

4. If Trina wants to have $800,000 in her retirement account in 40 years, how much should she deposit into the account each week, assuming she can earn 9.51%?

5. Carlos wants to start law school in 3 years, and figures that he needs to save up $12,500 between now and then to help pay for his tuition and living expenses. How much should he set aside at the beginning of each month to reach this goal, assuming his savings earn 4.05%?

B. Sinking Funds with Loans

6. A company borrowed $40,000 for 3 years at 6% compounded daily. It will not make any payments on this loan prior to maturity. Find the quarterly sinking fund payment needed to accumulate this maturity value, assuming a 5% rate.

7. Jason's uncle loaned him $37,500 to help him start a landscaping business. They agreed that Jason would pay off the entire loan plus 3½% annually compounded interest in 3 years, and that in the meantime he would make monthly deposits into a bank account to make sure that he has the amount needed when the loan comes due. The bank account will pay 3%. (a) Find the total amount Jason will need to repay the loan. (b) How much should each of his monthly deposits be?

C. Sinking Funds and Retirement Planning

8. Astrid figures that in the next 20 years she needs to build up a balance of $500,000 in a retirement account she has just opened. Deposits into this account will be made by deductions from her biweekly paychecks. If her account can earn 8.4%, how much should each deposit be?

9. Redit hopes to build up an account value of $1,600,000 over the next 40 years in an account that he thinks will earn 11.27%. Under these assumptions, how much should he invest each month?

D. Grab Bag

10. The business manager of the Rock City Central School District commented at a school board meeting that in 7 years the district will have a loan coming due that will require $996,423 to pay. He recommended that the district should prepare for this payment by setting funds aside each year in an account paying 3.09% to build up the needed balance. According to his plan, how much should each payment be?

11. I just borrowed $30,000, which I will have to pay off in 5 years, together with interest. Interest accumulates at an effective rate of 10.33%. How much should I deposit to a sinking fund each month to be able to pay off this loan, assuming that my account pays 3.68% compounded monthly?

12. Paul has nothing in his savings account, but he wants to have $8,000 in it 1 year from now. If the account pays 2½%, how much should he deposit each week?

13. How much would you need to deposit each week into an account earning 8.5% in order to have $1,000,000 in 40 years?

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14. Find the monthly sinking fund payment needed to accumulate \$5,000 in 3 years, assuming a 6.1% interest rate.

15. Kinzua County borrowed \$4,250,000 at an effective rate of 4.25% for 5 years. In order to make sure it has the money needed to repay the loan when it comes due, the county is making deposits into a sinking fund at the beginning of each quarter. The sinking fund pays them 3.21%. How much should each payment be?

E. Additional Exercises

16. Suppose that you want to have \$1,000,000 in an investment account when you turn 70. Based on your current age, how much would you need to deposit each week, starting on your next birthday, assuming that you earn 6% on your money. What if you earn 9%? 12%?

17. The town of Dettsville borrowed \$20,000,000 for 7 years at a 5⅜% simple interest rate. The town will not make any payments to its creditors until maturity, but it is setting up a sinking fund for this debt. Find the quarterly payment it needs to make, assuming that it earns 3½% in the sinking fund account.

4.4 Present Values of Annuities

So far, we have considered annuities whose payments and interest build up toward a future value. This covers plenty of situations, but there are many others that it does not fit so well. In Section 4.1 we saw that there are also many common examples of annuities where it is the present value, not the future value, that interests us. In this section, we will develop the mathematical tools to deal with annuity present values.

It seems reasonable to expect that we should be able to approach present values in much the same way that we did future values. This expectation is correct. Just as we had annuity factors for future values, we will have annuity factors for present values, though since present and future values are different we should expect that present and future value annuity factors will come out to be the same numbers.

There is no need to maintain any suspense about this.

Definition 4.4.1

For a given interest rate, payment frequency, and number of payments, the ***present value annuity factor*** *is the present value of an annuity at this rate, payment frequency, and number of payments if each payment were \$1. We denote this factor with the symbol* $a_{\overline{n}|i}$*, where n is the number of payments and i is the interest rate per payment period. (For convenience, this symbol can be pronounced "annie".)*

We will use these present value annuity factors in much the same way as we did future value annuity factors.

FORMULA 4.4.1
The Present Value of an Ordinary Annuity

$$PV = PMT a_{\overline{n}|i}$$

where
PV represents the PRESENT VALUE of the annuity,
PMT represents the amount of each PAYMENT,
and
$a_{\overline{n}|i}$ is the PRESENT VALUE ANNUITY FACTOR as defined in Definition 4.4.1

The formula for the present value of an annuity due should come as no surprise either.

FORMULA 4.4.2
The Present Value of an Annuity Due

$$PV = PMTa_{\overline{n|}i}(1+i)$$

Finding Annuity Factors Efficiently—Tables

Just as tables of annuity factors exist for future values, they also exist for present values. An example of such a table is given below:

SAMPLE TABLE OF ANNUITY FACTORS

Number of Payments (n)	RATE PER PERIOD (i)					
	0.25%	*0.50%*	*0.75%*	*1.00%*	*1.25%*	*1.50%*
24	23.2659796	22.5628662	21.8891461	21.2433873	20.6242345	20.0304054
36	34.3864651	32.8710162	31.4468053	30.107505	28.8472674	27.6606843
48	45.1786946	42.5803178	40.1847819	37.9739595	35.9314809	34.0425536
60	55.6523577	51.7255608	48.1733735	44.9550384	42.0345918	39.3802689

Just as with the future value annuity factor table, this table allows us to look up the value of the appropriate annuity factor for any interest rates and values of n that the table includes. For example, if we need the annuity factor for a 3-year monthly annuity at 6%, we would look in the n = 36 row and 0.50% column and see that the value is 32.8710162. Using our annuity factor notation, we would write $a_{\overline{36|}.005} = 32.8710162$.

The following example will illustrate the use of a table factor to find the present value of an annuity.

Example 4.4.1 ***Find the present value of $275 per month for 3 years at 6%.***

Following our present value formula we have:

$$PV = PMTa_{\overline{n|}i}$$
$$PV = (\$275)a_{\overline{36|}.005}$$

The annuity factor we are looking for can be found in the table; in fact, it is the factor we used as an example in our discussion above:

$$PV = (\$275)(32.8710162)$$
$$PV = \$9039.53$$

The use of tables presents the same issues as it did with future values. A table needs to be quite large to cover every reasonably possible value for n and i. However, there are a number of situations where tables may be a very useful approach for present value calculations. As we saw when first discussing annuities, one common application of annuities is payments on loans. The table given above might be used, for example, by the finance office of a car dealership. Payments on car loans are almost always monthly, and cars usually are financed only over 2, 3, 4, 5, or at most 6 years. Similarly, though almost any interest rate is possible for a car loan, it would not be unusual to have only certain selected interest rates in use at any given time. If there are only a limited number of possible values for n and i, a table of annuity factors can be kept to a reasonable size.

Example 4.4.2 ***A customer at a car dealership says that he can afford a $275 monthly car payment. He is looking for a used car and would be taking out a 3-year loan; on the basis of his credit rating, he would qualify for a 6% rate. How much can he afford to pay for the car?***

The monthly payments are an annuity, and the money to buy the car would be the present value of that annuity. This question, then, is the same as the one we worked out in Example 4.4.1. He can afford to pay $9,039.53.

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For convenience's sake, when a table is used in situations such as these, the columns will often be labeled with the nominal annual interest rates instead of the rates per period. This saves having to divide the rate by 12 to get the rate per month. So the table we are using might instead look like this:

SAMPLE TABLE OF ANNUITY FACTORS

Number of Payments (n)	**INTEREST RATE**					
	3%	**6%**	**9%**	**12%**	**15%**	**18%**
24	23.2659796	22.5628662	21.8891461	21.2433873	20.6242345	20.0304054
36	34.3864651	32.8710162	31.4468053	30.1075050	28.8472674	27.6606843
48	45.1786946	42.5803178	40.1847819	37.9739595	35.9314809	34.0425536
60	55.6523577	51.7255608	48.1733735	44.9550384	42.0345918	39.3802689

Be careful to make sure that you know whether it is the rate per period or the annual interest rate that is shown in the column headers, so that you get the correct factor.

Finding Annuity Factors Efficiently—Calculators and Computers

Just as with future value factors, some calculators and computer software have the built-in ability to calculate present value factors. The same comments made for future values apply for present values. If you are using this approach to get your future value factors, you will probably want to also use the approach for present value factors. You should consult your instructor and/or owner's manual for details of how to obtain these factors on the specific hardware you are using.

Finding a Formula for the Present Value Factors

It is not necessary to have a table or special calculator to find present value factors any more than it was necessary for future value factors. We can develop a formula for $a_{\overline{n}|i}$ just as we did for $s_{\overline{n}|i}$. To get things started, consider the following example, similar to some of the problems we considered in Section 4.3:

Example 4.4.3 ***Suppose that Jon borrows $8,000 from his uncle to buy a car. Rather than deal with the hassle of monthly payments to his uncle, they agree that Jon will repay the loan in full all at once at the end of 3 years. They agree that the loan will carry an interest rate of 4.2%, compounded monthly. To make sure that he has the money to pay off the loan when it comes due, Jon decides to set up a sinking fund at his company's credit union, into which he will make monthly payments. Coincidentally, the sinking fund also carries an interest rate of 4.2% compounded monthly. How much should Jon deposit each month?***

Remember that before we can determine the sinking fund payments, we must first determine how much Jon will actually need to pay his uncle:

$$FV = PV(1 + i)^n$$
$$FV = (\$8,000)(1 + .0035)^{36}$$
$$FV = \$9,072.26$$

Knowing that, we can determine the sinking fund payment:

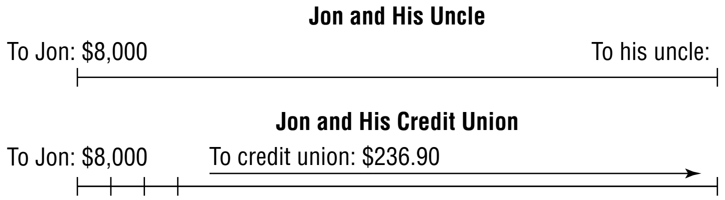
$$FV = PMTs_{\overline{n}|i}$$
$$\$9,072.26 = PMTs_{\overline{36}|.0035}$$
$$\$9,072.26 = PMT(38.29504816)$$
$$PMT = \$236.90$$

So Jon's monthly deposits should be $236.90.

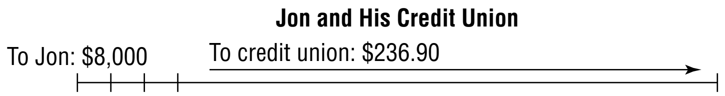
Now, let's make a few changes to the scenario described above. Suppose that instead of borrowing the $8,000 from his uncle, Jon borrowed the money from the same credit union where he set up his sinking fund. Would this change how we worked the problem? Of course not! It didn't matter in the slightest whether Jon was borrowing the money from his uncle, his credit union, or anyone else for that matter. Jon's monthly sinking fund payment is exactly the same regardless of who loaned him the $8,000.

But from a practical point of view, this does sound kind of strange. After all, the idea was that Jon wouldn't have to make any payments to his uncle until the very end. But if the lender is the same credit union to which he is making his deposits anyway, it would seem awfully awkward to keep the loan and the sinking fund separate given that both are deals between the same players: Jon and his credit union. There is no reason why the two deals couldn't be kept separate, but it would be far simpler to just have Jon's monthly payments go directly toward paying off his loan.

With that, we've made an important leap. The key to dealing with annuity *present values* is recognizing that



is the same as:



This realization enables us to develop formulas for annuity present values.

Formulas for the Present Value of an Annuity

Suppose that we know the payments, term, and interest rate for an annuity, and want to determine its present value. We can find the *future value* that this annuity could accumulate to by using:

$$FV = PMTs_{\overline{n}|i}$$

As we've just seen, we can relate this future value to the present value by means of the compound interest formula

$$FV = PV(1 + i)^n$$

In both cases, the FV we are talking about is the same amount of money. So then

$$PV(1 + i)^n = PMTs_{\overline{n}|i}$$

since both of these are equal to the FV.

To get the present value by itself, we can divide both sides of this equation by $(1 + i)^n$ to get:

$$PV = \frac{PMTs_{\overline{n}|i}}{(1 + i)^n}$$

Which we can adjust slightly to get:

$$PV = PMT\,\frac{s_{\overline{n}|i}}{(1 + i)^n}$$

Notice that this formula is essentially the ordinary annuity present value formula from the start of this section, except that $s_{\overline{n}|i}/(1 + i)^n$ is sitting where $a_{\overline{n}|i}$ sat. Since both formulas give us the present value, it must be that we have found our formula for $a_{\overline{n}|i}$!

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FORMULA 4.4.3
The Present Value Annuity Factor

$$a_{\overline{n}|i} = \frac{s_{\overline{n}|i}}{(1+i)^n}$$

Example 4.4.4 ***Use Formula 4.4.3 to find $a_{\overline{n}|i}$ when n = 36 and I = 0.0035.***

This formula depends on $s_{\overline{n}|i}$, so we will start by finding that. We calculate this just the same as we have been doing all along:

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} = \frac{(1+0.0035)^{36} - 1}{0.0035} = 38.29504816$$

Now, we add on the extra step of dividing. Following Formula 4.4.3, we get

$$a_{\overline{n}|i} = \frac{s_{\overline{n}|i}}{(1+i)^n} = \frac{38.29504816}{(1+0.0035)^{36}} = 33.76891092$$

For entry in the calculator, your best bet is to calculate the future value factor however you have been doing it, and then once you have that result on the screen, follow step 2 below:

Operation	*Result*
Steps to evaluate $s_{\overline{n}|i}$	39.29504816
/(1+.0035)^36=	33.76891092

Notice that the values of i and n here are the same as they were in the example we used in Example 4.4.3, Jon's car loan. We can confirm that our work here was correct by using this to find Jon's car payment, using the present value factor. We will hopefully end up with the same monthly payment!

Example 4.4.5 ***Use the annuity factor calculated in Example 4.4.4 to find the monthly payment on an $8,000 car loan at 4.2% for 3 years.***

$$PV = PMTa_{\overline{n}|i}$$
$$\$8{,}000 = PMT(33.7681092)$$

As we have done in many other similar situations, we divide both sides through to arrive at the payment:

$$\frac{\$8{,}000}{33.76891092} = \frac{PMT(33.76891092)}{33.76891092}$$
$$PMT = \$236.90$$

The fact that this is the same payment that we arrived at before should give us some confidence that we are doing things correctly.

An Alternative Formula for $a_{\overline{n}|i}$ (Optional)

Formula 4.4.3 is not the traditional formula for $a_{\overline{n}|i}$. It does, however, offer some advantages over the traditional formula, especially in that it builds on the work that we have already done to find future value factors. To calculate the present value factor of an annuity, we calculate the future value annuity factor (which at this point should be old hat), and then just add the one additional step of dividing it by $(1 + i)^n$.

While the author's personal preference is to take that approach, the more traditional formula can certainly be used just as well. We will briefly derive it here, and then demonstrate its use.[2] If we begin with the formula:

$$a_{\overline{n}|i} = \frac{s_{\overline{n}|i}}{(1+i)^n}$$

[2] The reader can skip the derivation and jump to the final formula without losing anything other than the pleasure of the mathematics.

and then replace $s_{\overline{n}|i}$ with its formula, we get:

$$a_{\overline{n}|i} = \frac{\frac{(1+i)^n - 1}{i}}{(1+i)^n}$$

We can simplify this formula by multiplying both top and bottom by $1/(1 + i)^n$:

$$a_{\overline{n}|i} = \frac{\frac{(1+i)^n - 1}{i}\left(\frac{1}{(1+i)^n}\right)}{(1+i)^n\left(\frac{1}{(1+i)^n}\right)}$$

which simplifies to:

$$a_{\overline{n}|i} = \frac{(1+i)^n - 1}{i(1+i)^n}$$

$$a_{\overline{n}|i} = \frac{1 - \frac{1}{(1+i)^n}}{i}$$

$$a_{\overline{n}|i} = \frac{1 - (1+i)^{-n}}{i}$$

Stating this formally, we write:

FORMULA 4.4.4
The Present Value Annuity Factor (Traditional Form)

$$a_{\overline{n}|i} = \frac{1 - (1+i)^{-n}}{i}$$

Admittedly, this does offer some advantages over our original formula, particularly in that is does not require calculating $s_{\overline{n}|i}$ first. There is also something to be said for following convention and tradition. On the other hand, at this point plenty of practice should have us all reasonably comfortable calculating $s_{\overline{n}|i}$, and so finding $s_{\overline{n}|i}$ on the way to $a_{\overline{n}|i}$ doesn't impose too heavy a burden. The conventional formula is not appreciably simpler, and it involves a negative exponent, something with which you may or may not be comfortable, depending on your prior algebra background and something that can easily lead to calculator errors if you are not careful.

Rather than insist on one formula or the other, we will present both. The difference between the two is a matter of taste and style, not mathematical necessity. Both formulas will give correct results in all situations. Your instructor may allow you the choice of using either, or may require you to use one or the other. If you are given the choice, you should try a few examples calculating the factor each way so that you can find which formula works best for you.

The following example will illustrate calculating $a_{\overline{n}|i}$ in each way.

Example 4.4.6 ***Find the present value annuity factor for a 5-year annuity with quarterly payments and a 9% interest rate.***

Using Formula 4.4.3:

We first find the future value annuity factor just as we have been:

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} = \frac{\left(1 + \frac{0.09}{4}\right)^{20} - 1}{\frac{0.09}{4}} = 24.91152003$$

Then plugging this in to Formula 4.4.3, we get:

$$a_{\overline{n}|i} = \frac{s_{\overline{n}|i}}{(1+i)^n} = \frac{24.91152003}{\left(1 + \frac{0.09}{4}\right)^{20}} = 15.96371237$$

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For entry in the calculator, once again your best bet is to calculate the future value factor however you have been, and then once you have that result on the screen, follow step 2 below:

Operation	*Result*
Steps to evaluate $s_{\overline{n}\rceil i}$	24.91152003
/(1 + .09/4)^20=	15.96371237

Using Formula 4.4.4:

Plugging in to the formula we get:

$$a_{\overline{n}\rceil i} = \frac{1-(1+i)^{-n}}{i} = \frac{1-(1+0.0225)^{-20}}{0.0225} = 15.96371237$$

On the calculator, this can be evaluated either step by step or by entering everything at once.

Step by step (the first steps):

Operation	*Result*
.09/4=	0.0225
+1=	1.0225
^−20=	0.640816472

Unfortunately, we want to subtract this value from 1, and with subtraction the order matters. This forces us hold this result (either by writing it down or by storing it in the calculator's memory) and then use it in the next step:

Operation	*Result*
1−0.640816472=	0.359183528
/(.09/4)=	15.96371237

Alternatively, entering it all at once we get:

Operation	*Result*
(1−(1+.09/4)^−20)/(.09/4)=	15.96371237

However you choose to enter these, be careful about the negative exponent. On many popular calculator models, there are two "−" keys, one for subtraction and one for negative numbers. You want to make sure that you are using the "−" key for negative numbers when you enter the exponent. Using the wrong key will most likely lead to an error.

Annuity Present Values and Loans

One of the most common applications of present values is using them to find loan payments. Most loans with a fixed payment schedule can be mathematically analyzed by using annuity present values (though there are a few exceptions, discussed in Chapter 10.) The following examples will illustrate this:

Example 4.4.7 ***Cienna is thinking about buying a condominium. She figures that she can afford monthly loan payments of $650, and that she would be taking out a 30-year loan with an interest rate of 8.4%. On the basis of these assumptions, what is the most she can afford to borrow? If she has $7,500 to use as a down payment, what is the most she can afford to pay for the condo?***

To begin with, we can see that $650 per month is an annuity, and the money that she receives to buy the condo is its present value.

Since this is a 30-year loan with monthly payments, $n = (30)(12) = 360$ and $i = 0.084/12 = 0.007$. We know that we will need the present value annuity factor for this, so we first will find that. This factor may be found from a table, from calculator programs, or by using one of our formulas. We will show the calculation for each of the two formulas; of course, when doing these problems yourself, you do not need to calculate the factor more than once.

Using Formula 4.4.3:

We first find the future value annuity factor just as we have been doing:

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} = \frac{(1+0.007)^{360} - 1}{0.007} = 1{,}617.137554$$

Then, plugging this into Formula 4.4.3, we get:

$$a_{\overline{n}|i} = \frac{s_{\overline{n}|i}}{(1+i)^n} = \frac{1{,}617.137554}{(1+0.007)^{360}} = 131.2615606$$

Using Formula 4.4.4:

Plugging values into the formula we get:

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i} = \frac{1-(1+0.007)^{-360}}{0.007} = 131.2615606$$

Whichever way we find the annuity factor, we can now use it to complete the problem.

$$PV = PMT\ a_{\overline{n}|i}$$
$$PV = \$650\ a_{\overline{360}|.007}$$
$$PV = (\$650)(131.2615606)$$
$$PV = \$85{,}320.01$$

Assuming that on an amount this large we can ignore the pennies, we conclude the most she can afford to borrow is \$85,320. In addition to what she borrows, she has her \$7,500 to put toward the purchase price, and so the most she can afford to pay is \$85,320 + \$7,500 = \$92,820.

In the remainder of this section, we will not show the work to calculate the annuity factors, but you should calculate them yourself and make sure that your factors agree with those used in these examples. The exercises at the end of this section also offer many opportunities to practice and get comfortable with calculating the present value annuity factor.

Finding Total Interest For a Loan

Finding the total interest paid on a loan is similar to, but not exactly the same as, finding the total interest in an annuity's future value.

Example 4.4.8 ***Pat and Tracy are buying a house, and will need to take out a \$158,000 mortgage loan. They plan to take out a standard 30-year mortgage loan, on which their interest rate will be 7.2%. If they make all their payments as scheduled, how much total interest will they pay over the course of the 30 years?***

First, we need to determine their monthly payments. (While the problem did not explicitly state that payments would be monthly, that is usually the case and can be assumed unless otherwise specified.)

We know that we will need the present value annuity factor. Since the payments are monthly for 30 years, $n = 30(12) = 360$ and $i = 0.072/12 = 0.06$. Using a table, calculator program, or one of the annuity factor formulas, we can calculate that $a_{\overline{360}|.006} = 147.3213568$. Putting this to use, we get:

$$PV = PMT\ a_{\overline{n}|i}$$
$$\$158{,}000 = PMT\ a_{\overline{360}|.006}$$
$$\$158{,}000 = PMT(147.321356802)$$
$$PMT = \$1{,}072.49$$

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This payment includes both principal and interest, but doesn't reveal how much goes to each category. To determine the total interest they will pay, we can use an indirect approach similar to the one we used with future values. This loan requires 360 payments of \$1,072.49, and so the total they will pay over the entire life of the loan is (360)(\$1,072.49) = \$386,096.40. Since they borrowed \$158,000, we know how much of that total must go toward principal, and so the rest must be interest. Thus:

$$\text{Total interest} = \$386{,}096.40 - \$158{,}000 = \$228{,}096.40$$

Over the course of the 30-year loan, Pat and Tracy will be paying an awful lot of interest. In fact, more of their money will go to pay interest than to pay for the house! It is good to be aware of this, even if the knowledge is depressing. The good news, though, is that knowing how to work with annuities allows us to consider alternatives and choose the most favorable options. Since debt is an unfortunate but unavoidable fact of both business and life, being able to work through the mathematics that underlies it holds obvious advantages—you generally do better at games where you know how to play.

Example 4.4.9 ***Suppose that Pat and Tracy from Example 4.4.8 decided instead to try to save some of the interest by paying off their loan more quickly. If they were to go with a 15-year loan rather than a 30-year loan, how much higher would their payments need to be (assuming the same interest rate)? How much interest would they save?***

Common sense says that to cut the term in half, they would need to double their payments. Sometimes, though, common sense is wrong.

$$\begin{aligned} PV &= PMT\ a_{\overline{n}|i} \\ \$158{,}000 &= PMT\ a_{\overline{180}|.006} \\ \$158{,}000 &= PMT(109.884466016) \\ PMT &= \$1{,}437.87 \end{aligned}$$

So they would need to increase their payments by \$1,437.87 − \$1,072.49 = \$365.38. While that is quite a bit more of a monthly payment, it is nowhere near double the original. (We will explore the reasons for this in Section 4.5.)

Under this scenario, their total payments would be (180)(1,437.87) = \$258,816.60, and so their total interest would be \$258,816.60 − \$158,000 = \$100,816.60. Compared to the 30-year schedule, this is an interest savings of \$228,096.40 − \$100,816.60 = \$127,279.80.

Interest is the incredibly powerful force that allowed us to accumulate astonishingly large future values from relatively small payments over long periods of time. But when we are the borrowers, instead of earning that interest, we pay it. By paying extra on the loan, we reduce the principal more quickly, and thus reduce the interest that we must pay dramatically. We will be able to see this all worked out in greater detail in Section 4.5.

Other Applications of Present Value

Loans are by no means the only uses for annuity present values, however. The following examples illustrate some other uses, which have nothing to do with loan payments.

Example 4.4.10 ***Charlie has accumulated \$557,893 in his 401(k) retirement savings account. Now that he has retired, he is planning to start using this fund to provide him with some income. He expects that the account can continue to earn 4½% interest, and plans to make monthly withdrawals from the account over the next 20 years. Under these assumptions, how much can he afford to withdraw each month?***

It is tempting to just take the \$557,893 and divide it by 240, but that would ignore the interest that the account will be earning over those 20 years. In essence, Charlie is talking about using his \$556,893 as the present value of a 20-year annuity. And so:

$$PV = PMT\, a_{\overline{n}|i}$$
$$\$557{,}893 = PMT\, a_{\overline{240}|.00375}$$
$$\$557{,}893 = PMT(158.065436810)$$
$$PMT = \$3{,}529.51$$

So Charlie's account will support a monthly income of $3,529.51 for 20 years.

So far our examples have all been ordinary annuities. There are, however, some cases where we need the present value of an annuity due, as this example shows:

Example 4.4.11 **The New York State Lotto Jackpot is advertised to be $52 million. However, this jackpot is not paid out all at once as a single lump sum. Rather, it is paid out in equal annual installments for 26 years (beginning immediately). The advertised $52 million is the total of all the payments. When you buy a ticket, you can choose to have the jackpot paid to you as a lump sum should you win, but in that case you receive the present value of the payments. If the interest rate used is 6%, what is the value of the jackpot as a lump sum?**

The payments do represent an annuity, and since they begin immediately they represent an annuity due. A lump sum received up front instead of the payments clearly would be their present value. The $52,000,000 is not the present value, but instead is the total of all the payments. Since the jackpot annuity is 26 equal payments totaling $52 million, each payment would be $52,000,000/26 = $2,000,000.

$$PV = PMT\, a_{\overline{n}|i}\,(1 + i)$$
$$PV = \$2{,}000{,}000\, a_{\overline{26}|.06}\,(1.06)$$
$$PV = (\$2{,}000{,}000)(13.003166187)(1.06)$$
$$PV = \$27{,}566{,}712.32$$

So winners who choose to take their winnings entirely up front will receive just a bit over half of the advertised jackpot. The practice of advertising the total payments of an annuity as the prize is very commonly used not only for lotteries but for other sweepstakes as well.

EXERCISES 4.4

A. Using Future Values to Find Present Values

1. To buy their new townhouse, Marc and Jun borrowed $116,509 from International National Mortgage Company for 30 years. The interest rate was 7.2% compounded monthly.
   a. Suppose that instead of making monthly payments on the loan, the entire balance together with all the interest it accumulates will be paid all at once at the end of the 30-year term. How much would that amount be?
   b. Suppose that Marc and Jun made monthly payments into a sinking fund earning 7.2% compounded monthly to accumulate the total from part (a). How much would the sinking fund payment be?
   c. Parts (a) and (b) are completely hypothetical; in reality, the mortgage company will require Marc and Jun to pay their loan off with monthly payments. How much will each of their monthly mortgage payments be?

B. Using Tables for Present Value Calculations

Questions 2 to 5 relate to the scenario below and the table given on the next page.

In the United States, the most common terms for a mortgage loan are 15 years or 30 years. In working with his clients, Jeff, a real estate agent, finds it convenient to have a table of annuity factors for monthly loan payments for these two terms at a selection of interest rates typical in the current market. He carries a small card with the table shown below in his briefcase:

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	INTEREST RATE				
Years	***7.0%***	***7.5%***	***8.0%***	***8.5%***	***9.0%***
15	111.255958	107.873427	104.640592	101.549693	98.5934088
30	150.307568	143.017627	136.283494	130.053643	124.281866

2. How much could someone afford to borrow on a 30-year loan with an 8.5% interest rate, assuming a $950 monthly payment?

3. A client says that he can afford a monthly payment of $1,200. On the basis of his credit, Jeff thinks he would qualify for a 7.5% rate. How much could this client borrow with a 15-year loan? A 30-year loan?

4. What would the monthly payment be for a $120,000 loan for 30 years at 9.0%?

5. Jeff took a couple out to look at a house on the market for $187,500. This couple could afford a $7,500 down payment, so they would need to borrow $180,000 to buy the house. Assuming that they would qualify for the loan at an 8% rate, what would their monthly payment be for a 30-year loan?

For another example of how tables like these are often used, see the Additional Exercises 37 to 39.

C. Finding Present Value Annuity Factors

Questions 6 and 7 deal with finding present value annuity factors, using Formula 4.4.3.

6. If $n = 120$ and $i = 0.00475$,
   a. Find $s_{\overline{n}|i}$.
   b. Find $a_{\overline{n}|i}$.

7. If $n = 96$ and $i = 0.076/12$,
   a. Find $s_{\overline{n}|i}$.
   b. Find $a_{\overline{n}|i}$.

Instructions for Exercises 8 through 12: Multiple different ways for calculating present value annuity factors have been presented in this chapter. If your instructor has told you that one of these methods is the one you should use, then find the factor in that way. If you are allowed your choice of methods, try working these problems with each method to see which one you like best.

8. Find $a_{\overline{n}|i}$ if $n = 60$ and $i = 0.005$.

9. Find $a_{\overline{n}|i}$ if $n = 15$ and $i = 0.02$.

10. Find $a_{\overline{n}|i}$ if $n = 12$ and $i = 0.035$.

11. Find $a_{\overline{120}|0.027}$.

12. Find $a_{\overline{360}|0.00725}$.

For each of Exercises 13 to 17:

- *a)* *Find n.*
- *b)* *Find i (expressed as either a fraction or a decimal).*
- *c)* *Find the value of* $a_{\overline{n}|i}$.
- *d)* *Find the present value of the annuity.*

13. The monthly payment is $150, the interest rate is 9%, and the term is 10 years.

14. The monthly payment is $184.39, the interest rate is 6¼%, and the term is 15 years.

15. The quarterly payment is $1,750.42, the interest rate is 5%, and the term is 5 years.

16. The weekly payment is $37.45, the interest rate is 11.74%, and the term is 25 years.

17. The monthly payment is $735.35, the interest rate is 12.59%, and the term is 7 years.

D. Present Values and Loans

18. I've figured out that the most I can afford to pay each month for a car loan is $250.00. My credit union will make me a 3-year used car loan with an interest rate of 8.4%. What is the most I can afford to borrow?

19. A replacement window company ran a newspaper ad stating that an average homeowner could replace all windows and pay nothing down and just $165 a month. The fine print said that this payment is based on a 10-year loan and a 7.8% interest rate. What total cost is the window company assuming for the windows?

20. Emilio's accountant told him that if he continues to pay $50 a month on his credit card, it will take him 42 years to pay off his current balance (assuming the interest rate doesn't change and assuming he doesn't charge anything else on that card). His credit card interest rate is 18.99%. What is his balance?

21. Geoff borrowed $23,990 for 7 years at 9.3% to buy a boat. How much is his monthly payment?

22. Warren bought a new house for $175,300. He made a $20,000 down payment, and financed the rest with a 30-year loan at an interest rate of 7.52%. What will his monthly mortgage payment be?

23. Find the monthly payment on a $17,500 loan if the term is 7 years and the interest rate is 11¼%.

E. Other Applications of Present Value

24. Gord is looking forward to retiring in a few years, and hopes that he will have enough in his retirement savings account to provide him with an income of $2,400 a month for 30 years. How much will he need in his account when he retires to do this, assuming that it will earn 4%?

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25. A company knows that it will need to pay $750,000 each year for the next 10 years to clean up toxic pollution at a former plant site. Management wants to set aside a lump sum now that will be enough to provide for those costs. If the money earns 5% interest how much would the company need to set aside today?

26. Jack won a sweepstakes jackpot, payable in the form of a 15-year annuity with equal annual payments beginning immediately and totaling $3,000,000. If he wishes, he can choose to instead receive the present value of this annuity, calculated by using a $7\frac{1}{2}$% interest rate. How much would he get under that option?

27. As the result of a personal injury lawsuit, Garret won a $1,800,000 judgment. However, he does not get the money all at once; the judgment will be paid to him in 20 equal annual installments totaling $1,800,000, starting right away. Garret doesn't want to wait for his money, though. A company offers to give him a lump sum immediately, in exchange for all of his payments from the judgment. If the interest rate is 8%, how much would this lump sum be?

F. Grab Bag

28. Find the present value of a 10-year ordinary annuity with $500 quarterly payments if the interest rate is 5%.

29. Miguel and Cindi borrowed $189,750 to buy their house, for which they took out a 30-year loan at 7.29%. (a) Find their monthly payment. (b) What would their payment have been if they had instead taken out a 15-year loan at the same interest rate?

30. If Emilio (from Exercise 20) does continue to only pay $50 a month, how much total interest will he pay before his credit card account is paid off?

31. Find the present value annuity factor for an 8-year annuity with semiannual payments and an interest rate of 4.49%.

32. A letter to the editor of the local newspaper complains that $775,000 "wasted" on a "useless" economic development study could have been better used. The writer claims that "if the money had been invested at just 4% interest, it could have provided enough money to keep an additional cop on the streets for the next 20 years!" Assuming a level cost per year over the next 20 years, what is the writer assuming it costs to pay a police officer for a year?

33. Jamie has just been accepted to pharmacy school. When she met with a college financial aid officer, he told her that, at the current 4.35% interest rate, she could expect that after graduating her student loan payments would be about $275 a month for 20 years. According to these figures, how much student loan debt will she have when she graduates?

34. Find the monthly payment on a 4-year, $11,500 loan at 11.97%.

35. Claudette owes $3,725.86 on her credit card. She has stopped using the card, and wants to get herself out from under this debt in 1 year. How much will she need to pay each month to do this, assuming the card carries a 15.99% interest rate?

G. Additional Exercises

36. Kathy's monthly payment on her 30-year mortgage is $725.14. Her interest rate is 6.59%.

    a. How much total interest would Kathy have saved if, instead of her 30-year loan, she had gone with a 15-year loan at the same interest rate?

b. Not only do shorter terms save you money by paying off the loan more quickly, they often (though not always) offer a lower interest rate. How much total interest would Kathy have saved if instead of her 30-year loan she had gone with a 15-year loan at a 5.92% interest rate?

37. In Exercises 2 to 5 we considered an abbreviated table of annuity factors used by a real estate agent to be able to quickly calculate mortgage payments. To find the amount to be borrowed based on a payment, you needed to multiply by these annuity factors; however, to find the payment based on a loan amount, you need to divide.

Sometimes an alternative to these tables is used, which, instead of giving the annuity factor, gives the monthly payment per $1,000 borrowed. This is sometimes preferred, since if we know this we do not need to divide.

Suppose that our real estate agent Jeff instead had this table:

MONTHLY PAYMENT PER $1,000 BORROWED

	INTEREST RATE				
Years	***7.00%***	***7.50%***	***8.00%***	***8.50%***	***9.00%***
15	$8.99	$9.27	$9.56	$9.85	$10.14
30	$6.65	$6.99	$7.34	$7.69	$8.05

Rework Exercises 4 to 5 using this table. Do your answers match?

38. Rework Exercises 2 and 3, using the table from Exercise 37.

39. Create a "payment per $1,000 borrowed" table like the one from Exercise 37, this time for a car salesman who wants to be able to find monthly payments at interest rates of 6%, 7%, 8%, and 9%, and terms of 3, 4, and 5 years.

40. As the result of litigation against the tobacco industry, Conesus County received a large judgment, payable as an annuity over 20 years. Rather than take its payments over time, the county decided to "sell" these payments to a finance company in exchange for a lump sum. Company A offered to give the county the present value calculated at a 9% interest rate. Company B offered a 10% rate. Which company offered the better deal? Why?

4.5 Amortization Tables

There were some surprises in Section 4.4. In Examples 4.4.8 and 4.4.9, we compared a 30-year mortgage to a 15-year mortgage, and found that, despite the commonsense expectation that halving the time would require doubling the payments, the 15 year loan's payments were not anywhere near twice the 30 year's. We attributed the difference to the shorter term's allowing for less interest, which generally makes sense, but it is a bit unsatisfying to have to leave it at that. Unfortunately, our annuity factor formulas allow us only to calculate payments from

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present value or vice versa. While we might have a vague and conceptual idea of what is going on as the payments are made—each payment chips away at the amount owed until the debt is entirely paid off—we don't yet have a way to look more specifically at the details. Being able to do so would provide a better means of understanding of how all of this works.

Amortization tables are a tool to do this. The word *amortization* actually comes from the French phrase *à morte*, meaning *toward death*, a fitting name for a look at how we kill off the debt![3] There are several different formats that can be used for amortization tables, but any amortization table will provide a payment-by-payment detail of how each dollar paid on a loan is allocated between paying the interest owed versus reducing the debt.

Setting Up an Amortization Table

The rows of an amortization table correspond to the scheduled payments for the loan. Normally, there are columns for the number of the payment, the amount of the payment, the amount of the payment that goes toward interest, the amount that goes toward principal, and the remaining balance after the payment is made. (Sometimes, the column for the payment amount is left out. If all payments are assumed to be the same, repeating it is a bit redundant. We will, however, include this column since later on we will be considering tables where the payments can vary.)

To illustrate, let's consider a loan of $1,000 at 8% interest for 4 years with payments due at the end of each year. The annual payment (calculated by using the techniques of Section 4.4) would then be $301.92. The amortization table for this loan would then look like this:

Payment Number	*Payment Amount*	*Interest Amount*	*Principal Amount*	*Remaining Balance*
1	$301.92	$80.00	$221.92	$778.08
2	$301.92	$62.25	$239.67	$538.40
3	$301.92	$43.07	$258.85	$279.56
4	$301.92	$22.36	$279.56	$0.00

We can see from the table how each payment is split up between interest and principal, and that the balance owed declines with each payment until it reaches zero after the last payment is made. How, though, is the split between interest and principal determined?

To answer this question, let's walk through the calculation of the entries in the first two rows. In the first year, the amount owed is the original $1,000, since no payments are made until the end of that year. Interest is assumed to compound annually, and so no compounding goes on during the year. The interest owed would be:

$$I = PRT$$
$$I = (\$1{,}000)(0.08)(1)$$
$$I = \$80.00$$

The first payment is $301.92, and since $80.00 of that must go to pay interest, the remaining $301.92 − $80.00 = $221.92 is left to go toward principal. Taking that off of the principal, the remaining balance would be $1,000 − $221.92 = $778.08, which is the amount in the Remaining Balance column. Notice that these amounts agree with those shown in the first row of the table.

Moving on to the second row, we repeat the same steps, though this time the principal is no longer $1,000 but instead the $778.08 left after the first payment. Thus:

$$\textbf{Interest: } I = PRT = (\$778.08)(0.08)(1) = \$62.25$$
$$\textbf{Principal: } \$301.92 - \$62.25 = \$239.67$$
$$\textbf{Remaining Balance: } \$778.08 - \$239.67 = \$538.40$$

[3] Or, alternatively and perhaps equally fitting, for looking at how debt can kill you.

You can (and should) verify the third and fourth rows of the table by repeating these steps for those payments.

Some Key Points about Amortization

Beyond being able to fill in the rows and columns of an amortization table, there are some key points worth noting about amortization.

- The amount of each payment is the same, but the split between interest and principal changes with each payment.
- As the balance is paid down, the portion of each payment dedicated to interest declines. A larger share of early payments will thus go toward interest than later payments.
- As interest's share of the payments decreases, principal's share of the payments increases. So not only does each payment reduce the amount owed, but also the pace of the reduction is accelerating. The first payment killed off $221.92, but the last payment wiped out $279.56.
- We might have expected that the interest would be compound interest, but notice that, since each payment must cover the entire amount of interest for the period, interest never gets a chance to compound upon itself.

Example 4.5.1 ***Pat and Tracy (from Examples 4.4.8 and 4.4.9) took out a 30-year loan for $158,000 at 7.2%. Their monthly payment was $1,072.49. Complete an amortization table for their first 12 monthly payments.***

For the first month, interest would be paid on the full $158,000, so the interest would be I = ($158,000)(0.072)(1/12) = $948.00. This leaves $1,072.49 − $948.00 = $124.49 to go toward reducing principal, and so after this payment they will owe $158,000 − $124.49 = $157,875.51.

The calculations for the next month are similar, except that instead of using $158,000 we instead use the slightly smaller $157,875.51. Because this is smaller, it requires less interest, increasing the amount that is left for principal. As expected, this trend continues in the ensuing months as well.

Payment Number	Payment Amount	Interest Amount	Principal Amount	Remaining Balance
1	$1,072.49	$948.00	$124.49	$157,875.51
2	$1,072.49	$947.25	$125.24	$157,750.27
3	$1,072.49	$946.50	$125.99	$157,624.28
4	$1,072.49	$945.75	$126.74	$157,497.54
5	$1,072.49	$944.99	$127.50	$157,370.04
6	$1,072.49	$944.22	$128.27	$157,241.77
7	$1,072.49	$943.45	$129.04	$157,112.73
8	$1,072.49	$942.68	$129.81	$156,982.92
9	$1,072.49	$941.90	$130.59	$156,852.33
10	$1,072.49	$941.11	$131.38	$156,720.95
11	$1,072.49	$940.33	$132.16	$156,588.79
12	$1,072.49	$939.53	$132.96	$156,455.83

We used Pat and Tracy's mortgage before, in the previous section, where we compared the payment on a 30-year mortgage to the payment on a 15-year loan. Looking at the amortization table now, we can start to see why the payment didn't need to be that much larger for the 15-year loan. Interest is based entirely on the amount owed, and so the "extra" money in the payment does not go toward any extra interest. The extra goes entirely to principal. An amortization table can help illustrate this:

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Example 4.5.2 ***Suppose Pat and Tracy took out a 15-year mortgage instead, also for $158,000 and at a 7.2% rate. Their monthly payment would then be $1,437.87. Construct an amortization table for their first 6 monthly payments.***

For the first month, interest would be paid on the full $158,000, so the interest would be $I = (\$158{,}000)(0.072)(1/12) = \$948.00$*, just as in the 30-year example. This leaves* $\$1{,}437.87 - \$948.00 = \$489.87$ *to go toward reducing principal, and so after this payment they will owe* $\$158{,}000 - \$489.87 = \$157{,}510.13$*. Note that, because of the larger payment, more progress is made toward reducing the balance than with the first payment of the 30-year example.*

The calculations for the next month are similar, except that instead of using $158,000 we instead use the slightly smaller $157,510.13, applying the same logic as before.

Payment Number	***Payment Amount***	***Interest Amount***	***Principal Amount***	***Remaining Balance***
1	$1,437.87	$948.00	$489.87	$157,510.13
2	$1,437.87	$947.25	$490.62	$157,019.51
3	$1,437.87	$946.50	$491.37	$156,528.14
4	$1,437.87	$945.75	$492.12	$156,036.02
5	$1,437.87	$944.99	$492.88	$155,543.14
6	$1,437.87	$933.26	$504.61	$155,038.53
7	$1,437.87	$930.23	$507.64	$154,530.89
8	$1,437.87	$927.19	$510.68	$154,020.21
9	$1,437.87	$924.12	$513.75	$153,506.46
10	$1,437.87	$921.04	$516.83	$152,989.63
11	$1,437.87	$917.94	$519.93	$152,469.70
12	$1,437.87	$914.82	$523.05	$151,946.65

Note that while the amount of interest in the *first* month is the same for both the 15-year and 30-year loans, in the second month this is no longer true. This is because the first payment does more to drop the balance of the 15-year loan, and so its smaller balance leads to less interest. This trend continues in the ensuing months, and in fact the difference becomes more pronounced as time goes by. The greater progress toward killing off the balance with the 15-year loan comes both from the larger payment and also from the interest savings of having a smaller balance. While the interest savings are small early on, they grow over time and become more and more significant as time goes by.

In these examples we compared 30-year and 15-year mortgage loans. While we assumed that the interest rates were the same, we did consider these as two separate potential loans. That did not necessarily have to be the case, however. In most circumstances, a borrower can choose to pay more than the required monthly payment on a loan and enjoy the interest saving benefits of doing so. The ability to do this is a consequence of an approach to interest on a loan sometimes referred to as the ***United States Rule***. For most financial situations in the United States, for each time period you pay the interest only on the amount that you actually owe. If you pay extra to reduce the balance, this reduces the amount of interest you pay.

Some loans, however, will have what are known as ***prepayment penalties***. When a loan has such a penalty, paying extra to reduce the term of the loan may mean that extra fees will be charged to the borrower. Once common, prepayment penalties are now fairly rare.

Also, some loans, or other financial transactions that closely resemble but may not technically speaking actually be loans (such as leases or “rent-to-own” plans), do not work in this way, and either forbid extra payments or apply them in some other way (such as treating any extra you pay this month as an early payment of the next month’s payment.) Again, this is not the norm, but it should be noted that that this sort of thing does sometimes happen. As with anything in the financial world, read the fine print!

We will use the term ***amortized loan*** to refer to any loan whose payments are calculated by using the annuity factors and whose progress toward payoff therefore follows what is illustrated in an amortization table. We will move forward assuming that all loans are amortized unless stated otherwise.

The Remaining Balance of a Loan

One very depressing fact about longer term loans is that, early on, very little progress is made toward paying off the debt. Most of the early payments go toward interest, with little left over to reduce the balance. In Example 4.5.1, after 6 months of payments, Pat and Tracy had reduced their debt from $158,000 down to $157,241.77. If that slow progress is not depressing enough in itself, we can note that they have paid a total of 6($1,072.49) = $6,434.94 to reduce their debt by a whopping $158,000 − $157,241,77 = $758.23. After 12 months, the progress was not much more impressive.

At some point, though, this has to change. Little by little, the principal is dropping, and with each passing month more and more of the payment goes toward killing it off. In the next 6 months, we can see from the amortization table that Pat and Tracy pay off $157,241.77 − $156,455.83 = $785.94. This *is* more than $758.23, and in the following 6 months it stands to reason they will pay off still more, and so on and so on. How far along will they be after, say, 5 years? Or 10 years? Or 20 years?

We could answer these questions by carrying our amortization table farther. This would certainly work, but it wouldn't be much fun. Even if we programmed a computer to do the calculation, an amortization that carried out monthly payments for 5, 10, or 20 years would be quite an undertaking.

We can determine the remaining balance of a loan at any point by giving a sly answer to the question "What do they still owe?" The answer we want to this question is of course the lump sum dollar amount that is owed, but another way of answering the question would be to observe that they owe the remaining payments on the loan! This seems like a smart-alecky and not very helpful answer until we realize that whatever the remaining debt, it must be *equivalent* to the remaining payments. To illustrate:

The remaining payments on the loan form an annuity, and the remaining balance is a single sum at the start of this annuity that is equivalent to it. In other words: the amount owed is the present value of the remaining payments. Even though this is not a formula per se, it seems significant enough to deserve special notice:

"FORMULA" 4.4.5
The Remaining Balance of a Loan

At any point in the term of an amortized loan, the amount owed is equal to the present value of the remaining payments.

We can test this out by using it to find the amount that Pat and Tracy would owe after 6 months of their 30-year mortgage. We know from the amortization table that this amount is $157,241.77. Using this new approach, we would observe that the original loan called for 360 payments, and after the sixth payment there would be 360 − 6 = 354 left. And so:

$$PV = PMTa_{\overline{n}|i}$$
$$PV = \$1{,}072.49a_{\overline{354}|.006}$$
$$PV = \$1{,}072.49(146.614395231)$$
$$PV = \$157{,}241.77$$

The fact that the answers come out the same both ways should gives us extra confidence in this approach.

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Example 4.5.3 ***Assuming that they make all their payments as scheduled, how much will Pat and Tracy owe on their mortgage after 5 years? 10 years? 15 years? 20 years? 25 years?***

Following the approach used above, we find after 5 years they will have made 60 payments. And so 360 − 60 = 300 payments remain. The amount they owe is the present value of those payments. So:

$$PV = PMTa_{\overline{n}|i}$$
$$PV = \$1072.49a_{\overline{300}|.006}$$
$$PV = \$1,072.49(138.96827618)$$
$$PV = \$149,042.09$$

The remaining calculations are essentially the same. The results are displayed in the table below.

Elapsed Years	*Remaining Payments*	*Annuity Factor*	*Remaining Balance*
5	300	138.96827618	$149,042.09
10	240	127.00843213	$136,215.27
15	180	109.88446602	$117,849.99
20	120	85.36656977	$91,554.79
25	60	50.26213003	$53,905.63

Note that, as expected, the balance declines much more slowly in the early years than in later ones. In the first 5 years, it drops from $158,000 down to $149,042, meaning that in the first 5 years only $9,958 of the original debt is eliminated. In each subsequent 5-year period, the amount of debt eliminated grows, and in the last 5 years $53,906 is paid off (since after the final payment the balance must be zero).

Extra Payments and the Remaining Balance

What if Pat and Tracy paid more than their required payments in one or more months? Suppose that they paid an extra $500 in the first month, but otherwise stuck with the payment schedule. Can we still use this approach to find out how much they owe later on? The answer to this question is yes—and no.

It is still true that the amount they owe is the present value of their remaining payments, and so in theory we could still approach the problem in this way. However, by making that extra payment, they reduced their balance, and thus shortened the amount of time that it will take to pay off the loan. After 5 years the number of remaining payments would not be 300, it would be something smaller. And in that "something" lies the difficulty. It is not easy to determine the actual number of remaining payments if we deviate from the original payment schedule. In theory the amount owed is still the present value of the remaining payments, but in practice, since we cannot readily determine how many payments actually remain, we will not normally be able to use this approach if there has been a deviation from the original payment schedule.

Loan Consolidations and Refinancing

In many cases it may be advantageous to replace one or more existing loans with a new loan. Suppose, for example, that someone started a construction business and borrowed money to buy needed equipment. Since he was just starting the business, lenders considered it risky to lend him money, and the best interest rate he could find was 15%. Three years later, though, his business has been very successful, and he finds that he can borrow at much better rates. In fact, the very same lender would now make the loan to him at 8%. Obviously, he would be much happier paying the lower rate.

In most cases, a borrower can take advantage of a more attractive opportunity by ***refinancing*** a loan. Refinancing means paying off an existing debt by taking out a new loan, using the new loan to borrow the amount needed to pay the old one off. In this example, he could pay off the balance of his 15% loan by borrowing the money he needs to do that at the more attractive 8% rate. This effectively cuts his interest rate to 8% for the remaining term of his loan. The following example will illustrate this in more detail:

Example 4.5.4 ***Suppose Kwame has 12 years remaining on a business loan at 15%, on which the quarterly payments are $2,531.00. If he refinances this debt with a new 12-year loan at 8%, what will his new monthly payment be? How much total interest will he save by doing this?***

First we need to determine how much he owes now. There are (12)(4) = 48 remaining payments on his loan, and so

$$PV = PMTa_{\overline{n}|i}$$
$$PV = (\$2{,}531)a_{\overline{48}|.0375}$$
$$PV = (\$2{,}531)(22.11112866)$$
$$PV = \$55{,}963.27$$

So this is the amount he needs to pay off the old loan.Borrowing this amount at 8% with quarterly payments for 12 years would require payments of:

$$PV = PMTa_{\overline{n}|i}$$
$$\$55{,}963.27 = (PMT)a_{\overline{48}|.02}$$
$$\$55{,}963.27 = PMT(30.67311957)$$
$$PMT = \$1{,}824.51$$

This results in a savings of $2,531.00 − $1,824.51 = $706.49 per quarter, or a total of (48)($706.49) = $33,911.52 over the remaining term of the loan. Since refinancing did not change the amount of principal he owed, this savings is entirely due to interest.

Obviously, Kwame needs no greater motivation to refinance than the potential to save nearly $34,000 in interest, and taking advantage of a lower rate is certainly one reason to replace an existing loan with a new one. Another reason for doing this may be to make things simpler by combining several smaller loans into a single loan. Doing this is sometimes referred to as ***consolidating*** the loans.

Example 4.5.5 ***Andrea has a car payment of $288.95 a month at 9% with 37 months remaining, a student loan of $353.08 at 5.4% with 108 months remaining, and a mortgage payment of $1,104.29 at 7.35% with 19 years remaining.***

A finance company suggests that she could refinance her mortgage and consolidate it with her other loans to lower her monthly payments. They propose that she take out a new 30-year mortgage loan at 7.74%, borrowing enough to pay off all three existing loans.

What would her new monthly payment be?

We first need to calculate the amounts she owed on her existing loans. For the sake of space we will not show all those calculations here. These amounts can be found as in our previous examples, by taking the present value of the remaining payments. Doing this, we find that she owes $9,306 on the car, $30,149 on the student loan, and $135,486 on the house.

To consolidate these, she would need to borrow $9,306 + $30,149 + $135,486 = $174,941 with the new mortgage. Calculating the monthly payment on this present value using 30 years and 7.75%, we can see that her new payment would be $1,253.30.

This sounds like a pretty good deal. Each month, Andrea is now paying a total of $288.95 + $353.08 + $1,104.29 = $1,746.32. Consolidating these loans will reduce her debt payments by nearly $500 each month!

There is more to the story, though:

Example 4.5.6 ***How much, in total, will Andrea save by doing this consolidation?***

With the new loan, she will make 360 payments of $1,253.30 each, totaling (360)($1,253.30) = $451,188.

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With her old loans, her remaining payments would total:

Loan	Payment	Remaining Payments	Total Remaining
Car	$288.95	37	$10,691
Student	$353.08	108	$38,133
Mortgage	$1,104.29	228	$251,778
	Total		**$300,602**

Surprisingly, we see that she will actually not save anything by consolidating. The total she would pay with the new loan is over $150,000 more than what she would pay on her existing loans!

How can this be? We just saw a moment ago that consolidating and refinancing would save Andrea nearly $500 a month. How can it be that she would end up paying so much more?

The reason lies in the number of months. By consolidating, she did lower her monthly payment, but she took debts that would be paid off in 19 years, 9 years, and just over 3 years and converted them into a debt that will take 30 years to pay off. Stretching this out, in fact, was the main reason why the new consolidated loan payment is so much lower.

Does this mean that consolidating and/or refinancing is a bad idea? Absolutely not. Andrea may decide that freeing up $500 cash in her monthly budget is well worth extending her final payoff farther into the future. Money may be tight now, and she may figure that things hopefully will not be so tight far in the future when those extra payments will be made. The decision of whether or not this sort of deal is a good one is complex, and it requires weighing the competing desires to get out of debt and to keep her payments down. There may be other factors to consider as well.[4]

There is no simple answer to whether or not consolidation is a good idea. At least in this example, there are both pros and cons to consolidating. What can be said in general, though, is that it is always worthwhile to make sure that you know what the deal is, and crunch the numbers to make sure you understand just exactly what you will be paying. Andrea may decide the greater overall payment is worth the lower monthly payment, but she should make that decision with her eyes open, understanding that the lower monthly payment comes at a price.

[4] For example: Mortgage interest is usually tax deductible, even for a refinanced loan, while interest for car loans is not and interest for student loans may or may not be. The new loan's interest rate may be much better, or much worse, than the rates on the old loans. And so on and so forth.

EXERCISES 4.5

A. Amortization Tables

1. Fill in the missing portions of the amortization table. The loan's initial balance was $23,450, payments are monthly, and the interest rate is 9%. (Note that these are only the first six rows of a much longer table; the balance will not reach zero in row 6.)

Payment Number	Payment Amount	Interest Amount	Principal Amount	Remaining Balance
1	$250.00			
2	$250.00			
3	$250.00			
4	$250.00			
5	$250.00			
6	$250.00			

2. Calculate the quarterly payment for a loan of $28,300 at 7.25% for 10 years. Construct an amortization table for the first four quarterly payments.

Quarter	Payment Amount	Interest Amount	Principal Amount	Remaining Balance
1				
2				
3				
4				

3. Suppose that you borrow $8,000 to pay to put a new roof on your house. The loan term is 6 years, payments are monthly, and the interest rate is 8.14%. Find the monthly payment on this loan, and determine how much of your first monthly payment goes toward interest and how much goes toward reducing the balance you owe.

4. Toran just graduated from college, and he owes $17,035 on his student loans. He is scheduled to make equal monthly payments intended to pay off the loan over 20 years. The interest rate is 5.75%. Construct an amortization table for his first three monthly payments.

5. Albert has a credit card on which he owes a balance of $3,765.42. The interest rate is 18.99%. His minimum monthly payment is $62.50, but Albert is planning on paying $250 each month to get the card paid off more quickly. Assume he makes no further charges to this card.

   a. If he pays the minimum this month, how much will go to interest and how much will go toward principal?

   b. If he pays $250 as he is planning on, how much will go to interest and how much will go to principal?

   c. Make an educated guess about how long it will take him to pay off his balance if he continues to pay $62.50 per month. Make an educated guess about how long it will take him if he continues to pay $250 per month. (*Note*: You do not need to try to actually calculate these values; you only need to try to make a reasonable guess.) The actual times required are given in exercise answers in the back of the book.

B. The Remaining Balance on a Loan

6. My mortgage payment is $734.55 and the interest rate is 6%. I have made all of my payments as scheduled, and I have 25 years of monthly payments left until the loan is paid off. How much do I owe today?

7. Suppose that you have 20 payments left on a personal loan. Your monthly payment is $188.75, and the interest rate is 8.91%. If you wanted to pay off the loan in its entirety today, how much would you need to do it?

8. Sylvain borrowed $200,000 at 7.39% for 10 years for start-up costs for his new business. Assume he makes all of his monthly payments as scheduled.

   (a) How much of his first payment will go toward interest, and how much will go toward principal?

   (b) How much of his 37th payment will go toward interest versus principal? (*Hint*: First, find how much he owes when he has 84 payments left to make).

   (c) How much of his last payment will go to interest versus principal?

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9. Phyllis took out a 6-year loan to buy a new car. She borrowed $18,000 and the interest rate was 7.56%. Assuming she makes all payments as scheduled, how much will she owe after 3 years, when she is halfway through the term of the loan?

C. Consolidation and Refinancing

10. Luke and Stacey have 21 years remaining on their mortgage, for which the interest rate is 8½% and the monthly payment is $915.72. They have made all of their payments as scheduled. Mortgage rates have recently come down quite a bit, and they are thinking about refinancing.
    a. How much do they owe today?
    b. If the refinance their loan by replacing it with a new, 21-year loan at 5.5%, what will their new payment be?
    c. How much would they save by doing this?
    d. Suppose that instead of a 21-year loan, Luke and Stacey decided instead to take out a new, 30-year loan at 6.5%. In that case, what would their new payment be, and how much would they save?

11. Kalpana has accumulated the following collection of loans:

Loan Type	***Monthly Payment***	***Remaining Payments***	***Interest Rate***
Mortgage	$875.19	24 years	7.5%
Car loan	$335.99	38 payments	8.95%
Personal loan	$101.49	45 payments	12.55%
Motorcycle loan	$202.15	17 payments	9.75%
Total	**$1,514.82**		

    a. She is planning on refinancing her mortgage. Suppose she refinances her mortgage with a new 24-year loan at a 6.3% interest rate. What would her new mortgage payment be? How much would she save by doing this?
    b. How much in total does she owe now for all of her loans?
    c. Instead of just borrowing enough with her new mortgage to pay off her old one, her mortgage broker has told her that she could borrow enough to pay off all of her loans with the new mortgage. If she does this, what would her new mortgage payment be?
    d. Your answer to (c) should be quite a bit lower than the total of her current monthly payments. The interest rate on the new mortgage is lower than the current rates on each of these loans. Does this necessarily mean that she will save money by consolidating all of these loans into her new mortgage?

D. Grab Bag

12. Ian took out an $80,000 business loan. His payments are quarterly, the interest rate is 9.75%, and the term of the loan is 8 years. He has made all of his payments exactly as scheduled, and now has 5 years remaining on the loan.
    a. Construct an amortization table for the first 3 quarters of his loan.

Month	***Payment***	***Interest***	***Principal***	***Remaining Balance***
1				
2				
3				

b. Determine how much he owes on this loan now.

c. Construct an amortization table for his next 3 monthly payments.

Month	***Payment***	***Interest***	***Principal***	***Remaining Balance***
13				
14				
15				

d. How much total interest will Ian have paid over the entire life of the loan, assuming that he continues to make all of his payments as scheduled.

e. Between today and the end of the loan's term, how much total interest will Ian pay, again assuming he makes all of his payments as scheduled.

f. How much total interest has Ian paid over the course of the first 3 years?

13. I have a student loan with 12 years left to go. My monthly payment is $188.75 and the interest rate is 8.5%. I have the opportunity to refinance this loan with a new 12-year loan at 5%. What will my new payment be? How much will I save by refinancing?

14. Complete the missing entries in the following (partial) amortization table. The interest rate is 7% and the initial balance was $207,913. Payments are monthly.

Payment Number	***Payment Amount***	***Interest Amount***	***Principal Amount***	***Remaining Balance***
1	$3,750.00			
2	$3,750.00			
3	$3,750.00			

15. To finance a new office building, Hydrotech Nanowidget Inc. borrowed $1,850,000 at 7.45% interest. The company is scheduled to make quarterly payments of $59,735 on this loan. Find the amount of the first payment that goes toward interest, the amount that goes toward principal, and the remaining balance on this loan after the company makes the first payment.

E. Additional Exercises

16. Jenny and Hamid took out a mortgage loan for $122,305 for 30 years at 6.39%. Assuming they make all their payments as scheduled, how much total interest will they pay in the first 5 years of the loan?

17. When she retired 5 years ago, Neryssa had $793,000 in her retirement account. She invested the money with an insurance company, which agreed to send her monthly payments for 30 years, using her $793,000 as the present value of the payments and an interest rate of 5.3%. Every month the company credits her account with the interest due, and then withdraws the amount of her monthly payment and sends it to her. What is the balance of her account today?

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18. Krystof took out a personal loan for $3,569.74, with monthly payments for 5 years. The interest rate was 11.39%.
    a. Construct an amortization table for the first 6 months of this loan.

Month	*Payment*	*Interest*	*Principal*	*Remaining Balance*
1				
2				
3				
4				
5				
6				

    b. Use the "present value of the remaining payments" method to find the amount he would owe after 6 months of payments.

    c. Why doesn't your answer to (b) agree with the balance shown in the last row of the amortization table?

4.6 Future Values with Irregular Payments: The Chronological Approach (Optional)

Annuities provide a powerful tool to work with streams of payments, but they have some serious limitations. The most serious of these is the requirement that all the payments must be the same. Our annuity formulas do not allow us to skip payments or change them, even though it is not hard to imagine realistic situations in which that might happen.

In this section, we will consider some of these situations. It turns out that the tools we have already can be used to find future vales of "annuities" whose payments change or stop. In particular, we will see that these problems can often be tackled by looking at the series of payments in pieces, in the order in which they are made. For convenience, we will refer to this method of attack as the ***chronological approach***.

"Annuities" Whose Payments Stop

Suppose that you start funding an investment or savings account by making equal payments at regular intervals (required for an annuity), but then at some point along the way decide to keep the account open but stop making any more payments (not allowed for an annuity.) This situation doesn't fit the requirements we demand of an annuity, but it's not hard to imagine it happening. Can we adapt the mathematics of annuities to figure out how much the account will grow to?

The account we are talking about here is not an annuity for its entire life, but until the payments stop it is. So we can at least use our annuity formulas to look at this first part of the story. Then notice that, once the payments stop, the account becomes a sum of money growing at compound interest for a period of time. That's something we can handle by using the compound interest formula. By looking at the account "as time goes by" we can break the problem up into two pieces, each of which can be handled with tools we already have. As noted, we will refer to this way of approaching problems as the *chronological approach*. The following example will illustrate:

Example 4.6.1 ***For 2 years, I deposited $175 monthly into a savings account. Then I stopped, but kept the account open for 3 more years. The interest rate was 3.6%. How much did I have in the account at the end of the full 5 years?***

For the first 2 years, this account was an annuity, and so we can find its future value by using the annuity formula:

$$FV = PMTs_{\overline{n}|i}$$
$$FV = \$175s_{\overline{24}|.003}$$
$$FV = \$175(24.84650638)$$
$$FV = \$4{,}348.14$$

So at the end of the first 2 years, the account had grown to $4,348.14.

Now for the last 3 years, no payments are being made, but that $4,348.14 does continue to grow at compound interest:

$$FV = PV(1 + i)^n$$
$$FV = \$4{,}348.14(1.003)^{36}$$
$$FV = \$4{,}348.14(1.113867644)$$
$$FV = \$4{,}843.25$$

So at the end of 5 years, the account was worth $4,843.25.

Don't be confused by the fact that $4,348.14 was the FV in the first step of this solution, but the PV in the second step. This amount came at the end of the annuity portion, so there it was the FV. In the plain compound interest portion it came at the beginning, and so there it was the PV. Using time lines can help keep this straight.

Breaking the life of an account up chronologically allows us to take on all sorts of situations like this one. Before moving on to try it out with other types of problems, let's use this approach on another problem similar to this example.

We have seen the surprising power of compound interest throughout Chapters 3 and 4, and we will see it again in Chapter 5. We have noted that compounding is especially powerful over long terms, where interest on interest (on interest on interest . . .) has the chance to really pile up. Comparatively modest sums can build up into shockingly large future values, and most of those large future values come from interest, not the payments themselves. This might lead us to wonder about questions like the one proposed in the following example:

Example 4.6.2 ***Wanda is 25 years old. She has decided to put $2,500 each year into a retirement account starting this year, but she plans to make these deposits for only 10 years. Her twin brother Wayne is also 25 years old. More of a procrastinator, he plans to wait 10 years before putting anything into his retirement account. But once he starts, he plans to keep up deposits of $2,500 per year until he is 65.***

Assuming that Wayne and Wanda both earn 8.4% interest, how much will each have at age 65?

Wanda's stream of payments is similar to Example 5.1.1. If we look at the first 10 years, her account is an annuity, since she is making equal payments every year. However, for the last 30 years no payments are being made at all, but compound interest does accumulate.

Again, we don't have a single formula that will allow us to find her future value at the end of the 40 years. But let's look at Wanda's account chronologically. The first 10 years of Wanda's account can be looked at as an annuity, and so we can find the future value for those:

$$FV = PMTs_{\overline{n}|i}$$
$$FV = \$2{,}500s_{\overline{10}|.084}$$
$$FV = \$2{,}500(14.76465595)$$
$$FV = \$36{,}911.64$$

Of course, this represents the value of her account at the end of the 10 years, or when she is 35. The account still will be open for another 30 years, with no payments made. While that isn't an annuity, that money will still be earning compound interest. And so the last 30 years can be dealt with by using the regular compound interest formula:

$$FV = PV(1 + i)^n$$
$$FV = \$36{,}911.64(1.084)^{30}$$
$$FV = \$414{,}993.99$$

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Wayne's future account value is less of a challenge. Once his payments start, they continue on to the end, and so Wayne's account is an annuity:

$$FV = PMTs_{\overline{n}|i}$$
$$FV = \$2{,}500s_{\overline{30}|.084}$$
$$FV = \$2{,}500(121.9393221)$$
$$FV = \$304{,}848.30$$

So even though he makes the payments for 30 years instead of Wanda's 10, Wayne winds up with less money in his account in the end!

The results of this example may be a little surprising. Wanda's account ran for 40 years versus Wayne's 30—that doesn't seem *that* much longer. But Wayne made contributions for 30 years versus Wanda's 10, so he contributed 3 times as much as she did—that *does* seem like a big difference. Yet Wanda came out ahead because of the power of compound interest over time. Another view of those future values may help explain Wanda's victory:

Example 4.6.3 ***How much of Wanda's future value came from interest? How much of Wayne's did?***

Wanda contributed \$2,500 a year for 10 years, so her total deposits were (\$2,500/year) (10 years) = \$25,000. Since her future value totaled \$414,993.99, her total interest earnings were \$414,993.99 − \$25,000 = \$389,993.99.

Wayne contributed \$2,500 a year for 30 years, so his total deposits were (\$2,500/year) (30 years) = \$75,000. His total interest earnings, then, were \$304,848.30 − \$75,000 = \$229,848.30.

Those extra 10 years that Wanda had compound interest working for her made all the difference, even making up for her much smaller total contributions. This is interesting mathematically, but it also interesting from a practical point of view: if you want to have an account grow into a large future value, how soon you start putting money to work can be far more important than how much money you put to work.

"Sinking Funds" Whose Payments Stop

What if, instead of letting the payments determine the future value, we let a target future value determine the payments? This is basically the same idea as a sinking fund, except that now we are open to the possibility that the payments might stop at some point along the way.

Example 4.6.4 ***Dario wants to have \$1,000,000 in his retirement account when he reaches age 70, 45 years from now. Dario thinks that his account can earn 9.6%, and he'd like to reach his goal by making monthly deposits for just the next 10 years. How much should each deposit be?***

Here we need to work backwards. For the last 35 years, the account simply earns interest, and so at the start of the period its value must have been enough to grow to the target \$1,000,000. So:

$$FV = PV(1 + i)^n$$
$$\$1{,}000{,}000 = PV\left(1 + \frac{0.096}{12}\right)^{420}$$
$$PV = \$35{,}202.74$$

So the payments for the first 10 years must accumulate to this amount:

$$FV = PMT\ s_{\overline{n}|i}$$
$$\$35{,}202.74 = PMT\ s_{\overline{120}|.096/12}$$
$$PMT = \$175.82$$

As the examples of this section hopefully demonstrate, a change in the payment stream need not stop us from calculating an account's future value. Such changes do, however, require us to break the problem into parts. The chronological approach that we have used in this section is an effective way of doing this in many cases. In Section 4.7, we will see an alternative way of looking at things that may prove more useful in other situations.

EXERCISES 4.6

A. "Annuities" Whose Payments Stop

1. a. Suppose that you invest $350 a quarter for 10 years at 8%. How much will you have at the end of the 10 years?
   b. Suppose that you then stop making payments, but keep the account open and your money continues to earn 8% compounded quarterly for the next 25 years. How much will your account grow to?

2. Destiny has been depositing $3,000 each year into an account earning 9.25% for the last 15 years. She has 20 years to go until retirement.
   a. How much does she have in her account today?
   b. Assuming she decides to stop making new deposits but keeps her account open, earning the same rate, how much will she have when she retires in 20 years?
   c. Assuming she decides to keep making $3,000 annual deposits, and her account continues to earn 9.25%, how much will she have when she retires in 20 years?

3. If you deposit $100 each month into an account earning 7% for 20 years, and then stop making deposits but keep your money invested, earning this same rate for another 10 years, how much in total will you have in the end?

4. Based on the scenario in Exercise 3, how much total interest would you earn?

5. When he was 3 years old, Toby's parents started investing $25 each week in an investment account for him, thinking he would use the money for college. When he was 18, his parents stopped making these deposits, but Toby didn't need to money for college since he won a football scholarship. He kept the money invested, but didn't put any more money in the account.

   Assuming that his account earns 7.8%, how much will he have in it when he turns 65? How much of that total will be from interest?

6. I deposited $25 each week into a savings account earning 5% for 1 year. Then I stopped making deposits and just let the account grow at compound interest for the next 3 years. How much total interest did I earn?

B. "Sinking Funds" Whose Payments Stop

7. I want to have $200,000 in an investment account 20 years from now and plan to reach this goal by making equal quarterly deposits for the next 12 years, and then stopping the deposits and just letting the account grow. My account earns 10%. How much do I need to have in the account when I stop making deposits? How much should each of my quarterly deposits be?

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C. Grab Bag

8. Anja and Ted both have just set up investment accounts. Suppose that Anja deposits $1,000 each year for 10 years and then stops, while Ted deposits nothing for 10 years and then starts and continues depositing $2,000 each year. How much will each person have at the end of 25 years, assuming they both earn 8.2%? How much total interest will each person earn?

9. Eigen's Valuemart needs to have $20,000,000 in a special account 20 years from now in order to meet an employee benefits commitment. The company plans to make equal quarterly deposits for the next 8 years and then leave the money invested to grow to the needed future value. If their account will earn $8\frac{3}{4}$%, how much should each quarterly deposit be?

10. Amy wants to have $1,000,000 in her retirement account in 40 years. She believes her account will earn 9% compounded monthly.
    a. Assuming she keeps making the same monthly deposit for the entire 40 years, how much should she deposit each month to her account?
    b. Assuming she wants to make monthly payments for the next 20 years and then stop her deposits, how much should she deposit each month?
    c. Assuming she wants to make monthly payments for the next 10 years and then stop her deposits, how much should she deposit each month?

11. How much will I have in my retirement account when I'm 64, assuming that I invest $375 each month beginning at age 27 and then stop making investments at age 39. Assume that my account earns 11%.

D. Additional Exercises

12. Suppose that you deposit $120 each month into an account earning 8% for 5 years. You continue your deposits at this same pace, but the interest rate is increased to 10% for the next 15 years. How much in total will you have at the end of 20 years?

4.7 Future Values with Irregular Payments: The Bucket Approach (Optional)

The previous section's approach works well in some circumstances, but in others it falls short. It will handle annuities whose payments stop, but it isn't adequate to handle "annuities" whose payments change, or accounts that begin with some amount of money already in them. We can address those sorts of problems with another technique, which we will refer to as the ***bucket approach***.

"Annuities" That Don't Start from Scratch

Suppose you have $5,000 in an account that earns an effective rate of 4.35% for 8 years. This account will grow to $7,029.26 at the end of the 20-year term. Now, suppose that instead of one account with $5,000, you opened up *two* accounts, each with $2,500. Would you have earned more by doing this? Less? The same?

A deposit of $2,500 at 4.35% for 8 years grows to $3,514.63. Each of your two accounts would have grown to this. In total you would have $7,029.26, the same as if you had kept the entire amount in a single account. As we noted back when we were trying to find a future value annuity formula, it makes no difference whether you kept everything in a single account, or split the money up. In the end, you still have every single one of your $5,000, together with all the compound interest each dollar accumulates.

This observation is the key to the bucket approach. Since it makes no difference whether or not the money is all kept in a single account, it will not change anything if we pretend that an account is split into separate "buckets" whenever it helps us to do this.

The following example will illustrate:

Example 4.7.1 ***Brandon has $1,457.06 in his savings account, which pays 3.6% interest compounded monthly. Each month he deposits $75 into this account. How much will he have in his savings account after 2 years?***

Brandon's monthly payments are an annuity, but we also have to deal with the $1,457.06 which was in the account from the start. To handle this, we will pretend that the original $1,457.06 is kept in a separate account from his new deposits.

Bucket 1: The original $1,457.06, earning compound interest.

$$FV = PV(1 + i)^n$$
$$FV = \$1,457.06\left(1 + \frac{0.036}{12}\right)^{24}$$
$$FV = \$1,565.67$$

Bucket 2: The monthly $75 payments.

$$FV = PMTs_{\overline{n}|i}$$
$$FV = \$75s_{\overline{24}|.036/12}$$
$$FV = \$1,863.49$$

Of course, Brandon's actual account contains all the money we pretended was separated into those two buckets. So his total future value will be $1,565.67 + $1,863.49 = $3,429.16.

"Annuities" with an Extra Payment

The same approach works nicely to deal with additional payments made to an annuity as well.

Example 4.7.2 ***Kevin deposited $1,000 each year into a savings account that earned a 5% effective rate for 10 years. In the third year, though, he was able to deposit $5,000 instead of his usual $1,000. How much did he have at the end of the 10 years?***

We need to break Kevin's payments up into buckets that can be handled either as an annuity or as a single sum of money growing at compound interest. There are several ways to do this, but the most efficient one is to make the first bucket an annuity of $1,000 each year (pretending that the third-year payment was $1,000, like all the others). The second bucket, then, would be an extra $4,000 in the third year. (The second bucket is $4,000 and not $5,000 because $1,000 in the third year was included in the first bucket annuity).

Bucket 1: $1,000 per year for 10 years.

$$FV = PMTs_{\overline{n}|i}$$
$$FV = \$1,000s_{\overline{10}|.05}$$
$$FV = \$12,577.89$$

Bucket 2: The extra $4,000 paid in year 3. This money will earn compound interest for the last 7 years. So:

$$FV = PV(1 + i)^n$$
$$FV = \$4,000(1.05)^7$$
$$FV = \$5,628.40$$

Putting the two buckets together, we see that Kevin's future value was $12,577.89 + $5,628.40 = $18,206.29.

"Annuities" with a Missing Payment

What if instead of an extra payment, a payment is missed? This approach can handle that sort of situation as well.

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Example 4.7.3 ***For the last 5 years, the management of Watermill Corp. has put $25,000 from its profits each quarter into a fund intended to provide capital for future expansion plans. Unfortunately, in the last quarter of the third year the company had some financial difficulties, and skipped making this deposit. If the account earned 4.8%, how much did the company have at the end of the 5 years?***

There are quite a few different ways to break this account into buckets, but, just as in Example 4.7.2, the most efficient solution is to start with a bucket that assumes that the regular annuity payments were made for the entire term.

Bucket 1: $25,000 per quarter for 5 years.

$$FV = PMTs_{\overline{n}|i}$$
$$FV = \$25{,}000s_{\overline{20}|.012}$$
$$FV = \$561{,}321.59$$

In the previous example, the second bucket was the extra money, but here we don't have any extra money—instead money is missing. Here, though, we can make a second bucket out of the missing money, and calculate what it would have grown to with interest if the payment hadn't been missed.

Bucket 2: The missing $25,000 from the end of the third year. This money was missing for the last 2 years, so we calculate what it would have grown to in those last 2 years:

$$FV = PV(1 + i)^n$$
$$FV = \$25{,}000(1 + 0.012)^8$$
$$FV = \$27{,}503.26$$

Bucket 1 assumes that all the payments were made, Bucket 2 contains the missing payment together with all the interest it would have earned. The company's actual account was bucket 1, take away the contents of bucket#2. So the actual future value is $561,321.59 − $27,503.26 = $533,818.33.

Annuities with Multiple Missing or Extra Payments

What if an "annuity" has more than one extra and/or missing payment? In that case, we can follow the approach we used in the previous two examples. Start out assuming that all of the payments were the same, and then create a separate bucket for each extra or missing payment. Then at the end, add the extra and subtract the missing. A few of the exercises will require this.

What if instead of just a few deviations up or down, the "annuity" payments actually change?

Example 4.7.4 ***For 15 years I deposited $2,500 each year into an investment account that earned 7½%. Then, for the next 5 years, I kept making payments, but only $1,000 each. How much was my account worth at the end of 20 years? How much total interest did I earn?***

Bucket 1: Once again, we start by assuming the $2,500 payments continued for all 20 years.

$$FV = PMTs_{\overline{n}|i}$$
$$FV = \$2{,}500s_{\overline{20}|.075}$$
$$FV = \$108{,}261.70$$

Bucket 2: In each of the last 5 years, $1,500 was missing compared to what we assumed for bucket 1. These missing payments form an annuity:

$$FV = PMTs_{\overline{n}|i}$$
$$FV = \$1{,}500s_{\overline{5}|.075}$$
$$FV = \$8{,}712.59$$

The overall future value, then, is the bucket 1 annuity less the missing deposits and their interest. So the future value is $108,261.70 − $8,712.59 = $99,549.11.

The second question, how much interest I earned, can be answered by subtracting the total payments from the future value. I made 15 payments of $2,500 each, and 5 payments of $1,000 each, so the total payments were 15($2,500) + 5($1,000) = $42,500. So the total interest earned was $99,549.11 − $42,500 = $57,049.11.

"Sinking Funds" That Don't Start from Scratch

Sometimes we have a set goal for a sinking fund, but already have some funds in the account working toward our goal. The following example will illustrate how we can address this sort of situation.

Example 4.7.5 ***Alexis will be retiring in 5 years, and wants to have a balance of $600,000 in an investment account at that time. Right now, she has $375,000 in this account. She expects her account to earn 8.4%. How much does she need to deposit each month to reach this goal?***

Since Alexis already has quite a bit of money in her account, her payments only need to make up the difference between what she will already have and the $600,000 target. The money she has, though, will keep earning interest for these next 5 years, and so we need to see how much it will grow to.

Bucket 1: The $375,000 with its interest.

$$FV = PV(1 + i)^n$$
$$FV = \$375{,}000(1.007)^{60}$$
$$FV = \$569{,}901.11$$

Since she can expect this much in her account without making payments, the payments just need to make up the difference. So their future value must be $600,000 − $569,901.11 = $30,098.89.

Bucket 2: A sinking fund with a $30,098.89 future value.

$$FV = PMTs_{\overline{n}|i}$$
$$\$30{,}098.89 = PMTs_{\overline{60}|.007}$$
$$PMT = \$405.38$$

So monthly payments of $405.38 would be enough to fill the gap.

Dealing with problems like the ones in this section can be a challenge. Take your time and think each one through; these are not the sort of problems that you can expect to coast through on autopilot. Keep in mind that, when you break a problem up into buckets, every dollar and its interest must be included in one, but only one, of the buckets. Also, don't be afraid to look back at the examples; while not every problem will match up with one of them, the ideas exploited in the examples can be put to use in other situations as well.

EXERCISES 4.7

A. "Annuities" That Don't Start from Scratch

1. HuperMart Convenience Stores Corp. has $1,598,375 in an account set aside for longer term employee benefit commitments. The company is depositing $37,500 each quarter into this account. Assuming these deposits continue, and assuming the account earns 7½%, how much will it have in this account in 10 years?

2. Trevor has $296,101.43 in his 401(k). Assuming he keeps up deposits of $275 each month and his account continues to earn 7.825%, how much will he have in his 401(k) when he retires in 20 years?

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B. "Annuities" with an Extra Payment

3. Suppose Kerry deposits $2,000 each year into an investment account every year. The only exception to this schedule is that for her 10th payment she deposits $5,000 instead of her usual $2,000 deposit. If her account earns 6.98%, how much will she have at the end of 30 years?

4. Allyson deposits $3,000 at the beginning of each year into an account earning 10% for 25 years, except that in the 15th year she deposits an additional $5,000 above and beyond her usual $3,000 deposit. How much will she have at the end of the 25 years? How much total interest will she earn?

C. "Annuities" with a Missing Payment

5. Mike deposited $2,500 into his individual retirement account (IRA) every year for 25 years. He missed his deposit in the eighth year because he was temporarily laid off, but other than that he has kept his deposits up every year. If his account earned 7.72%, what was his account value at the end of 25 years?

6. Suppose you deposit $1,250 each year for 30 years, except that in year 21 you deposit only $750 instead of your usual $1,250. How much do you have at the end of the 30 years, assuming your account earns 5.25%?

D. "Annuities" with Multiple Missing/Extra Payments

7. Suppose that the Cauchy-Schwartz Equalization Corp set aside $150,000 each quarter for 12 years into a special fund to cover potential environmental cleanup costs. For the next 8 years, the company reduced its deposits to $80,000 per quarter. Find the value of this account at the end of 20 years, assuming it earned a 5.3% interest rate for the entire period.

8. Kelly deposited $135 per month into her retirement account for 20 years. For the next 12 years she increased her deposits to $175 per month. Assuming she earned 8.13%, find her account's future value. Find the total interest earned on this account.

E. "Sinking Funds" That Don't Start from Scratch

9. Bud has $1,357.15 in a special vacation savings account. He hopes to have $2,500 in this account 1 year from now. If his account earns 6% interest, how much does he need to deposit each month to reach his goal?

10. The Business Department at McKeanville Community College is trying to raise money for a scholarship fund. The department's goal is to have $25,000 in the scholarship fund in 5 years. Right now there is $7,346.19 in the fund. Assuming that the fund earns 7.15%, and that no scholarships will be paid from this fund until the goal is reached, how much needs to be deposited to this fund each quarter to reach the goal?

F. Grab Bag

11. a. Lucy has $275,935 in her individual retirement account right now. She wants to have $1,500,000 in this account in 20 years. Assume that her account earns 6.85%. How much does she need to deposit each month to reach this goal?

    b. If she lowers her goal to $750,000 in 20 years, how much does she need to deposit each month to reach this goal?

12. I have $1,575.19 in my savings account right now. It earns 3.6% interest. If I deposit $200 into this account each month, how much will I have in 3 years?

13. Find the future value of an "annuity" of $500 per year at 9% for 20 years, assuming that the 12th and 15th annual payments are $850 instead of $500.

14. Leslie deposits $400 each month for 10 years into an investment account. She then increases her deposits to $500 per month. If her account earns 8.4%, how much will she have at the end of 16 years?

15. Pat deposited $2,000 per year for 25 years into an account earning 9.35%. He missed his scheduled payment in years 10 and 15, and deposited an extra $3,000 in year 20. Find the future value of his account.

G. Additional Exercises

16. Suppose I deposit $125 per month for 10 years into an account earning 6%. For the next 15 years I increase my deposits to $150 per month, then for the next 5 years after that I decrease my deposits to $80 per month. Find my total future value.

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CHAPTER 4 SUMMARY

Topic	Key Ideas, Formulas, and Techniques	Examples
Annuities, Their Uses, and Terminology, pp. 141–143	Any series of regular payments can be analyzed as an annuity. Key terms: present value, future value, ordinary annuity, annuity due.	See discussion in Chapter 4.1
The Future Value of an Ordinary Annuity, p. 149	• The Future Value Ordinary Annuity Formula: $FV = PMTs_{\overline{n}\rceil i}$ • $s_{\overline{n}\rceil i}$ is the future value annuity factor. • The annuity factor may be found from tables, calculator/computer programs, or formulas.	How much will I have as a future value if I deposit $3,000 at the end of each year into an account paying 6%? (Example 4.2.1)
The Future Value Annuity Factor Formula, pp. 153–154	• $s_{\overline{n}\rceil i} = \frac{(1+i)^n - 1}{i}$ • n is the number of time periods, i is the interest rate per period.	Find the future value annuity factor for a monthly annuity with a 15-year term and a 7.1% interest rate. (Example 4.2.5)
Finding Total Interest Earned, pp. 154–155	• To find the total interest earned in an annuity, subtract the total payments from the future value.	Carrie deposits $250 each month into an account paying 4.5% for 5 years. How much interest does she earn? (Examples 4.2.6 and 4.2.7)
The Future Value of an Annuity Due, p. 156	• The Future Value Annuity Due Formula: $FV = PMTs_{\overline{n}\rceil i}(1 + i)$ • The calculation is done the same way as an ordinary annuity's, but then multiply by (1 + i) at the end.	Mariano deposits $2,500 at the start of each year at 8¼%. How much will he have after 40 years. (Example 4.2.8)
Sinking Funds, p. 163	• A sinking fund is just an annuity whose payments are determined by the desired future value. • Use the future value annuity formula, find the annuity factor, and then use algebra to find the payment amount.	How much should Shauna set aside at the start of each quarter to accumulate $10,000 in 2 years, assuming her account earns 4.8%? (Example 4.3.2)
Sinking Funds and Retirement Planning, p. 160	• To accumulate a large future value, the period of time makes an enormous difference in the size of the payments needed.	To accumulate $1,000,000 by age 70, how much would Joe need to deposit semimonthly assuming he starts at age 65, 50, 35, 25, 18, or age 2? (Example 4.3.6)
The Present Value of an Ordinary Annuity, p. 169	• The Present Value Ordinary Annuity Formula: $PV = PMTa_{\overline{n}\rceil i}$ • $a_{\overline{n}\rceil i}$ is the future value annuity factor • The annuity factor may be found from tables, calculator/computer programs, or formulas.	How much can you afford to borrow on a car loan if your payments will be $275 per month for 3 years at 6%. (Example 4.4.2)
The Present Value Annuity Factor Formula, pp. 173–174	• $a_{\overline{n}\rceil i} = \frac{s_{\overline{n}\rceil i}}{(1+i)^n}$ • $a_{\overline{n}\rceil i} = \frac{1-(1+i)^{-n}}{i}$ • Either formula can be used; both give the same results.	Find the present value annuity factor for a 5-year annuity with quarterly payments and a 9% interest rate. (Example 4.4.6)

(Continued)

Topic	Key Ideas, Formulas, and Techniques	Examples
Finding Loan Payments, pp. 175–176	• Use the present value annuity formula, find the annuity factor, and then use algebra to find the payment amount.	Pat and Tracy took out a 30-year, $158,000 mortgage at 7.2%. How much will their monthly payment be? (Example 4.4.8)
Finding Total Interest Paid, pp. 175–176	• To find the total interest paid in an annuity, subtract the present value from the total of the payments.	Pat and Tracy took out a 30-year, $158,000 mortgage at 7.2%. How much total interest will they pay? (Example 4.4.8)
The Present Value of an Annuity Due, p. 177	• The Present Value Annuity Due Formula: $PV = PMTa_{\overline{n}\mid i}(1+i)$ • The calculation is done the same way as an ordinary annuity's, but then multiply by $(1+i)$ at the end	A lottery jackpot is paid out as annual $2 million payments for 26 years. If you choose instead to receive a lump sum payment all at once, how much will you receive, assuming a 6% interest rate? (Example 4.4.11)
Amortization Tables, p. 183	• An amortization table shows payment by payment the amount of interest, the amount of principal, and remaining balance on a loan. • The amount to interest is calculated by using the simple interest formula. • The amount to principal is the difference between the payment and the interest amount. • The remaining balance is the previous period's balance, less the amount paid to principal.	Pat and Tracy took out a 30 year, $158,000 mortgage at 7.2%. Create an amortization table for their first twelve monthly payments. (Example 4.5.1)
The Remaining Balance of a Loan, p. 186	• To find the remaining balance on an amortized loan, calculate the present value of the remaining payments.	Assuming that they make all their payments as scheduled, how much will Pat and Tracy owe on their mortgage after 10 years? (Example 4.5.3)
Consolidation and Refinancing, p. 187	• A loan may be paid off by borrowing the amount needed to do so with a new loan. • Refinancing is replacing a single loan with a new one. Consolidation is replacing several loans with a single new loan. • Determine the amount owed on each loan, and then use the result as the present value of the new loan.	Kwame has 12 years remaining on a business loan at 15%. His payments are $2,531. What would his new payment be if he refinances with a 12-year loan at 8%? (Example 4.5.4)
The Chronological Approach to Future Values with Irregular Payments (Optional), pp. 192–193	• The future value of a stream of payments that is not an annuity can be calculated by building the future value chronologically.	For 2 years, I deposited $175 monthly. For the next 3 years there were no payments made. If my account earned 3.6%, what was my future value? (Example 4.6.1)
The Bucket Approach to Future Values with Irregular Payments (Optional), p. 197	• The future value of a stream of payments that is not an annuity can be calculated by imagining the payments broken up into separate accounts ("buckets").	Kevin deposited $1,000 each year into an account earning 5% for 10 years. In the third year, though, he deposited $5,000 instead of $1,000. Find his future value. (Example 4.7.2)

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CHAPTER 4
EXERCISES

The following exercises are a mixture of problems primarily from the topics covered in Chapter 4. One of the objectives of these exercises is to be able to correctly identify which topics and tools are needed for each problem. While the emphasis is on material covered in Chapter 4, some problems covering material from Chapters 1 to 3 may also be included. All of the material covered in this chapter is "fair game," except for optional topics, which are not included in these exercises.

1. Zeno bought his house with a 30-year mortgage at 9% interest. His monthly payment for the loan is $813.25. How much did he borrow to buy his house? If he made a down payment of $8,927.77, how much did he pay for the house?

2. George is participating in his company's 401(k) plan by contributing $25 every week (at the end of the week) to his account. He expects his investments to earn 9%. How much will he have in his account after 30 years? How much total interest will George earn?

3. A school district issues a bond to provide for a new addition to the local high school. The terms of the bond call for the district to repay a total of $5,243,816 at the end of 5 years, and also require that the district establish a sinking fund to accumulate the amount to be repaid. The sinking fund will be funded by deposits made at the beginning of each quarter into an account that pays 5.6% interest. How much will the district have to contribute each quarter?

4. Lex just won his lawsuit against the Amber Rainbow Restaurants Corp. for injuries sustained when he poured a carafe of hot coffee on himself at one of their restaurants. The court awarded him $5,000,000 to be paid as a structured settlement with payments at the beginning of each year for the next 10 years. The $5,000,000 is the total of all the payments, not the present value. A private investor offers to purchase these payments with a lump sum payment today, calculated at an interest rate of 14.6%. How much would he receive if he accepted this offer?

5. Hiroshi wants to retire in 45 years with $2,000,000. He plans to achieve his goal by making annual deposits at the end of each year into an IRA account, on which he believes he will earn 11.2%. How much does he need to deposit each year in order to achieve his goal?

6. Marat bought a car, which he financed with a 5-year loan at 11.3% interest. He made a down payment of $2,000 and his monthly payment is $317.98. How much did he pay for the car?

7. MostlyXAct Testing Labs borrowed $2,500,000 to expand its facilities by taking out a loan at 8.4%. The lender requires the labs to pay off the loan by making payments at the end of each quarter for 7 years. How much will each payment be?

8. Vern wants to buy a new car that costs $14,362.50. His credit union will loan him the money with a 4-year loan at 7.8% interest. How much will his monthly payments be? How much will his total payments be? How much total interest will he pay?

9. Tom's property taxes are $3,300 per year. The bank that holds his mortgage requires him to make monthly deposits into a sinking fund at the beginning of each month in order to accumulate the total for his taxes at the end of the year. If the account earns 4.8% interest, how much should each monthly payment be?

10. The Ampersand Corporation borrowed $2,500,000 from a group of private investors in order to finance an expansion of its manufacturing facilities. The investors will not receive any payments during the course of the loan, but under terms of the loan, the company must make quarterly payments into a sinking fund for 10 years.

    a) If the loan interest rate is 12% compounded quarterly, how much will Ampersand have to accumulate in order to pay off its creditors at the end of 10 years?

    b) If the sinking fund earns 6.6% interest, how much will each of Ampersand's quarterly payments have to be?

11. Internet vendor EPottingSoil.com borrowed $1,000,000 from a group of investors for 5 years at a 9.74% effective rate. The company will not make any payments to these investors until the maturity date of the note. However, loan covenants require the company to establish a sinking fund for the purpose of accumulating the maturity value, into which PottingSoil.com must make equal deposits at the end of each quarter. If the sinking fund earns 4.5% interest, how much must each deposit be?

12. Chris pays $15 each week into a Christmas Club account at his credit union. The account pays 3.7% interest. How much will he have in the account at the end of 1 year?

13. Ubiquitous Advertising Associates financed its new office building with a 10-year loan on which payments of $3,142.00 are due at the beginning of each month. The interest rate of the loan is 8.1%. How much did the company borrow? How much total interest will it pay?

14. Seven years ago, Taneisha took out a 30-year mortgage at 7.83%. Her monthly payments are $603.80. She has made all of her payments as scheduled to date.

    a) How much did she originally borrow?

    b) How much does she owe today?

    c) Suppose that Taneisha decided to refinance her mortgage today with a new 30-year loan at 7.5%. What would her new monthly payment be? How does the new payment compare to the old one? Would she save money by doing this?

15. Steve bought a house for $137,000 with a down payment of $16,000. He financed it with a 30-year mortgage at 7.83%. How much total interest will he pay over the life of the loan? Suppose that he instead took out a 15-year loan at the same interest rate. How much higher would his monthly payment be? How much total interest would he pay over the life of the 15-year loan?

16. Julia deposits $175 at the beginning of each quarter into an account earning 6.7% for 30 years. How much money will she have at the end of those 30 years? How much total interest will she earn?

17. Rhonda has just graduated from Wassamatta U. and has student loans totaling $12,375. She must now start making payments monthly. Her loans' interest rate is 8.88%, and she will pay the loan off over 18 years. How much will her monthly payments be? Of her first payment, how much will go for interest and how much for principal?

18. Mark is saving up for a tuition deposit for veterinary school. He will need $4,500 in 3 years, and is trying to build up this amount by making monthly deposits into an account earning 3.93%. How much should each monthly deposit be?

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19. Aaron would like to have $2,000,000 when he retires 42 years from now. How much should he set aside each week if his investments can earn 9.1%?

20. McCabe Industrial Corp. borrowed $750,000 for an upgrade of its plant, borrowing the money at 8.5% compounded daily for 5 years. McCabe is not required to make any payments to the lender until maturity, but is required to set up a sinking fund for the purpose of accumulating the loan's maturity value. If they make quarterly deposits into an account earning 4.8%, how much should each payment be?

21. Dawn deposited $3,000 into an investment account paying 5.29% compounded daily for 5 years. How much total interest will she earn?

22. I have a $3,031.59 balance on my credit card. The interest rate is 21.95%. If I pay $300 this month, how much of my payment will go toward interest? What will my balance be after my payment (assuming I make no additional charges).

23. Nicole is about to retire, and has $800,000 saved in a 401(k) plan, from which she wants to take monthly payments for the next 25 years. If the account can be expected to earn 6% interest, how much will each monthly payment be?

24. Mary just won the Rhode Island Lottery, with a total jackpot of $5,200,000. She will receive the money spread out as equal payments at the beginning of each year for the next 26 years. The $5,200,000 is the *total* of all payments, *not* the present value. So she is entitled to 26 annual payments of $5,200,000/26 = $200,000 per year. If she decided instead to take her winnings as one lump sum, the interest rate used would be 6%. How much would she receive as a lump sum? (Do not take taxes into account in this calculation.)

25. Harald put $2,000 into his retirement savings account at the end of each year for 40 years. The account earned an interest rate of 7.2%.
    a) How much did his account grow to at the end of the 40 years?
    b) Suppose that Harald decides to use the money in his account to provide a monthly income in retirement. If his account earns 5.4% during his retirement, how much can he withdraw each month if he wants his account to last 20 years?

26. Find the future value of $25 a week for 30 years at 7.23%.

27. Walt borrowed $30,000 for 7 years at 8.4%. Find his monthly payment, and construct an amortization table for his first three payments.

Month	***Payment***	***Interest***	***Principal***	***Remaining Balance***
1				
2				
3				

28. Daniel and Shauri have a 30-year mortgage. They borrowed $177,259 and their monthly payment is $1,205.79. If they make all of their payments exactly as scheduled, how much total interest will they pay?

29. Suppose that $425 is invested each quarter into an account paying 6% for 20 years. Find the total interest earned.

30. If $n = 100$ and $i = 0.0045$, find $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$.

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CHAPTER 5

Spreadsheets

"A wise man should have money in his head, but not his heart."

—Jonathan Swift

Learning Objectives

LO 1 Create basic spreadsheets as tools for financial calculations.

LO 2 Use spreadsheets to find future values of annuities or series of irregular payments.

LO 3 Create amortization tables using spreadsheets.

LO 4 Use spreadsheets as a tool to solve annuity and other financial problems for values that cannot be readily determined with formulas and algebra.

Chapter Outline

5.1 Using Spreadsheets: An Introduction

The tools we developed in Chapter 4 allow us to work mathematically with all sorts of streams of payments. But if there are a lot of extra or missing payments, or a lot of changes in the amount of the regularly scheduled payments, these approaches can get awfully tedious. ***Spreadsheet*** programs provide an alternative way to handle these problems. As we will also see in this chapter, spreadsheets are a very useful tool for a wide range of business and financial calculations.

The most commonly available spreadsheet program on the market today is ***Microsoft Excel***. There are alternatives, but other spreadsheets are similar enough that you can adapt to them quickly if you know how to use Excel. This section will discuss the use of spreadsheets for financial calculations, and demonstrate how this can be done, using Excel.

The Layout of a Spreadsheet

A spreadsheet is a rectangular arrangement of individual packets of information. Each of these packets is called a ***cell***. A cell can contain words (text), numbers, or formulas.

A group of cells that falls into a line horizontally is called a ***row***, a group of cells in a vertical line is called a ***column***. The rows of a spreadsheet are numbered, starting from the top. The columns are identified by letters, starting from the left. Each cell is identified by the letter of its column and the number of its row. Figure 5.1 at the bottom of this page is an example of a simple spreadsheet, giving the sales of the three wonderful products offered by the InterGlobal Incredibly Useful Products Company.

Spreadsheets are widely used in the business world. © The McGraw-Hill Companies, Inc./Jill Braaten, photographer/DIL

A cell is identified by the column and row that it falls in; usually the column is given first. For example, in this spreadsheet, A5 is the cell in column A, row 5. A5 contains the text "Totals". B3 is the cell in column B, row 3, which contains the number 880. The total dollar value of all the product sales is contained in cell D5.

It is important to understand that the contents of a cell are not always what they appear to be. For example, cell C2 appears to contain the text "$23.95" but in fact it only contains the number 23.95. The dollar sign shown is not actually part of the contents of this cell. In all the problems we have worked so far in this text, we have recognized that in calculations the $ symbol is not important, and we have ignored it when working through problems. We only put a "$" in front of an answer to make it clear that the answer was an amount of money. Likewise, the spreadsheet does not need the $ for its calculations, but, since the contents of the cell are supposed to be an amount of money, we do want it to be displayed on the screen.

How does the computer know that the contents of the cell C2 should be shown with a dollar sign ("$23.95" not "23.95"), while the contents of cell B2 should not be given a dollar sign ("1800" rather than "$1800.00")? Each cell can be given a ***format***, instructions to the program about how to display the information in the cell. Because the numbers in cells C2 through C4 and D2 through D5 represent money, these cells have been formatted to display their contents with a $ sign in front. The formatting does not matter from a strictly mathematical point of view, but it does make the spreadsheet easier to read and therefore more user-friendly. There is a lot that can be done with formatting, such as use of different colors, fonts, highlighting, and so on, to make a spreadsheet more attractive and user-friendly. In this text we are mostly interested in spreadsheets as tools for calculations, and so we will not pay too much attention to formatting. We won't completely ignore it, though, because, without decent formatting a spreadsheet can be difficult to read, and difficult to use as a result.

Formatting is not the only way that a cell can contain something different than what it appears to. Cell D2 appears to contain the number 43,110, but actually that number is not what is in that cell at all. D2 should represent the total dollar amount of sales of solar fish gutters, which is $43,110. Now, if we didn't already have this figure and wanted to figure out just how much those sales total, we would need to multiply the number of items sold (which is contained in B2) times the price per item (contained in C2). We could do this ourselves on the calculator and then type the result into D2, but spreadsheets allow another option. Since the spreadsheet contains the two values to be multiplied, we can instead put a formula in the cell that tells the program to multiply B2 times C2. By entering formulas like this, we allow the spreadsheet program to do the calculations for us.

FIGURE 5.1

	A	B	C	D
1	**Product**	**Units Sold**	**Unit Price**	**Product Sales**
2	Solar Singing Fish Gutter	1800	$23.95	$43,110
3	Home Crash Test Kit	880	$127.95	$112,596
4	Twelve Tone Door Chime	500	$44.99	$22,495
5	**Totals**	**3180**	**N/A**	**$178,201**

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Now that we have discussed some of the basics behind this spreadsheet example, let's start off by recreating it. Open Excel on a computer, and follow the instructions spelled out below to recreate this spreadsheet yourself.

Creating a Basic Spreadsheet

When you open up Excel, the program will start you out with a blank spreadsheet, which should look something like the one shown here.

	A	B	C	D	E
1					
2					
3					
4					
5					

The number of columns and rows that you'll see, the specific colors that show up, and other such details may vary depending on the version of the program you are using. However, your starting point will most likely be in cell A1, and this will be apparent by that cell being highlighted in some way. If you begin typing, what you type will appear in that cell. We want cell A1 to contain the word "Product" so let's type that now, and then hit Enter. The result should look like this.

	A	B	C	D	E
1	Product				
2					
3					
4					
5					

You can move to other cells either by using the arrow keys on the keyboard, or by moving the cursor with your mouse and clicking on the cells you want to go to next. Now, let's move to each of the remaining cells in row A and column 1 and type in the text for each. The result will probably look similar to this.

	A	B	C	D	E
1	Product	Units Sold	Unit Price	Product Sales	
2	Solar Singing Fish Gutter				
3	Home Crash Test Kit				
4	Twelve Tone Door Chime				
5	Totals				

This is a bit of a mess, because the text is too long to fit in some of the cells. The text in A2, for example, in addition to looking sloppy as it overruns its cell, will cause problems when we put the items sold information into B2, since both the text from A2 and the numerical value from B2 will be trying to show up in the same place on the screen. To fix this, we need to adjust the width of the cells to fit their contents.

- With the mouse, click on cell A1 and hold the button down when you do.
- Then with the mouse, move the cursor all the way to D5. (This will highlight the area we are using for the spreadsheet.)
- Select (click on) the Format menu from the menu bar at the top of the screen.
- Select Column from that menu.
- This will bring you to yet another menu, and from that one select Autofit Selection. This will automatically adjust the widths of the columns to fit the contents of the cells.

The result should be a much more manageable looking, as you can see here.

	A	B	C	D
1	Product	Units Sold	Unit Price	Product Sales
2	Solar Singing Fish Gutter			
3	Home Crash Test Kit			
4	Twelve Tone Door Chime			
5	Totals			

In the original version, which we saw back in Figure 5.1, all of the text was centered inside the cells. We can make this happen by once again highlighting the area we are using. Most likely there will be three buttons in the toolbar at the top of your screen, one of which shows text all justified to the right, one of which shows it centered, and one of which shows it justified to the left. If you click on the one with the text centered, it will format the cells you have highlighted so that their contents will be centered. If those buttons are not on your toolbar, you can do this by selecting the Format menu, then Cells, then Alignment, then Horizontal, then Center. (From here on in, we'll indicate a string of menu selections like this: Format ➞ Cells ➞ Alignment ➞ Horizontal ➞ Center.)

To make the text in row 1 bold, you can highlight that row, and then either hit the bold "**B**" button from the toolbar, or use Format ➞ Cells ➞ Font ➞ Font Style ➞ Bold. You can bold A5 in the same way.

Now, to the numbers. Type in the Units Sold (cells B2, B3, and B4) and Unit Price (cells C2, C3, and C4) values. Your resulting spreadsheet will look like the one here.

	A	B	C	D
1	**Product**	**Units Sold**	**Unit Price**	**Product Sales**
2	Solar Singing Fish Gutter	1800	23.95	
3	Home Crash Test Kit	880	127.95	
4	Twelve Tone Door Chime	500	44.99	
5	**Totals**			

To format the cells in the Unit Price column to look like money, highlight these cells and then use Format ➞ Cells ➞ Number ➞ Currency ➞ 2 decimal places.

Now, moving on to the formulas in column D, in D2, we want the program to multiply the contents of B2 by the contents of C2. To do this, enter "=B2*C2" in that cell. (The "=" is necessary to tell the program that we are entering a formula. Without it, the program will think you want it to store the text "B2*C2" and will display that text in the cell.) Once you hit enter, the cell should display the result, 43200.

For D3 and D4 we want the spreadsheet to do the same thing. Rather than type a similar formula into each cell, we can copy what we did in D2. To do this, highlight D2 and then Edit ➞ Copy (or click on the copy icon from the toolbar.) Then, highlight D3 and D4 and

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Edit → Paste (or click on the paste icon.) You should see the correct amounts now in the product sales column.[1]

Here, we exploited of one of the biggest advantages of spreadsheet programs. When we copied D2 to D3, we didn't mean we wanted an *exact* copy of D2 in D3. We didn't want D3 to contain "=B2*C2". We wanted it to have a formula that would do the same sort of thing, but one row down: "=B3*C3". Fortunately for us, when you copy a formula the spreadsheet program automatically assumes that if you are moving the formula down one row, you must want all the cells in that formula to be moved down a row as well. If you highlight D3 and look in the edit line, you will see that the formula in D3 is what we wanted, "=B3*C3". Likewise, the formula in D4 is "=B4*C4". The advantage of being able to do this should be obvious; even if we were working with 1,000 different products in 1,000 rows, we could still set up a spreadsheet to calculate the product sales for each and every row easily and quickly just by copying and pasting.

The spreadsheet should now look like this.

	A	B	C	D
1	**Product**	**Units Sold**	**Unit Price**	**Product Sales**
2	Solar Singing Fish Gutter	1800	$23.95	43110
3	Home Crash Test Kit	880	$127.95	112596
4	Twelve Tone Door Chime	500	$44.99	22495
5	**Totals**			

Lastly, we want to set up the program to put the totals in the last row. We can once again do this with a formula. In B5, enter the formula "=SUM(B2..B4)". This formula tells the program to add up all the values in the range of cells from B2 to B4. We can then copy this into cell D5 to get the sum of the product sales. (We won't do this in column C, because a sum of all the unit prices is probably not something anyone would be interested in, so we can just put the text "N/A" in that cell.)

At this point, you may want to format column D. Since the dollar amounts are fairly large, it makes sense to format this column for 0 decimal places (so it will display the amounts rounded to the nearest dollar.) You may want to bold the total row as well to make it stand out more. The result, shown here is the table we set out to create.

	A	B	C	D
1	**Product**	**Units Sold**	**Unit Price**	**Product Sales**
2	Solar Singing Fish Gutter	1800	$23.95	$43,110
3	Home Crash Test Kit	880	$127.95	$112,596
4	Twelve Tone Door Chime	500	$44.99	$22,495
5	**Totals**	**3180**	**N/A**	**$178,201**

Making Changes in a Spreadsheet

Suppose that, after building this spreadsheet, we realize that we had inaccurate information. There were actually 1,500 door chimes sold, not the 500 we put in our original spreadsheet. This will change the total sales for that product, the total units sold, and the total overall sales of the company. Just this one change will result in many other changes.

[1]There are also shortcut ways to copy and paste, depending on the version of the program you are using. In this text, we are giving the steps to do most things from menu selections, since those steps will work on just about any version of Excel. However, your instructor may show you some shortcuts that can be used with the specific spreadsheet program that you are using.

Here is where using formulas in a spreadsheet really pays off. Fortunately, since we set the spreadsheet up to calculate those values with formulas, updating the spreadsheet is quite easy. Simply go to cell B4 and type in the number 1500 to replace the 500 that is already there. As you can see here, the values in B5, D4, and D5 will automatically change as a result.

	A	B	C	D
1	**Product**	**Units Sold**	**Unit Price**	**Product Sales**
2	Solar Singing Fish Gutter	1800	$23.95	$43,110
3	Home Crash Test Kit	880	$127.95	$112,596
4	Twelve Tone Door Chime	1500	$44.99	$67,485
5	**Totals**	**4180**	**N/A**	**$223,191**

You can experiment for yourself by making changes in units sold, or changes in unit prices. No matter how many changes you make to the units sold or unit prices, the spreadsheet automatically updates all the results that depend on these values.

We can now easily make changes to the sales and prices for the three products we included in our sheet, but what if we realize that we needed to add a fourth product to the list? Suppose we now realize that we forgot that the company also sold 1,200 Widget Detectors at $8.95 each and need to add those to the sheet. This requires a little more effort, since we need to insert an entire new row above the total column.

To accomplish this, click on cell A5, and then use Insert → Row. This will insert a row above the Total row, moving the Totals down to row 6. (Once again, the formulas automatically update to reflect this change.) In the new A5, type in the product name, in the new B5 type in the number of units sold, and in the new C5 type in the unit price. We can copy the formula for D5 from D4.

Since the row was inserted into the range of cells included in the sum formulas, those formulas should automatically update to include the values from the new rows. There should be no need to adjust the formulas in B6 and D6.[2] Once we make any needed formatting adjustments, the result should be a spreadsheet like the one here.

	A	B	C	D
1	**Product**	**Units Sold**	**Unit Price**	**Product Sales**
2	Solar Singing Fish Gutter	1800	$23.95	$43,110
3	Home Crash Test Kit	880	$127.95	$112,596
4	Twelve Tone Door Chime	1500	$44.99	$67,485
5	Widget Detectors	1200	$8.95	$10,740
6	**Totals**	**5380**	**N/A**	**$233,931**

Before moving on, let's work through another spreadsheet example similar to the one we've created here.

Example 5.1.1 ***A company has four employees who all work different hours and are paid different hourly rates. Adam earns $12.75 per hour, Betty earns $11.85 per hour, Carole makes $13.95 per hour, and Dario earns $12.50. Last week each person worked the following hours: Adam: 20, Betty: 28, Carole: 36, and Dario: 27.5.***

Set up a spreadsheet to find the total gross (i.e., before deductions) pay for each person and the total gross pay for all four employees for last week.

To create this spreadsheet, we follow essentially the same steps as in the first example. In fact, the formulas are essentially the same as in that example, though column D should be

[2]You should make sure to check that this actually does happen though. In some versions of Excel this does not consistently work as expected.

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formatted to give dollars and cents instead of showing values rounded to whole dollars. The end result should look similar to this:

	A	B	C	D
1	**Employee**	**Hours**	**Rate**	**Gross Pay**
2	Adam	20	$12.75	$255.00
3	Betty	28	$11.85	$331.80
4	Carole	36	$13.95	$502.20
5	Dario	27.5	$12.50	$343.75
6	Totals	111.5	N/A	$1,432.75

Once you have a spreadsheet set up for a particular application, you can reuse it over and over again, as the next example illustrates.

Example 5.1.2 ***In the next week, Adam worked 22.5 hours, Betty worked 25, Carole put in 32, and Dario worked 35. Find the total gross payroll for the company for that week.***

To find this, all we need to do is change the hours in the spreadsheet we created in Example 5.1.1. The program then automatically calculates the desired values.

	A	B	C	D
1	**Employee**	**Hours**	**Rate**	**Gross Pay**
2	Adam	22.5	$12.75	$286.88
3	Betty	25	$11.85	$296.25
4	Carole	32	$13.95	$446.40
5	Dario	35	$12.50	$437.50
6	Totals	114.5	N/A	$1,467.03

The total gross payroll for the company this week is $1,467.03.

Rounding in Spreadsheets

In this example, the formatting we used hid something from us. If Adam worked 22.5 hours and made $12.75 per hour, his total gross pay would be (22.5)($12.75) = $286.875. Of course, we know enough to round this to $286.88, and that is the value that was displayed as Adam's gross pay on the spreadsheet.

Unfortunately, the spreadsheet *displays* $286.88, but that is not actually what the program "thinks" Adam's gross pay for the week should be. While you and I know that money must be rounded to two decimal places, the spreadsheet doesn't. It displays the value to two decimal places because that is what we told it to. However, if this value is used in any further calculations, the spreadsheet will use 286.875, not 286.88. Now, the discrepancies that this may cause are not large, but they can be aggravating. Further, Adam's actual gross pay really will be rounded, and so it actually is not correct for the program to think his gross pay is 286.875.

We can, and should, address this by including an instruction in the formula to tell the program that it actually should round to two decimal places—not just in the display, but in the actual numerical value as well. The instruction to do this in Excel may look a bit strange at first. To indicate that something should be rounded, you use the formula "=ROUND(number, number of decimal places)". So in the case of Adam's pay, the formula in that cell should be "=ROUND(B2*C2,2)". This formula should likewise be copied into the gross pay column for each of the other employees.

Do we need to alter the total formula in D6? Actually, we don't. Since each employee's gross pay is now rounded to dollars and cents, there is no way that the total of all of the gross pay amounts could have any extra decimal places in it. Of course, it doesn't hurt anything to tell the program to round it, and you may want to get in the habit of including rounding of any formulas whose answers should be dollar amounts.

While failing to actually round did not make any difference in this payroll example, in a large spreadsheet that involves many repeated calculations, rounding can become an issue. For that reason, from here on in we will adopt the practice of rounding all formulas that calculate money values.

Illustrating Compound Interest with Spreadsheets

In Chapter 3, we first looked at compound interest by using a table and watching the account value grow with each compounding. Then we developed the compound interest formula, and so we had a much more efficient way of getting from present value to future value, and we abandoned the compounding-by-compounding table approach.

When we just use the formula, though, we lose the ability to see how that growth occurs. While the huge final account values that compound interest can create over time should no longer be surprising to us, it was probably hard to believe at first, which is why having a table that shows the growth year by year was helpful to convince us that the figures really were correct. This is of course useful if you need to convince someone else that a figure that seems unrealistic is actually correct. Also, if we want to be able to see where the account value is at different points along the way, a table makes it easy to see, while getting those answers with the formula would require using the formula over and over again.

Of course, creating these sorts of table by hand is tedious, but it is not too hard to do with a spreadsheet. In the following example, we will use a spreadsheet to create such a table.

Example 5.1.3 ***Tom deposited $5,000 at 7% compounded annually for 4 years. Use a spreadsheet to find his future value and illustrate the compound growth by showing the interest earned with each compounding.***

Our goal here is to create a table similar to the ones we saw at the start of Chapter 3. So as a starting point, we will want to label columns for time, starting balance, interest earned, and ending balance:

	A	B	C	D
1	**Time**	**Starting Balance**	**Interest Earned**	**Ending Balance**
2				
3				
4				

You may need to adjust the column widths to fit these labels.

We should then have rows for each year, so we will want to label our rows for time 1, 2, 3, and so on. We could do this by typing that into each row. But we can avoid all that typing by use of formulas. We will enter 1 in cell A2, and then, since the cell below it should be 1 more, we can enter into that cell the formula "=A2+1". Then, we copy this formula, highlight cells A4 through 5, and paste into those cells. The result should be:

	A	B	C	D
1	**Time**	**Starting Balance**	**Interest Earned**	**Ending Balance**
2	1			
3	2			
4	3			
5	4			

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Since Tom's starting balance was $5,000, we will enter 5000 in cell B2. The interest he earned in the first year can be found by using I = PRT = $5,000(0.07)(1) = $350. However, rather than just enter 350 in C2, we will use a formula. "=B2.07*1" would work to let the spreadsheet program do the calculation. However, as discussed above, we need to include an instruction in our formula so that the value will be rounded appropriately. The formula we need then is "=ROUND(B2*.07*1,2)".*

The ending balance for the first year should be $5,000 + $350 = $5,350, but once again, we'll let the spreadsheet do the work. Since the ending balance is the starting balance plus the interest earned, entering the formula "=B2+C2" in cell D2 will do the trick. We don't need to include rounding instructions in this formula, since the numbers we are adding will already be in dollars and cents terms. However, as we discussed above, it doesn't hurt to round it either, so "=ROUND(B2+C2,2)" would also be acceptable.

Your spreadsheet should now look like this:

	A	B	C	D
1	**Time**	**Starting Balance**	**Interest Earned**	**Ending Balance**
2	1	$5,000.00	$350.00	$5,350.00
3	2			
4	3			
5	4			

Now let's look at year 2. The starting balance for year 2 is the same as the ending balance for year 1, so in cell B3 we can use the simple formula "=D2". Interest is calculated the same way in year 2 as in year 1, and so we can just copy cell C2 and paste it into C3. Similarly, we want D3 to do the same thing as D2 did, and so we can just copy and paste cell D2 into D3. The result should be:

	A	B	C	D
1	**Time**	**Starting Balance**	**Interest Earned**	**Ending Balance**
2	1	$5,000.00	$350.00	$5,350.00
3	2	$5,350.00	$374.50	$5,724.50
4	3			
5	4			

In each of the remaining rows, we want essentially the same thing as we have in row 3. Since the formulas will adjust the cell references automatically, we can complete our table simple by highlighting cells A3 through D4, copying them, then highlighting the range of cells from A4 down to D5, and then pasting.

The final result should look like this:

	A	B	C	D
1	**Time**	**Starting Balance**	**Interest Earned**	**Ending Balance**
2	1	$5,000.00	$350.00	$5,350.00
3	2	$5,350.00	$374.50	$5,724.50
4	3	$5,724.50	$400.72	$6,125.22
5	4	$6,125.22	$428.77	$6,553.99

If we wanted to have the account run for a longer period of time, we would only need to copy the formulas from the last row into as many rows as we like. Doing this, we can create a table for 40 years as easily as 4. In fact, now that we mention it . . .

Example 5.1.4 ***Tom deposited $5,000 at 7% compounded annually for 40 years. Use a spreadsheet to find his future value and illustrate the compound growth by showing the interest earned with each compounding.***

This is the same problem as Example 5.1.3, just for 36 more years. However, carrying the table out 36 years further is not much work. Each year works essentially the same as the one before. So highlight cells A5 to D5 and copy. Then, highlight from A5 to D41 and paste. This will add all the additional rows needed. To save space, rather than show the entire table, we will show only the last row here.

	A	B	C	D
41	40	$69,974.20	$4,898.19	$74,872.39

Two things deserve comment before we leave this example. First, in Example 5.1.4, without much explanation the solution said to take the table as far as row 41. The reason for going to row 41 was that, since the first row is a header, the number of the row is always one ahead of the number of the year. Row 3 contained year 2, row 4 contained year 3, and so on. For that reason, we could have seen that year 40 would be in row 41. However, if we didn't notice that it would not have been too big of a deal. If you are unsure about how far to copy, you can simply copy "a ways" down. If you copy too many rows, it is no problem; you can either delete or ignore the extras. If you do not copy enough, it is a simple matter to just copy some more. It never is a bad thing to first figure out the exact number of rows you need, but it is not something that is worth worrying too much about either.

Secondly, if you go ahead and find the future value of $5,000 at 7% compounded annually for 40 years, using our compound interest formula, you certainly would have every reason to expect the result to be $74,872.39. Yet, *it isn't!* Calculating the future value with the compound interest formula gives $74,872.29. The reason for this discrepancy lies in the rounding. In our spreadsheet calculation, we rounded the interest every year. But there was no such year-to-year rounding in the formula, and so it is possible that the formula-calculated value will differ from the spreadsheet-calculated one.

If you change the formulas in the spreadsheet, taking out the rounding instructions, your final answer will agree with the formula's. Why then did we do the rounding? In reality, when a bank pays you interest, it pays you interest in dollars and cents—rounded amounts. A strong case can be made that $74,872.39 is the better answer to this question, since what you see in your spreadsheet is a simulation of what actually would happen in an actual account.

In any case, though, as we have seen repeatedly in the past, small discrepancies due to rounding are not enough to worry about. The difference between these two answers raises a question that we should be able to answer, and we should also recognize that it is not a big deal. However, it is a good thing to be aware of this consideration, since, as we'll see in the rest of the chapter, discrepancies due to rounding often show up in places such as amortization tables, and when they do it is important to be able to recognize and explain the source of the discrepancy.

More Formatting and Shortcuts

In the examples we have done so far, we have worried about formatting only so far as it affects the ability to reasonably read the spreadsheet. Making it pretty has not really been a goal. There is plenty more that we can do with formatting. We can adjust fonts, highlight cells, use colors, and so on to make the table's appearance more professional and visually appealing. However, as we stated at the outset, our goal in this chapter is only to use spreadsheets as a tool for calculations. We will limit our formatting efforts only to the minimum needed to make the table reasonably easy to understand. You may, though, want to explore and play around with some other formatting options on your own.

Excel and other spreadsheet programs also offer a number of shortcuts, quicker ways to do many of the things we did above without having to navigate through menus. There are, for example, tricks for copying and pasting in Excel that can accomplish what we've done here with fewer mouse clicks. Your instructor may have shown you some of these, or

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you may already know some yourself if you have used Excel before. As with formatting, though, our goal here is only to introduce spreadsheets and use them as a calculation tool; becoming a "power user" falls outside the scope of this book.

Classes or training sessions, whether for credit or not, are widely offered by many colleges, adult education programs, and businesses, and you may want to consider taking advantage of some of those offerings, since expertise with spreadsheets can be a valuable skill for many different jobs.

EXERCISES 5.1

A. Creating a Basic Spreadsheet

Exercises 1 to 4 are based on the following situation: Some electric utilities offer their customers a "time-of-use" rate plan, where the rate charged for electricity varies depending on the time of day. Often, a higher rate is charged for electricity used during peak hours of the business day, when demand is high and power plants are struggling to produce enough, while a lower rate is charged for off-peak use, when demand is lower than the power plants' capacity.

Suppose that the Jamesboro Valley Municipal Electric Company offers such a plan to its business customers. Monday through Friday, the peak rate is $0.1385 per kilowatt hour (kWh) and the off-peak rate is $0.0545 per kWh. On the weekend, the rate is $0.0825 regardless of time of day. (It is not unusual to have a rate go beyond two decimal places, though the overall bill would be rounded to the usual two.)

1. Create a spreadsheet to match the one shown below.

	A	B	C	D
1	**AnyCorp Manufacturing Inc.**		**7/06 Electric Bill**	
2	**Time of Use**	**Kwh Used**	**Rate Per Kwh**	**Total**
3	Weekday Peak	28595	0.1385	$3,960.41
4	Weekday Off-Peak	14675	0.0545	$799.79
5	Weekend	8503	0.0825	$701.50
6	Totals	51773	N/A	$5,461.70

2. Use the spreadsheet you created in Exercise 1 to find the company's total electric bill for August, in which it used 35,642 weekday peak, 16,754 weekday off-peak, and 11,402 weekend kWh.

3. Use the spreadsheet you created in Exercise 1 to calculate the company's total electric bill for September. In that month, it used 27,043 weekday peak, 12,998 weekday off-peak, and 9,017 weekend kWh. Also, in September, the electric company lowered its weekday peak rate to $0.1318 per kWh and raised its weekend rate to $0.0885 per kWh.

4. (This problem builds on Exercise 3.) Suppose that in September the electric company had also changed its rate structure, establishing peak and off-peak rates for the weekend as well. Suppose that the weekend peak rate is $0.0995 per kWh and the weekend off-peak rate is $0.0475 per kWh. AnyCorp's total weekend use of 9,017 kWh in September was split between 4,003 peak kWh and 5,014 off-peak kWh. Calculate the company's total electric bill for September.

Exercises 5 to 8 all are based on the following scenario. Each problem in this group builds upon the previous problems (i.e., Exercise 6 is based on Exercise 5 and so on.)

Odenbach Industrial Refrigeration Corp. produces and sells a line of commercial refrigeration units. Their product line consists of four models: The A-780, the B-1000, the C-2750, and the D-365.

5. Suppose that the prices for these units are as follows: each A-780 costs $14,500, each B-1000 costs $17,900, each C-2750 costs $25,800, and each D-365 costs $46,000.
   a. Construct a spreadsheet similar to the one from Example 5.1.1 to calculate the total dollar amount of sales for each product, and for the product line as a whole.
   b. Use your spreadsheet to find the total dollar amount of sales for October, when the company sold 128 A-780s, 75 B-1000s, 12 C-2750s, and 34 D-365s.

6. Use the spreadsheet you created in Exercise 5 to find the total dollar sales generated by sales representative Will Lohman, who sold six A-780s, four B-1000s, no C-2750s, and two D-365s.

7. Suppose that the company pays its sales representatives a 2½% commission on the A-780 and B-1000, and a 1¾% commission on the C-2750 and D-365 models. Modify your spreadsheet so that it will calculate Will's commission earnings for the month of October.

8. In November, Will sold seven A-780s, nine B-1000s, four C-2750s, and none of the D-365s. The company raised its price on the A-780s to $15,525 and lowered the price on the C-2750 to $24,075. Also, it increased the sales commission on the B-1000s to 2¾%. Use a spreadsheet to calculate Will's commission for that month.

B. Illustrating Compound Interest with Spreadsheets

9. Set up a spreadsheet similar to the one from Example 5.1.3 to calculate the future value of $3,407.19 invested at 5.25% compounded annually for 37 years.

10. Set up a spreadsheet to calculate the future value of $14,278.19 invested at a 9½% effective rate, and use it to determine the value of this account after (a) 10 years, (b) 20 years, (c) 30 years, and (d) 40 years. Also use the spreadsheet to determine the amount of interest earned in the (e) seventh year, (f) twenty-third year, and (g) fortieth year.

11. Suppose that I invested $2,375.17 at 7.35% compounded monthly for 20 years. Set up a spreadsheet to illustrate how my account will grow with this compound interest each month. How much will I have at the end of the 20 years? How much interest will I earn in the first month? How much interest will I earn in the last month?

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12. Calculate the future value for Exercise 11, using the compound interest formula. Does the answer agree with the value calculated by the spreadsheet? Should it?

C. Grab Bag

13. Don invested $2,000 at 6% compounded annually for 10 years. Then, he found that he could earn a higher rate and for the next 30 years his account earned 8%. Use a spreadsheet to find the future value of this investment at the end of 40 years.

14. A bicycle store wants to use a spreadsheet to determine the retail prices for the bikes that it sells. The store uses the following pricing formula: begin with the wholesale cost for the bike, add a 27.5% markup, and add on $20.

    a. Set up a spreadsheet to allow the store manager to quickly determine the retail price of any bike, based on cost.
    b. Use your spreadsheet to calculate the retail prices for these bikes (and wholesale costs given in parentheses): model A ($349.02), model B ($189.08), model C ($201.05), model D ($505.05) and model E ($372.75).
    c. Use your spreadsheet to calculate the total bikes sold for these models in a month when the store sold 13 Model A's, 17 B's, 6 C's, 8 D's, and 11 E's. (*Note:* If you need to add columns to your spreadsheet, they can be added in the same way that you add rows.)
    d. Use your spreadsheet to determine the store's total sales for the month.
    e. Use your spreadsheet to determine the store's total gross profit for the month. (i.e., the difference between the total retail sales and the total wholesale cost of the products sold.)

15. A law firm has three associate attorneys and two paralegals, each of whose time is billed at a different rate. The associates (and hourly rates) are Dewey ($175/hour), Cheatham ($215/hour), and Howe ($275/hour). The paralegals (and hourly rates) are Yu ($85/hour) and Bette (also $85/hour).

    a. Set up a spreadsheet to determine the total bill for each client, based on the billable hours for each member of the firm.
    b. Use your spreadsheet to calculate the total bill for a corporate client that had billable hours as follows: Dewey 3.5, Cheatham 0, Howe 11.5, Yu 6.0, and Bette 1.5.
    c. Use your spreadsheet to calculate the total bill for a corporate client that had billable hours as follows: Dewey 1.0, Cheatham 0.5, Howe 1.75, Yu 3.0, and Bette 0.
    d. The firm offers a discounted rate to one of its largest clients. For this client, Dewey and Howe bill $165/hour and Cheatham bills $160/hour. Yu bills $65/hour and Bette's rate is $60/hour. Use your spreadsheet to calculate the bill for a month in which this client had billable hours as follows: Dewey 24.0, Cheatham 17.5, Howe 32.5, Yu 49.5, and Bette 8.0.

D. Additional Exercise

16. Suppose that I deposit $750 per quarter into an account earning 8.13% compounded quarterly for 30 years.

    a. Set up a spreadsheet to calculate the future value of this account.
    b. Suppose that each year I increase my quarterly deposits by $10 each. (So I deposit $760 per quarter in year 2, $770 per quarter in year 3, and so on.) What would the future value of my account be then?

5.2 Finding Future Values with Spreadsheets

Spreadsheets can be effectively used to calculate future values for annuities, as the following example will show:

Example 5.2.1 ***Use spreadsheets to find the future value of an annuity of $1,000 per year for 8 years at an interest rate of 8%.***

We will calculate this future value by setting up a spreadsheet that shows how the value of the annuity grows with each payment, similar to the tables used in Section 4.2 when we first encountered annuities and their future values. In Section 4.2 we developed the future value of an annuity in two different ways. We could use either the bucket or the chronological approach here equally well, but since the chronological approach probably seemed more natural, we will set up our spreadsheet that way. (In other words, we will set up our spreadsheet showing how the account grows year by year.)

We start as follows:

- *Use the first row for titles of columns for Year, Starting Balance, Interest Earned, Payment, and Ending Balance*
- *Set up the first row of values by entering 1 in A2, 0 in B2, and 1000 in D2. In C2 enter the formula "=ROUND(B2*.08*1,2)". In E2 enter the formula "=B2+C2+D2".*
- *In the second row, enter the following formulas: In A3, enter "=A2+1", in B3 enter "=E2" (since the starting balance of year 2 is the same as the ending balance of year 1), and in D3 enter "=D2". Copy C2 into C3 and E2 into E3.*

The result should look like this:

	A	B	C	D	E
1	**Year**	**Starting Balance**	**Interest Earned**	**Payment**	**Ending Balance**
2	1	$0.00	$0.00	$1,000.00	$1,000.00
3	2	$1,000.00	$80.00	$1,000.00	$2,080.00

The entries for the rows for years 3 to 8 will be set up in precisely the same way as for year 2.

So we can simply copy this row into the six below to complete our task.

	A	B	C	D	E
1	**Year**	**Starting Balance**	**Interest Earned**	**Payment**	**Ending Balance**
2	1	$0.00	$0.00	$1,000.00	$1,000.00
3	2	$1,000.00	$80.00	$1,000.00	$2,080.00
4	3	$2,080.00	$166.40	$1,000.00	$3,246.40
5	4	$3,246.40	$259.71	$1,000.00	$4,506.11
6	5	$4,506.11	$360.49	$1,000.00	$5,866.60
7	6	$5,866.60	$469.33	$1,000.00	$7,335.93
8	7	$7,335.93	$586.87	$1,000.00	$8,922.80
9	8	$8,922.80	$713.82	$1,000.00	$10,636.62

So the future value is $10,636.62.

Calculating the future value from the annuity formula as a check, we would get $10,636.63 as the future value. As with the compound interest spreadsheets in Section 5.1, the reason for this slight difference once again lies in the rounding. In the spreadsheet we rounded the interest calculation each year; this rounding doesn't happen when we use the annuity formula. These discrepancies are far too small to make any meaningful difference in the results.

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Building a Future Value Spreadsheet Template

Both in working through the examples of this (and the following) sections and in doing the homework, we will be creating a lot of future value spreadsheets. While it is of course fine to create a new spreadsheet from scratch for each problem we work, we can save a lot of pointless and redundant work by saving a future value spreadsheet with the basic features required, and then modifying that basic spreadsheet as needed for each problem.

If you take the spreadsheet we created in Example 5.2.1 and copy the last row down to another 30 or 40 more rows, it will serve as a good basic template for future value spreadsheets. It is highly recommended that you do this before continuing with this section.

Spreadsheets for Nonannual Annuities

Suppose we want a spreadsheet to illustrate an annuity whose payments are not annual. The approach is essentially the same, but we have to be careful to make sure that we don't mistakenly apply the entire annual interest rate every period. In other words, don't forget the T in PRT.

Example 5.2.2 ***Use spreadsheets to find the future value of an annuity of $83.33 per month for 8 years at an interest rate of 8%.***

We start from the Example 5.2.1 spreadsheet as a template, making the following changes:

- *Change the header of column A to read Month instead of Year*
- *Change the payment amount in D3 to 83.33 instead of 1000.*
- *Change the formula in C2 to =ROUND(B2*.08*1/12,2)*

We also need to change the interest calculation in all the other cells of column C. But we have another problem as well; since this is monthly, to get to 8 years we need n = 8(12) = 96 rows, more than we have in the template spreadsheet to start off.

We can attack both of these issues at the same time though, since if we copy more rows, we can copy them with the "new" interest formula.

We don't want to copy the first row, however, since it is special in that its Starting Balance entry is not a formula the way it is in other rows. So we will do the following:

- *Copy cell C2 to C3.*
- *Highlight row 3, copy, and then paste the contents of row 3 as far down as (at least) row 97. This will overwrite the existing rows so that they contain the new interest formula.*

The resulting spreadsheet should look something like this one:

	A	B	C	D	E
1	**Month**	**Starting Balance**	**Interest Earned**	**Payment**	**Ending Balance**
2	1	$0.00	$0.00	$83.33	$83.33
3	2	$83.33	$0.56	$83.33	$167.22
	Rows Omitted				
97	96	$10,998.61	$73.32	$83.33	$11,155.26

So we conclude that the future value would be $11,155.26.

We have not shown all of the rows of this spreadsheet because the space required to show them would be so large, and because we really are interested only in the answer from the last row anyway. One major disadvantage of using spreadsheets for nonannual annuities is the number of rows that they usually require. These extra rows don't require any extra effort on our part—the computer is doing all of the heavy lifting—but can make a spreadsheet awkward to work with.

Notice that Example 5.2.2 is really the same as Example 5.2.1, except that the payments have been changed from annual to monthly. Not surprisingly, the future value with the monthly payments is not all that far off of the future value with annual payments; getting the money in a bit sooner and having interest compound more often leads to a larger future value, but not *that* much larger. Since nonannual spreadsheets are so awkwardly large, and because the results are so close, it is common practice to use annual payments with future value spreadsheets, especially when the purpose is a prediction or a projection. So, for example, even if it were an annuity of $83.33 per month we wanted to calculate and illustrate with a spreadsheet, we might choose to instead treat it as though the total $83.33(12) = $999.96 (essentially $1,000) was paid annually.

There are, of course, situations where it would be inappropriate to do this. In the exercises, you should set your spreadsheet up according to the payment schedule specified in the problem. You will, however, find that many of the exercises and remaining examples use annual payments.

Finding Future Values When the "Annuity" Isn't

Why would someone want to go to the effort of creating a spreadsheet when we could have just as well have found the future value with an annuity formula? One advantage of using a spreadsheet is that it allows you to see *how* the payments and interest build the account value. This can be especially helpful when the future value that will accumulate seems "too good to be true".

The annuity formula gives essentially the same end result, but it does nothing to illustrate the account's growth. We chose to use the annuity formulas for practical reasons—they were a lot less work than the alternatives. Even though the row-by-row chronological approach seemed logical and illustrated things nicely, we had to abandon it because of the effort required to do something like this line by line with pencil, paper, and calculator. Spreadsheets, though, allow us to create a line-by-line buildup of the future value with a far more reasonable amount of effort.

A second advantage of using a spreadsheet will be illustrated in the following example.

Example 5.2.3 ***Rework Example 5.2.1, this time assuming that the account started out with a $5,000 balance, and that the last three payments were increased to $2,000 instead of $1,000.***

This new stream of payments is not an annuity, since the payments are not all equal. However, we can readily find the future value simply by changing the appropriate cells in the payment column. The required changes are these:

- *Change the starting balance for the first year. Change cell B2 from 0 to 5,000.*
- *Change the payments for the last 3 years. Since the spreadsheet is set to assume that each year's payment is the same as the previous year's, if we change cell D7 from 1,000 to 2,000, this will change the payments for the last 3 years.*

The resulting spreadsheet should look like this:

	A	B	C	D	E
1	**Year**	**Starting Balance**	**Interest Earned**	**Payment**	**Ending Balance**
2	1	$5,000.00	$400.00	$1,000.00	$6,400.00
3	2	$6,400.00	$512.00	$1,000.00	$7,912.00
4	3	$7,912.00	$632.96	$1,000.00	$9,544.96
5	4	$9,544.96	$763.60	$1,000.00	$11,308.56
6	5	$11,308.56	$904.68	$1,000.00	$13,213.24
7	6	$13,213.24	$1,057.06	$2,000.00	$16,270.30
8	7	$16,270.30	$1,301.62	$2,000.00	$19,571.92
9	8	$19,571.92	$1,565.75	$2,000.00	$23,137.67

(If your spreadsheet includes rows beyond these, they are harmless and can be ignored.)

So the future value comes to $23,137.67.

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The future value found in Example 5.2.3 could also have been calculated by using the techniques developed in Sections 4.6 and 4.7. But doing that would require us to break the stream of payments into several separate buckets, calculate the future values separately, and then add them all together. The effort required to set up and use a spreadsheet to do this compares pretty favorably to that approach.

The advantage of a spreadsheet becomes more obvious when dealing with a stream of payments that change frequently, as shown in the following example:

Example 5.2.4 ***Erika is setting up a retirement account. She plans to make a deposit into this account every year, but, rather than invest the same amount each year, she intends to deposit $2,000 this year and then increase her annual deposits by 4% each year. If her account earns 7.25%, how much will she have in 30 years?***

Even though there is a clear pattern to her deposits, Erika's account is not even close to being an annuity. Every year's payment is different! Even though it is theoretically possible to find her future value without using spreadsheets, it would be a nightmare. Using a spreadsheet, though, this is hardly any more trouble than if the payments all stayed the same.

*We set up the spreadsheet just like the previous examples. For the first payment, we enter 2,000. Then, for the next year's payment we use the formula "=ROUND(1.04*D2,2)". Then, we can simply copy rows until we reach year 30. To save space, only the first 2 years and the last year are shown below:*

	A	B	C	D	E
1	**Year**	**Starting Balance**	**Interest Earned**	**Payment**	**Ending Balance**
2	1	$0.00	$0.00	$2,000.00	$2,000.00
3	2	$2,000.00	$145.00	$2,080.00	$4,225.00
	Rows Omitted				
31	30	$276,539.29	$20,049.10	$6,237.34	$302,825.73

Erika's future value is then $302,825.73.

From these examples, it should be clear that using spreadsheets for these sorts of calculations can offer advantages over using formulas, especially when the payments are not all exactly equal. The way we have set them up makes it comparatively easy to make changes to the stream of payments. But, what if we also want to be able to change the interest rate easily? The following example will illustrate a way to make interest rate changes conveniently.

Example 5.2.5 ***Rework Example 5.2.4 to find Erika's future value if her account earned (a) 8% and (b) 10%.***

We could make this change by editing the formula in C2, changing it to "=ROUND (B2.08*1,2)" and then copying that formula all the way down. Then, to do 10%, we could just do another change and copy. Instead, though, we will take a different approach and set the spreadsheet up so that we can easily change the rate to whatever we want more easily.*

To accomplish this, insert a new row above the column titles. In the new cell A1, enter the text "Rate:", and then in the new B1 enter the number 0.08. (You may want to format this to look like a percent: Format → Cells → Number → Percentage.) We will now change our interest formulas so that instead of having a particular interest rate in them, they instead will use whatever has been entered in cell B1.

*At first thought, it seems that the formula to put in the new C3 would be "=ROUND (B3*B1*1,2)". That would work fine in C3, but when we copy the formula down to the next row, the spreadsheet would also move the address "B1" down a row, and look for the interest rate in B2. This will obviously not work. We need a way to tell the spreadsheet* *that the cell address "B1" should not change when we copy the cells down. This is done by using a dollar sign in the cell address. By using "$B$1" instead of "B1", the program understands that no matter where we may copy the cell, the address of the interest rate always stays in column B and row 1. So instead the formula should be "=ROUND (B3*$B$1*1,2)". We can now copy this cell down the rest of the row.*

Doing this creates a spreadsheet like this:

	A	B	C	D	E
1	**Rate:**	8.00%			
2	**Year**	**Starting Balance**	**Interest Earned**	**Payment**	**Ending Balance**
3	1	$0.00	$0.00	$2,000.00	$2,000.00
4	2	$2,000.00	$160.00	$2,080.00	$4,240.00
	Rows Omitted				
32	30	$309,932.07	$24,794.57	$6,237.34	$340,963.98

So at 8%, Erika's future value would be $340,963.98.

To find the future value for 10% (or any other rate for that matter,) we can just change the cell with the rate in it. Doing so for 10%, we get that Erika's future value at that rate would be $473,534.93.

Using the $ to indicate that the cell being referenced in a formula should not change as the formula is copied to other locations is called an ***absolute cell reference***. As the previous example has illustrated, there are times where absolute references are necessary in a spreadsheet. When you enter a formula into a spreadsheet that refers to other cells, it is a good idea to ask yourself whether or not those cell references should change if the formula is copied, so that you make the appropriate choice as to whether an absolute reference should be used.

You may want to modify your future value spreadsheet template now so that the interest rate is given as a cell reference, as we did in this example. While not essential, having the interest rate so easily changeable can be an advantage.

The following exercises offer you the opportunity to get comfortable with using spreadsheets to find future values. (If you have previously covered Sections 4.6 and 4.7 you may recognize a number of these exercises from those sections. You may want to compare the solution techniques from those sections to the spreadsheet approach we are using here; each approach has its advantages and disadvantages.)

EXERCISES 5.2

A. Finding Future Values Using Spreadsheets

1. Khalil deposits $5,000 annually into an investment account that earns 8.3% compounded annually. Assuming he keeps this up for 37 years, calculate his account's future value (a) using the annuity formulas and (b) using a spreadsheet.

2. Cristina deposits $1,750 annually into an investment account earning 9.35%. How much will she have in this account in (a) 10 years, (b) 20 years, (c) 30 years, (d) 40 years, and (e) 50 years?

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3. *This question is based on Exercise 1*. How much interest will Khalil earn in the 20th year? How much interest will he earn in the 37th year? How much total interest will he earn?

4. *This question is based on Exercise 2.* How much interest will Cristina earn in the (a) 10th year, (b) 20th year, (c) 30th year, (d) 40th year, (e) 50th year?

5. Suppose that Traci deposits $100 each month into an account earning 7½% (compounded monthly). Set up a spreadsheet to illustrate the growth of her account over 20 years. According to this spreadsheet, how much will she have in her account in 20 years?

6. Suppose that Valeriy deposits $20 every other week into an account earning 9.15% (compounded biweekly). Set up a spreadsheet to illustrate the growth of his account. How much will he have in his account after 32 years?

B. Finding Future Values with Changing Payments

7. Wally is planning to deposit $2,350 each year into an account in which he expects to earn 7%.
   a. Use a spreadsheet to determine his account value based on these assumptions after 30 years.
   b. Suppose that after 10 years, Wally decides to increase his deposit to $3,000 per year. What would his account value be after 30 years?
   c. Suppose that in addition to what we assumed in part b, Wally already had $11,046.19 in this account. What would his account value be after 30 years?

8. a. Find the future value of an annuity of $1,750 per year for 20 years, assuming a 6.75% interest rate.
   b. Suppose that instead of $1,750 per year, the deposits were $1,750 per year for the first 5 years, $2,000 per year for the next 7, and then $2,700 per year for the last 8. What would the future value be then?

9. Suppose I have $34,072.16 in a retirement account right now, and I think that the account will earn 9.25%. I plan to deposit $3,000 into my account this year, and then increase my payments by 3% every year. For these assumptions, how much will I have in my account 25 years from now?

10. Lindsey is going to deposit $2,000 this year into an account paying 8%. In the future, she plans to increase her annual deposits by $250 each year, so that next year she will deposit $2,250, the following year she will deposit $2,500, and so on. Assuming she keeps this up, how much will she have in her account in 30 years? How much will she have deposited in total? How much total interest will she have earned?

C. Finding Future Values with Different Interest Rates

11. Fay deposits $50 each month into an investment account. Assuming she keeps this up, find her future value in 25 years if her account earns (a) 6.45%, (b) 8.25%, (c) 9%, (d) 10.75%, (e) 12½%. (*Note:* Use a spreadsheet; do not find these future values by using the annuity formulas!)

12. Micah plans to deposit $2,500 into a retirement account this year. In future years, he plans to increase his deposit amount by 4% each year. He also already has $16,902.11 in this account. Find his account's future value in 30 years under these assumptions, assuming his account earns (a) 6%, (b) 8%, (c) 10%, and (d) 12%.

D. Grab Bag

13. HuperMart Convenience Stores Corp. has $1,598,375 in an account set aside for longer term employee benefit commitments. The company is depositing $150,000 each year into this account. Assuming these deposits continue, and that the account earns 7½%, how much will the company have in this account in 10 years?

14. Trevor has $296,101.43 in his 401(k). Assuming he keeps up deposits of $3,300 per year and his account continues to earn 7.825%, how much will he have in his 401(k) when he retires in 20 years?

15. Suppose Kerry deposits $2,000 each year into an investment account every year. The only exception to this schedule is that for her 10th payment she deposits $5,000 instead of her usual $2,000 deposit. If her account earns 6.98%, how much will she have at the end of 30 years?

16. Mike deposited $2,500 into his individual retirement account (IRA) every year beginning at the end of 1982. He missed his deposit in 1991 because he was temporarily laid off, but other than that he has kept his deposits up every year. If his account earned 7.72%, what was his account value at the end of 2006?

17. Find the future value of an annuity of $4,000 per year for 28 years if the interest rate is (a) 6.5%, (b) 7.89%, and (c) 11.76%.

18. Suppose that the Cauchy-Schwartz Corp. set aside $150,000 each quarter for 12 years into a special fund to cover potential environmental cleanup costs. For the next 8 years, the company reduced its deposits to $80,000 per quarter. Find the value of this account at the end of 20 years, assuming it earned a 5.3% interest rate for the entire period.

19. Kelly deposited $1,620 per month into her retirement account for 20 years. For the next 12 years she increased her deposits to $2100 per month. Assuming she earned 8.13%, find her account's future value. Find the total interest earned on this account.

20. Find the future value in 22 years for an investment account if deposits are made annually, the first annual deposit is $750, and deposits increase by $75 per year in each subsequent year. *The interest rate is 8%.*

21. Zaxarinax Corp. has $475,901 in an account set aside for future financial goals. The company plans to deposit $125,000 in the account this year, and then each year after that increase its deposits by 3.75%. The company expects that this account will earn 8.19%. Under these assumptions, project the value of this account in 10 years.

22. Leslie deposits $400 each month for 10 years into an investment account. She then increases her deposits to $500 per month. If her account earns 8.4%, how much will she have at the end of 16 years?

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23. a. Suppose I deposit $1,500 per year for 10 years into an account earning 6%. For the next 15 years I increase my deposits to $1,800 per year, then for the next 5 years after that I decrease my deposits to $960 per year. Find my total future value.

b. Suppose I deposit $125 per month for 10 years into an account earning 6%. For the next 15 years I increase my deposits to $150 per month, then for the next 5 years after that I decrease my deposits to $80 per month. Find my total future value.

E. Additional Exercises

24. Suppose that I deposit $2,000 *at the start* of each year for 25 years. My account earns 7.5%. Create a spreadsheet to calculate my account's future value.

25. Suppose that you set up an investment account today, and the following occurs over the next 40 years:

- You deposit $750 a year for 10 years, then $1,250 a year for the next 10 years. In the 21st year you deposit $2,000, and increase your deposits by 5% every year thereafter.
- For the first 30 years you invest aggressively and earn a rate of 11% on your investments. In the last 10 years, you move your money to lower risk investments that earn a lower rate of 7%.

Find the future value of this account.

26. Alexander has set up a 401(k) account at work. He is paid every other week, and this year he will deposit $30 per paycheck into his 401(k). Next year, he plans to increase his biweekly deposit by 4% (to $31.20 per week), and every year thereafter he will increase his deposit by 4%. If his account earns 9.05%, find his future value.

5.3 Amortization Tables with Spreadsheets

In Section 4.5 we talked about amortization tables, which illustrate how the payments made on a loan kill off the loan balance, step by step. While the calculations done to complete an amortization table aren't all that complicated, there are a lot of them, which makes building an entire amortization table with a pencil, paper, and calculator an unrealistic project. It should be apparent, though, that this is exactly the sort of task that a spreadsheet can handle beautifully.

Example 5.3.1 ***Construct an amortization table spreadsheet for a $5,000, 2-year loan at 8.59% interest. (Assume payments are monthly.)***

As a first step, we'll set up columns, using the same format that we used in Section 4.5. Also, even though this problem doesn't ask us to consider different interest rates, we'll store the interest rate in a cell at the top as we did in Example 5.2.5 so that if we want to change the interest rate later, we can. We also may find it useful to store the initial loan amount in a cell, so we'll do that at the top as well.

	A	B	C	D	E
1	**Rate:**	8.59%	**Initial Balance:**	$5,000.00	
2	**Month**	**Payment**	**To Interest**	**To Principal**	**Ending Balance**

Next, we can calculate the amount of each payment, using the annuity formulas from Chapter 4. That works out to be $227.48. In A3 we enter "1", in B3 we enter "227.48", in C3 we enter "=ROUND(D1*B1/12,2)", in D3 "=B3-C3", and in E3 "=D1-D3". In the next row we enter in A4 "=A3+1", in B4 "=B3", in C4 "=ROUND(E3*$B$1/12,2)", in

D4 "=C4-B4", and in E4 "=E3-D4". The second month's row can then be copied for the months up to the 24th. The resulting spreadsheet will be:

	A	B	C	D	E
1	**Rate:**	8.59%	**Initial Balance:**	$5,000.00	
2	**Month**	**Payment**	**To Interest**	**To Principal**	**Ending Balance**
3	1	$227.48	$35.79	$191.69	$4,808.31
4	2	$227.48	$34.42	$193.06	$4,615.25
	Rows Omitted				
26	24	$227.48	$1.62	$225.86	$0.13

An amortization table is supposed to show how the loan is paid off in its entirety. For this reason, we probably would have expected that the ending balance in the 24th month would have been 0. Thirteen cents isn't much, but it isn't zero. Why isn't it zero?

When we calculated the monthly payment, it didn't come out to be *exactly* $227.48. Actually, the payment comes out to be $227.4842828 each month for us to pay off the entire loan over the 24 months. Of course it would be ridiculous to actually try to pay $227.4842828, but rounding means that with each payment we are actually $0.0042828 short. This difference is less than a penny—not worth worrying about—but over the course of 24 payments this tiny monthly shortfall adds up enough to actually appear. Most of the time, we just ignore these small amounts, just as we ignore the fraction of a penny in the monthly payment. In some cases, though, the final payment actually will be adjusted slightly, to make the final balance come out to be zero. (This is sometimes legally required.) In this case, if we raised the last payment to $227.61, the extra 13 cents of the payments would take the balance to actually $0.00.

Sometimes, the end of an amortization table will show a negative balance instead of a positive one. This happens when the payment is rounded slightly up; for example, if the payment from a formula comes out to be $393.48867102 we would round that up to $393.49, even though this means each payment is just a tiny bit more than it theoretically should be. In that case, instead of raising the last payment to make everything balance out to zero, we would lower it.

As recommended with future values, it is a good idea to save a basic amortization table as a template to work from, so that you do not have to always create them from scratch. The spreadsheet from Example 5.3.1 would be a good one to use as a template, though you might want to add more rows before saving it (since most amortization tables will run more than 24 rows.)

Using Amortization Tables to Find Payoff Time

So far, we've used amortization tables to *illustrate* how a loan gets paid off, payment by payment. When we set one up, we have already calculated the payment based on the loan's original balance, interest rate, and term. The table sheds light onto how the loan is paid off, but it doesn't really tell us anything that we didn't already know.

But amortization tables can also be used as a tool to answer questions that we weren't able to before. In Chapter 4, we saw that our annuity formulas could be used to find the payment or present value, but manipulating the formula to solve for an interest rate or term was not feasible. Setting up an amortization table with a spreadsheet, though, provides a way to solve those sorts of problems. The following example will illustrate this.

Example 5.3.2 ***Suppose Javad took out a mortgage for $172,500. The term is 30 years, and the interest rate is 7.5%. The monthly payment is scheduled to be $1,206.15 , but he figures he can pay $1,800 each month. How long will it take him to pay off the loan if he does this?***

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We set up an amortization table for Javad's loan just as in the previous example, using $1,800 as the payment. Unfortunately, we don't know how many rows we will need. We do, though, know that it will be less than the 360 we would need for the full 30 years, so we'll use 360, since we know that will definitely be enough.

The result looks like this:

	A	B	C	D	E
1	**Rate:**	7.50%	**Initial Balance:**	$172,500.00	
2	**Month**	**Payment**	**To Interest**	**To Principal**	**Ending Balance**
3	1	$1,800.00	$1,078.13	$721.87	$171,778.13
4	2	$1,800.00	$1,073.61	$726.39	$171,051.74
	Rows Omitted				
362	360	$1,800.00	-$4,958.93	$6,758.93	-$800,187.04

The values in row 362 (for month 360) are absurd, though, showing negative interest and an enormous negative balance. Since Javad's higher-than-required payment would have finished off the loan long before the 360th month, carrying the table out that far gives us nonsensical values. We can, though, use this table to answer our original question. The mortgage is paid off when the balance reaches $0, so we can scroll through the table to find the point where this happens. Rows 148 and 149 look like this:

148	146	$1,800.00	$18.37	$1,781.63	$1,158.30
149	147	$1,800.00	$7.24	$1,792.76	-$634.46

We see that if Javad pays the full $1,800.00 for the 147th payment, that would take his mortgage balance below zero. From this we can conclude that his loan will be paid off entirely by the time of his 147th monthly payment. We can also conclude, though, that he does not need to make a full $1,800 payment in the last month to bring his balance to zero.

How much should his last payment be? We can determine this in either of two ways. Since making the full payment would bring the balance to $634.46 below zero, his 147th payment will need to be only $1,800.00 − $634.46 = $1,165.54. Another way of seeing this is to note that his last payment needs to cover a balance of $1,158.30 from the prior month, plus $7.24 of interest, for a total of $1,158.30 + $7.24 = $1,165.54. Either way the conclusion is the same: his last payment should be $1,165.54. You can verify this by changing the 147th payment to $1,165.54 to get:

148	146	$1,800.00	$18.37	$1,781.63	$1,158.30
149	147	$1,165.54	$7.24	$1,158.30	$0.00

To answer the original question: It will take Javad 147 months to pay off the loan at this rate, with his last payment being $1,165.54.

While this last example assumed a consistent payment schedule made from the beginning of the loan, there is no reason why we couldn't do the same sort of thing starting from some point other than the loan's beginning, or using an irregular payment schedule. The following example will illustrate this.

Example 5.3.3 ***Ted and Kirsti owe $94,372.57 on their mortgage right now. Their monthly payment is $845.76 and their interest rate is 6.49%. They have inherited some money, and they are thinking about making a one-time $7,000 payment to reduce their mortgage debt. They also expect to be able to pay $1,200 a month for the next year, and then go back to paying $845.76 after that.***

(a) How long will it take to pay off the loan if they only pay the scheduled $845.76?

(b) How long will it take to pay off the loan if they make the extra payments?

To answer (a), we can set up an amortization table for Ted and Kirsti, using their current balance, interest rate, and payments. We do not need to worry about what their original balance might have been, since we are interested only in the future of their loan, not its past.

	A	B	C	D	E
1	**Rate:**	6.49%	**Initial Balance:**	$94,372.57	
2	**Month**	**Payment**	**To Interest**	**To Principal**	**Ending Balance**
3	1	$845.76	$510.40	$335.36	$94,037.21
4	2	$845.76	$508.58	$337.18	$93,700.03

Rows Omitted

	A	B	C	D	E
173	171	$845.76	$6.81	$838.95	$420.64
174	172	$845.76	$2.27	$843.49	-$422.85

By the same reasoning used in the previous example, we conclude that their loan will be paid off after 171 more payments of $845.76 and a 172nd payment of $422.91.

To answer (b) we just need to adjust the payments in the table we created for part (a). We change the first payment to $8,200 ($7,000 plus $1,200). Since the spreadsheet is set up to copy the payments to all future ones, this makes ***all*** *of the payments $8,200. But directly entering $1,200 for the second payment does not affect the first, while making all subsequent payments $1,200. To return Ted and Kirsti to their $845.76 payment, we change the 13th payment, which also changes all of the subsequent ones. The result looks like this:*

	A	B	C	D	E
1	**Rate:**	6.49%	**Initial Balance:**	$94,372.57	
2	**Month**	**Payment**	**To Interest**	**To Principal**	**Ending Balance**
3	1	$8,200.00	$510.40	$7,689.60	$86,682.97
4	2	$1,200.00	$468.81	$731.19	$85,951.78

Rows Omitted

	A	B	C	D	E
17	15	$845.76	$419.54	$426.22	$77,147.03
18	16	$845.76	$417.24	$428.52	$76,718.51

Rows Omitted

	A	B	C	D	E
143	141	$845.76	$4.78	$840.98	$43.32
144	142	$845.76	$0.23	$845.53	-$802.21

From this, we see that the 142nd payment would be the last, and that payment would only need to be $43.55.

Negative Amortization

Even though we can now mathematically handle any payment schedule for a given loan, there is usually a practical limit to what we can do with actual payments. While you are usually allowed to pay more than the scheduled payment, you are not normally allowed

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to pay less. It is fine for Ted and Kirsti (from Example 5.3.3) to pay any amount they like in any given month, but it must be at least $845.76. While it is mathematically possible to determine how long it would take to pay off their loan if they paid just $800 a month, in reality their lender would probably not allow this.

But sometimes a lender might allow, or even encourage, a borrower to pay less than the payment required to pay off the loan in the expected amount of time. For example, a lender might allow smaller payment for someone who has a temporary financial strain, such as loss of a job, military deployment, or completing a graduate degree. There are also more sinister possibilities: a lender might permit an initially low payment to make a borrower feel like the amount he is borrowing is not as large as it really is, or might encourage lower payments so that more interest will pile up on a debt. Regardless of the reasons for such a situation, though, the financial consequences can be serious, as the following example will illustrate.

Example 5.3.4 ***Jennifer owes $2,538 on a personal loan with a 12½% interest rate. She was originally supposed to make payments of at least $100 a month, but the lender has told her that from now on she can pay as little as $20 a month if she wants. If she decides to reduce her payment to just $20 a month, how long will it take her to pay off the remaining balance?***

Looking at an amortization table for this loan with a $20 payment, we get:

	A	B	C	D	E
1	**Rate:**	12.50%	**Initial Balance:**	$2,538.00	
2	**Month**	**Payment**	**To Interest**	**To Principal**	**Ending Balance**
3	1	$20.00	$26.44	-$6.44	$2,544.44
4	2	$20.00	$26.50	-$6.50	$2,550.94

Notice that the amount going to principal is a negative number, and that the ending balance is increasing. Because $20.00 is not enough to cover the first month's interest, the $6.44 that it falls short must be added to the balance. In the second month, the interest is a bit higher, so the $20.00 payment is a bit further short, and so the balance grows by a little bit more. Looking out to further months, we can see that the situation just continues to get worse and worse; after 20 years, for example, Jennifer's debt will have grown to $9,352.10 at this rate. And there is no opportunity for things to ever get better.

We can conclude from this that at $20.00 a month, Jennifer will ***never*** *manage to pay off this balance.*

When the amount of the payment falls below the interest charges, the shortfall causes the balance to increase instead of decreasing. This sort of situation is referred to as ***negative amortization***.

Example 5.3.5 ***What is the minimum monthly payment Jennifer (from Example 5.3.4) needs to schedule in order to avoid negative amortization?***

From the amortization table, we can see that if her payment is less than $26.44, we will have negative amortization. If she pays exactly $26.44 she won't encounter negative amortization, but her balance will never go down either, since that would be just enough to pay the interest, and nothing more. If she pays $26.45 (or more) she will be reducing interest by a little bit with each payment. The $26.45 a month won't accomplish much with each payment, but at least it will accomplish some reduction in the balance.

Her minimum to avoid negative amortization then is $26.45.

Of course, if she only pays a penny over the minimum, it will take a very long time to pay off the debt. But at that rate it will at least eventually get paid off. As the cliché goes, little things mean a lot. Slight changes in a payment schedule can make an enormous difference.

Example 5.3.6 ***If Jennifer pays $25 a month, how much will she owe in 20 years? If she pays $27.50, how much will she owe in 20 years? What if she pays $30.00?***

$25 a month is not enough to avoid negative amortization. Using the amortization table with this monthly payment, we can see that after 20 years her debt will have grown to $4,059.34.

Paying $27.50 is enough to avoid negative amortization, but it is just a bit more than a dollar above the $26.45 minimum. Nonetheless, this small difference is enough to reduce the balance to $1,413.53 after 20 years.

At $30.00 a month, just a bit more, the question becomes irrelevant, because the debt would be paid off in its entirety after the 206th payment.

Negative amortization does not necessarily always occur, though, even if the payment is not enough to cover the monthly interest on a loan's balance. In some cases, laws or other regulations, or even the terms of the loan itself, may prohibit compounding of interest on the loan. In such cases, interest does continue to be charged on the outstanding balance, but no interest is charged on any unpaid interest. The lender may still, though, assess late payment fees or other financial penalties, which may actually add up to more than compounding interest would have.

In the exercises, you should assume that interest will be charged on the loan's entire outstanding balance, without any distinction between original principal and unpaid interest. In other words, in the exercises you should assume that negative amortization may occur exactly as it has in the examples above.

EXERCISES 5.3

A. Basic Amortization Tables

1. Les is borrowing $18,350 to buy a new car. The loan interest rate is 6%, the term is 5 years, and his monthly payment is $354.76. Assume he makes all of his payments as scheduled.
   a. Construct an amortization table for Les' car loan.
   b. How much should his last payment be in order for the final balance to be exactly $0.00.
   c. How much total interest will Les pay on this loan.

2. Jasmine has just graduated from pharmacy college, and owes $38,965 in student loans that she must now start making payments on. Her monthly payments will be based on a 20-year term at a 5.39% interest rate. Assume she makes all of her payments as scheduled.
   a. Calculate her monthly payment.
   b. Construct an amortization table for her loan using this payment.
   c. According to this schedule, how much of her first payment will go to interest? To principal?
   d. According to this schedule, how must of her 121st payment (when she is halfway through the term) will go to interest? To principal?
   e. How much should her last payment be to bring her final balance to exactly $0.00.

3. Anica has taken out a business start-up loan for $75,000. The interest rate is 9.3% and she will make monthly payments for 10 years. Assume she makes all payments as scheduled.
   a. Calculate the monthly payment for this loan.
   b. How much total interest will she pay in the first year of this loan?
   c. How much should her last payment be to bring her final balance to exactly $0.00.
   d. How much will Anica owe halfway through this loan's term?

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B. Using Spreadsheets to Find Payoff Time

4. Brad's mortgage has an $89,902.49 balance right now. The interest rate is 7.59%. How long will it take him to pay off this loan if he pays $1,000 per month? How much will his last payment be?

5. Suppose you owe $8,502.25 on a credit card. Realizing that this is not a good thing, you vow to make no more charges to the card and work hard to pay the balance off. The interest rate is 18.75%. How long will it take to pay off this balance assuming you pay (a) $125 per month, (b) $250 per month, (c) $500 per month?

6. How long will it take me to pay off a $10,595 debt at 16% if I pay $250 per month?

7. Rick has taken out a $125,000 small business loan to be able to launch his new property maintenance business. The loan carries an interest rate of 7¼% compounded quarterly. If he makes quarterly payments of $7,500, how long will it take to pay off the loan, how much will his last payment need to be, and how much total interest will he pay?

C. Negative Amortization

8. Britt owes $1,984.92 on a personal loan. The interest rate is 15.4%. How long will it take to pay off this loan if she pays $25 a month? If she pays $50 a month? What is the minimum she needs to pay each month in order to avoid negative amortization?

9. What is the minimum monthly payment necessary to avoid negative amortization on a debt of $48,500 at 7.75%?

D. Grab Bag

10. I borrowed $21,500 for 6 years at 8.48%. Assuming that I make all of my monthly payments as scheduled, how much will I owe at the end of 3 years?

11. Erin and Scott have just taken out a mortgage loan for $168,308. The interest rate is 7.13% and the term is 30 years. Calculate their monthly payment for this loan. If they pay twice the monthly payment you calculated, how long will it take them to pay off their loan? If they pay half the monthly payment you calculated, how long will it take them to pay off this loan?

12. How long will it take to pay off a $100,000 debt at 10.44% interest with a monthly payment of $1,200? What would the final monthly payment need to be?

13. Suppose I have a loan with a $25,804.56 balance on which I am paying $825 a month. The interest rate is 7.9%. How long will it take me to pay off this loan at this rate?

14. What is the minimum you need to pay each month on a $81,575 debt at 10½% in order to avoid negative amortization?

E. Additional Exercises

15. All of the examples we have been considering have had interest rates that do not change. Many loans do have **fixed** rates, which don't change, but other loans carry **adjustable** rates, which may change.

    Suppose that you take out a 30-year adjustable rate mortgage with a balance of $208,900. The interest rate is 3.99% for the first year, and then increases by 1% (to 4.99%) in the second year, and in each additional year until the rate reaches 10.99%, where it remains for the rest of the mortgage's term. If you pay $1,000 a month, how long will it take to pay off this loan? If you pay $2,000 a month, how long will it take?

16. As mentioned in the text of this section, some loans do not allow negative amortization. If a payment is made that is not enough to cover the interest due in a month, the entire payment is applied toward the interest owed. While interest continues to accumulate on the balance, the unpaid interest is not included in that balance. Instead, the unpaid interest is considered "on hold"; future payments are not applied to any later interest or to reduce the loan's principle until all on-hold interest is paid.

    Suppose that Ed and Carol owe $96,575.18 on their mortgage. The interest rate is 6.75%. For the next 3 months, they pay only $250. After that, they increase their monthly payments to $1,000 each month and keep this up until the loan is paid off.

    Their loan does not allow negative amortization; too-small payments are handled as described at the start of this exercise. Also, any payment that is less than the scheduled payment is assessed a fee of $35.

    How long will it take for Ed and Carol to pay off their mortgage?

5.4 Solving Annuity Problems with Spreadsheets

In the previous sections we have seen how spreadsheets can illustrate what we calculate with annuity formulas while also allowing us to "crunch the numbers" for situations that we could not reasonably handle with our formulas. In this section, we will further explore the use of spreadsheets to solve problems that we would not be able to otherwise.

Example 5.4.1 ***Suppose that I deposit $3,000 each year into an investment account that earns 9%. How long will it take before my account balance reaches $1,000,000?***

Our annuity formulas for future value don't give us any reasonable way to find the term any more than we could for present value. We can, though, adapt the approach of Section 5.3 to a future value table to answer this question. Using the spreadsheet template saved from Section 5.2, we can adjust the rate and payments to fit this question, and then look for the year when the balance goes over $1,000,000.

	A	B	C	D	E
2	Year	Starting Balance	Interest Earned	Payment	Ending Balance
3	1	$0.00	$0.00	$3,000.00	$3,000.00
4	2	$3,000.00	$270.00	$3,000.00	$6,270.00
	Rows Omitted				
42	40	$927,199.65	$83,447.97	$3,000.00	$1,013,647.62

From this table, we see that in year 40 my account balance will reach $1,000,000.

Similarly, we can deal with more complicated situations by combining this idea with the work we did in Section 5.2.

Example 5.4.2 ***Miyako has $47,593 in her retirement account. She plans to deposit $2,000 this year, and will increase her payment by 3% each year. If her account earns 9%, how long will it be before her account balance reaches $500,000?***

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We set up an amortization table, and look for her account balance to hit the target value.

	A	B	C	D	E
1	**Rate:**	9.00%			
2	**Year**	**Starting Balance**	**Interest Earned**	**Payment**	**Ending Balance**
3	1	$47,953.00	$4,315.77	$2,000.00	$54,268.77
4	2	$54,268.77	$4,884.19	$2,060.00	$61,212.96
	Rows Omitted				
24	22	$434,552.51	$39,109.73	$3,720.59	$477,382.83
25	23	$477,382.83	$42,964.45	$3,832.21	$524,179.49

So we see that the target account value is reached in year 23.

Solving for Interest Rates

We have solved for time with present values in Section 4.3, and with future values in this section. So far, however, we haven't looked at the question of finding a needed interest rate. As with time, algebraically manipulating the annuity formulas to find a required interest rate is not a practical goal, but spreadsheets can provide us the ability to handle this.

Example 5.4.3 ***Bryce has $28,500 in his retirement account, and he plans to contribute $2,500 each year to this account. He wants to have $1,000,000 in this account 35 years from now. What interest rate does he need to earn to reach this goal?***

We start by setting up a spreadsheet to illustrate Bryce's account. Since we don't have an interest rate, we will set the spreadsheet up with an educated guess. You could use anything reasonable for this; for now, let's set things up assuming 8%. We get:

	A	B	C	D	E
1	**Rate:**	8.00%			
2	**Year**	**Starting Balance**	**Interest Earned**	**Payment**	**Ending Balance**
3	1	$28,500.00	$2,280.00	$2,500.00	$33,280.00
4	2	$33,280.00	$2,662.40	$2,500.00	$38,442.40
	Rows Omitted				
37	35	$786,735.37	$62,938.83	$2,500.00	$852,174.20

From this, we see that at an 8% rate, Bryce's ending value falls short of the target. Fortunately, though, we've set up our spreadsheet so that changing the interest rate only requires changing one cell.

If we bump the rate up to 9% we get:

	A	B	C	D	E
1	**Rate:**	9.00%			
2	**Year**	**Starting Balance**	**Interest Earned**	**Payment**	**Ending Balance**
3	1	$28,500.00	$2,565.00	$2,500.00	$33,565.00
4	2	$33,565.00	$3,020.85	$2,500.00	$39,085.85
	Rows Omitted				
37	35	$1,026,215.61	$92,359.40	$2,500.00	$1,121,075.01

An interest rate of 9% reaches the goal, but overshoots it by quite a bit. Bryce would probably be fine with having more money, but we don't want to tell him he needs 9% when in fact he doesn't need quite that much. We know the needed rate lies somewhere between 8% and 9%. So by trial and error, we try different rates to see what we get, adjusting upward when the result is too low, and downward when it is too high.

At 8.5%, we get $977,074.09 in year 35. At 8.75%, the result is $1,046,513.97. The 8.5% result was a bit closer, so we'll try something between the two, maybe a bit closer to 8.5%. Trying 8.6% we get $1,004,258.77, which is awfully close to the target. Moving down just slightly to 8.59% we get $1,001,505.37. Dropping down just a bit more to 8.58% we get $998,759.80

Unless we take our interest rate out to more than the customary two decimal places, we won't hit $1,000,000 exactly. Both 8.58% and 8.59% give results that are as close to the target as one could reasonably expect; 8.58% gives a result that is closer, but it falls a little bit short, which might make 8.59% the better call. We can argue back and forth for either answer, but there is not much difference between them, and in any case the answer is approximate anyway.

For a final answer: Bryce needs to earn somewhere between 8.58% and 8.59%.

The whole trial-and-error approach to Example 5.4.3 may seem a little rough, but it is actually a perfectly appropriate and highly effective solution method. Our spreadsheet is set up in a way that makes changing the rate easy, and so even though this approach is not exactly elegant, it is still an efficient way of arriving at the answer we needed.

What about solving for interest rates with a present value? This can be accomplished in a very similar way.

Example 5.4.4 **A contractor is offering a payment plan for home improvement projects. The contractor is advertising that with its plan, you can finance $15,000 worth of improvements for $250 a month for 10 years. What is the interest rate?**

We start with an amortization table, plugging in $15,000 for the initial balance, $250 for the payment, and a guess for the interest rate. Using these values with an 12.5% guess, we get:

	A	B	C	D	E
1	**Rate:**	12.50%	**Initial Balance:**	$15,000.00	
2	**Month**	**Payment**	**To Interest**	**To Principal**	**Ending Balance**
3	1	$250.00	$156.25	$93.75	$14,906.25
4	2	$250.00	$155.27	$94.73	$14,811.52
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	Rows Omitted				
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
182	180	$250.00	-$349.18	$599.18	-$34,120.86

We are interested in the balance at the end of the 180th month, because we know that when we have found the correct rate this balance should be zero (or at least as close as we can get to zero with a two decimal place interest rate). At 12.5%, the $250.00 monthly payment would more than pay off the loan; the actual rate must therefore be quite a bit higher.

You probably have already noticed one thing that is going to be annoying about trying different interest rates and checking for the result: since your computer monitor can't display 182 rows on a single screen, you have to scroll down to see what happens in the 180th month. You can get around this difficulty by hiding the rows in between that you don't need to see. Click on any cell in the 5th row, hold the mouse click down and scroll down to the 181st row. Then, with this section of the spreadsheet highlighted, choose Format → Row → Hide. Those rows will now be hidden; they are still there, but don't display on the screen, so you don't have to scroll past them to get to the row you actually want to see.

We now need to try a higher rate, and on the basis of the result at 12.5% it seems like we need to try something much higher. There is no way of knowing precisely how much higher.

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We'll just have to try something and see how it works, basing our future guesses on how it works out. Jumping up to 20%, we get:

	A	B	C	D	E
1	**Rate:**	20.00%	**Initial Balance:**	$15,000.00	
2	**Month**	**Payment**	**To Interest**	**To Principal**	**Ending Balance**
3	1	$250.00	$250.00	$0.00	$15,000.00
4	2	$250.00	$250.00	$0.00	$15,000.00
5	3	[illegible]	[illegible]	[illegible]	[illegible]
	Rows Omitted				
182	180	$250.00	$250.00	$0.00	$15,000.00

At 20%, a $250.00 monthly payment is just enough to cover the interest, making no progress against the balance. So the rate must be lower, somewhere between 12.5% and 20.00%. We continue the process of making educated guesses and refining them. It may take many guesses, but eventually we discover that an interest rate of 18.77% leaves an ending balance of −$80.17, while 18.78% leaves a balance of $27.13. So we can conclude that the rate is somewhere between 18.77% and 18.78%.

Using Goal Seek

Many versions of Microsoft Excel include a feature called *Goal Seek*, which can help in solving these sorts of problems. In simple terms, what Goal Seek does is the same "guess-and-check" approach that we used in the previous two examples; the advantage is that the computer does all of the guessing and checking. (Depending on the version of Excel or other spreadsheet program you are using, the instructions given below may not work; if they don't, consult your software's manuals or ask your instructor about if, and how, you can do the same thing with your program.)

Let's return to Example 5.4.4, but now suppose that we want to work with a $275.00 monthly payment. Obviously this will change the interest rate answer as well. If you take the spreadsheet from that example with the 18.78% interest rate we found, but change the payment to $275.00, your spreadsheet should look like the one shown here.

	A	B	C	D	E
1	**Rate:**	18.78%	**Initial Balance:**	$15,000.00	
2	**Month**	**Payment**	**To Interest**	**To Principal**	**Ending Balance**
3	1	$275.00	$234.75	$40.25	$14,959.75
4	2	$275.00	$234.12	$40.88	$14,918.87
5	3	[illegible]	[illegible]	[illegible]	[illegible]
	Rows Omitted				
182	180	$275.00	-$373.57	$648.57	-$24,518.57

The large negative balance reflects the fact that this payment at this interest rate would more than pay off the loan. What we *want* is a zero balance in cell E182, and we hope to find it by changing the interest rate in cell B1.

From the Tools menu on the toolbar, select Goal Seek. This will cause the pop-up box shown in Figure 5.2 (shown on the next page) to appear on your screen. The first option in that box will ask you to "Set Cell". The cell we want to set is B182, so type that in. The next option asks "To Value". We want B182 to contain 0, so that is what we type into this box. The last option asks "By changing Cell". The cell we want to change is B1, so we type that in. If you then click on "OK", the program should quickly change cell B1 to the desired solution: 21.04%.

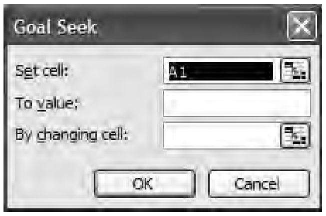

FIGURE 5.2

At the time of this writing, the current version of Excel's Goal Seek feature is not completely reliable. It will often work, but may crash unpredictably, and so it cannot be entirely relied on. At best, it is worth trying on any given problem, but if it fails to work, you will have to revert to the trial-and-error approach.

Changing Interest Rates

It is possible, by adapting the techniques of Sections 4.6 and 4.7, to deal with changing interest rates in future value calculations, but doing it that way gets tedious very quickly. Trying to work through those sorts of calculations for present values using the annuity formulas is even worse. With a little adjustment, our spreadsheets can handle these types of problems efficiently and effectively.

We will illustrate this with a present value example; it should be clear from this present value example how to make similar adjustments to a future value spreadsheet.

Example 5.4.5 ***Viveca borrowed $75,000 to start up a small business. The loan carries a variable interest rate. For the first year, the rate is 4.99%. In the second year, the rate increases to 6.99%. Thereafter, the rate will be based on a national index of interest rates, but it is guaranteed never to go above 9.99%. She will be required to make payments of at least $2,000 each quarter. Assuming that she makes the minimum quarterly payment, and that the interest rate will always be the highest it can be, how long will it take her to pay off the loan?***

We start with a basic amortization spreadsheet, and make a few changes.

- *We insert two rows at the top of the sheet to allow room for more than one interest rate.*
- *In the top rows we insert each of the interest rates that we will be using.*
- *We change the header of the time column from Month to Quarter.*

The header now looks like this:

	A	B	C	D	E
1	**1st Rate:**	4.99%	**Initial Balance:**	$75,000.00	
2	**2nd Rate:**	6.99%			
3	**3rd Rate:**	9.99%			
4	**Quarter**	**Payment**	**To Interest**	**To Principal**	**Ending Balance**

*We now adjust the interest formulas. In the first four rows, the formula should be set to use the first rate and calculate the interest quarterly. So in cell C5 we change the formula to "=Round(D1*$B$1/4,2)"; in cell C6 we change it to "=Round(E5*$B$1/4,2)" and copy this formula into cells C7 and C8 to complete the first year.*

*In cell C9 we cover the fifth quarter, which is in the second year and so needs to reflect the second-year rate. So we set cell C9 to "=Round(E8*$B$2/4,2)", and copy that into cells C10 through C12. Likewise, we set C13 to be "=Round(E12*$B$3/4,2)" and copy that formula into all the remaining cells below it.*

Once we have done this, we simply read the spreadsheet the same way as we have in prior examples.

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	A	B	C	D	E
1	**1st Rate:**	4.99%	**Initial Balance:**	$75,000.00	
2	**2nd Rate:**	6.99%			
3	**3rd Rate:**	9.99%			
4	**Quarter**	**Payment**	**To Interest**	**To Principal**	**Ending Balance**
5	1	$2,000.00	$935.63	$1,064.37	$73,935.63
	Rows Omitted				
9	5	$2,000.00	$1,234.82	$765.18	$69,896.99
	Rows Omitted				
13	9	$2,000.00	$1,686.32	$313.68	$67,206.60
	Rows Omitted				
87	83	$2,000.00	$53.43	$1,946.57	$192.65
88	84	$2,000.00	$4.81	$1,995.19	-$1,802.54

So we see that under these assumptions, Viveca will have the loan paid off in a bit less than 22 years. Of course, this assumes the minimum payment and the maximum interest rate. If the rate stays below the maximum, or if Viveca pays more than the minimum, the loan will not take nearly as long to pay off; we know how to work out the payoff time under any other assumptions that we like. What we have here is a worst-case scenario. We conclude that it will take at most 84 quarters (or 22 years) to pay off the loan.

Very Complicated Calculations

Of course, the more complicated the situation, the more complicated the spreadsheets need to be to deal with it, but the methods we have been using here will allow us to manage enormously complicated situations with a reasonable effort. We will work through one such example here to illustrate how this can be done; the exercises provide the opportunity to work out other situations.

Example 5.4.6 ***A high school alumni association has established a scholarship fund for graduates of the school. The association plans to raise funds through an annual campaign in each of the next 10 years, and then use the accumulated fund to pay out $25,000 in scholarships each year. The fund presently has $38,536 in it, and the leaders expect that they can raise $15,000 this year and increase the amount raised by 4% each year. They believe that the money in the fund will earn 7.5% during the fundraising period, and 6% during the period when it is being used for scholarships.***

Under these assumptions, how long will the fund be able to pay out scholarships?

There are two parts to this problem. We need to find the future value of the accumulation period, which will then become the present value of the scholarship payment period.

First, the accumulation period. Working from one of our future value spreadsheets, we work this out in a way similar to that shown in Example 5.2.4:

	A	B	C	D	E
1	**Rate:**	7.50%			
2	**Year**	**Starting Balance**	**Interest Earned**	**Payment**	**Ending Balance**
3	1	$38,536.00	$2,890.20	$15,000.00	$56,426.20
4	2	$56,426.20	$4,231.97	$15,600.00	$76,258.17
	Rows Omitted				
12	10	$285,565.66	$21,417.42	$21,349.68	$328,332.76

Having found the accumulated future value, we now move to an amortization table for the payments. We could use a separate spreadsheet, typing in the $328,332.76 as the starting balance. However, there is an advantage to putting the amortization table in the same spreadsheet as our future value table: if we want to work out some other "what-ifs" and change the assumptions for the accumulation period, we would then need to calculate the new future value and then type the value into the spreadsheet. If we combine both tables in the same spreadsheet, we can use a cell reference to transfer the future value to the amortization table, allowing it to change automatically when we change our assumptions.

To do this, we'll start with a basic amortization table, highlight it in its entirety, and copy. Then, in the spreadsheet we used for the accumulation, we'll go to cell G1 and paste. In J1 we will enter the formula "=E12". Then we just need to change the headers to suit annual payments and to describe the "use of a fund" scenario instead of the more common "paying off a loan" one, and also adjust the interest calculation for annual rather than monthly payments. The result should look like this:

	G	H	I	J	K
1	**Rate:**	6.00%	**Initial Balance:**	$328,332.76	
2	**Month**	**Payment**	**From Interest**	**From Principal**	**Ending Balance**
3	1	$25,000.00	$19,699.97	$5,300.03	$323,032.73
4	2	$25,000.00	$19,381.96	$5,618.04	$317,414.69
	Rows Omitted				
29	27	$25,000.00	$888.11	$24,111.89	-$9,309.98

In the 27th year, the scholarship fund runs out of money under these assumptions. So we can conclude that under these assumptions the fund will last 26 years. In the 27th year, paying out the full $25,000 would drop the fund below $0, so there will only be enough money left in that year to pay $25,000 − $9,309.98 = $15,690.02.

Of course, the answer we arrive at is only as good as the assumptions it is based on. If the assumptions about fundraising, investment returns, or annual scholarship payouts differ from the ones we used, the results will differ as well. Fortunately, though, we can make those changes with not much effort because of the way we set up our spreadsheet.

The exercises that follow will present you with the opportunity to work out other, similar problems on your own by using spreadsheets like the ones we have used in the examples. Hopefully, this chapter has demonstrated the power of spreadsheets as a computational tool, and has equipped you with the basic ability to work with them. We have actually, though, barely scratched the surface. You may want to take advantage of further opportunities, whether in academic coursework, workplace or continuing education training sessions, or independent study, to strengthen your skills with this powerful business tool.

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EXERCISES 5.4

A. Finding Time with Future Values

1. Thierry has $43,925 in his retirement account right now. He deposits $3,000 each year, and intends to continue to do so. If his account earns 8.55%, how long will it be before his account value reaches $1,000,000?

2. Jess plans to open an investment account at the end of this year with a $2,500 deposit. She plans to increase her deposits by 4% each year, and thinks that her account can earn 10%. How long will it take for her account value to reach $750,000?

B. Solving for Interest Rates

3. At a presentation for her company's 401(k) plan, Sahlia was told that, if she deposits $2,400 per year into the plan, she will have $1,000,000 in 30 years. What growth rate was the presenter assuming in making this claim?

4. If you invest $250 per month, what rate would your account have to earn to reach $1,000,000 in 20 years? How about in 40 years?

5. Mike and Nancy are in the market for a new house. They expect to have to borrow $200,000, and they hope to keep their monthly payment at (or below!) $1,650 a month. What interest rate would they have to get for a 30-year loan in order to be able to do this?

C. Changing Interest Rates

6. Anna plans to invest $200 each month in a retirement account. She already has $24,043.25 in her account. If her account earns 9% for the first 10 years, and then 8% for the next 10 years, how much will she have at the end of the 20 years?

7. ParmOgden Cheese Company borrowed $2,500,000 with a variable interest rate loan. According to the loan's terms, the interest rate will be 4.99% for the first year, and the rate may increase by 1.50% each year (6.49% maximum in year 2, and so on), up to a maximum of 8.99%. All rates are compounded quarterly.

   The company will make quarterly payments of $75,000. Assuming that the maximum interest rates are charged on this loan, how long will it take to pay off this debt?

D. More Complicated Situations

8. Magda has $935,277 in her 401(k) account and plans to retire in 5 years. She is depositing $3,577.09 each year into the account and plans to increase this amount by 4% each year until she retires. She believes her account will earn 7¼% until she retires.

   In retirement, she expects her account to earn 5¾%, and plans take $72,500 in the first year, increasing this amount by 3% per year. If she does this, how long will her money last?

E. Grab Bag

9. Bob borrowed $174,500 on a 30-year mortgage loan. His monthly payments are $1,349.95. What interest rate is he paying on this loan?

10. How long will it take to accumulate $20,000 in a savings account with $125 monthly payments if the account earns 5%?

11. Elyjah is putting $5,000 each year into an investment account earning 8.5%. How long will it take for his account balance to reach $500,000? What rate would he need to earn on his investments to reach this goal in 20 years?

12. A $20,000 loan has quarterly payments of $1,250. The term is 7 years. What is the interest rate?

13. According to a financial commentator, anyone who puts $2,500 a year into an aggressive investment fund "can be a millionaire in just 25 years!" What interest rate is this projection assuming?

F. Additional Exercise

14. Malik just borrowed $20,000 with a personal loan. The interest rate will be 5.99% for the first year, 7.99% for the second year, and 9.99% for the remainder of the loan's term. Mike wants to pay the same amount each month. If the term of the loan is 10 years, what should his monthly payment be?

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CHAPTER 5 SUMMARY

Topic	Key Ideas, Formulas, and Techniques	Examples
Spreadsheets as a Mathematical Tool, pp. 213–214	• Spreadsheets can be set up with formulas to automatically perform calculations based on values that you enter.	A company has four employees who earn different hourly rates. Adam earns $12.75 per hour and worked 20 hours, Betty $11.85 and 28, Carole $13.95 and 36, and Dario $12.50 and 27.5. Use a spreadsheet to calculate the gross pay for each person and for the company as a whole. (Example 5.1.1)
Future Values with Spreadsheets, p. 223	• Spreadsheets can be used to find the future values of annuities, or of streams of irregular payments. • Spreadsheets may be a more efficient method of doing this accurately and quickly than using formulas. • Spreadsheets are especially efficient to use if the payments vary a lot.	Erika plans to deposit $2,000 this year into her retirement account and then increase her deposits by 4% per year. If her account earns 7.25%, how much will she have in 30 years? (Example 5.2.4)
Amortization Tables with Spreadsheets, pp. 228–231	• Spreadsheets can be used to build amortization tables, making it easy to create a full. amortization table for a loan without repetitive calculations. • Amortization spreadsheets are an effective tool for analyzing loans when the payments are irregular.	Ted and Kristi owe $94,372.57 on their mortgage. Their monthly payment is $845.76. If they pay $7,000 extra right now, and $1,200 a month for the next 12 months, and then go back to their regular payment after that, how long will it take to pay off their loan? (Example 5.3.3)
Solving for Time with Spreadsheets, pp. 235–236	• Given a set of payments and an interest rate, we can solve for the time required to reach a set future value. • Use a future value spreadsheet, enter the payments and rate, and then scroll down to see when the future value is reached. • Amortization tables with spreadsheets can be used to solve for time with present values.	Miyako has $47,593 in her retirement account. She plans to deposit $2,000 this year, and increase her payments by 3% each year. If her account earns 9%, how long will it be before her balance reaches $500,000? (Example 5.4.2) Ted and Kristi's mortgage (shown above) is an example of solving for time with present value.
Solving for Interest Rates with Spreadsheets, pp. 236–237	• Given a set of payments, period of time, and future value, we can solve for the interest rate required to achieve this result. • Enter the payments and then use educated guesses for the interest rate until the spreadsheet shows the desired future value at the desired time. • Interest rates for present values can be solved for similarly, using an amortization spreadsheet; use educated guesses for the interest rate until the balance is $0 at the desired time. • Goal Seek can be used as an alternative to guess and check.	Bryce has $28,500 in his retirement account and plans to contribute $2,500 each year. He wants to have $1,000,000 in his account 35 years from now. What interest rate must he earn to reach his goal? (Example 5.4.3)

1. a. Anycorp has regional sales offices in Des Moines, Omaha, Denver, Boise, and Spokane. In the third quarter, the sales for these offices were $1,845,275, $2,087,416, $1,878,080, $3,567,029, and $2,502,135, respectively. The company set a $2,000,000 sales target for each of its offices for this quarter.

   Create a spreadsheet to record the sales for each of the five offices for the third quarter. Your spreadsheet should also show the sales target for each office, and how far that office's sales fell above (or below) their target. Your spreadsheet should also total the sales, target, and amount above (or below) target for the company as a whole.

   b. Suppose that in the fourth quarter, Anycorp raised its sales targets to $3,000,00 for Boise and $2,500,000 for Spokane, leaving the other offices' targets at $2,000,000. Sales in the fourth quarter came in at $2,534,026 for Des Moines, $1,546,032 in Omaha, $1,994,032 in Denver, $3,075,075 for Boise, and $2,403,716 for Spokane. Modify the spreadsheet created in part a to find each office's performance against their sales target, as well as the overall company's sales and performance against the sales target.

2. What rate of return do I need to earn to turn a $250 monthly investment into $450,000 in 20 years? In 30 years? 40 years?

3. Shandora owes $7,325.19 on a personal loan. The interest rate is 11¾%, and she is paying $125 a month. How long will it take for her to pay off the debt entirely?

4. For a fundraiser, the students at School No. 19 are collecting labels from Tastyland Farms soup products. Each condensed soup label is worth 1.5 Tastyland points, each premium soup label is worth 2.5 points, each label from canned pastas or stew is worth 3 points, and each label from a club-pak of microwave soup bowls is worth 5 points.

   In October, the students collected 1,289 condensed soup labels, 835 premium labels, 1,572 pasta or stew labels, and 72 club-pak microwave soup labels.

   a. Set up a spreadsheet to calculate the total number of points they earned in October.

   b. Use the spreadsheet to find the total number of points earned in November, when they collected 1,572, 1,099, 3,059, and 42 labels of each type (listed by type in the same order as before.)

5. Robbie has $19,772.59 in his 401(k) account now. He plans to deposit $125 a month for the next 3 years, then increase his deposits to $225 a month for the next 7 years, and then stop making deposits altogether. If his account earns 9%, how much will he have 35 years from now?

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6. Mark and Jennie have a mortgage balance of $119,002.78. Their interest rate is 8.53%, and they currently are scheduled to pay $1,195.06 per month on their loan. Jennie just got a major promotion, and they plan to use some of her raise to help pay off their mortgage more quickly. Suppose that they increase their monthly payment to $1,500 per month. How long will it take them to pay off their mortgage at that rate?

7. If I invest $3,000 this year in a retirement account, and then increase my annual investments by 5% each year, how much will I have in this account in 25 years, assuming it earns a 10.2% rate of return?

8. Suppose you borrow $20,000 and plan to pay it off with monthly payments of $250 for the next 10 years. What interest rate would your loan need to carry for this to work?

9. Suppose that an investment manager proposes that you invest $2,000 per year with his firm this year, and increase your investment amount by $150 each year (so you invest $2,150 next year, $2,300 the next, and so on), and says that in 40 years you will be a millionaire. What rate of return on your investment is he assuming?

10. According to the Rule of 72, it would take a little more than 20 years for a sum of money earning 3.5% compounded monthly to double. Use a spreadsheet to determine, correct to the nearest month, how long it would actually take.

11. Ashok has a business loan on which he currently owes a $28,095 balance. The interest rate is 7.99% compounded quarterly. If he makes quarterly payments of $1,250, how long will it take for him to pay off the loan entirely? How much would his last payment need to be? How much total interest would he end up paying between now and when the loan is finally paid off?

12. Set up an amortization table for a loan of $50,000 at 8.85% with monthly payments for 20 years. (Calculate the correct monthly payment yourself.) Use this table to answer the following questions:
    a. How much of the first payment goes toward interest? Toward principal?
    b. At what point does the amount of each payment going to principal become larger than the amount going to interest?
    c. Suppose that for the first year I pay $1,800 a month, and then go back to the payment you calculated for this loan. How long will it take to pay the loan off if I do this?
    d. What is the minimum amount that I would need to pay in the first month to avoid negative amortization?

13. Paul owes $6,320.45 on his credit card. The interest rate is 18.99%. He has vowed to make no more charges to this card until the balance is paid off. If he pays $100 a month, how long will it take to pay it off?

14. Paco's Taco Hut has seven employees. Andy, Bill, Craig, Desiree, and Frances earn $7.35 per hour; Emma and Gerardo earn $9.25 per hour.
    a. Set up a spreadsheet to determine each employee's gross pay, as well as the restaurant's overall gross payroll for the week, based on the hours each employee works.

b. Use your spreadsheet to calculate these payroll figures for a week in which the employees worked the following number of hours: Andy 35, Bill 28, Craig 11, Desiree 38, Emma 28.5, Frances 37.5, and Gerardo 31.

15. I have $58,053 in my retirement account, which I hope will be worth $700,000 when I retire in 24 years. I don't plan on putting any more money into this account between now and then. Using the Rule of 72, I determined that I would need to earn around 10½%, though I don't have a lot of confidence in that figure since this does not involve an even number of doublings and the Rule of 72 is an approximation anyway.

    Use a spreadsheet to determine the actual rate I would need to earn to reach this goal.

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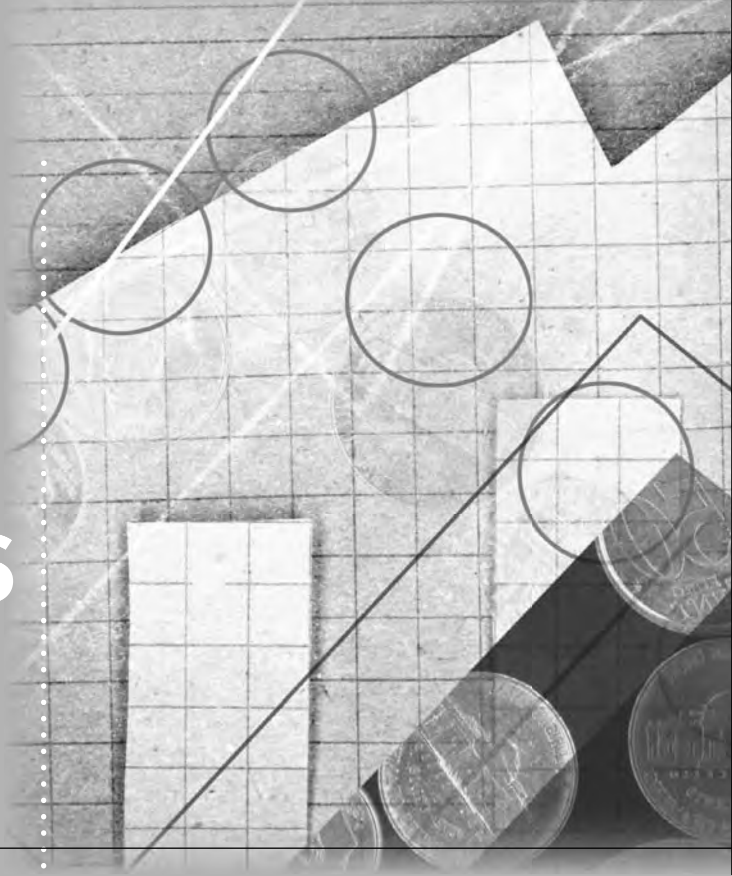
NOTES

PART **two**

SPECIFIC APPLICATIONS



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CHAPTER 6

Investments

"October: This is one of the peculiarly dangerous months to speculate in stocks. The others are July, January, September, April, November, May, March, June, December, August and February."

—Mark Twain, "Pudd'nhead Wilson's Calendar for 1894"

Learning Objectives

LO 1 Identify the key characteristics of different types of investments, including stocks, bonds, futures and options, and mutual funds.

LO 2 Correctly use technical terminology related to various types of investments.

LO 3 Calculate values used to measure the financial results, including dividend rates, dividend yields, compound annual growth rates, and total rates of return.

LO 4 Recognize how investment concepts such as risk, volatility, diversification, and leverage can affect investment choice and investment performance.

LO 5 Assess a reasonable rate of return expectation for an investment portfolio based on the types of investments it contains.

Chapter Outline

6.1 Stocks

In our work so far, we have put a great deal of effort into the mathematics of money invested growing over time. When money is put to work as a loan, this growth is due to the payment of interest. We have also recognized, though, that there are other ways that money can be put to work aside from simple notes and bank deposits. We have seen that present and future values can be calculated in the same way regardless of whether the growth really comes from interest or from other types of investment gain, and so we have not spent much time discussing the details of these other sorts of possible investments. In this chapter we will.

Money can be put to work by loaning it to someone else, but it can also be put to work by buying something that we hope will provide a good return on the money invested. We could buy a piece of real estate or gold coins in the hope that they will increase in value. One of the main ways to put money to work, though, is by using it to start or buy part of a business. We will begin this section by looking at money invested in the ownership of a business.

Businesses can legally be set up in a range of different ways. In a ***sole proprietorship*** the business is owned entirely by one individual, the person who runs it. Many small businesses are set up in this way. For example, if Tom owns a snow plow and runs a business plowing driveways in the winter, his business may well be a set up as a sole proprietorship. A ***partnership*** is a business owned by two or more people, again typically the people who actually run the business. If Lisa and Sunita work together preparing tax returns, their business may be set up as a partnership.

There are other ways that a business can be structured though. You may be familiar with such terms as ***limited partnership, limited liability company*** (LLC), and ***corporation***. Each type of structure has its advantages and disadvantages, and the choice of how to structure a business requires weighing considerations such as taxes, exposure to risk and liability, paperwork requirements, and other concerns.

One of the most commonly used structures for a business is a ***corporation***. A corporation is a legal entity that can be thought of in many ways as an artificial legal person, able to own property, enter into contracts, borrow money, and conduct business and financial affairs just as an actual person could. To most people the word *corporation* suggests big business, like Home Depot, Exxon Mobil, or General Electric. Those businesses are all corporations, but small businesses like Tom's snowplowing business, or Lisa and Sunita's tax service, could equally well be set up as corporations. Setting up a business as a corporation is generally more complicated than a sole proprietorship or simple partnership, but it can offer significant advantages to its owners. One of the biggest advantages of a corporation is that it exists as a separate legal entity from its owners; if things go badly and business is sued or suffers severe financial losses, generally its owners can not be held *personally* responsible for those liabilities.

A sole proprietorship is owned by its sole proprietor, and a partnership is owned by its partners, but who owns a corporation? The ownership of a corporation is divided up among its ***stockholders***. Each individual piece of this ownership is called a ***share*** of the company's ***stock***. How much of the business each share of stock represents depends on how many shares of stock the company has issued. A corporation can have any number of shares, so it is impossible to know how large a percent of the ownership one share represents unless you know the total. If you own one share of stock in a company with just 10 shares, then you own 1/10 (or 10%) of the company. If the company has issued one billion shares, then one share equates to owning 1/1,000,000,000 (or 0.0000001%) of the company.

When stock is first issued by a corporation, it may be issued with a ***par value***. Loosely speaking, a stock's par value is a reflection of a portion of the money paid by the original shareholders into the corporation. Par value used to be considered more important than it usually is today, and in fact it is not unusual today for a stock to have ***no par value***, or to have a par value which is absurdly low compared to the realistic value of the stock.

Some companies also issue several different types of stock. As its name suggests, ***common stock*** is the most common type, though ***preferred stock*** is another. While both types of stock represent ownership of a piece of a corporation, the two types differ in how their owners share in the company's profits, who has first claim to the corporation's remaining assets in the event it goes into bankruptcy, and in their voting rights in the election of the ***board of directors*** (the group of people who direct the corporation's activities). Except as noted, we will be discussing only common stocks in this text. Small businesses seldom issue preferred stock, and even among very large corporations it is not unusual to see little, it any, preferred stock issued.

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Dividends

The profits earned by a corporation properly belong to its shareholders, the people who own the corporation. However, since the corporation is a separate legal entity, a shareholder does not have the right to access the corporation's funds directly. Each shareholder receives a portion of the corporation's profits when they are paid out in the form of ***dividends***. A corporation's board of directors will periodically evaluate the business's performance and decide how much money should be paid out to its shareholders (this is known as ***declaring*** a dividend.) Corporations normally hold on to at least some of their profits to use in growing the business for the future, and some corporations don't pay out any dividends at all. On the other hand, sometimes a corporation will pay more in dividends than it earns if it has a significant amount of unused cash on hand. In either case, the dividends paid out by a corporation are seldom exactly equal to the corporation's profits.

Dividends are divided up among the shareholders according to the number of shares each owns.

> ***Example 6.1.1*** ***Zarofire Systems earned $743,000 in the last quarter, and the company's management has declared a dividend of $450,000. The company has 1,000,000 shares of stock issued. If you own 200 shares of the company's stock, how much will you receive as a dividend?***
>
> *The $450,000 total dividend must be distributed among the shareholders based on the number of shares each one owns; $450,000/1,000,000 shares works out to $0.45 per share. Since you own 200 shares, you will receive (200 shares)($0.45 per share) = $90.00.*

Rounding can be an issue with dividend rate calculations. In Example 6.1.1 the dividend rate came out evenly, but this will not always happen. Since the dividend rate per share often comes out to be a fairly small amount of money, it is not uncommon to see dividend rates carried out to a tenth of a cent. For example, a company might pay a dividend of 12.5 cents per share, or $0.125 per share.

Dividend calculations work out the same way with small businesses as with large ones.

> ***Example 6.1.2*** ***Jason and Dave's dry cleaning business is set up as a corporation. There are 100 shares of stock; Jason owns 51 shares and Dave owns 49. In the last quarter the business earned $39,750 in profits, and the company declared a dividend of $35,000. How much will Jason and Dave each get?***
>
> *The $35,000 profit will be distributed based on the number of shares each person owns; $35,000/100 shares = $350 per share. Since Jason owns 51 shares, he will receive (51 shares) ($350 per share) = $17,850.*
>
> *Dave will receive (49 shares)($350 per share) = $17,150.*

In this example, Jason and Dave both own nearly the same number of shares, and so they receive nearly the same amounts in dividends. Financially speaking, they are close to being equal owners of the business. However, in another important respect they are not equal at all. Since Jason owns 51% of the shares of the business, whenever any decisions need to be made his vote will always beat out Dave's. With respect to control of the business, they are not equal at all. While Dave is entitled to his nearly equal share of the dividends, the decision of how much to pay out in dividends is entirely Jason's.[1]

If you compare Example 6.1.1 with 6.1.2, the difference in the dividend payable per share is striking. Jason and Dave's dry cleaning business pays a much larger dividend per share than Zarofire Systems. Directly comparing these numbers can be very misleading though, among other reasons because the number of shares is largely arbitrary. The dry cleaning business has only 100 shares. If they wanted to, Jason and Dave could have structured the company to have 100,000 shares instead, with Jason owning 51,000 and Dave owning 49,000. The

[1]This assumes that the corporation's bylaws require that votes will be decided by a majority vote. Corporation bylaws can be set up so that votes require a larger majority (such as a two-thirds majority). Also some corporations have different classes of shares, where some shares carry more votes than others.

dividend rate would then be $0.35 per share. Yet Jason would still own 51%, Dave would still own 49%, and each man would still receive the same overall dividend. The dividend per share can be a misleading measure of how desirable a stock investment is.

Distributing Profits of a Partnership

If a business is not set up as a corporation, how are the profits to be distributed among its owners? The technical and legal details of how other businesses are structured fall outside the scope of this book, but however the details are set up, there must be some agreement as to how much each owner is entitled to receive.

If a business is set up as a partnership, the partners may agree to distribute all profits equally. In that case, dividing them up is just a matter of dividing the profits to be distributed by the number of equal partners. This would not be unusual if all the partners of a business contribute essentially the same effort, skills, and capital to the business. However, it is also not unusual for the partners in a business to agree that some should receive a larger share than others. One partner might work more hours or contribute more valuable expertise or more financial capital than another. In those cases, there is no formula that must be used to determine how the profits would be split up; it is a matter of whatever distribution the partners can agree is fair.

The partners may agree on a split of the profits based on the percent each will receive. In that case, finding each individual's take is simple a matter of applying her percent to the total profit. Sometimes, rather than use percents, the partners may decide to split the profits based on "parts" each is to receive. Mathematically, we treat each part as though it were a share of stock. The following example will illustrate.

Example 6.1.3 ***Suppose that TJ, Rudy, Eric, and Kevin have a band, which they have set up as a business partnership. They agree to distribute their profits in unequal shares, because Rudy owns most of the band's equipment and Eric wrote most of the songs. They agree to distribute the profits as follows: TJ gets 3 parts, Rudy 5 parts, Eric 4 parts, and Kevin gets 3 parts. (This sort of distribution can be abbreviated as 3:5:4:3 split, as long as it is clear which number goes with which person.)***

The band earned $7,250 last month, and all four partners agree to distribute this entire amount among them. How much does each person receive?

Even though this is not a corporation, we can pretend that each part is a share of stock for the purposes of distributing the profits. So there are a total of 3 + 5 + 4 + 3 = 15 "shares," and so each share should receive $7,250/15 = $483.33.

Thus, TJ would receive 3($483.33) = $1,449.49, as would Kevin. Similarly, Eric gets 4($483.33) = $1,933.32, and Rudy gets 5($483.33) = $2,416.65.

This could also be done with percents. Since he gets 3 parts out of 15, we could instead say that TJ gets 3/15 = 20%. Applying this 20% to the total profit, we get (20%)($7,250) = $1,450. (The penny difference is due to rounding.) Likewise, the other three bandmates would receive 33.3%, 26.7%, and 20%, respectively, and their shares could be calculated with percents in the same way.

Dividend Yields

It is often desirable to express a company's dividend rate as a percent, to make comparisons more meaningful. This rate is called the stock's ***dividend yield***.

The difficulty here is what that percent should be *of*. The most common way to do this is to express the dividend as a percent of the current fair market value of a share of the company's stock. It is also common practice to express this as a rate per year, making it more comparable to an interest rate.

The shares of many large corporations can be bought and sold through any stock broker. Large companies, with many shares of stock outstanding, typically have their shares ***listed*** on a major ***stock exchange*** such as the New York Stock Exchange or NASDAQ. There are

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also major stock exchanges outside the United States, in London, Paris, Toronto, and Tokyo, for example, as well as many smaller exchanges both in the United States and outside. If a company is listed on an exchange, the exchange maintains a market for buyers and sellers of that stock to buy and sell its shares.

If you want to buy shares in a company such as, say, Walmart or Coca-Cola, you can do that on any business day simply by opening a brokerage account and placing an order to buy the shares on the open market. The stockbroker sends your order to one of the exchanges, and, assuming shares are available for sale at a price you are willing to pay, you can become an owner of part of the company. Shares can be sold just as easily. For large companies whose shares are publicly traded, the fair market value is easy to determine—it is simply the price for which shares are selling on the open market. You need only look at what people who want to buy the stock have been paying people who want to sell it. Stocks that can be readily bought and sold in the open market are often referred to as ***liquid***. The market prices of many large stocks can be found listed in most daily newspapers, and can also be readily found online.

Example 6.1.4 The market price per share of Zarofire Systems is currently $49.75. Calculate the stock's dividend yield.

The company is currently paying $0.45 per share quarterly (see Example 6.1.1). This works out to a rate of (4)($0.45) = $1.80 per year. As a percent of the stock price, this works out to a dividend yield of $1.80/$49.75 = 3.62%.

As in this example, most corporations in the United States pay dividends quarterly. Most try to maintain a reasonably steady dividend rate, but the dividends paid can vary from one quarter to the next. The dividend yield we calculated in this example was based on the assumption that the dividend being paid in the current quarter would hold up for an entire year. This is common practice in calculating dividend yields.

However, sometimes a dividend yield will be calculated on the basis of the total dividends paid out by the company over the prior 12 months.

Example 6.1.5 Zarofire Systems has paid dividends totaling $1.75 in the past 12 months. Calculate the stock's dividend yield.

$1.75/$49.75 = 3.52%.

When someone talks about a stock's "dividend yield," it is not always clear which of these ways of calculating the dividend yield has been used. The term ***current dividend yield*** is sometimes used for the yield calculated on the basis of the current dividend in distinction from a ***trailing dividend yield*** based on the actual past year's dividends. Unfortunately, this distinction is not always clearly drawn in practice. Using these terms, though, we would say that Zarofire Systems' current dividend yield is 3.62%, and the company's trailing dividend yield is 3.52%.

Zarofire Systems is meant to be representative of the stock of a listed, openly traded company's stock. Not all companies are listed, however. Smaller companies and other companies whose shares seldom change hands would not normally be listed on a stock exchange, and as a result it is not as easy to determine what price a share of such a stock could be bought or sold for. Jason and Dave's dry cleaning business is surely not listed on any stock exchange. If you want to buy a piece of their company, you cannot do it by just contacting your broker and placing an order. All of the shares of stock are owned by Jason or Dave; there is no open market for shares of this corporation's stock. If you wanted to buy stock in their company, you would need to contact Jason or Dave and see if you can convince them to sell you some of their shares. They may or may not be interested in selling. Likewise, if Jason or Dave decides that he wants to sell his shares, he can't do this simply by placing an order with the neighborhood stockbroker either; he needs to find an interested buyer.

Stocks that are not readily available to be bought or sold are referred to as ***illiquid***. Determining the market value of an illiquid stock is more difficult than for a liquid one, because there are no other open market sale prices to compare with. Companies (whether they are legally set up as corporations or not) are commonly referred to as ***private companies*** if they are owned by a few individuals and other people are not generally presented with the opportunity to buy into the company's ownership.

In such cases calculating a dividend yield requires an educated guess of the stock's fair market value.

> ***Example 6.1.6*** ***Dave believes that each share of stock in the dry cleaning business is worth $8,000. Based on this estimate, what is the dividend yield?***
>
> *$350 per quarter annualizes to $1,400 per year. So the dividend yield is $1,400/$8,000 = 17.5%.*

The dividend yield calculated in this example is only as good as the estimate of the business's value. If Jason thinks the business has a higher value than Dave does, and believes each share is worth $14,000, he would calculate the yield to be just 10%. Even though a 17.5% seems completely different from a 10% rate, they are both based on the same $350 quarterly payout.

Capital Gains and Total Return

Dividends are one way that an investor hopes to profit from a stock investment, but they are not the only one, or even necessarily the main one. Often an investor buys shares in a business in the hopes that the business itself will grow and become more valuable, and that down the road she can sell her shares for more than she paid for them. Profits due to the increase in value of an investment are called ***capital gains***. Historically, publicly traded stock prices have tended to rise on average—though the performance of *individual* stocks varies wildly—and a large percentage of the profits from stock market investments have come from capital gains. Similarly, the owner of a private company normally would expect (or at least hope) that over time the value of the business will increase, so that, if and when he decides to sell, it will command a higher price.

While both are a way of profiting from an investment, there are significant differences between capital gains and dividends. If you own shares in Zarofire Systems, for example, you receive any dividends the company declares while you own the stock. You get the dividends in cash and can do with them as you please. If you bought your shares for $25 a share, and the price of the stock rises to $100 a share, on paper you have a profit of $75 per share. But you don't actually have this as money you can spend unless you sell your stock. If the market price of the stock declines, your "profit" can disappear.

There are some advantages to capital gains over dividends, though. When dividends are paid, they are income to you, and so they are subject to income tax. Capital gains are not considered income until you make them income by selling, and so you do not pay income taxes on those gains until you sell the stock. It is also the case that capital gains may be taxed at a different (usually lower) rate than dividends. Some companies and investors prefer that a stock not pay large dividends and invest almost all its profits in growing the business, the idea being that this will provide the opportunity for greater capital gains.

Calculating a *rate* of return from capital gains requires a bit of algebra. The compounded rate of growth can be approximated by the Rule of 72, but to get a more exact measurement we can manipulate the compound interest formula:

$$FV = PV(1 + i)^n$$

Dividing both sides by PV, we get:

$$\frac{FV}{PV} = (1 + i)^n$$

The next step requires a bit of more advanced algebra. (Readers not familiar with the rules of exponents will have to take this next step on faith . . . .)

$$\left(\frac{FV}{PV}\right)^{1/n} = 1 + i$$

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Now subtracting one from both sides gives us:

FORMULA 6.1.1
Compound Annual Growth Rate (CAGR) Formula

$$\mathbf{i = \left(\frac{FV}{PV}\right)^{1/n} - 1}$$

where
i represents the COMPOUND GROWTH RATE
FV represents the FUTURE VALUE
PV represents the PRESENT VALUE
and
n represents the NUMBER OF YEARS

Let's try this formula out with an example.

Example 6.1.7 ***You bought shares of Zarofire Systems 7 years ago for $12.50 per share. The current stock price is $50 per share. If you sell your stock today, what compound annual growth rate will your capital gain represent?***

Apply the formula:

$$i = \left(\frac{FV}{PV}\right)^{1/n} - 1$$

$$i = \left(\frac{\$50}{\$12.50}\right)^{1/7} - 1$$

$$i = 0.2190137 = 21.90\%$$

The fractional exponent we used in this calculation can be a problem with a calculator. It seems logical that to evaluate $i = (50/12.5)^{1/7} - 1$ you would just enter "(50/12.5)^1/7-1=", but this will not give a correct result. The problem lies with order of operations. Since exponents take priority over division, the calculator evaluates that expression by raising (50/12.5)^1 and then dividing the result by 7. In order to make it clear to the calculator that the exponent actually is 1/7, we need to place it in parentheses. The calculator steps to evaluate this are as follows:

Operation	*Result*
(50/12.50)^(1/7)-1=	0.2190137

Alternatively, you could convert the fraction to a decimal and then use the decimal, though this is a bit messier:

Operation	*Result*
1/7=	0.1428571
(50/12.50)^0.1428571-1=	0.2190137

This formula can also be used to find compound interest rates, as a more exact alternative to Rule of 72. This idea is explored in a few of the exercises at the end of the section.

In this example, we calculated a rate of return based on the per-share price of a stock. We need to be a bit careful when doing this. Occasionally, a corporation may ***split*** its stock. A split occurs when a corporation increases the number of its shares by issuing new shares to its existing shareholders. For example, in a 2-for-1 stock split, the company issues new shares so that each shareholder has 2 shares for each 1 she previously owned. This would affect the return on investment calculations. If Zarofire Systems had split its stock 2 for 1, each share you bought 7 years ago for $12.50 would have become two shares now worth $50 each, for a total of $100. If that had happened, you would need to use $100 as the future value in your CAGR calculation, and your return would have been 34.59%. While stock splits are not an everyday occurrence, you do need to be careful to take any splits into account. Splits are not an issue, though, if our calculations are based on the total value of the investment, rather than on the per-share value.

Of course, it is possible to lose money on a stock investment as well. This results in a ***capital loss***, which is indicated by a negative growth rate.

Example 6.1.8 ***Five years ago, I invested $8,400 in the stock of Sehr-Schlecht Investment Corp. I sold the stock today for $1,750. What compound annual growth rate does this represent?***

Applying the formula:

$$i = \left(\frac{FV}{PV}\right)^{1/n} - 1$$

$$i = \left(\frac{\$1,750}{\$8,400}\right)^{1/5} - 1$$

$$i = -0.2692787 = -26.93\%$$

Work through this example on your calculator now to make sure that you have the correct steps. If you work it through and come up with a different answer, make sure that you used the parentheses correctly as discussed after the prior example.

Total Rate of Return

The rate of return that we have calculated puts capital gains in terms of a rate of return, which can be very helpful in evaluating an investment's performance. However, it completely ignores any dividends that might have been received along the way. Most investors are concerned with the overall financial return on their investments, including both capital gains and dividends. This overall return can be referred to as the ***total return*** on the investment.

Unfortunately, the total *rate* of return can be a slippery concept. In Example 6.1.7 we found that the capital gains on your investment in Zarofire Systems resulted in a 21.90% rate of return, but we also know that the company has paid some dividends along the way. In Example 6.1.3 we found that the stock's dividend yield was 3.62%. But that yield is a moving target; it is based on the market price of the stock, which changes, and also is based on the dividend at a particular point in time, which has probably not been the same over the entire 7 years. Also, since you are calculating the return on your original investment, perhaps we should look at the dividends as a percent of your original investment, not the current market value when those dividends were paid. To make matters worse, the "total rate of return" that we are seeking needs to be a compound growth rate that equates to the return from capital gains, which are compounded, together with the return from dividends, which are not.

There are a number of different ways of dealing with this, some of them quite complicated. One crude but reasonable approximation for the total return is simply to add the compound growth rate from capital gains to a rate that represents an "average" dividend yield for the stock over time. This is only an approximation of the total return rate, and can not be taken too literally, but does provide a reasonable enough estimate of total return for many purposes.

Example 6.1.9 ***The dividend yield rate on Zarofire Systems has been on average around 3½% over the time you've owned it. What is the approximate total rate of return you have received on this investment?***

Using the approach discussed above, we find the total rate of return is approximately 21.90% + 3.50% = 25.40%.

The total return idea would be much simpler if the dividends were paid in company stock instead of in cash. In that case, the dividends would earn compound growth, as dividends are paid on the stock dividends, and so on. Many companies offer ***dividend reinvestment plans***. The money-dividends earned on stocks owned in these plans are not paid out to the stockholder, but are instead used to purchase more stock at the market price. (The dividends are still taxed as income, though, since you still had the option of taking them in cash.)

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Example 6.1.10 ***Ten years ago I invested $2,000 in a dividend reinvestment plan offered by my local electric utility company. The value of my original investment, including reinvested dividends, has grown to $3,525.18. What was my total rate of return?***

We do not need to look at capital gains and dividends separately; the values we are given include everything. Applying the compound rate of growth formula once again gives:

$$i = \left(\frac{FV}{PV}\right)^{1/n} - 1$$

$$i = \left(\frac{\$3{,}525.18}{\$2{,}000}\right)^{1/10} - 1$$

$$i = 0.0583154 = 5.83\%$$

Of course, if I invested additional money along the way (which people often do with these plans) we would once again be dealing with an unpleasantly complicated situation. The most efficient way to find the rate of return in those situations would probably be to set up a spreadsheet reflecting the payments, and then use guess and check to find the rate.

Volatility and Risk

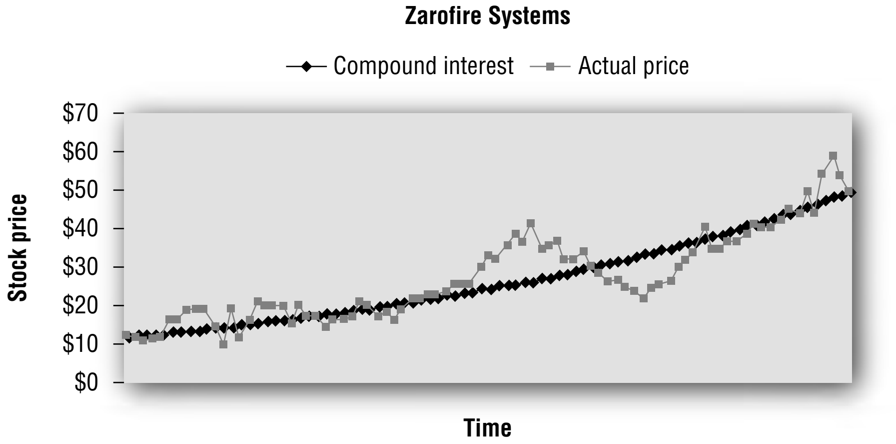
If you put your money in a bank certificate to earn compound interest, the value of your account can be expected to grow steadily and consistently, with essentially no risk of losing money on your investment.[2] Investments in stocks carry much greater risks. No matter how solid a company might look when you first invest in it, there is always the real possibility that things will go badly for the business and your stock decline in value, or even become worthless. This risk is present whether you are looking at a publicly traded stock like Zarofire Systems, or looking at the stock of a small business like Jason and Dave's Dry Cleaning. Even good businesses can fall on hard times or fail. While money invested in owning a business has the potential to offer much greater gains than money snugly deposited in a bank account, it also has the potential for loss. When you deposit money in a CD, you know the rate of return on your money; when you buy stock in a company, you're not even certain of the return *of* your money.[3]

Even if all works out well and you earn a good return on your investment, though, there is another important difference in *how* that growth occurs. In Example 6.1.5 we found that your investment in Zarofire Systems paid off handsomely, earning a 21.90% rate of return. It would be a mistake, though, to think of this as happening smoothly. When we calculate a rate of return on a stock investment, we are saying that the end result was equivalent to earning that rate of compound growth. It should never be assumed that the journey looked like a smooth and even rate though. Over the years, the price of Zarofire's stock may have gyrated wildly up and down, and in fact it probably did. This variation in the price along the way is referred to as ***volatility***. Volatility is not necessarily a bad thing, especially if you have the judgment and courage to buy on the lows and sell on the highs, but it can make for a bumpy ride. As you watch the stock price swing up and down from day to day it is not hard to fall victim to greed or fear and make unwise decisions, buying on greed at the highs and selling on fear at the lows as a result.

The route taken from your original $12.50 investment to the eventual $50 sale in a stock might look quite a bit different than the route at a steady 21.90% compound interest rate. The graph on the next page illustrates how Zarofire's stock price may have actually behaved over time, contrasted with a 21.90% compound interest rate. Notice that the stock price, instead of going up smoothly, goes up, and down, at varying rates.

[2]Even in the unlikely event that the bank collapses, most accounts are covered by FDIC insurance, through which the federal government guarantees your accounts at the bank up to $100,000. Of course, it is *possible* that both the bank and the federal government could collapse, but if that happened we'd all have much bigger worries than our CDs.

[3]With apologies to Mark Twain.



Given the choice, it would be preferable to earn 21.90% compound interest instead of the wilder ride offered by Zarofire's stock. (Of course, that statement assumes that you could find an investment paying a 21.90% compound interest rate.)

Volatility is less of an issue with illiquid stocks. It is certainly true that the value of stock in Jason and Dave's Dry Cleaning varies over time, but unless they are looking to sell, neither owner is likely to be thinking about that on a daily basis. Even there, though, if either owner is thinking about selling his shares in the business he would be wise to think about doing this when things are going well and it is likely that his shares will command a high price, rather than try to sell in an unfavorable business climate where the shares will probably be much harder to sell at a good price.

While stock investments can be much more financially rewarding than other, safer investments, they carry much greater risks as well. People contemplating putting their money to work in this way should make sure that they understand and can accept this risk and volatility. (In the remaining sections of this chapter we will discuss alternative investments as well as ways to balance and manage these risks.)

EXERCISES 6.1

Percents should be rounded to two decimal places (when necessary). Dividend rates should be rounded to the nearest tenth of a cent (where necessary).

A. Calculating Dividends

1. Metanational Globalnet Corp. has declared a dividend of 84 cents per share. How much will Mike receive if he owns 257 shares of this stock?

2. Find the amount that the owner of 1,378 shares would receive if the corporation declared a dividend of 35 cents per share.

3. Cattarauqua Ginseng Enterprises is a corporation with 3,000 shares. For the current quarter, the company will pay out dividends totaling $37,560.

   a. Find the dividend per share.

   b. If Ray owns 1,445 shares, how much will he receive?

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B. Distributing Profits of a Partnership

4. An accounting firm has three partners: Pacioli, Mellis, and Dafforne. The three partners have agreed to distribute the firm's profits in an 18:14:16 split. This quarter, the firm made $127,309 and this entire profit will be distributed among the partners.

   a. How much will each receive?

   b. If the profits were split equally among the partners, how much would each have received?

5. Cindy, Jim, and Shawn are partners in a catering business. They have agreed to distribute profits with a 6:5:4 split. Last month, the business earned $3,645, which will be entirely distributed to the partners.

   a. What is each partner's share of the profits?

   b. How much would each have been paid if the profits were distributed equally.

C. Dividend Yields

6. Metanational Globalnet Corp. has just declared a quarterly dividend of 84 cents per share. The current stock price is $42.45 per share. Calculate the current dividend yield.

7. Salamanca Semiconductors pays a quarterly dividend of 57.5 cents per share. The current stock price is $108.45. Calculate the current dividend yield.

8. Aeolus Wind Systems has just declared a quarterly dividend of $1.35 per share. Over the last 12 months, the company has paid out a total of $5.25 per share in dividends. The company's stock is currently selling for $157.03 per share.

   a. Calculate the current dividend yield.

   b. Calculate the trailing dividend yield.

9. Zovaxacquin Pharmaceuticals pays a quarterly dividend of 22 cents per share. Axerixia Corp. pays 59 cents per share quarterly. The stock of Zovaxacquin sells for $17.35 per share, while Axerixia sells for $68.45. Which company has the higher dividend yield? Justify your answer.

D. Compound Annual Growth Rates

In this section, growth rates should be calculated from the capital gain or capital loss on the investment only. Any dividends that may or may not have been paid should be ignored.

10. Three years ago, I bought stock in Salamanca Semiconductors for $25.09 per share. The stock is now worth $108.45 per share. There were no stock splits. Calculate the compound annual growth rate for my investment.

11. Alan invested $25,000 in his brother's home improvement business 5 years ago. The business has done well, and his brother has offered to buy out Alan's share of the business for $52,000. If Alan takes this offer,

    a. How much of a capital gain would he have earned?

    b. What compound annual growth rate would his investment have earned?

12. Ten years ago Collette invested $15,409.18 in the stock of Metanational Globalnet Corp. Today, that stock is worth $30,902.17. Find the compound annual growth rate for this investment.

13. Three years ago, the stock of Malidee Corp. sold for $145.82 per share. Today, the stock sells for $35.15 per share. There have been no stock splits. What is the compound annual growth rate for this company's stock?

14. Three years ago, the stock of Pasmalidee Corp. sold for $145.82 per share. Today, the stock sells for $35.15 per share. The stock had a 5-for-1 split last year. What is the compound annual growth rate for this company's stock?

E. Total Rate of Return

15. My stock in Honeoye Gas and Electric has grown at a CAGR of 4.75% and the stock has paid a dividend yield averaging 4.15%. Approximately what total rate of return have I earned on this investment?

16. Suppose that over the past 3 years, Salamanca Semiconductor (from Exercise 10) has paid a dividend rate averaging 1.25%. Approximately what was the total rate of return on this investment over the past 3 years?

17. Corey invested $8,453.19 in a company dividend reinvestment plan 12 years ago. He has not made any additional investments or withdrawals, but has kept reinvesting the dividends. Today, his account is worth $15,003.02. Calculate the total rate of return on his investment.

18. Over the past 10 years, Metanational Globalnet Corp. (from Exercise 12) has paid an average dividend yield of 2½%. What is the approximate total rate of return from this stock over this time period?

F. Volatility and Risk

19. Five years ago, Sam deposited $5,000 in a 5-year certificate of deposit, and also invested $5,000 in the stock of Plancktonol Inc. Today, his CD is worth $6,834.37. His stock is worth $9,130.94.
    a. Calculate the compound annual growth rate for each of Sam's investments.
    b. How much was Sam's CD worth 2 years after he opened it?
    c. How much was Sam's stock worth 2 years after he bought it?

G. Grab Bag

20. Max bought a collection of rare stamps at an auction 6 years ago for $11,300. A stamp dealer offered to buy the collection from him today for $18,500. If Max takes this deal, what compound annual growth rate would he have earned from this investment?

21. Suppose that a stock pays a $1.42 quarterly dividend, and the stock's price is currently $92.05. Find the current dividend yield.

22. Last month, a boat rental business earned $17,200, which will be distributed among the partners who own and run the business. They distribute their profits as follows: Adrian gets 10 shares, Steve gets 12 shares, and Rene gets 8 shares. How much will each partner receive?

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23. The stock of Big L Corporation sold for $15.35 per share 2 years ago. Today, it sells for $0.14 per share. There have been no stock splits. Find the compound annual growth rate which this represents.

24. Waterdown's Will Micro-Breweries stock sells for $27.95 per share. The company is currently paying 12.5 cents per share in quarterly dividends. Over the last 12 months, the company has paid dividends totaling 42.5 cents per share. Calculate the current dividend yield and the trailing dividend yield.

25. Five years ago, I invested $10,000 in my nephew's landscape business. The business fared poorly, and now he is closing it down. I will receive $2,500 for my share of the company. Without considering any dividends that may have been paid, what compound annual growth rate did I earn on this investment?

26. CombinaTaco Restaurants pays a 34.5 cents per share quarterly dividend. The stock sells for $48.66 per share. What is the current dividend yield?

27. Tastee Lard Donut Shoppes earned $498,000 last quarter, and the board of directors has determined that $275,000 should be paid out as a dividend to the shareholders. The company has 846,832 shares issued.

    a. Calculate the dividend rate per share. Round your answer to the nearest tenth of a cent.

    b. Find the dividend that would be paid to the owner of 475 shares.

28. Suppose you invested $5,000 in a stock, and 4 years later it is worth $8,500. The stock paid no dividends. What CAGR does this represent?

29. Edith invested $1,600 in her local electric company's dividend reinvestment plan 8 years ago. She has made no additional investments, nor has she withdrawn any money from the plan. Right now, her account is worth $2,849.15. What is the total rate of return she has earned from this investment?

30. On the basis of its profits last quarter, a small computer software company will pay out $27,345 to its shareholders. The company has 3,500 shares of stock, and Rashan owns 850 shares. How much will he receive?

H. Additional Exercises

31. A stock pays a $0.57 per share quarterly dividend, and has a 2.48% current dividend yield. What is the stock's price per share?

32. a. Three years ago, Cipherbound Systems stock sold for $45 a share. Things have not gone well for the company, and the stock is now worthless. What is the compound annual growth rate for this investment?

    b. Rework this problem assuming that the time is 10 years instead of 3.

33. Seven years ago, Dwayne bought stock in Barrytronic Corporation for $73.50 per share. The stock has split 2 for 1 three times in those 7 years. The stock price is now $34.79 per share. What CAGR has Dwayne earned on this investment?

6.2 Bonds

Back in Chapter 1, we discussed promissory notes. Recall that a promissory note is simply a written agreement between a borrower and a lender, specifying the terms of a loan. The notes that we discussed in Chapter 1 were fairly simple ones; a borrower borrows some amount of money and then repays the entire loan together with (simple) interest at the maturity date. We have seen many variations on this theme: the simple discount notes of Chapter 2, loans made at compound interest in Chapter 3, and the loans that we analyzed as annuities in Chapters 4 and 5. In this section, we will take a look at another, much more complex, generalization of the idea of a promissory note.

Speaking generally, a ***bond*** can be thought of as a type of promissory note. A bond is also a written agreement between a borrower and a lender, specifying the terms of the loan. The term *bond* is more commonly used when the borrower makes periodic interest payments to the lender, instead of repaying the entire principal and interest all at once at maturity (though there are exceptions to this, as we will see). Also, while simple interest or simple discount promissory notes tend to be fairly short term loans, bonds often extend for longer terms.

The Language of Bonds

Before proceeding any further, we need to define certain terms that are commonly used with bonds.

The ***issuer*** of the bond is the entity that creates the bond as a means to borrow funds. Put more simply, the issuer is the borrower. While a bond could theoretically be issued by just about any individual, business, or other organization, bonds are most commonly issued by large companies or federal, state, or local governments or agencies. When a bond is first sold, its ***buyer*** pays the issuer some amount of money in exchange for the bond. More simply put, the initial buyer of a bond is the lender. The original buyer of a bond may sell it to someone else, just as we saw with promissory notes in Chapter 2.

The issuer of the bond is obligated to make certain payments to its owner. The ***par value*** of a bond is the amount that will be paid to the bond's owner at maturity. This can also be called the ***face value*** of the bond. The most common par value for a bond is $1,000, though other amounts could be used. However, if the amount that the issuer needs to borrow is large, rather than create higher par value bonds, the issuer may just issue however many bonds are necessary to raise the needed amount.

A bond is said to be ***redeemed*** when the borrower pays the bond's owner its par value. For this reason, the par value is also sometimes referred to as the ***redemption value***. Normally, redemption occurs at the bond's maturity date, though some bonds include a provision that the debt can be paid off early, possibly at either the issuer or the owner's option. When a bond is redeemed early, we say it is ***called***; bonds that may be called are said to be ***callable***. We will not consider callable bonds in this section, other than mentioning their existence.

The ***coupon rate*** is the interest rate for the bond's periodic interest payments, as a percent of par value. This name may seem strange; we usually think of "coupons" as clippings from the Sunday paper to save 50 cents on peanut butter. The reason for this term is that there are two types of bonds: ***registered bonds*** and ***coupon bonds***.

The fact that bonds can be bought and sold poses a challenge for a bond's issuer: if the bond is sold, how does the issuer know to whom the interest payments should be made? With a registered bond, the owner is recorded by the issuer and interest payments are sent to this registered owner. If that type of bond is sold, the issuer must be notified. This solves the problem of where to send the payments, but it creates the burden of reporting and keeping track of every sale. With a coupon bond, there are actual coupons attached to the bond, which the owner clips off and submits when interest payments are due. The issuer sends the interest to whoever submits the coupon for payment. The term "coupon rate" comes from this situation (even though the term is used even for registered bonds).

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Example 6.2.1 ***On May 25, 2007, Zarofire Systems issued a $1,000 par value bond with an 8% coupon rate and a May 25, 2019, maturity date. Interest will be paid semiannually. What payments will the owner of this bond receive?***

The owner of this bond will receive the par value, $1,000, on the maturity date of May 25, 2019. Interest payments will be made semiannually, on November 25 and May 25 of each year until maturity. The amount of the interest payments will be:

$$I = PRT$$
$$I = (\$1{,}000)(0.08)(1/2)$$
$$I = \$40$$

When a bond is issued, the par value and coupon rate are set in advance. The amount that the issuer will receive for each bond is not. In an ideal situation (from a mathematical point of view at least), each bond would sell for its par value, so that the coupon rate would be equal to the actual interest rate. When this happens, we say that the bond is sold ***at par***. This seldom actually happens though. If the market decides that the bond's issuer is a good credit risk (there is not much concern that the issuer will be able to actually make the promised payments) and the coupon rate is attractive, buyers may be willing to pay more than the par value for the bond. We say then that the bond sells ***at a premium***. On the other hand, if the issuer is not considered quite so good a risk and the coupon rate is not so attractive, the issuer may not be able to sell the bond for its full par value. In that case, we say that the bond sells ***at a discount***.

Example 6.2.2 ***Suppose that Zarofire Systems is financially sound and the consensus of the investment community is that the company's prospects are excellent. Right now, investors can earn around a 7% rate of return by buying bonds with similar maturities issued by similarly sound companies. Would you expect Zarofire's bonds (from Example 6.2.1) to sell at par, at a premium, or at a discount?***

If you buy the bond at par, you will earn an 8% rate of return when comparable investments pay only 7%. Since Zarofire's bonds offer a better deal, investors would be thrilled to be able to buy these bonds at par, but in the open market they will almost certainly sell at a premium. If you aren't willing to pay more than par, someone else will, and for obvious reasons the company will sell its bonds to whoever is willing to pay them the most.

Current Yield and Bond Tables

The ***current yield*** of a bond is the interest rate that the bond pays as a percent of the current market price.

Example 6.2.3 ***Suppose that one of the May 25, 2019, 8% Zarofire Systems bonds sells for $1,094. What is the current yield?***

We have already determined that the interest payments are $40 semiannually. Calculating this as a rate based on the selling price gives:

$$I = PRT$$
$$\$40 = (\$1{,}094)(R)(1/2)$$
$$R = 7.31\%$$

While the par value and the coupon rate are set when the bond is issued, the amount the bond will sell for both initially and in any subsequent sales will be determined by the open market. The selling price of a bond can change from day to day, or minute to minute, as market conditions change. Suppose that some time goes by, and you want to know what has happened to the price of your Zarofire Systems bond. The current market selling prices of bonds can be found in some newspaper's financial pages, on the Internet, and also from a stockbroker (brokerage offices normally handle bond transactions as well as stocks). A typical quote might look something like this:

CORPORATE BONDS

Company	*Coupon*	*Maturity*	*Current Yield*	*$ Volume (000s)*	*Last Price*
Zarofire Systems	8.000	5/25/19	7.235	25,073	110.573

To be able to read this table, you need to be aware of certain assumptions. The coupon rate and current yield are understood to be percents, even though they are often printed without a % sign. The Volume column indicates the dollar amount of trading of this particular bond that has taken place in the last day; this can be taken as an indication of how liquid this bond is, or in other words how easy it would be to find a seller if you want to buy and a buyer if you want to sell. The Last Price is also a percent; in this case it is a percent of the par value that the bond is presently selling for.

Example 6.2.4 ***Based on the quote shown above, what is the current selling price for one of Zarofire's May 25, 2019, 8% coupon bonds?***

The bond is selling for 110.573% of par value. So the selling price is:

$$(110.573\%)(\$1{,}000) = 1.10573(\$1{,}000) = \$1{,}105.73$$

Yield to Maturity

Based on the quote shown in the table, the current yield of this bond is 7.235%. Remember that this means that if you buy this bond the semiannual interest payments that you will receive work out to a 7.235% rate based on the amount you would currently pay for it. It is easy to misunderstand this to mean that this means that you are earning a 7.235% rate on your investment. That is not correct! The interest payments work out to 7.235% of the price paid, but really we can't entirely overlook the fact that you are paying more than the par value. You pay $1,105.73 for the bond, but at maturity you will receive only $1,000 for it. We cannot overlook this when assessing the overall rate of return you are earning on this investment.

The ***yield to maturity*** of a bond is a measurement of the actual interest rate that will be earned, assuming that the bond is held to maturity. Unlike the current yield, the yield to maturity takes into account the effect of any premium (or discount) to maturity value. Calculating this by means of a formula is quite complicated, and falls outside the scope of this book. It should be clear, though, that for the Zarofire Systems bond we have been using as an example, the yield to maturity would be lower than the current yield. This is because the current yield does not take into account the $105.73 "loss" between selling price and par value. Likewise, it should be clear that a bond sold at a discount would have a higher yield to maturity than current yield, because of the gain between purchase price and par value.

For our purposes, the most efficient means to calculate the yield to maturity is using a spreadsheet with guess and check. The following example is provided as an illustration of how this can be done.

Example 6.2.5 ***Use a spreadsheet to determine the yield to maturity for the Zarofire Systems May 25, 1919, 8% coupon bond from the table given above.***

The $1,105.73 selling price should be the present value of the payments that the seller will receive. So we set up an amortization table. We will use the current yield of 7.235% as our initial guess; even though we know it is not the yield to maturity, it is a good starting point to work from. Since the remaining term is 12 years, the table should run to the 24th half year.

	A	B	C	D	E
1	**Rate:**	7.235%	**Initial Balance:**	$1,105.73	
2	**Half Year**	**Payment**	**From Interest**	**From Principal**	**Ending Balance**
3	1	$40.00	$40.00	$0.00	$1,105.73
4	2	$40.00	$40.00	$0.00	$1,105.73
	Rows Omitted				
26	24	$1,040.00	$40.00	$1,000.00	$105.73

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It may not be entirely clear how to interpret the entries in this table. What we are doing is imagining that the $1,105.73 is a debt being paid off by the coupon and redemption payments made by the issuer. If the interest rate is correct, we know that the ending balance should work out to be $0.00, reflecting the fact that with the correct interest rate the issuer's payments should "pay off" the $1,105.73 present value. Using guess and check, we eventually find that the closest we can get this value to zero occurs when the rate is 6.70% (even though the rates quoted in the table went to three decimal places, we will continue our practice of only carrying interest rate calculations out to two.)

	A	B	C	D	E
1	**Rate:**	6.70%	**Initial Balance:**	$1,105.73	
2	**Half Year**	**Payment**	**From Interest**	**From Principal**	**Ending Balance**
3	1	$40.00	$37.04	$2.96	$1,102.77
4	2	$40.00	$36.94	$3.06	$1,099.71
5	3	$40.00	$36.84	$3.16	[illegible]
	Rows Omitted				
26	24	$1,040.00	$33.69	$1,006.31	-$0.71

The Bond Market

We've repeatedly noted that bonds can be bought and sold, and in fact the amount of money that changes hands in the bond market is enormous. Governments, banks, corporations, insurance companies, pension funds, investment companies, individual investors, and others regularly buy and sell bonds on the open market. As with stocks, some bonds are more liquid than others. The bonds of large, well-known corporations naturally tend to be more liquid than those of smaller companies, and likewise the bonds issued by the U.S. federal government are naturally more liquid than those issued by a small-town water and sewer district. Overall, though, the bond market represents an enormous financial market, by most estimates far larger in size than the stock market, even though the stock market gets paid greater attention.

Just like stocks, bond prices fluctuate from moment to moment according to market conditions. However, bond prices tend to be less volatile than stock prices. This is partially because bond prices are heavily dependent on prevailing interest rates expected in the market, and interest rates usually change at a fairly slow pace over time. Also, if a company's prospects dim, its bondholders have the consolation that, if the company does end up in bankruptcy, the bondholders' claims against the company's remaining assets generally come ahead of stockholders' claims. The bondholders of a bankrupt company usually end up receiving some compensation from the company's remaining assets, while the stockholders often end up with worthless stock and little else. In addition, stocks tend to move more significantly with changes in a business's outlook, since the owners' (i.e., stockholders) future investment return is more closely tied to the business's profits or losses, while its creditors (i.e., bondholders) receive the same coupon and redemption payments regardless of profits or losses (unless of course things go very badly and the company actually ends up in bankruptcy).

Even though bonds generally carry less risk and volatility than stocks, they are by no means risk- and volatility-free. Every bond carries with it some ***credit risk***, the risk that the issuer will not be able to make the required payments of the bond. There are several ***rating agencies***, companies whose business it is to evaluate how great this risk is with any given bond issuer. Each agency has its own methods, rating systems, and standards, but even though there may be differences from one rating agency to another, the different agencies usually agree reasonably closely in their assessments of different issuers' creditworthiness.

The highest credit rating belongs to the U.S. federal government, since there is effectively no risk of the federal government failing to pay its debt obligations.[4] Corporations, state and local governments, and other issuers carry varying bond ratings depending on the agencies' assessments of their financial strength and prospects. Bonds of issuers whose ratings fall within the range that are typically considered low to moderate risk are sometimes referred to as ***investment grade***. Bonds of issuers whose ratings fall in a range suggesting that there is a realistic cause for concern are sometimes referred to as ***junk bonds***.

A bond issuer's credit rating can have a significant impact on the interest rates that it will have to pay to borrow money, just as consumers with good credit ratings can borrow more easily at more attractive rates than consumers with poor ratings. After a bond has been issued, changes in a company's credit rating can affect the value of its bonds. If you own a Zarofire Systems bond and one or more of the major rating agencies upgrades the company's bond rating, it is reasonable to expect that the price of your bond will rise. When a company's credit risk profile improves, its bonds become more attractive to buyers and hence command higher prices. On the other hand, if an issuer's ratings drop, it is equally reasonable to expect that the prices of its bonds will drop, as a poorer credit risk is obviously less attractive to buyers.

Bonds also carry what is known as ***interest rate risk***. The interest rates that prevail in the market change over time. In the early 2000s, it was not unusual for a savings account to pay an interest rate of only 1% or even less, whereas the same type of account 20 years earlier might have carried a rate of 8% or more. Through the mid 2000s interest rates have been rising. Rates change over time depending on an enormous number of factors at work in the economy. These changes in interest rates pose both risks and opportunities for owners of bonds. The Zarofire Systems 8% coupon rate bond sold for a premium because the 8% rate was higher than the prevailing market rate for similar bonds, and so the bond commanded a higher price. If, however, market conditions change so that the prevailing rate for bonds rises to 9½%, what do you think will happen to the market value of this bond? Naturally, if the market rate rises, buyers will no longer be willing to pay such a premium for a bond with an 8% coupon rate. In fact, the bond will go from selling at a premium to selling at a discount.

As interest rates rise, the selling prices of bonds tend to go down; as rates decline, bond prices tend to rise. When two quantities move in opposite directions as these do, we say that they are ***inversely correlated***.

Example 6.2.6 ***Jack has a lot of money invested in government bonds. His financial advisor tells him that interest rates are rising. Is this good news for Jack?***

At first blush, it might sound as though higher interest rates are good news for Jack, but this is not correct. If interest rates are rising, that means that the value of the bonds that Jack already owns will go down. Jack will not be pleased to hear this news.

Example 6.2.7 ***Jill has no bond investments, but she expects to take out a mortgage loan to buy a new house soon. A report on the morning news says that bond prices are expected to rise over the next few months. Will she be happy to hear this news?***

Since bond prices and interest rates are inversely correlated, rising bond prices mean declining interest rates. Since Jill has not yet taken out the loan she should be happy to hear this, because it suggests that she will be able to get her loan at a lower rate.

The astute reader will notice that Jack and Jill are getting contradictory information about the direction of interest rates. This really isn't all that surprising. Future interest rates are quite difficult to predict, and so it really isn't all that unusual to get contradictory predictions from two different sources. In the words of the noted philosopher Yogi Berra: "Predictions are difficult to make. Especially about the future."

[4]Even if worst came to worst and the federal government were unable to collect enough in taxes to pay its debts, it has a printing press. Printing money to pay debts might have disastrous economic consequences, but the bondholders would still get paid (though the dollars they would get paid with would probably be worth a lot less than the dollars are worth today). The risks this would pose are enormous, but they are not *credit* risks.

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When a bond is sold, the buyer usually compensates the seller for the interest that was earned but not paid while the seller owed the bond. In addition, the buyer may pay a small brokerage commission for handling the transaction, though this commission is often included in the quoted prices for the bond. We will not discuss this in the text, though it is reflected in one of the Additional Exercises at the end of this section.

Special Types of Bonds

The preceding discussion has presented the basics of bonds in general terms. In a market as large as the bond market, though, it should not be terribly surprising that there exist many, many special types of bonds with unique features. It would be impossible to cover these all here, or even to cover many of them in any depth, but before closing this section we will mention some common examples.

Savings bonds are one type of bond that may be familiar to many readers. We discussed these bonds briefly in Chapter 2. Savings bonds are issued by the United States federal government and can be purchased through many banks and other financial institutions. There are several different types (called ***series***) of savings bonds. The most familiar types of savings bonds are sold for half of their face value, and are issued with a guaranteed minimum interest rate. When they are issued, they have a maturity date based on the time it would take to grow to maturity value at this guaranteed minimum rate. The interest rate that is actually paid on these bonds, though, is based on an index of market rates for certain government bonds, and in fact the actual interest rate paid is normally higher (and hence the time to reach maturity is usually lower) than what is specified when they are purchased. Savings bonds can also be held beyond their maturity, and continue to earn interest. Interest on savings bonds is not normally paid prior to maturity, but instead compounds over time. (There are types of savings bonds that do make semiannual interest payments; while these types of bonds still exist, they are not available for purchase as of this writing.) Savings bonds cannot be bought and sold. They can, however, be cashed in for their accumulated value at any time, subject to some limitations.

Inflation-protected securities are bonds, almost always issued by the federal government, whose coupon rates (and/or redemption value) vary depending on the inflation rate. If the rate of inflation increases, so does the bond interest rate. Likewise, if the inflation rate declines then so do the bond rates. ***Treasury inflation-protected securities (TIPS)*** are a bond of this type sold on the open bond market. ***Series I savings bonds*** are a form of savings bond whose rates vary with the rate of inflation.

Zero coupon bonds are long-term bonds which do not make any interest payments prior to maturity. These bonds carry the significant disadvantage that IRS regulations require the bondholder to pay taxes on the interest these bonds earn prior to maturity, even though the interest is not paid until maturity. For this reason, these bonds are generally unpopular with individual investors, though they are used in tax-advantaged retirement accounts and in other accounts where taxes do not have to be paid on an ongoing basis.

Municipal bonds ("munis") are bonds issued by state and local governments. In many cases, the interest paid on these bonds is exempt from federal income taxes. Since the income is tax-free, these bonds often carry much lower interest rates than similar bonds of other issuers.

When evaluating the rate of return earned on bonds such as savings and zero coupon bonds, where the interest accumulates and/or the rate varies over time, we can use the same formula we used in Section 6.1 to find rates or return on stocks.

Example 6.2.8 ***Laurie bought a \$50 face value savings bond for \$25. Sixteen years later, she cashed the bond in for \$79.36. What effective rate of compound interest did she earn on this bond?***

$$i = \left(\frac{FV}{PV}\right)^{1/n} - 1$$

$$i = \left(\frac{\$79.36}{\$25.00}\right)^{1/16} - 1$$

$$i = 0.0748648 = 7.49\%$$

Bonds and Sinking Funds

When a company issues bonds, it takes on the obligation to make the promised payments. The ongoing interest payments are actually a comparatively small part of these obligations. Looming ahead at the maturity date is the obligation to pay off the bond's par value.

Good planning requires the bond issuer to be preparing for the bond's redemption over the course of its term. As we discussed in Chapter 4, setting up a sinking fund is a reasonable way to build up the funds needed. Sometimes the terms of a bond will require the issuer to have a sinking fund in place, giving the buyers of the bonds greater confidence that the redemption value will indeed be paid at maturity, and hopefully therefore allowing the bonds to be sold at a better price for the issuer.

Example 6.2.9 The City of Summerfield issued seven thousand 10-year, $1,000 par value bonds with a 4.7% semiannual coupon. The city set up a sinking fund into which it will make semiannual payments to accumulate the bonds' redemption value. The sinking fund earns 4%. How much money does the city need semiannually to meet its obligations under this bond issue?

Since the city issued 7,000 bonds, the total it will need for redemption is 7,000($1,000) = $7,000,000. Calculating the sinking fund payment as we did in Chapter 4 gives:

$$
\begin{aligned}
FV &= PMTs_{\overline{n}|i} \\
\$7{,}000{,}000 &= PMTs_{\overline{20}|.02} \\
\$7{,}000{,}000 &= PMT(24.2973698) \\
PMT &= \$288{,}097
\end{aligned}
$$

In addition, the city must pay the coupon on these bonds. The coupon payable for each bond is:

$$
\begin{aligned}
I &= PRT \\
I &= (\$1{,}000)(0.047)(1/2) \\
I &= \$23.50
\end{aligned}
$$

The total of all the coupon payments will then be 7,000($23.50) = $164,500. In total, the city will need $288,097 + $164,500 = $452,597 semiannually to service this debt.

The amount that an organization needs to pay periodically to cover its debts is sometimes called its ***debt service***.

EXERCISES 6.2

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A. The Language of Bonds

1. The coupon rate for a $1,000 par value bond is 7½%. Find the semiannual interest payment for this bond.

2. The coupon rate for a $5,000 par value bond is 4¾%. Find the semiannual interest payment.

3. An investor paid $9,467 for ten $1,000 par value bonds carrying a 5.25% coupon rate. How much will he receive in semiannual interest payments from this investment? How much will he receive at maturity?

4. A government bond with 12 years to maturity carries a coupon rate of 6%. Similar government bonds with the same remaining term sell on the open market with a 7.1% interest rate. Would you expect this bond to sell at a premium or at a discount?

5. A $10,000 par value bond is sold for $9,765.19. Did the bond sell at a premium or at a discount? Find the amount of the premium or discount.

B. Current Yield and Bond Tables

6. A $1,000 par value bond is sold for $1,056.17. The coupon rate is 8⅛%. What is the current yield?

7. An investor purchased a bond for $4,875.35. The par value is $5,000, and the coupon rate is 6¾%. Find the current yield.

8. Jordan bought a $1,000 par value bond with a 9% coupon rate. She paid a premium of $182.15 for the bond. What is the current yield on this investment?

9. A $1,000 par value bond is sold at a $45.36 discount. The coupon rate is 5.75%. What is the current yield?

10. Suppose that you are interested in the bond whose current market quote is given in the table below:

CORPORATE BONDS

Company	*Coupon*	*Maturity*	*Current Yield*	*$ Volume (000s)*	*Last Price*
Ralyd Pharmaceuticals	7.750	8/15/14	8.214	2,171	94.352

a. Suppose the par value of this bond is $1,000. How much would you pay if you bought one of these bonds at the price listed in the table?

b. What is the semiannual coupon payment for one of these bonds?

c. Does this bond sell at a premium or at a discount? What is the amount of the premium or discount?

d. If you buy 20 of these bonds, how much would you have invested? How much would you receive in each semiannual interest payment? How much would you receive when the bond is redeemed?

11. Suppose you are considering investing some money in government bonds. You look up a rate quote on your broker's Web site, and get the following quote:

UNITED STATES TREASURY BONDS

Par Value	*Coupon*	*Maturity*	*Current Yield*	*$ Volume (000s)*	*Last Price*
$10,000	6.250	10/31/12		275,998	103.516

a. What is the selling price for one of these bonds for the rates shown in the table.

b. There was an error—the current yield was left blank. What should have been shown in that spot of the table?

C. Yield to Maturity

12. On July 1, 2007, an investment manager purchased five-hundred $1,000 par value bonds with an 8.75% coupon rate for $467,000. The bonds mature on July 15, 2015.

    a. According to this information, would you expect that the rates being offered by similar investments on the open market carry a rate that is higher, or lower, than the coupon rate?

    b. Find the current yield and the yield to maturity.

13. A $1,000 par value bond with 8 years until maturity sells for $1,017.11. The coupon rate is 6.33%.

    a. Find the current yield.

    b. Find the yield to maturity.

14. A bond sells at a premium. Is the yield to maturity more, less, or the same as the coupon rate? Is the yield to maturity more, less, or the same as the current yield?

D. The Bond Market

15. Answer the following questions, using the table below.

CORPORATE BONDS

Company	*Coupon*	*Maturity*	*Current Yield*	*$ Volume (000s)*	*Last Price*
AnyCorp	7.000	7/9/18	9.329	5,043	75.034
SomeOtherCo	7.000	7/9/18	7.139	156,094	98.050

    a. Which, if any, of these two bonds sells at a discount? Which, if any, of these two bonds sells at a premium?

    b. For the information in this table, which bond issue would you say is more liquid?

    c. For the information in this table, which company would you expect has a better credit rating?

    d. For the information in this table, which company's bonds would you think would be more likely to be considered junk bonds?

16. Edgar has quite a bit of money invested in the bonds of the Port Gibson Widget Company. He hears on the morning business news that a major rating agency has upgraded the company's credit rating. On the basis of this news, would you expect that each of the following would increase, decrease, or remain the same?

    a. The market value of his bonds.

    b. The current yield of his bonds.

    c. The par value of his bonds.

    d. The semiannual interest payments on his bonds.

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17. Sybil owns several U.S. Treasury Bonds (bonds issued by the U.S. federal government). On the radio she hears a well-known financial commentator predict that interest rates are going to rise significantly over the next few months. Would this be good news or bad news for the market value of her investments?

E. Zero Coupon Bonds

*Recall that a **zero coupon bond** is a note with a fixed future value and maturity date, on which the borrower makes no payments until maturity, and on which compound interest is earned. Zero coupon bonds resemble simple discount notes, except that the interest is calculated by using compound interest instead of simple discount. For example, to find the selling price for a $10,000 zero coupon bond with 5 years to maturity and an interest rate of 7.2% compounded daily (bankers' rule):*

$$\$10{,}000 = PV(1.0002)^{1{,}800}$$
$$\$10{,}000 = PV(1.433277823)$$
$$PV = \$6{,}977.01$$

So the zero coupon bond would sell for $6,977.01 today.

Exercises 18–22 deal with zero coupon bonds.

18. Find the selling price for a zero coupon bond with 8 years to maturity and a $100,000 maturity value if:
    a. The interest rate is 7.7% compounded monthly
    b. The interest rate is 5.5% compounded daily.
    c. The effective interest rate is 6.3%.

19. Find the selling price for a zero coupon bond with 3½ years to maturity and a $10,000 maturity value if
    a. The interest rate is 6.25% compounded quarterly
    b. The interest rate is 7.35% compounded daily (bankers' rule)
    c. The APY is 12.72%

20. Find the selling price for a zero coupon bond with 28 years to maturity and a $10,000 face value if:
    a. The effective rate is 5.25%
    b. The effective rate is 6.83%
    c. The effective rate is 12.43%

21. Suppose that you buy a zero coupon bond, with the intention of selling it on the secondary market before it matures. On the morning business news you hear that interest rates are going up. Would this be good news or bad news for you? Explain. (*Hint:* Look at what happened in Exercises 19 and 20.)

22. Tom and Jerry both invested in zero coupon bonds. Tom bought a bond with 30 years to maturity, while Jerry bought a bond with 2 years to maturity. After both bought, interest rates dropped dramatically. Who would be happier, Tom or Jerry? Whose investment was riskier? (*Hint:* Compare Exercise 19 to 20. In which case did the changes in interest rates make a bigger difference in the bond prices?)

F. Bonds and CAGR

23. Eighteen years ago, Porshia's grandmother bought her a $1,000 face value savings bond for $500. Today, she can cash this bond in for $1,514.67. What is the effective interest rate earned by this bond?

24. A $10,000 par value zero coupon bond with 8 years to maturity sells for $5,632.12. What is the effective interest rate for this bond?

25. An investment fund manager bought a $25,000 par value zero coupon bond for $20,053.10. It matured 3 years later. What was the CAGR for this investment?

G. Bonds and Sinking Funds

26. To fund a major facilities expansion, the Fullamport Congregational Church sold 1,000 twenty-year $1,000 par value bonds with a semiannual coupon. The church set up a sinking fund to accumulate the bonds' redemption value, which pays 5% interest. How much does the church need semiannually to meet the obligations of this bond issue if the bond's interest rate is 4%?

27. The City of Winterplain issued 8,000 fifteen-year, $1,000 par value bonds with a semiannual coupon. The city set up a sinking fund into which it will make semiannual payments to accumulate the bonds' redemption value. The sinking fund earns 4.25%. How much money does the city need semiannually for its debt service if the bond's interest rate is 5.35%?

H. Grab Bag

28. A $10,000 par value bond sells for $9,503.19. The coupon rate is 5%. The bond matures in 8 years. Find the current yield.

29. Driving home from work, you hear on the radio evening news that "bonds fell in today's trading." Does this mean that interest rates in the market rose, fell, or stayed the same?

30. A $10,000 par value bond sells for $9,503.19. The coupon rate is 5%. The bond matures in 8 years. Find the yield to maturity.

31. Eleven years ago, I bought a zero coupon bond with a $10,000 par value for $4,992.03. What effective interest rate has my investment earned?

32. Dechele and Tony are looking at buying a house. They will need to take out a mortgage to pay for the purchase. If they hear on the evening news that "the bond market is rising, and the trend is likely to continue for the next few months," does this mean that the interest rates they are likely to see when they find the house they want to buy will likely be higher or lower than the rates available right now?

33. Suppose this quote is listed in your morning newspaper:

CORPORATE BONDS

Company	*Par*	*Coupon*	*Maturity*	*Current Yield*	*$ Volume (000s)*	*Last Price*
Steron Magnetics Corp.	$1000	5.250	12/1/21	6.517	38,055	80.559

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a. How much would one of these bonds cost?

b. If you bought one of these bonds, what payments would you receive?

c. Is this bond selling at a premium, at a discount, or at par?

d. Cattarauqua Corp. also has another bond issue with the same maturity date but with a coupon rate of 7½%. Would you expect the 7½% coupon bonds to sell for a higher price, a lower price, or the same price as the 5.25% coupon bonds?

34. A $10,000 par value bond with a 7⅜% coupon is sold for $11,110.55. What is the current yield?

35. A $50,000 par value bond with a 10% coupon is sold for $12,043.19. What is the current yield?

36. Someco Inc. issued three thousand five hundred $1,000 par value bonds with a 10.25% semiannual coupon rate. These bonds mature in 8 years. The company set up a sinking fund, paying 6%, to accumulate the redemption value of the bonds. Calculate the semiannual debt service required by this bond issue.

I. Additional Exercises

37. Suppose the following quote is listed in the financial pages of your local newspaper:

Company	***Par***	***Coupon***	***Maturity***	***Current Yield***	***$ Volume (000s)***
Major Localbiz Corp.	$10,000	7.250	1/1/11	7.135	19,024

This table does not list the last price for this bond issue. Determine it for the information given.

38. As mentioned in the text of this section, a buyer must often reimburse the seller of a bond for the simple interest due for the period since the last interest payment during which the seller has owned the bond.

Suppose that you buy a $1,000 U.S. Treasury bond with an 8% coupon rate for $1,135.19. The last interest payment was made 126 days ago, and in addition you are required to reimburse the seller for 56 days of interest. How much in total will you actually pay for this bond?

6.3 Commodities, Options, and Futures Contracts

Stocks (ownership of businesses) and bonds (loans) represent the two most significant categories of investments, but they are by no means the only way that money can be invested. In this section we will discuss some other common types of investment, and how these investments can be used both for investment and business management purposes.

Commodities are goods that are bought and sold in bulk. As you might imagine, since commodities are things that are produced and used in great quantities, the dollar volume of commodities trading is quite large. This dollar volume is made even larger by trading among ***speculators***. Commodities speculators are individuals or financial institutions who buy and sell commodities solely for the purpose of making a profit from movements in commodity prices. A sale of pork bellies[5] might represent a hog farmer selling his animals to a meatpacker. But it also might represent one speculator selling to another, where both are simply engaged in the business of buying things and hoping to make a profit by selling them at a higher price. It may well be that neither speculator has the slightest interest in either raising pigs or processing pork.

Commodities can be sold with the idea that the seller will provide the commodity in question to the buyer right away. Such deals are often referred to as ***for immediate delivery*** or ***spot*** (short for "on the spot") transactions. Commodities also may be sold, however, under an agreement where the seller agrees to provide the buyer with the specified commodity at some specified future date. Not at all surprisingly, this sort of arrangement is called a ***futures*** transaction, and the agreement which the buyer and seller enter into is called a ***futures contract***. We call the date on which the commodity is to be provided to the buyer by the seller the ***delivery date***.

Prices for commodities can be extremely volatile, sometimes swinging wildly over a short period of time. In part, this is due to the nature of the commodities themselves. If a spell of weather starts to look like it is turning into a drought in a major coffee-producing area, the price of coffee is likely to rise as both buyers and sellers begin to expect that a poor crop will cause supply to fall short of demand. If rain then suddenly arrives and the threat of drought subsides without major damage to the crop, the price may quickly drop as the fear of a supply shortfall subsides. Weather forecasts change rapidly, and thus so may the prices for agricultural commodities. Other commodities are not as subject to the weather, but may be subject to other factors such as economic growth predictions, mining accidents, labor strikes, and world politics. These moves are often exaggerated by the actions of speculators. Buying and selling by speculators may affect the supply and demand balance beyond actual physical supply and demand, exaggerating movements in commodity prices.

Hedging With Commodity Futures

Suppose that you are a soybean farmer looking ahead to harvesting your crop. Though the harvest may be many months away, you are concerned about the price that you will be able to get for your soybeans at harvest. If the price at harvest time is high, you may be able to sell your crop for a large profit. On the other hand, though, if the price turns out to be low, you may be faced with the prospect of selling at a loss, or putting your crop in storage in hopes of higher prices later, assuming that you can do without the proceeds until that later date and can absorb the cost of storage.

In order to limit the risk and uncertainty of what the market price for soybeans will be down the road, you may choose to sell some of your expected crop on the futures market. Suppose that, on the futures market today, soybeans for November delivery are selling for $6.50 per bushel. You can then enter into a futures contract that will obligate you to sell an agreed to number of bushels for that price in November. You will not be paid for those future bushels of soybeans today. Instead, you have made a deal that obligates you to sell the agreed number of bushels for this $6.50 price to whoever is on the other side of this contract when November rolls around.

If the price of soybeans on the spot market is only $4.00 a bushel in November, the owner of the contract will be nonetheless obligated to buy your soybeans for the agreed $6.50. In that case you would be very happy to have made this deal. On the other hand, if the spot market price is $10 a bushel in November, you are still obligated to sell them for the agreed $6.50. In that case, you might not be so happy to have made this deal. The chance that you might have to sell at a below-market price is the price you pay for the

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[5]Pigs sold on the commodities market are traditionally referred to as "pork bellies." Many commodities are referred to professionally with names that seem comical at first.

privilege of being protected from having to sell at an unattractively low market price. The overall advantage of selling your soybeans in the futures market is the ability to eliminate price uncertainty.

This example imagined a futures contract from the point of view of a seller; what about a buyer of a commodity? On the other side of the deal, a food processor that needs plenty of soybeans for its products has the exact opposite concerns—but that is precisely why it might want to buy this futures contract. While the farmer hopes for $10 a bushel and fears $4 a bushel, the processor fears $10 and hopes for $4. The futures contract provides the opportunity for both parties to protect against the potential for an unfavorable price by giving up the potential for a favorable one. The use of futures contracts to lock in a price in advance and protect against unfavorable prices is called ***hedging***. Hedging can be a true win-win for both parties to the deal. Even though it is certain in advance that one side will end up with a less attractive price than it could have obtained on the spot market, both sides benefit from the certainty of knowing the price to be used in advance.

There is some technical lingo that is used with futures contracts. The party that is obligated to sell the commodity is said to be ***short*** that commodity. The party who is obligated to buy is said to be ***long***.

Example 6.3.1 Andi is a futures trader who believes that there will be an extremely large cotton crop this year, and that the price being quoted on the market for March delivery is too high. Would Andi want to be long or short this futures contract?

Andi believes that as March approaches the price of cotton will go lower. Thus, she would want to lock in the right to sell cotton at the currently quoted (higher) price. She would want to be short this contract.

In this example, Andi is not a cotton farmer looking to lock in a good price for her crops. She is not hedging; she is a speculator hoping to profit from an expected drop in the price of cotton. The party on the other end of the contract may be a business such as a clothing company that wants to hedge against the risk of rising cotton prices, or it may be another speculator who thinks that the price of cotton is going to rise. This should raise a question in your mind: if Andi does not actually have any cotton, how will she fulfill her obligation to sell it in March? We will address this question shortly.

The Futures Market

The term ***commodities market*** refers to the overall global activity between buyers and sellers of commodities. This includes but is not entirely limited to the trading that occurs at major ***exchanges***, such as the New York Mercantile Exchange, Chicago Mercantile Exchange, or the Chicago Board of Trade. There are many other commodity exchanges operating in the United States and abroad; each of these exchanges trades only certain commodities. Some examples of commodities that may be traded include natural resources (such as gold, silver, platinum, copper, aluminum, lumber), crops (such as corn, soybeans, cotton, coffee, cocoa beans), fuels (such as oil, natural gas, coal), and other agriculture products (such as cattle, pigs, milk, butter).

Physical commodities like those mentioned above are certainly not the only things that can be bought and sold, and futures trading occurs for many other things, such as kilowatt-hours of electricity, U.S. government bonds, foreign currencies, Internet bandwidth, and greenhouse gas emissions. The ***futures market*** and ***futures exchanges*** are broader terms that are used to include trading in futures contracts for things that are not physical commodities.

A futures contract could be made between any willing buyer and seller. The owner of a coal mine and an electric utility company could directly talk with each other and make a business agreement today for the sale of a thousand tons of coal 6 months from now at a set price. However, the various futures exchanges function to provide a convenient way of bringing together buyers and sellers. Rather than make direct contacts with all sorts of potential buyers, a mine owner looking to sell coal can simply offer it for sale on an exchange. Likewise, an electric utility looking to purchase coal can do so at an exchange rather than make direct contacts with coal producers.

How is the price for a commodity determined? As with anything else in a free market, the price is whatever willing buyers will pay and willing sellers will accept. If a farmer hopes to sell his corn crop for $3.50 per bushel, but other farmers are willing to sell for $2.75, buyers will naturally choose to buy at the lower price. Likewise, if a buyer hopes to buy corn for $2.00 a bushel but other buyers are willing to pay $2.75, sellers will naturally choose to sell to that other buyer. In this case, the ***market price*** for a bushel of corn would end up at $2.75, since it is the highest price that buyers will pay and lowest price at which sellers will sell. If demand increases and/or supply decreases, the market price will most likely rise; on the other hand, if demand fades and/or supply swells, the market price will most likely drop.

To keep things simpler for all involved, the contracts made on commodity exchanges are generally only certain standardized types and only for certain standardized sizes. Each contract will provide for the sale of a certain set quantity of the commodity in question, and contracts will be offered for sale only with certain specific dates on which the sale is supposed to actually take place. Most exchanges specify the delivery date by month only; a "July contract" means that delivery must be made in July, though the exact date in July is not usually mentioned.

The market prices of some commodities (especially petroleum and gold) are often reported on daily TV and radio business news reports; daily farm reports in rural areas will usually give updates on the prices for agricultural commodities. Quotes given in the news usually only give the spot price, though, or price for a limited number of delivery dates. Price quotes for other commodities are sometimes listed in newspapers, though this is becoming less common as people looking for these quotes rely more and more on the Internet as a source.

A futures quote will generally contain the information shown in the example below:

SOYBEANS DELAYED FUTURES FRIDAY AUGUST 23 1:35 PM

Contract	*Last*	*Change*	*Open*	*High*	*Low*	*Previous*
October '06	612-5	+3-6	608-2	615-4	603-1	608-9

This quote gives the market price information for the October 2006 soybean contract as of August 23. Interpreting this quote requires some background information. You would first need to know the units. Each commodity trades in a set unit amount. While the units involved are often familiar, some may not be to the average consumer; milk is commonly sold by the "hundredweight," for example. Soybeans commonly are traded in bushels. Also, the prices for soybeans are usually given in cents, with the hyphen used in place of a decimal point. So this quote tells us that at 1:35 PM the current market price of soybeans for October delivery was 612.5 cents (or $6.125) per bushel. Quotes often will show the ***open*** (the price of the first trade of the day), ***high***, and ***low*** (as the name suggests, the highest and lowest prices seen in trading during the day so far), and the ***previous close*** or ***previous settlement*** (the price as of the close of trading on the prior day).

Because these quotes require some background knowledge, interpreting them can be frustrating at first. However, anyone with a professional or other serious interest in a commodity will be aware of that information, and anyone trading commodities or futures will both fall into that category and be working though a broker who has that knowledge as well.

Note also that while it is unlikely that anyone would be buying or selling a futures contract for a single bushel of soybeans, the price is generally quoted per bushel, not the price for the entire quantity of a contract. We will see in the following how the mathematics works.

Profits and Losses from Futures Trading

Example 6.3.2 ***Suppose that Luis believes that the price of soybeans will decline. He takes a short position in October soybeans for 5,000 bushels, at 612.5 cents per bushel. At the delivery date, the spot market price per bushel is 543.0 cents per bushel. Calculate his profit.***

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Luis has purchased the right to sell 5,000 bushels for 612.5 cents per bushel, which we can work with more conveniently as $6.125 per bushel. So the total price he can sell them for is:

(5,000 bushels)($6.125 per bushel) = $30,625

While it may have felt strange to write a dollar amount with three decimal places, the issue disappears when we find the total.

The market price for his soybeans at the delivery date is:

(5,000 bushels)($5.43 per bushel) = $27,150

Since he can sell his soybeans for more than the market price, Luis can be seen as making a profit of $30,625 − $27,150 = $3,475.

We don't know here whether or not Luis is a soybean grower with actual beans to sell, or a speculator simply hoping to profit from changes in soybean prices. If he is as speculator, won't he need to go out and buy soybeans on the spot market in order to be able to deliver what was promised to the buyer? If the buyer was a speculator with no actual interest in actually purchasing any actual soybeans, won't the buyer then have to sell the beans on the market after he buys them from Luis? It certainly seems that there could need to be an awful lot of buying and selling going on to wrap up the deal in the end. Is all of this really necessary?

In fact, it isn't. Luis had a contract that allowed him sell 5,000 bushels of soybeans for $30,625, when the spot market price was $27,150. It seems that in order to realize his profit, he would have to actually sell 5,000 bushels of soybeans for $30,625. He would then have to actually have 5,000 bushels of soybeans. But there is an alternative. Instead of actually buying the soybeans and selling them, Luis could equally well just agree to take $3,475 in cash from the other party. If he is a grower with actual beans to sell, he can sell them for $27,150, which together with $3,475 gives him $30,625. If he is a speculator, just taking the $3,475 makes matters far simpler.

This works out equally well for the other party. If the buyer actually wants the soybeans, he can then go buy them on the spot market for $27,150. The price, plus the $3,475 paid to Luis, would mean a total cost of $30,625, which is what he agreed to in the first place. If the buyer is a speculator, it is easier to just handle things with a cash payment. By settling the contract in cash, we get to the same financial result for both parties with less buying and selling required. Since this ***cash settlement*** approach is far simpler, it is the way in which futures contracts are normally actually settled. In fact, many futures contracts are specifically ***cash settlement contracts***, meaning that right from the start both parties agree that cash settlement will be used.

In this example, things worked out well for Luis. He was right about what would happen to soybean prices, and he made a nice profit from this. Of course, no one gets it right every time. What if someone takes a futures position and then decides before the delivery date arrives that he was wrong? He cannot bail out of the contract early; that would not be fair to the person on the other side. He can, however, enter into a new contract with someone else to offset his obligation under the first contract. The following example will illustrate.

Example 6.3.3 ***Jimmy believes that petroleum prices will rise. In March, he went long a September contract for 1,000 barrels of oil at $75.09 per barrel. By May, though, the price of oil for September delivery had declined to $68.35 per barrel, and Jimmy decided he was wrong. Fearing that the price would decline further before the delivery date, he went short a 1,000-barrel contract. Calculate his loss.***

The total dollar value of the initial (long) contract was:

(1,000 barrels)($75.09 per barrel) = $75,090.

The total dollar value of the later (short) contract was:

(1,000 barrels)($68.35 per barrel) = $68,350.

Jimmy's long contract obliges him to buy 1,000 barrels for $75,090, but his short contract allows him to sell those 1,000 barrels for $68,350. No matter what happens to the price of oil in the interim, his buying and selling prices are now locked in. So he will lose $75,090 − $68,350 = $6,740.

In theory, Jimmy has two contracts running, but no actual financial stake in the oil, since the two contracts cancel each other out. Rather than keep those two contracts running, the exchange will normally instead match the short and long contracts, require Jimmy to pay the difference between them, and remove him from the picture entirely. In this case we say that Jimmy has ***closed his position***.

Margins and Returns as a Percent

Determining the *percentage* return on a commodities investment can be tricky. It was not hard to see that Luis made $3,475, but if we want to look at it as a percent, what is it a percent *of*? Luis really did not have to invest any actual money—it appears that he made a profit without putting any money on the table. In reality, though, the futures exchange requires each party to a contract to put up a certain amount of money to be held in reserve while the contract is in place. This is called ***posting a margin*** for the contract. This prevents someone from making promises he can't keep. Both parties to a futures contract need to know that the other party has the means to settle the contract. The margin provides that assurance.

The margin that must be posted for a given contract can vary, depending on the commodity in question and the rules of the exchange on which the contract is being traded. It is common for the ***initial margin***, the margin that must be posted up front, to be 5% of the eventual value of the contract. If prices start to move against you, you can be required to post additional margin (called a ***margin call***) or close your position.

Example 6.3.4 ***Suppose that Luis (from Example 6.3.2) was required to post a 5% initial margin, and there were no margin calls along the way. Also, suppose that the time between when he opened his position and the contract's expiration was 47 days. Find (a) his percent profit and (b) his profit as a rate of return.***

The margin Luis had to post was (5%)($30,625) = $1,531.25.

(a) Luis made a $3,475 profit on a $1,531.25 investment. This amounts to a $3,475/$1,531.25 = 2.2694 = 226.94% profit. While it does express the profit as a percent of the amount invested, this figure does not take into account the time involved.

(b) If we look at this as his principal and calculate a simple interest rate using it, we find that his percent rate of return is:

$$I = PRT$$
$$\$3,475 = \$1,531.25(R)(47/365)$$
$$R = 17.6240 = 1,762.40\%$$

(We chose to calculate this as simple rather than compound interest because the length of time is short and the simple interest formula is easier to work with in this case. See the Additional Exercises for an example of this calculation using a compound rate.)

This percent rate of return is astounding, far higher than anything we might have imagined from stocks, bonds, and similar lower-adrenaline investments. This sort of return is made possible by the fact that the amount that Luis had to actually invest is quite small in relation to the overall value of the soybeans involved. Thus, even a fairly small percent movement in the value of the soybeans translates into a large percent measured against Luis's margin. This effect is referred to as ***leverage***.

Before we get too excited about the potential leverage offers to earn such eye-popping percent returns, it is important to realize that just as the margin is small in comparison to the potential gain, it also quite small in comparison to the potential loss.

Example 6.3.5 ***Suppose that Jimmy (from Example 6.3.3) was required to post an initial margin of 5%, and that the time between his initial long position and his closing short position was 52 days. Calculated Jimmy's return as (a) a percent and (b) as a simple rate of return.***

Jimmy's initial margin would have been (5%)($75,090) = $3754.50.

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(a) $-\$6{,}740/\$3{,}754.50 = -1.7952 = -179.52\%$. *Since this is a loss, it should be expressed with a negative number, so the answer is* -179.52%.

(b)

$$I = PRT$$
$$-\$6{,}740 = \$3{,}754.50(R)(52/365)$$
$$R = -12.6008 = -1{,}260.08\%$$

His rate of return was thus an astonishing $-1{,}260.08\%$.

Just as a leveraged investment can produce astonishingly high percentage profits, it can produce astonishingly high percentage losses. This example also reveals something especially worrisome about futures speculation: notice that if we consider Jimmy's investment to have been his initial margin, he actually lost *more* than the amount of that investment! Once the loss on his investment became larger than his margin, Jimmy most likely would have had to put up additional money for a margin call. While Example 6.3.3 suggested that it was Jimmy's choice to close his contract, it may have actually happened that Jimmy was forced to do this as a result of an unwillingness or inability to meet a margin call.

Options

Options are similar to futures contracts in that they are based on the idea of selling something for a specified price at specified future date. They differ from futures in one very important respect, though. With a futures contract, both parties are obligated to buy and sell at the specified date at the specified price. With an option, one party owns the contract, and the owner has the choice of whether or not she wants to buy/sell the thing in question. With futures, both parties are obligated to complete the deal; with options, only one party is obligated. The owner of the contract has the *option* (hence the name) of completing the sale or not, and so logically will only do so if the doing so works to her benefit. Options also differ in that the contract's owner can choose to complete the transaction (called ***exercising the option***) on the contract's expiration date *or at any time prior* to the expiration date.[6]

With futures, neither party really needs to pay the other anything for the contract, since it is a mutual agreement that offers the same opportunities and risks to both sides of the deal. Options are a bit more one-sided. Since the option's owner has the right to exercise the option when it benefits her and no obligation to do so when it doesn't, the owner will have to pay the other party (called the option's ***writer***) something for agreeing to the contract. This payment is called the option's ***premium***.

There are two types of options. A ***call*** entitles its owner to buy something at a specified price. A ***put*** gives its owner the right to sell something at a specified price. Options are widely available for the stocks of large companies. Suppose that the price of the stock of Yoyodyne Corp. is currently $48.00 per share. You buy a call option expiring at the end of this year, which allows you to buy the stock for $60 per share. If the stock price never rises above $60, you will never have occasion to exercise this option, and so it will expire worthless. If, however, the stock prices rises to $75 per share you can exercise your option and buy it for $60.

If you believe that the stock is going to rise in price, why not just buy the stock? Either way, you would turn a profit if you are right. One of the advantages of options is that, like futures, they offer the potential for a great deal of leverage, offering an exaggeration of the rate of return of the underlying security. The following example will illustrate this:

Example 6.3.6 ***You believe that Yoyodyne's stock will rise in value between now and the end of the year. The stock price is now $48 and you buy a $60 call option for 100 shares. The option premium is $1.50 per share. How much will you make if the stock price rises to (a) $75 per share, (b) $64.50 per share, or (c) $55 per share?***

Whatever the stock price does, this option costs you ($1.50 per share)(100 shares) = $150.

[6]This is true of ***American-style options***. Another type, ***European-style options***, can be exercised only on a specific date. Since the overwhelming majority of options in the United States are American-style, we will assume this in this section are as well.

(a) If the stock price rises to \$75, you can exercise your option to buy 100 shares at \$60, costing you \$6,000. But you can then immediately turn around and sell them for \$75 per share, or \$7,500, providing yourself with a profit of \$1,500 − \$150 = \$1,350. You've grown your money 10-fold!

(b) In this case you can still exercise your option for \$6,000 and then immediately sell the stock for \$6,450, a \$450 − \$150 = \$300 profit. You've tripled your money.

(c) There is no point in exercising your option to buy at \$60 if the open market price is lower. The contract expires worthless. You've lost your entire investment.

We can see how this leverage works if we compare the scenarios of Example 6.3.6 to simply buying 100 shares. Buying 100 shares would tie up \$4,800 in the investment, enormously more than the option approach required. If the stock rose to \$75 a share, you could sell your stock for a \$7,500 − \$4,800 = \$2,700 profit. This is larger than the \$1,500 profit made with the option, but it is a far smaller profit in comparison to the size of the investment. With the option you grew your money 10-fold, with the actual stock you did not even come close to doubling it. In scenario (b), where the stock rose to \$64.50 a share, you would profit \$1,650 by owning the actual stock. Once again this profit is larger in absolute terms, but in comparison to the amount invested it does not even come close to the tripling that the option provided.

On the other hand, if the stock rises to \$55 a share, by owning the stock you could earn a \$700 profit, while with the option you actually lose all your money despite the fact that the stock price rose in value. While options offer the potential for greatly magnified returns based on the money invested, they also offer a much greater risk of enormous losses as well. While options contracts sometimes pay off handsomely, in actuality most options contracts expire worthless.

Puts work similarly.

Example 6.3.7 ***The stock of Global Consolidated Meganormo Corp. presently sells for \$73 per share. You believe that the stock price is likely to drop, and so you buy a \$60 put option for 500 shares. The option premium is \$2. How will you make out if the stock price drops to (a) \$50 and (b) \$60.***

In either case, the total cost of the options contract is (500 shares)(\$2 per share) = \$1,000.

(a) If the stock drops to \$50 per share, you could buy 500 shares at this price for a total of \$25,000, and then exercise your put and sell them for \$60 per share, or \$30,000. This would give you a gain of \$30,000 − \$25,000 = \$5,000. Since you paid \$1,000 for the option, your profit is \$4,000.

(b) If the stock price drops to \$60 per share, an option to sell at that price is nothing special, and so the contract expires worthless. Even though you were right about the stock price drop, you will lose your entire \$1,000.

Writers of options hope to profit by pocketing the premiums and then having the options written expire worthless. By some estimates, anywhere from 85 to 95% of all options do in fact expire unexercised.

Profits or losses from options investments can be calculated as rates in much the same way as we did with futures. The difference is that for options it would be the option premium that should be used as principal.

Example 6.3.8 ***Calculate the simple rates of return for both scenarios in Example 6.3.7. Assume that the term of the investment was 3 months.***

(a)

$$I = PRT$$
$$\$4{,}000 = (\$1{,}000)(R)(3/12)$$
$$R = 1{,}600\%$$

(b)

$$-\$1{,}000 = (\$1{,}000)(R)(3/12)$$
$$R = -400\%$$

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The Options Market

There is really no theoretical reason why an option could not be written for any number of shares at any price someone might want to buy an option for. However, for the same reasons as for futures, there are many benefits to having only a certain set of standardized options available. Such options are listed on one or more options exchanges, such as the Chicago Board Options Exchange (CBOE).

Options are offered with certain prices at which they can be exercised (called ***strike prices***) and certain expiration dates (options often expire on the third Friday of each month). If you are interested in buying an option, you may select from any of the various combinations of strike prices and expiration dates that are available. A quote listing these is known as an ***option chain***. An example of such a chain is shown below:

GANARGUA HYDRO CORP—CALLS

Date	*Strike*	*Last*	*Bid*	*Ask*	*Vol*	*Open Int*
Oct '06	30	9.23	9.20	9.25	66	350
Oct '06	35	4.18	4.16	4.21	221	1087
Oct '06	40	2.25	2.23	2.27	107	502
Oct '06	45	1.18	1.18	1.20	35	205
Oct '06	50	0.65	0.61	0.65	0	4
Nov '06	30	10.37	10.34	10.39	11	87
Nov '06	35	5.06	5.04	5.09	105	935
Nov '06	40	3.55	3.53	3.57	67	772
Nov '06	45	2.17	2.17	2.19	195	602
Nov '06	50	1.08	1.04	1.08	2	21

The ***last*** price shown in the table is, as one would expect, the last price at which a sale of the given contract occurred. The ***bid*** price is the highest price that is currently being offered for a contract with the given expiration and strike price; the ***ask*** price is the lowest price that is currently being offered for such a contract. The ***volume*** gives the current day's number of contracts traded; the ***open interest*** is the total number of contracts which are currently in existence.

The prices quoted for options are normally a price per share of the underlying stock. Each contract, though, is usually the right to buy or sell 100 shares of the underlying stock.

Example 6.3.9 Kevin bought three November contracts to buy shares of Ganargua Hydro Corp. at 35 at the best price available for the option chain shown above. How much did he pay?

For the November contract, the ask price was 5.09. Since each contract represents the right to buy 100 shares, the price for each contract was $509. The total amount paid would then be 3($509) = $1,527.

You may have heard of stock options on the news in stories about their use as part of the compensation paid to executives of large companies. Many companies offer their executives call options on the company's stock either as part of their compensation or at attractive prices. The theory behind this is that if the executive stands to benefit from a rise in the company's stock price, she will have a strong incentive to manage the company in a way that makes this happen, benefiting all of the shareholders. Critics of this practice complain that these options are often set up in such a way, though, that executives can reap huge financial rewards from fairly modest moves in the stock price, due to the leverage options offer, and that if the price declines the executive's risk is limited to her options expiring worthless.

Abstract Options and Futures

Futures are usually settled in cash; options are typically settled in the same way. Consider the scenario from Example 6.3.6, where you had a call option for 100 shares of Yoyodyne

Corp. at 60, and the market price was \$75. We said there that you could exercise your option, buy the shares for \$6,000 and then sell them for \$7,500, netting a \$1,500 profit. This could be settled more simply, though, if the writer of the option just directly paid you \$1,500 in cash. If you actually want the stock, you can buy it for \$7,500 on the open market, applying your \$1,500 options profit toward the cost, leaving you a net cost of \$7,500 − \$1,500 = \$6,000.

While cash settlement makes matters simpler, it has some unusual side effects. In the futures example we discussed, note that it really wasn't necessary that either party either own or want to buy any actual cotton. Likewise, in the options examples it wasn't really necessary that either the writer or the owner of the option actually ever own, or even want to actually own, any stock. With cash settlement, options and futures essentially become bets on the future price of a thing, no longer requiring that anyone actually have physical possession of the thing in question.

Removing the need for any actual physical possession of the things in question allows options and futures to be written even for abstract things that actually *cannot* actually be owned! ***Index options*** are one common example of this sort of thing. ***Index futures*** also exist. There are a number of ***indexes***[7] that are calculated and reported as a way of measuring the performance of some group of investments *overall*. Some well-known examples include the Dow Jones Industrial Average (the Dow 30), calculated on the basis of prices of the stocks of 30 large companies, and the Standard and Poor's 500 (the S&P 500), based on the stock prices of a larger number of companies. You have no doubt heard these indexes mentioned on the news. There are also indices for foreign stocks (such as the Nikkei index for Japanese markets and the FTSE, called the "footsie," for British stocks), bonds of various types, commodities, interest rates, and so on.

Obviously, you cannot *own* an index, and you can't buy or sell it either. It is an abstract thing, a number calculated by a mathematical formula. Nonetheless, you can "bet" on whether the index will rise or fall over a period of time. With cash settlement, we can calculate what one party must pay the other by calculating the settlement amount just as if the index were a thing that could actually be owned.

Since the numerical value of an index may not be a number that would be a reasonable price for a stock, index options are not always based on 100 "shares." It is necessary to know the number of shares (called the ***multiplier***) in use for an index option.

The following example will illustrate:

Example 6.3.10 ***Suppose that the GlobalInvestrex 375 index currently stands at 617.506 and you believe that it will rise in the near future. You buy five call options at 620.000. The option price is 3.850. The index rises to 630.239 and you exercise your option. The option multiplier is 10. What will you receive when you do this? How much profit will you make?***

Since the option multiplier is 10, each contract is equivalent to 10 "shares" of this index. So you pay 5(10)3.850 = \$192.50.

The difference between the strike price and the price when you exercise is 630.239 − 620.000 = 10.239. So you get 5(10)(\$10.239) = \$511.95.

Your profit, therefore, is \$511.95 − \$192.50 = \$319.45.

Since the results of options and futures contracts are derived from the price performance of other things they are often referred to as ***derivatives***.

Options on Futures and Other Exotica

Options contracts can be created at an even higher level of abstraction. Options on future contracts can be written or bought. It is possible to construct options that do not simply profit from a rise or fall in price but instead pay off if prices fall within a certain range. The

[7]Technically, the correct plural of *index* is *indices*. The term *indexes* is in common use, though, and so we will use either term as the mood strikes.

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possibilities are nearly unlimited. These more exotic possibilities fall far outside the scope of this chapter. We mention them here only to make sure the reader is aware of the immense possibilities that exist in the derivatives markets.

Uses and Dangers of Options and Futures

Options and futures can provide a valuable tool for both businesses and investors. They allow businesses to hedge against unfavorable changes in prices, enabling them to control and limit the risks that price fluctuations can pose. They can also help limit investment risks; if an investor has a large investments in stocks and is concerned about the possibility that the market may decline, he can buy put options on the stocks he owns, or on some market index. Instead of having to bail out of his investments altogether, he can use options as a kind of insurance. A company doing business overseas can "insure" itself against the risk of foreign currency fluctuations with foreign currency futures or options. In each of these cases, and in many, many others, derivatives can help reduce risk.

Unfortunately, options and futures also present the potential to create immense financial risk. Their leverage makes it possible for an investor to make huge bets with comparatively little money, and the potential for astounding profits can lead to the temptation to overlook the potential for astounding losses. While there are many legitimate uses of derivatives, they can easily be misused, with financially disastrous consequences. These risks are magnified by the incredible complexity that derivatives can entail. Even fairly basic derivatives can be confusing, and some of the more exotic possibilities can make quantum physics look like a game of tic-tac-toe. Tragic errors can easily occur because of misunderstanding or miscalculation.

There are a number of companies that promote derivatives trading through late night infomercials, airport hotel seminars, and on the Web as an easy path to immense wealth. It is true that some people have become rich trading futures and options, and some people do successfully speculate in these markets for a living. It is also true that many people have wiped themselves out financially in the same way. Estimates vary, but most experts agree that the vast majority of people who try to get rich speculating in options and futures end up poorer for the effort.

Of course, some people do succeed in becoming rich in these markets. *Some* people have become rich betting on sports or playing high stakes poker, but that does not mean that *most* people who do these things make a good living at it. Getting involved in derivatives trading presents enormous risks, and it is not a game for amateurs. It is a serious and highly complex business. While this chapter and the following exercises provide an introduction to the basics of derivatives, they are by no means a thorough treatment of the subject. It should be clear, though, that anyone contemplating getting financially involved in this market should exercise extreme caution.

EXERCISES 6.3

A. Futures Terminology

1. In each of the following scenarios, determine whether the person described would want to be long or short the commodity in question.
   a. Travis expects the price of natural gas to rise.
   b. A wheat farmer wants to lock in a price to sell his harvest.
   c. An electronics company wants to lock in a price for the copper it will need to buy in the future.
   d. Robbie thinks that the price of gold is going to drop.

2. In each of the following scenarios, state whether the person described is betting that the price of the commodity in question is going to rise or fall.

a. Howard is long cotton futures.

b. Jill is short cocoa.

3. Seramsis Corp. is looking to buy 250 tons of coal, which they need right away.
   a. Would Seramsis look to buy this on the spot market or the futures market?
   b. Is Seramsis speculating, hedging, both, or neither?

4. Cattarauqua Ginseng Enterprises sells herbal supplements. The company has a large number of Korean customers. The company is concerned that the value of the Korean won (the unit of currency) may decline and adversely affect its business, and so it enter into a futures contract for won.
   a. Is this company speculating, hedging, both, or neither?
   b. Would the company long or short the won?

B. Calculating Profits/Losses from Futures Trades

As in the examples of the text, assume that any broker's fees or other transaction costs are minimal and can be ignored. Also, as in the text, returns should be calculated as a percent of the initial margin.

5. Don took a long position for 8,000 bushels of December soybeans at 575 cents per bushel. He did not close his position prior to delivery. In December, the spot price was 593 cents per bushel.
   a. Did Don make or lose money on this deal?
   b. Calculate the amount of his profit or loss.
   c. Assume that the required initial margin was 5%. Calculate the initial margin.
   d. Calculate Don's return as a percent.
   e. Calculate Don's rate of return (as a simple interest rate), assuming that this position was open for 113 days.

6. Mike took a long position for 10,000 bushels of March wheat at 335 cents per bushel. He did not close the position prior to delivery. In March, the spot price was 307 cents per bushel.
   a. Did Mike make or lose money on this deal?
   b. Calculate the amount of his profit or loss.
   c. Assume that the required initial margin was 5%. Calculate the initial margin.
   d. Calculate Mike's return as a percent.
   e. Calculate Mike's rate of return (as a simple interest rate), assuming that this position was open for 74 days.

7. Veronica took a short position for 2,000 barrels of October crude oil at $78.35 per barrel. She did not close the position prior to the delivery date. In October, the spot price was $84.14 per barrel.
   a. Did Veronica make or lose money on this deal?
   b. Calculate the amount of her profit or loss.
   c. Assume the required initial margin was 5%. Calculate the initial margin.
   d. Calculate Veronica's return as a percent.
   e. Calculate Veronica's rate of return (as a simple interest rate), assuming that her position was open for 30 days.

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8. Scott took a short position for 500 ounces of July gold at $653.25 per ounce. He did not close the position prior to the delivery date. In July, the spot price was $735.19.
   a. Did Scott make or lose money on this deal?
   b. Calculate the amount of his profit or loss.
   c. Assume the required initial margin was 5%. Calculate the initial margin.
   d. Calculate Scott's return as a percent.
   e. Calculate Scott's rate of return (as a simple interest rate), assuming his position was open for 63 days.

9. Kenny took a position short 10,000 pounds of February copper at $2.53 a pound. By December, the price of February copper had risen to $2.85 a pound, and Kenny took a position long 10,000 pounds at this price.
   a. Did Kenny make or lose money on this deal?
   b. Calculate the amount of his profit or loss.
   c. Is it likely that Kenny got a margin call?
   d. On the basis of these contracts, what will Kenny need to do when February arrives?

10. Leila took a position long 15,000 gallons of June unleaded gasoline at $1.83 per gallon. By May, the price of June gasoline had risen to $1.92, and she took a position short 15,000 gallons at this price.
    a. Did Leila make or lose money on this deal?
    b. Calculate the amount of her profit or loss.
    c. Is it likely that Leila got a margin call?
    d. On the basis of these contracts, what will Leila need to do when June arrives.

11. Suppose that I am long September orange juice and I decide I want to close my position. Describe what I need to do to accomplish this.

C. Options Terminology

12. Tracy believes that Zarofire Systems stock will drop in value. Would she be more likely to buy a put or a call on this stock?

13. Dom thinks that Triloquant Logistics Corp.'s stock will rise in value. Would he be more likely to buy a put or a call?

14. An investment manager has a portfolio that includes a large investment in Ganargua Hydro Corp. While she believes the company's prospects are excellent, she is concerned about the risk to the portfolio if something she is not expecting happens and drives down the stock price. Would she be likely to buy puts or calls on the company's stock to protect against this risk?

15. Lacourtney wants to buy stock in Burali-Forti Corp. but is concerned that the stock market as a whole is at risk of dropping. The stock now sells for $35 a share, and she would like to be able to have the option of buying the stock for no more than $40. Would she want to (a) buy calls, (b) sell calls, (c) buy puts, or (d) sell puts?

D. Calculating Profits/Losses from Options Trades

16. Gustavo bought 12 call options on the stock of Cartswell Carts. The stock price is currently $61.75 per share. The strike price was $65, and the option premium was $2.50.
    a. Do these options give Gustavo the right to buy or sell shares of the company? How many shares does he have the right to buy/sell?
    b. Calculate the total amount Gustavo paid for these options.
    c. Calculate his profit or loss if the price rises to $75 a share.
    d. Calculate his profit or loss if the price rises to $65 a share.
    e. Calculate his profit or loss if the price drops to $55 a share.

17. Kerry bought 20 put options on the stock of Global Consolidated Megacorp. The stock currently sells for $17.96 per share, and the strike price is $17.50. The option premium was $5.18.
    a. Do these options give Kerry the right to buy or sell the company's shares? How many shares does she have the right to buy/sell?
    b. Calculate the total amount she paid for these options.
    c. Calculate her profit or loss if the price rises to $22.50.
    d. Calculate her profit of loss if the price rises to $20.00.
    e. Calculate her profit or loss if the price drops to $15.00.

18. Repeat Exercise 16c, d, and e, but calculate the profits or losses as percents.

19. Repeat Exercise 17c, d, and e, but calculate the profits or losses as percents.

Question 20 refers to the option chain for Ganargua Hydro given in the text of this section.

20. Suppose that Jeff thinks that Ganargua Hydro's stock will rise, and wants to buy options for 500 shares. The company's current share price is $35.59.
    a. How many options contracts would Jeff buy?
    b. Calculate the total cost for Jeff to buy a November 2006 call option for 500 shares at a strike price of $45.
    c. Suppose that Jeff is right that the price will rise. If the price rises to $45 a share and he exercises his option, calculate his profit or loss.
    d. If the price rises to $50 a share and he exercises his option, calculate his profit or loss.
    e. Suppose that Jeff exercises his options 55 days after buying the contract, when the stock price is $49.93. Calculate his profit as a percent rate of return.
    f. What is the minimum share price at which Jeff can exercise his options without losing money?

21. The GlobalInvestrex 250 Index presently stands at 1,135.09. The option premium for January calls at 1,150 on this index is 17.35. The index multiplier is 10.
    a. Calculate the amount you would pay for 15 of these option contracts.
    b. If the index rises to 1,184.32 and you exercise your options at that point, calculate your profit or loss.
    c. If the index rises to 1,162.35 and you exercise your options at that point, calculate your profit or loss.

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d. What is the minimum level for this index at which you can exercise your options without losing money?

22. A company whose share price is currently $74.25 grants its CEO options to buy 10,000 shares at $80 per share. Suppose you buy 250 shares of this company's stock at the current market price.

    a. If the stock price rises to $100 per share, how much will the CEO make from these options? How much will you make from your investment?

    b. If the stock price drops to $60 per share, how much will the CEO lose? How much will you lose from your investment?

E. Grab Bag

23. Reggie went short futures on the GlobalInvestrex 5000 Index. If he had used options instead, would he have bought puts or calls?

24. Caldin bought call options on the GlobalInvestrex 5000 Index. If he had used futures instead, would he have gone long or short?

25. DJ bought eight put options on the stock of AnyCorp. The stock price is currently $51.75 per share. The strike price was $50, and the option premium was $4.70.

    a. Do these options give DJ the right to buy or sell shares of the company? How many shares does he have the right to buy/sell?

    b. Calculate the total amount DJ paid for these options.

    c. Calculate his profit or loss if the price rises to $60 a share.

    d. Calculate his profit or loss if the price drops to $50 a share.

    e. Calculate his profit or loss if the price drops to $40 a share.

26. Howard took a position long 5,000 bushels April wheat at $3.22 a bushel. By March, the price of April wheat had risen to $3.75 a bushel, and Howard decided to close his position.

    a. Did Howard make or lose money on this deal?

    b. Is it likely that Howard closed his position because of a margin call?

    c. Calculate Howard's profit or loss.

27. Suppose that I bought 10 put options on AnyCorp with a strike price of $40. The option premium was $4.78 and I exercised by options when the price had fallen to $30. Calculate my return as a percent. If the time for this investment was 82 days, calculate my return as a percent rate.

28. Suppose that you buy call options on Zarofire Systems for $8.25. The strike price is $45. What is the minimum price at which you can exercise without losing money?

29. Debbie took a position short 10,000 ounces of August silver at $10.43 an ounce. At the delivery date, the spot price was $10.72 an ounce.
    a. Did Debbie make or lose money on this deal?
    b. Calculate Debbie's profit or loss.
    c. Assuming the usual initial margin requirements, calculate Debbie's return as a percent.
    d. Assuming a term of 114 days, calculate Debbie's return as a simple interest rate.

30. If you expect nickel prices to rise, would you want to be long or short nickel futures? If you believe that a stock price is likely to decline, would you want to own puts or calls on that stock?

F. Additional Exercises

31. In Example 6.3.4 we calculated Luis's rate of return from a futures transaction as a simple interest rate. Calculate his rate of return from this example as an effective compound interest rate.

32. Suppose that the stock of Ganargua Hydro presently trades for $35 a share. You believe that between now and October the stock will rise in value, and have $2.500 you want to put into an investment based on this belief. Suppose that you consider three different alternatives: investing all of your money in the company's actual stock, buying calls with a strike price of $35, or buying calls with a strike price of $50. Calculate the number of shares or options contracts you could buy with $2,500, using the prices shown in the options chain for this company given in the text. Then, for each of these alternatives, evaluate how things would work out for you if the stock price dropped to $30, if it stayed at $35, if it rose to $45, and if it rose to $60. Discuss any conclusions you can draw from this.

6.4 Mutual Funds and Investment Portfolios

Savings and wise investments provide a solid road to financial well-being. Yet that broad statement hides an awful lot of difficult details. Obviously if you knew the future it would be easy to pick the best investments today and become fabulously wealthy. Sadly, knowing the future is not an option for any of us. So when we make investment decisions (or any other ones for that matter), we have to settle for basing our choices on informed and educated guesses of what the future holds. No matter how informed and educated our guesses may be, we have to deal with the possibility that they will turn out to be wrong.

In Section 6.1, we considered an investment in the stock of Zarofire Systems, and found that this investment worked out extremely well. Over the course of 7 years, we would have earned an approximately 25% overall rate of return. This is absolutely wonderful! From our previous work we know that this kind of rate, compounded over time, can result in fabulous investment gains.

Now if we had known from the start how things would work out, we would have been wise to pour all of our money into Zarofire stock. The problem, of course, is that there is no way that we could have known this in advance. When we first bought the stock, we must have thought that the company had good prospects, that it would be a successful business and profitable investment—otherwise, why buy its stock? But there was no way that you could know *for sure* that the company's management would not make misguided decisions, that a competitor with even better products and better prices would not

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steal all of Zarofire's customers, that the company would not be crippled by unexpected product failures, lawsuits or scandals, that the economy would not fall into a recession destroying the demand for Zarofire's products, that the stock market would not fall overall dragging stock prices down in general, or that some other misfortune would not strike the company from out of the blue. As it turns out, none of those things happened and the stock turned out to be an excellent investment. But at the time you bought, and during the time you owned the stock, these risks were still possibilities that you could not have ruled out.

Diversification

There is no way to entirely avoid risk, in finance or in life. However, there are ways to manage risk. One simple but powerful way to do this is through ***diversification***. Diversification can be summed up by the old adage "Don't put all of your eggs in one basket." To illustrate how diversification can work with investments, suppose that you are considering investing in one of four different companies: Axerixia Corp., Blunderbluff Industries, Caughdenoy Container Corp., and DDKind Cuts Hair Salons. You have every reason to expect that each of these four companies will be very successful, and that investing in them will be profitable. How do you decide which of the four stocks to invest in?

Let's gaze into the crystal ball (obviously impossible in real life, but we can pretend for now). Suppose that the table below gives the prices of each stock at the end of each of the next 12 months:

	Axerixia	*Blunderbluff*	*Caughdenoy*	*DDKind*
Start	$20.00	$20.00	$20.00	$20.00
1	$17.77	$22.50	$22.47	$21.25
2	$18.03	$26.75	$19.76	$20.17
3	$23.98	$18.05	$21.99	$26.55
4	$24.24	$14.44	$28.65	$25.99
5	$28.55	$12.65	$33.13	$21.06
6	$26.27	$11.64	$37.41	$21.11
7	$25.03	$9.47	$39.63	$24.44
8	$28.02	$10.43	$41.56	$20.01
9	$32.77	$6.18	$42.46	$19.11
10	$35.09	$3.61	$43.43	$18.50
11	$31.82	$2.05	$49.23	$18.77
12	$28.52	$1.54	$50.98	$20.00

Now, if you did have this crystal ball, you would have known that you were wrong about Blunderbluff's prospects, and that in fact that company's stock would be a dreadful investment. You would also have known that Caughdenoy would be the best performer of the group, and would thus have known that you should invest all of your money in that company's stock.

But this is of course all a fantasy—no one actually has this sort of information in advance. Given this reality, what is your best course of action? You could just pick one of the four at random and cross your fingers. Doing that, you have a 1 in 4 chance of picking the best performer—but you also have a 1 in 4 chance of picking the worst. You could try to select the best one, but there is still a good chance that you will be wrong, either because of an error in judgment or because something that you didn't know about or couldn't have anticipated may happen.

An alternative, very reasonable, choice might be to divide your investment among these four stocks. What if, instead of investing your entire $10,000 in just one of the stocks, you decide to invest $2,500 in each of the four? In this example at least, that would have worked out quite well.

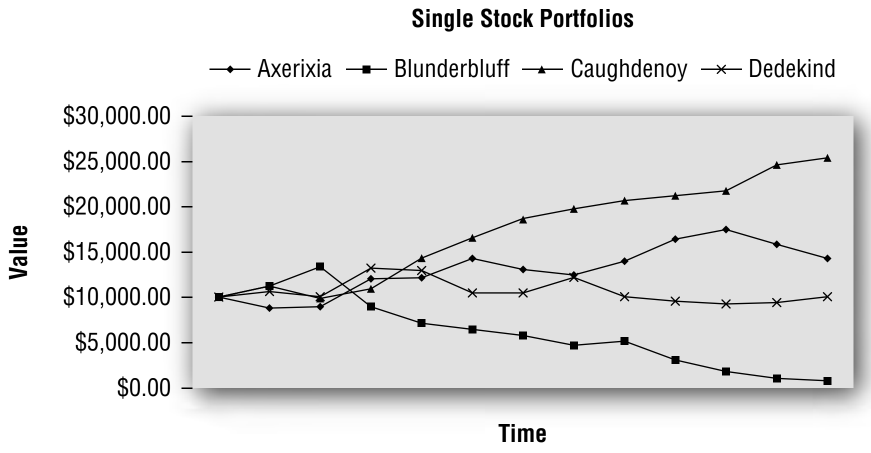
The following table compares $10,000 invested in each of the four stocks (500 shares each) with $10,000 invested by putting $2500 (125 shares each) into each stock.

	$10,000 in Axerixia	***$10,000 in Blunderbluff***	***$10,000 in Caughdenoy***	***$10,000 in DDKind***	***$2500 in each***
Start	$10,000.00	$10,000.00	$10,000.00	$10,000.00	**$10,000.00**
1	$8,885.00	$11,250.00	$11,235.00	$10,625.00	**$10,498.75**
2	$9,015.00	$13,375.00	$9,880.00	$10,085.00	**$10,588.75**
3	$11,990.00	$9,025.00	$10,995.00	$13,275.00	**$11,321.25**
4	$12,120.00	$7,220.00	$14,325.00	$12,995.00	**$11,665.00**
5	$14,275.00	$6,325.00	$16,565.00	$10,530.00	**$11,923.75**
6	$13,135.00	$5,820.00	$18,705.00	$10,555.00	**$12,053.75**
7	$12,515.00	$4,735.00	$19,815.00	$12,220.00	**$12,321.25**
8	$14,010.00	$5,215.00	$20,780.00	$10,005.00	**$12,502.50**
9	$16,385.00	$3,090.00	$21,230.00	$9,555.00	**$12,565.00**
10	$17,545.00	$1,805.00	$21,715.00	$9,250.00	**$12,578.75**
11	$15,910.00	$1,025.00	$24,615.00	$9,385.00	**$12,733.75**
12	$14,260.00	$770.00	$25,490.00	$10,000.00	**$12,630.00**

By putting equal amounts into each stock, you would end up with $12,630. This is not as good as the $14,260 you would have ended up with if you have put all your money into Axerixia, or the $25,490 you would have had from Caughdenoy, but it beats the $10,000 you would have had with DDKind and positively kills the $770.00 you would have wound up with in Blunderbluff. By spreading your investment across these four stocks, you guarantee that you will not do as well as if you have put all your money in the best performers, but you also ensure that you will not do as badly as if you hand put all your money into the poorer performers.

Of course Blunderbluff's losses eat up some of Caughdenoy's gains. Owning a mix averages the performance of all the stocks together. This makes it very unlikely that you will end up with an extraordinary return (any mix is bound to contain mediocre performers like DDKind and clunkers like Blunderbluff). But it also makes it unlikely that you will end up with an appallingly bad return (the mix is likely to contain good performers like Axerixia and Caughdenoy to balance to duds).

Diversification can not only affect the end result of an investment, it can also affect the volatility along the way. From month to month, the prices of each of these stocks varied, as did the value of the mixture. However, from the table above we can see that the month-to-month variation was greater for the investments in individual stocks than for the combined investment. This can also be seen by looking at a graph of the month-by-month values:



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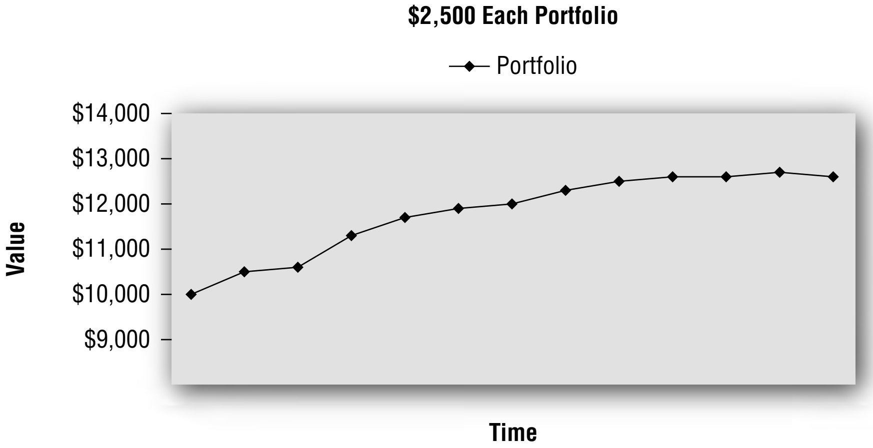

An investment ***portfolio*** is a mix of investments in different securities. Our four-stock mix above is a simple example of a portfolio. Portfolios can include stocks and/or other investments such as bonds and commodities, and real estate as well. By investing in a portfolio, instead of any one single security, an investor seeks to obtain a good return on the investment while limiting the risks posed by "putting all your eggs in one basket."

In our illustration above, we created an investment portfolio by splitting our investments equally among four stocks. There is not requirement, though, that the investment be equally divided; if we felt more strongly about some of the stocks than others we could have put more money into some and less into others. Also, it should be noted that, while we used only four stocks in our portfolio here (to keep the illustration manageable), in practice a diversified portfolio is likely to contain many more different investments—often hundreds or even thousands. The greater the number of different investments that a portfolio is spread out among, the less impact that any one particular investment can have.

A portfolio that spreads its investments around a broad range of investments is referred to as a ***diversified portfolio***.

Asset Classes

Diversifying across different stocks allows us to mitigate the risk that any given company's stock may perform badly. But what if the overall stock market takes a slide? In our example, while Blunderbluff's stock was a terrible performer, the other three did reasonably well. Even DDKind, while a laggard, still didn't fare too badly. Sometimes, though, the stock market as a whole drops—taking down both good stocks and bad. No matter how many different stocks you have in a portfolio, if the stock market as a whole fares badly, your stock portfolio as a whole is not likely to do all that well, even if it contains terrific companies.

But stocks are not the only game in town. An investment portfolio can be further diversified by including other types of investments, which may behave differently than stocks. After the September 11 terrorist attacks, the U.S. stock market, not surprisingly, fell into a slump. Yet prices for U.S. government bonds generally rose, as frightened investors looked to put their money into "safe haven" investments, driving up the demand for, and hence the prices of, government bonds. A portfolio that included both stocks and bonds would most likely have lost money on its stock investments during that period, but this loss could have been at least partially offset by gains from its bond investments.

Of course, stocks and bonds don't always move in different directions. During much of the 1990s, both stock and bond prices were strong. While both can go up, it is also true that both can go down. Investors seeking greater diversification can construct portfolios to include a wider range of investments, including such things as foreign stocks and bonds, commodities, and real estate. But no amount of diversification can insure against all risk. Still, the general idea at work is that the greater the diversification, the greater the likelihood that when all the investments are put together they will *on average* produce a reasonable return with an acceptable degree of risk.

Investment professionals often speak of ***asset classes***, a term for general categories of similar investments. The three main asset classes are ***equities***, ***fixed income***, and ***cash***. "Equities" are investments that represent ownership of businesses – essentially, equities means stocks. "Fixed income" investments provide a set stream of payments to their owners. This class can include corporate and government bonds, as well as mortgages and long-term certificates of deposit. The term "cash" can be a little misleading. It does include the usual meaning of cash, dollar bills in you wallet, but it also is taken to include short-term investments with little or no risk, such as bank accounts, short-term certificates of deposit or government bonds, and short-term corporate bonds of companies with very solid credit ratings.

These three asset classes differ in terms of both the degree of risk they carry and the rates of return they have historically tended to earn. Not surprisingly, the very low risk investments in the cash asset class do not offer particularly high rates of return. The somewhat riskier loans included in the fixed income asset class tend to offer higher rates of return overall. This only stands to reason; if a higher risk investment could not be hoped to produce a higher return than a lower one, there would be no incentive for any sane investor to put money into the riskier class. Finally, equities offer the greatest potential for both price volatility and risk, but they compensate for this by offering the greatest potential for investment gains.

There is quite a bit of controversy about the rates of return that can be expected from each asset class overall. Of course, there is a great deal of diversity among the investments included in each of these broad classes, and there is no definitive way to decide just what should be included in each of these classes. Should we base equities only on the stocks of solid, established American companies or should we include smaller, higher risk, less established businesses? Should we include foreign stocks or not? It should be apparent that a portfolio restricted to large, well-established (so called ***blue chip***) company stocks would be expected to behave differently from a portfolio that also includes developing biotechnology companies and Vietnamese nanotech solar panel manufacturers. Should the fixed income class be based only on investment-grade bonds, or should junk bonds be included? How about foreign bonds? How much should be based on short-term bonds, and how much on long-term?

The values given in the table below are intended to be a reasonable representation of the long-term average rates of return from each of these classes in the United States. They offer a range of commonly cited figures based on the past, not the future. Even if we all agreed on what mix of investments to use as the basis for each category's overall average risk and return, no one can know what the future returns will be on any investment or class of investments. Also, no one can know in advance whether any given selection of investments from an asset class will perform better or worse than the overall average return for investments of that type. Taking all of this into account, it is still reasonable to use these figures as a basis for an *assumption* of what might be reasonable to expect.

Asset Class	*Degree of Risk*	*Historical Average Rate of Return*
Cash	Minimal	2–5%
Fixed Income	Moderate	4–7%
Equities	High	8–12%

Do not misunderstand the ranges shown in this table. The 8–12% range shown for equities here does not mean that, in each year, equities can be expected to return between a low of 8% and a high of 12%. This is absolutely not the case; there have been many years when equities returned less than 8% or more than 12%. The 8–12% range means that, depending on what sorts of equities you include and how long a historical period you look at, the long-term average rate of return would probably fall somewhere between these two values.

Asset Allocation

How an investor decides to divide a portfolio among different asset classes is referred to as ***asset allocation***. An investor looking to achieve the highest overall rate of return might

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want to allocate all of his money to equities. But that rate of return does not come with a guarantee; rather, it comes with a high degree of risk, and the near certainty that, even if the journey ends with the hoped for rate of return overall, it will be a bumpy ride. An investor primarily concerned with safety might want to put all of his money into cash. But that safety comes at the cost of a very uninspiring investment return; cash investments do not run much risk of losing money, but they don't risk making much either.

Asset allocation is a matter of balancing the investor's desire for return with his tolerance for risk. A couple trying to save enough money for a down payment on a house that they hope to buy in a few years does not have the time to ride out volatility or make up for losses, and their money is not being invested for long enough for compounding to really work its magic. In their situation, they may choose to allocate most, if not all, of their money to cash.

A 20-year-old setting aside money for retirement has plenty of time to ride out any investment volatility, or to make up for any losses along the way to a far-in-the-future retirement. Having studied business math, she also no doubt realizes that, with compounding over time, a high overall rate of return will produce much larger future value. Such an investor may choose to allocate most of her investment dollars to equities. Yet if she is the type of person who would lose sleep worrying about the stock market or, worse yet, buy more when prices are high and sell out when prices are low, she might choose to lighten up on the equity investments just to be able to sleep at night. Even in identical financial situations, someone who spends vacations cliff diving will likely choose a somewhat different allocation than someone whose idea of a wild time is ordering *both* an appetizer and desert. Determining the right asset allocation for a given situation is an art, not a science, and it can have as much to do with personality as with mathematical formulas.

It is possible, though, to get a sense of the overall rate of return that might be expected from a given asset allocation. This can be reasonably estimated by a ***weighted average***. A weighted average is a mathematical tool in which each item being averaged is given a different importance, or weight, based on how much of the given item is included in the mix. The following example will illustrate how this can work.

Example 6.4.1 ***On the basis of how long he has until retirement and his comfort with investment risk, Matt has decided that he wants to allocate the money in his retirement account as follows: 60% to equities, 30% to fixed income, and 10% to cash. If he assumes that each asset class provides the low end of the rates of return shown in the table above, what overall rate of return would he expect to earn over the long term? What if he assumes that each asset class provides the high-end rate of return?***

We calculate the weighted average by using 60% of the equities rate, 30% of the debt rate, and 10% of the cash rate.

Using the low-end returns we get: $(60\%)(8\%) + (30\%)(4\%) + (10\%)(2\%) = (0.60)(0.08) + (0.30)(0.04) + (0.10)(0.02) = 0.062 = 6.2\%$

Using the high-end returns we get:

$(60\%)(12\%) + (30\%)(7\%) + (10\%)(5\%) = (0.60)(0.12) + (0.30)(0.07) + (0.10)(0.05) = 0.062 = 9.8\%.$

According to this, it would be reasonable for Matt to expect that his portfolio will earn on average somewhere between 6.2% and 9.8%. That is a pretty wide range, but even this is by no means certain. Again, these figures are reasonable guesses, based on historical investment returns, but there is no way of knowing what will actually happen in the future. His actual investments might earn a much higher, or much lower, rate of return.

If Matt wants to make projections, these rates give him some sense of what sort of rate might be reasonable to assume. If he wants to be cautious, he might use the 6.2% rate; if he wants to be optimistic, he might use the 9.8%. A good middle ground might be to average these two rates and use 8%.

Mutual Funds

Mutual funds are a popular way to build a diversified investment portfolio. © Royalty-Free/CORBIS/DIL

Even though there are plenty of good reasons to diversify, creating a diversified portfolio of investments can pose some practical problems for an investor. In the previous example Matt decided to invest 60% of his money in stocks; but now, how does he decide *which* stocks? Unless Matt is knowledgeable about how to evaluate a business's financial strength and prospects—not an easy task—he will not be in a position to make good decisions about which specific stocks to buy. How will he decide which specific stocks to buy? It might be a good thing for Matt to invest time and effort in learning how to select the best stocks and bonds, but doing that requires *lots* of time and effort. If he has an aptitude for this and finds it interesting, great; it can be a very rewarding hobby. But most people find they have other things they'd rather do with their time.

Also, unless Matt has a large amount of money to invest, creating a diversified portfolio of individual securities may not be practical. Suppose he has $3,000 in his investment account right now; 60% of $3,000 works out to $1,800. If he divides this money evenly among a dozen different stocks, that works out to $150 in each stock. This is simply too small an amount to practically invest in an individual stock. After taking into account the commissions (stock brokers do not work for free) and other fees involved in buying stocks, it really isn't practical to invest less than a few thousand dollars in an individual stock.

Mutual funds are a type of investment vehicle that addresses these difficulties. A mutual fund is an investment portfolio that pools money from many different investors. The decisions of which securities to invest in are made by the fund's ***portfolio manager***. By owning shares in the mutual fund, each investor effectively owns a piece of the overall investment portfolio. If Matt invests $1,800 in a stock mutual fund, he may find his investments diversified among hundreds of different stocks owned by the fund.

Like a stock, the ownership of a mutual fund is divided into shares. Each share is entitled to an equal percent of the overall fund's assets.[8] The value of one share of a fund is determined by dividing the value of the fund's overall assets by the total number of shares outstanding. This called the fund's ***net asset value*** (or ***NAV***).

Example 6.4.2 The Nesoh Capital Equity Fund has assets totaling $15,735,962. There are 845,362 shares. What is the fund's NAV?

To find the NAV we simply divide $15,735,962/845,362 to get $18.61.

An ***open-end*** mutual fund does not have a fixed number of shares. New shares are created whenever someone invests money in the fund, old shares cease to exist when someone pulls money out of the fund. Most mutual funds in the United States today are open-end funds. ***Closed-end*** funds, on the other hand, have a fixed number of shares. When someone wants to invest in a closed-end fund, no new shares are created; instead, that investor must buy existing shares from someone else. Closed-end funds can be bought and sold just like stocks. Unlike open-end funds, though, they do not necessarily sell for their net asset value. The selling price of a closed-end fund is determined by what the market will bear. Closed-end funds may sell for more, or less, than their net asset values.

Example 6.4.3 If Matt invests $1,800 in the open-end Nesoh Capital Equity Fund, how many shares will be created?

The investment of $1,800 divided by $18.61 per share works out to 96.7221937. A mutual fund can have fractional shares; the number of shares is often carried out to three decimal places. Assuming this, Matt would have 96.722 shares in this fund.

[8]Some mutual funds do have different classes of shares, which may not all be of equal value, but this is the exception, not the rule.

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When funds are withdrawn, the investor receives the value of the withdrawn shares, calculated at the then-current NAV.

Example 6.4.4 ***Six months later, Matt decides to sell his shares of the fund. When he sells, the fund's total assets are $16,509,362 and the total number of shares is 903,444. How much does Matt receive?***

The value of a share is $16,509,362/903,444 = $18.27. Multiplying this NAV by 96.722 shares works out to $1,767.11.

Mutual funds are required to distribute the dividends they receive and capital gains they earn (less any capital losses) to their shareholders each year. The shareholders may choose to receive these payouts in cash, but in most cases they choose to ***reinvest*** these payments in more shares of the mutual fund. In that case, the number of shares owned may grow over time, even if no new money is invested in the fund.

Mutual funds charge their shareholders a number of fees, to pay for the expenses of running the fund and to allow the fund company to make a profit. These expenses are normally deducted from the fund's overall assets, and so the fund's shareholders do not pay these directly, though they do pay them in the form of a lower NAV than if there were no such fees. A fund's ***expense ratio*** indicates the amount of fees charged as a percent of overall fund assets. Obviously, funds with low expense ratios have an advantage over funds with higher ratios. It may be worth paying a high expense ratio, though, if the mutual fund is an exceptionally strong performer.

Some mutual funds charge ***loads***. A load is a fee paid when the fund is purchased. Loads are expressed as a percent of the total invested in the fund. Loads are most commonly charged on funds sold through stockbrokers. A **no-load** fund is a fund that does not charge a load.

Example 6.4.5 ***The NAV of the Macedon Mutual Global Fund is $43.79. The fund charges a 5% load. How many shares will I own if I invest $3,000 in this fund.***

The 5% load works out to (5%)($3,000) = $150. This leaves $3,000 − $150 = $2,850. Then $2,850/$43.79 per share = 65.083 shares.

Some funds also charge fees when you sell shares, or if you sell your shares within a certain period of time of buying them.

There are literally thousands of mutual funds available in the United States today. Some are very general in nature: ***equity funds*** invest in stocks, ***bond*** (or ***fixed income***) ***funds*** invest in bonds, and ***money market funds*** invest in the cash asset class. Within each of these categories there are special types of funds that invest in certain specified ways. You can find equity funds that invest only in stocks of companies in certain industries, or in foreign stocks, or in stocks that pay high dividends. If you want to invest your money specifically in renewable energy companies, or in Japanese companies, or in companies considered "socially responsible,"[9] you can find mutual funds to meet your interests. You can find bond funds that invest only in U.S. federal government bonds, or only in municipal bonds or junk bonds, or only in the bonds of foreign companies, and so on.

Two particular types of mutual funds deserve particular mention. ***Index funds*** are mutual funds that are managed to match the investment performance of some market index, such as the S&P 500. Index funds are quite popular. Even though it might seem that buying an index fund is aiming for a mediocre return, in fact very few mutual funds that are managed to do better than the index consistently succeed at this goal. ***Balanced funds*** (or ***asset allocation funds***) invest in a mix of different asset classes. They are intended to provide "one-stop shopping" for investors seeking an investment portfolio diversified across different asset classes.

[9]There is also at least one mutual fund which caters to "socially *irresponsible*" investors. Really.

Measuring Fund Performance

Determining the rate of return from a mutual fund investment is very much the same as determining the rate of return on a stock. Just as with stocks, we have to confront the issue of capital gains versus dividends; however, with mutual funds, since most investors reinvest their dividends in new fund shares, the overall rate of return is usually calculated on the assumption that all dividends are reinvested.

Example 6.4.6 ***On June 5, 2000, the net asset value of the Aloquin Commonwealth International Fund was $172.59. On June 5, 2007, the NAV was $154.08. If all dividends were reinvested, each share of the fund on June 5, 2000, would have grown to 1.973 shares on 6/5/07. Calculate the average annual rate of return on this investment.***

Suppose that you owned one share of this fund on June 5, 2000. This would have been worth $172.59. On June 5, 2007, this would have grown to 1.973 shares, each worth $154.08, for a total of (1.973)($154.08) = $304.00. Using the rate of return formula from Section 6.1, we get:

$$i = \left(\frac{FV}{PV}\right)^{1/n} - 1$$

$$i = \left(\frac{\$304.00}{\$154.08}\right)^{1/7} - 1$$

$$i = 0.1019479 = 10.19\%$$

Sometimes we may not know how much an investment would have grown to with dividend reinvestment, but do have access to the annual rate of return for each of several years. From the annual rates of return, we can calculate an average rate over a number of years; this "average," though, may not work out quite the way you might expect.

Example 6.4.7 ***A mutual fund had annual returns of +10%, +5%, −8%, +15%, and +3% in each of the past 5 years. What was the average rate of return over this period?***

At first, it seems as though the average could be found in the usual way, by adding up the five annual rates and dividing by 5. This would give an average return of (10% + 5% − 8% + 15% + 3%)/5 = (25%)/5 = 5%. Unfortunately, this is not really correct.

Suppose that you invested $100 in this fund at the start of the period (the problem can be worked by assuming any starting value). At the end of that year, you would have $100(1.10) = $110. In the next year, your $110 would grow to ($110)(1.05) = $115.50. Carrying this idea further, we can see that at the end of the 5 years your $100 would grow to:

$$FV = \$100(1.10)(1.05)(0.92)(1.15)(1.03) = \$125.86$$

Now, using the rate of return formula, we calculate the rate of return to be:

$$i = \left(\frac{FV}{PV}\right)^{1/n} - 1$$

$$i = \left(\frac{\$125.86}{\$100.00}\right)^{1/5} - 1$$

$$i = 0.0470827 = 4.71\%$$

The correct average rate of return is actually 4.71%.

If you have doubts about this average, you can calculate the future value of $100 at 5% for 5 years. It works out to be $127.63. If you calculate the future value using 4.71%, you get the correct $125.86 future value. The 5% figure, calculated in the usual way, is known in mathematics as an ***arithmetic mean***. This is a fancy word for what we normally mean when we use the term *average*. The 4.71% figure is an example of a ***geometric mean***,[10] an

[10] It is actually the geometric mean of the (1 + i)'s, less 1, not the geometric mean of the i's. We should more technically say, then, that the average rate *is based on* a geometric mean.

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average based on multiplication instead of addition. The arithmetic mean does not work out quite right in this case because it is based on addition, whereas compound growth is based on multiplication.

This "average" issue can lead to misunderstandings about investment performance.

Example 6.4.8 ***A mutual fund that I own had a really bad year in 2005; it lost 40% of its value that year. In 2006 things went much better though, and the fund gained 40%. Over the course of the 2 years, what was the fund's overall performance?***

Common sense says that if I lost 40% one year and gained 40% the next, I wound up where I started. This is not correct, though; in saying this I am using the logic that +40% and −40% average out to 0%.

In actuality, $100 invested in this fund at the start would have become:

$$FV = \$100(0.60)(1.40) = \$84$$

Overall, I lost $16 on a $100 investment, and overall loss of 16%. Though the question did not ask for an average rate of return, we can calculate it to be −8.35%.

EXERCISES 6.4

A. Portfolio Diversification

Exercises 1 to 6 are based on the table below, showing the prices of the stocks of four different companies on January 1, 2006 and January 15, 2007. No stock splits occurred for any of these companies.

	East Corp.	West Corp.	North Corp.	South Corp.
1/15/06	$40.00	$25.00	$10.00	$100.00
1/15/07	$45.00	$20.05	$15.50	$81.50

1. On a percentage basis, which of the four companies listed in the table above was the best performer? Which company was the worst performer?

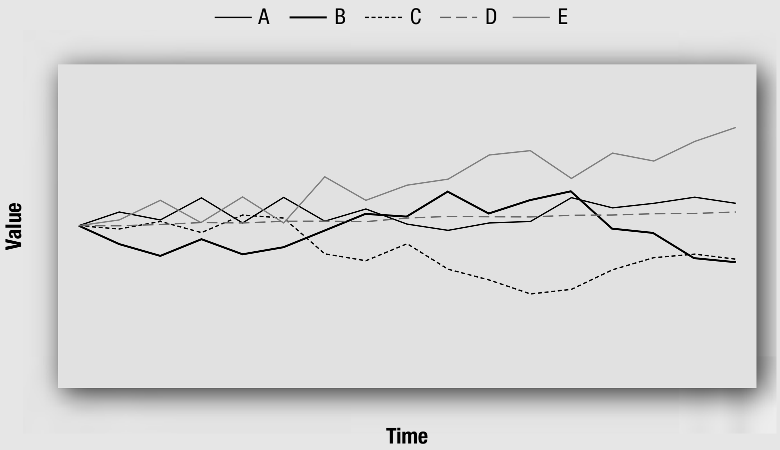
2. Suppose that I invested $5,000 in East Corp. on January 15, 2006. How many shares would I get for this investment? How many shares would I have got for $5,000 invested in West Corp.? North? South?

3. Calculate the value on January 15, 2007, of a $5,000 investment made in East Corp. on January 15, 2006. Do the same for each of the other three companies.

4. Suppose that instead of investing $5,000 in just one of the companies, I instead invested $1,250 in each of the four. What would the value of this portfolio have been on January 15, 2007?

5. Suppose that instead of investing $5,000 in just one of the companies, I instead decided to invest in a mix of the four. However, since I felt a little stronger about some of the companies than others, I did not invest equal amounts in all four. I invested $1,500 each in East and South, and $1,000 each in West and North. What was the value of my portfolio on January 15, 2007?

6. Four of the lines in the graph below show the value of $5,000 invested in each of the four individual companies. One of the lines shows the value of a portfolio composed of $1,250 in each stock. Which line represents the portfolio?



B. Asset Allocation

Use the table of return expectations by asset class from this section to answer the following questions.

7. Suppose that a portfolio is invested 65% in equities, 25% in fixed income, and 10% in cash. If we assume that each asset class earns the low-end rate of return from the table in this section, what would we expect the overall portfolio's rate of return to be? What if each asset class earns the high-end rate of return?

8. Jayeth intends to invest $200 a month into an investment account for the next 30 years. He expects his account value to grow to $700,000 in that time. If he allocates his investments 50% to stocks, 30% to fixed income, and 20% to cash, is this a reasonable expectation? Justify your answer.

9. If you invest in a portfolio of 45% equities, 30% fixed income, and 25% cash, by what percent will your investment grow in value over the next year? (Be careful in answering this one.)

10. Since equities have historically tended to provide a much higher rate of return than fixed income or cash investments, why doesn't everyone put all investment money in the stock market?

C. Basic Mutual Fund Calculations

11. The Hopewell Diversified Value Fund has total assets of $142,575,814. There are 3,475,912 shares outstanding. What is the NAV per share?

12. The Hopewell International Equity Fund is a no-load open-end fund whose NAV is $42.66 per share. If Todd invests $3,000 in this fund, how many shares will be created?

13. The Hopewell Emerging Markets Bond Fund is an open-end fund whose NAV is $103.59. The fund charges a 4.75% load. If Brenna invests $2,500 in this fund, how many shares will she own?

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14. The Hopewell Asian Markets Telecommunications Fund is a closed-end fund. There are 2,814,000 shares outstanding, and the fund's total assets are $89,035,047. Shelly bought 100 shares of this fund. How much did she pay for them?

D. Measuring Fund Performance

15. Mehmet invested $4,000 in an open-end, no-load mutual fund when the NAV was $34.35 per share. He has reinvested all dividend and capital gains distributions. Five years later, his account statement showed that he owned 124.570 shares, and the NAV had grown to $57.02. Determine:
    a. The total profit Mehmet has earned on this mutual fund.
    b. The average annual rate of return (CAGR) on this investment.

16. Louise invested $5,000 in an open-end, no-load mutual fund. The NAV was $14.47. All distributions have been reinvested. Three years later, she owns 472.516 shares. The NAV is $14.45. What is the CAGR she has earned on this investment?

17. I invested $3,257.09 in an open-end mutual fund with a 3.5% load. The NAV was $59.95. All distributions were reinvested. Seven years later, I owned 72.58 shares with an NAV of $52.08. What CAGR did I earn on my investment?

18. The Hopewell Aggressive Growth Fund has achieved returns of +47.5%, −32.1%, +11.3%, +9.2% and +13.2% in the last 5 years. What is the average rate of return over this period?

19. The Hopewell U.S. Government Bond Fund had returns of −20%, +8%, and +12% in the last 3 years. If I invested in this fund 3 years ago, is the value of my investment more than, less than, or the same as it was 3 years ago? What is the average rate of return over this period?

E. Grab Bag

20. Find the NAV of a mutual fund portfolio if the total assets are $103,569,101 and the number of shares is 8,504,219.

21. True or false:
    a. Investing in a diversified portfolio guarantees that you will not lose money in any given month.
    b. Investing in a diversified portfolio guarantees that you will not lose money in the long run.
    c. Investing in a diversified portfolio guarantees that you will not do as well as if you had invested all your money in the best stock from the portfolio.
    d. Investing in a diversified portfolio guarantees that you will not do as badly as if you had invested all your money in the worst stock from the portfolio.
    e. The growth of a diversified portfolio will be as smooth as it would be with a constant rate of compound interest.

22. Rick's investment advisor has told him that he should invest 70% in stocks, 25% in bonds, and 5% in cash. Using the values given in the table, determine a reasonable rate that he might expect this portfolio to return, using (a) the low-end rates and (b) the high-end rates.

23. Suppose that 20 years ago you invested $12,000 in a 5% load mutual fund with a $78.35 net asset value. All distributions have been reinvested. Today you own 356.753 shares with an NAV of $114.03. Find the average annual rate of return for your investment.

24. The Hopewell Regional Real Estate Fund has a $100,579,043 in assets and 945,155 shares. Mikka owns 58.348 shares of this fund. What is the value of her investment?

25. The Hopewell Cash Reserves Money Market Fund's returns in the last 4 years have been +3.49%, +3.76%, +4.55%, and +6.13%. If you had invested $3,000 in this fund 4 years ago, how much would you have today? What is the average annual rate of return for this fund?

26. Over the past 10 years, the Hopewell Horizons Fund's worst year has been a 3.7% loss, and its best year has been an 8.1% gain. The fund name doesn't tell you what asset class this fund invests in. On the basis of these performance figures, is this fund most likely a stock fund, a bond fund, or a money market fund?

27. Last year, Stephanie's investment portfolio lost 8.5%. What rate of return does she need to earn this year in order to get back to where she was at the start of last year?

F. Additional Exercise

*The **cumulative rate of return** of an investment over some period of time is the total return, expressed as a percentage, over that period of time. It is not an annualized rate. Mutual fund performance figures often give cumulative returns either in addition to, or instead of, annualized rates.*

28. The Hopewell New Generations Fund reports that for the past 10-year period, the cumulative rate of return of the fund has been 124.65%.
    a. If I had invested $5,000 in this fund 10 years ago, how much would I have today?
    b. What is the average annualized rate of return for this period?
    c. If the average annual rate of return for the last 5 years has been 5.89%, what is the cumulative rate of return for the last five years?
    d. For the values given, what were the cumulative and average annual rates of return for the first 5 years of this ten year period?

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CHAPTER 6 SUMMARY

Topic	Key Ideas, Formulas, and Techniques	Examples
Calculating Dividends For Each Shareholder, p. 252	• The total dividend amount is divided by the number of shares to determine a dividend per share. • Multiply the shares owned by the dividend per share.	Jason and Dave's dry cleaning business is a corporation with 100 shares. Jason owns 51, Dave owns 49. If the company declares a $35,000 dividend, how much will each receive? (Example 6.1.2)
Calculating Dividend Yields, p. 254	• Determine the total dividends paid per share per year. • Divide this by the market value of each share. • If this is calculated by annualizing the current rate, it is the current yield; if the actual past 12 months' dividends are used it, is a trailing yield.	Zarofire Systems pays a quarterly dividend of $0.45 per share. The market price of the stock is $49.75. Calculate the current dividend yield. (Example 6.1.4)
Calculating a Compound Annual Growth Rate, p. 257	• The CAGR formula: $i = \left(\frac{FV}{PV}\right)^{1/n} - 1$	5 years ago I invested $8,400 in the stock of Sehr-Schlecht Investment Corp. I sold the stock today for $1,750. What compound annual growth rate does this represent? (Example 6.1.8)
Total Rate of Return, p. 257	• Return from an investment may include both dividends and capital gains. • Total rate of return earned can be approximated by adding the CAGR from capital gains to an approximate average dividend yield. • If dividends are reinvested, this can be calculated by using the CAGR formula.	The dividend yield of Zarofire Systems has averaged 3½% while you have owned it. Capital gains from the stock work out to a 21.90% CAGR. What total rate of return have you earned? (Example 6.1.9) A $2,000 investment has grown to $3,525.18 in 10 years with reinvested dividends. Find the total rate of return. (Example 6.1.10)
Volatility and Risk, pp. 258–259	• Investments in stocks carry risk of losing some or all of the amount invested. • Unlike compound interest, growth of stock investments does not occur at a steady pace.	See discussion at end of Chapter 6.1.
Calculating Bond Coupons, p. 264	• The coupon rate is a percent of the par value. • Calculate the coupon amount using the simple interest formula.	A $1,000 par value bond has an 8% coupon rate. Interest is paid semiannually. What payments will its owner receive? (Example 6.2.1)
Current Yield of a Bond, p. 264	• Calculation of current yield is based on the bond's current market price. • Use the periodic coupon amount with the current market price in the simple interest formula.	A $1,000 par value bond with an 8% coupon rate sells for $1,094. Calculate the current yield. (Example 6.2.3)

(Continued)

Topic	Key Ideas, Formulas, and Techniques	Examples
Yield to maturity, pp. 265–266	• Yield to maturity is the interest rate equivalent to the coupons and maturity value. • Use a present value spreadsheet and guess-and-check to find the rate.	An 8% coupon rate $1000 par value bond with 12 years to maturity sells for $1105.73. Find the yield to maturity. (Example 6.2.5)
Inverse Correlation of Bond Rates and Prices, p. 267	• As market rates increase, the prices of bonds decrease and vice versa.	Jill is planning to take out a mortgage to buy a house soon. She hears that bond prices are rising. Is this good news for her? (Example 6.2.7)
Total Debt Service, p. 269	• A bond issuer needs to both make periodic interest payment and prepare for payment at maturity. • Calculate the sinking fund payment needed to accumulate the maturity value and add it to the periodic interest payments.	A city issues bonds with $7 million maturity value and a 4.7% semiannual coupon. Their sinking fund earns 4%. How much does the city need semiannually to service this debt? (Example 6.2.9)
Cash Settlement of a Futures Contract, pp. 277–278	• Calculate the total price for the items based on the contract price. • Compare this to the spot market price at the delivery date.	Luis takes a short position for 5,000 bushels of October soybeans at 612.5 cents per bushel. At the delivery date, the spot market price per bushel is 543.0 cents per bushel. Calculate his profit. (Example 6.3.2)
Gain or Loss on a Futures Contract as a Rate, pp. 278–279	• Calculate the profit/loss from the contract. • Use the simple interest formula to calculate the rate of return, using the initial margin requirement as principal.	Jimmy went long 1,000 barrels of September oil at $75.09. 52 days later he closed his position at $68.35. His initial margin was 5%. Calculate his rate of return. (Examples 6.3.3 and 6.3.5)
Cash Settlement of an Options Contract, p. 281	• Calculate the total price of the stock at its price when the option is exercised. • Calculate the total price using the strike price. • Find the difference between the two. • Subtract the option premium.	You believe that a stock price is likely to drop, and so you buy a $60 put option for 500 shares. The option premium is $2. How will you make out if the stock price drops to (a) $50 and (b) $60. (Example 6.3.7)
Gain or Loss on an Options Contract as a Percent, p. 281	• Calculate the profit/loss from the contract. • Use the simple interest formula using the premium as the principal.	Calculate the rate of return for the options investment described above. (Example 6.3.8)
Projecting Rates of Return Based on Asset Allocation, p. 294	• Use a realistic long-term rate of return projection for each asset class. • Calculate a weighted average of these rates based on the percent allocation for each class.	Matt has invested 60% in equities, 30% in fixed income and 10% in cash. What range of rates of return would be reasonable for him to expect over the long term? (Example 6.4.1)

(Continued)

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Topic	Key Ideas, Formulas, and Techniques	Examples
NAV of a Mutual Fund, p. 296	• The NAV is the total assets divided by the total number of shares.	Matt owns 96.722 shares of a mutual fund. The total assets are $16,509,362 and there are 903,444 shares. How much is Matt's investment worth? (Example 6.4.4)
Mutual Fund Performance, p. 297	• Use the initial value of the investment as the PV, the ending value as the FV, and the CAGR formula.	In 7 years, a share of a mutual fund originally worth $172.59 grew to 1.973 shares worth $154.08 each. Calculate the average annual rate of return. (Example 6.4.6)
Average Rate of Return, p. 297	• The average rate of return is not calculated in the usual "average" way. • Find the future value of $100 applying the rates in succession, then use the CAGR formula.	A mutual fund had annual returns of +10%, +5%, −8%, +15%, and +3% in each of the past 5 years. What was the average rate of return over this period? (Example 6.4.7)

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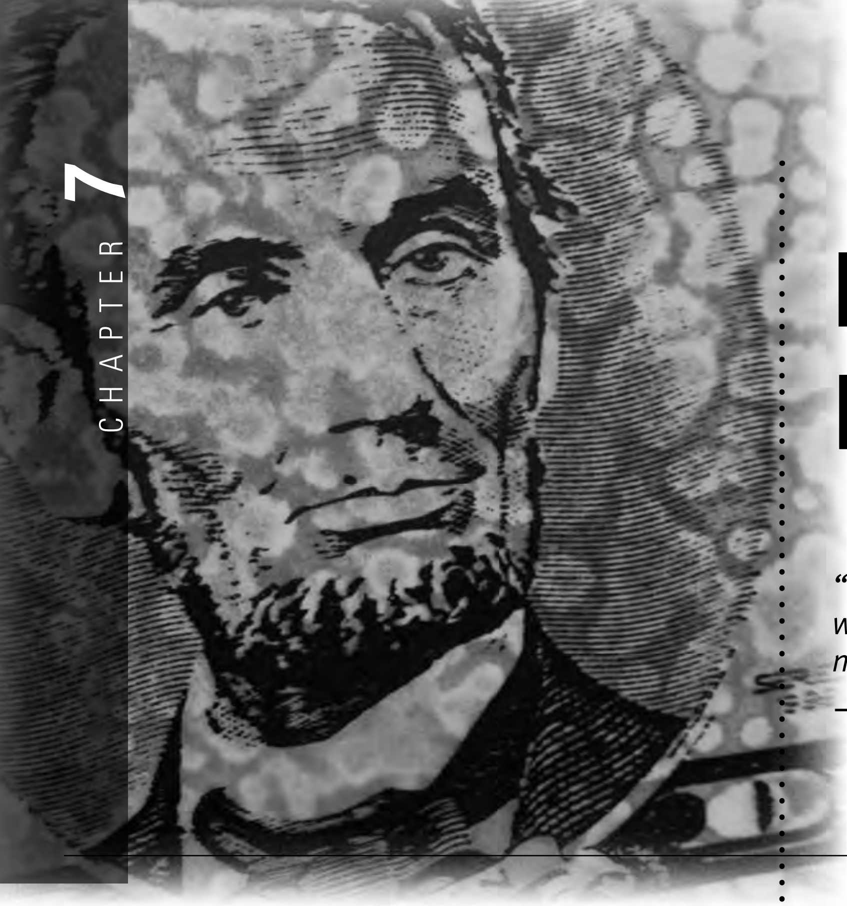

CHAPTER 7

Retirement Plans

"When I was young I thought that money was the most important thing in life; now that I am old I know that it is."

—Oscar Wilde

Learning Objectives

LO 1 Recognize the features and distinguish the key differences between defined benefit and defined contribution retirement plans.

LO 2 Calculate employer matching and vested balances for retirement plans.

LO 3 Perform basic calculations to assess the difference between traditional and Roth IRAs.

LO 4 Make projections for defined contribution plans, and use those projections to determine reasonable contribution levels.

LO 5 Make projections for retirement plans that take into account the impact of inflation.

Chapter Outline

7.1 Basic Principles of Retirement Planning

For many students using this book, retirement planning may not seem like a very pressing issue. It's hard to be overly concerned about ending a career when you are still going to school working toward getting one started. Still, there are some excellent reasons to be thinking about retirement planning now.

For one, we've already seen that small amounts of money can build to large amounts with compound growth—but for really impressive growth to occur we need *time*. Someone who starts planning for retirement when she is 50 does not have this advantage to anywhere near the same extent as someone who starts when she is 20. Planning ahead and starting early can make the task much easier.

Second, retirement planning is a business issue that goes well beyond preparing for your own retirement. Not all retirement programs are directly connected to employment, but many are, and whether and what kind of retirement plan a business offers its employees can be a significant issue for the business. A company that offers good retirement benefits has an advantage over one that doesn't in attracting and keeping the best employees. On the other hand, a company that offers generous retirement benefits can find that the costs of providing those benefits may place it at a competitive disadvantage, as many large American companies (such as the major automakers) have found in recent years.

Last, with the "baby boomers" now reaching retirement and with the responsibility for funding retirement falling increasingly to individual Americans, the already massive financial services industry is likely to become an even larger part of the business landscape. Many of the readers of this book may find that the financial planning industry provides favorable career opportunities.

Many Americans take for granted that they will be able to enjoy a financially comfortable and secure retirement at a fairly young age. Meanwhile, news stories with tones ranging from cautionary to alarming report that most people are not doing nearly enough to reach this goal. Whether or not this is true, a basic understanding of retirement planning would help all of us to assess what we can and should be doing.

A financially secure retirement requires sound planning. © Steve Mason/Getty Images/DIL

Defined Benefit Plans

Most retirement programs fall into one of two categories: ***defined benefit plans*** and ***defined contribution plans***. A ***defined benefit (DB) plan*** is a retirement program that provides a set income in retirement, which may be calculated on the basis of preretirement earnings, years worked, retirement age, and/or the period of time for which benefits are guaranteed to be paid. Defined benefit plans used to be considered the standard type of retirement plan; when people talk about having a ***pension***, they usually mean a plan of this type (though technically the term ***pension*** does not have to mean a defined benefit plan). We will consider DB plans first.

The income that a defined benefit plan provides in retirement is determined by a formula set out in the documents that govern the plan (not surprisingly called the ***plan documents***). There are some legal restrictions on what types of formulas can be used, but within these there is still quite a bit of variation from one plan to another. There are, though, certain general principles that are typically reflected in benefit formulas. Benefit formulas usually award a larger income to retirees who have more years of service than to those with fewer; this is often accomplished by making ***years of service*** (the number of years the retiree has worked while covered by the retirement plan) a part of the formula itself. Years of service usually refer only to years of full-time service; part-time work may earn partial credit or may not count at all, depending on the specific plan.

The following example will illustrate this type of formula.

Example 7.1.1 ***A union retirement plan promises an income to union members beginning at age 65 and continuing until death. The formula states that retirees who have at least 5 years of service will receive a monthly income of $80 for each year of service, up to a maximum monthly income of $2,000. Find the benefit that would be payable to a member who retires at age 65 with (a) 4 years of service, (b) 10 years of service, and (c) 40 years of service.***

(a) Since the formula requires a minimum of 5 years of service, someone retiring with fewer than 5 years would not receive any benefit.

(b) Following the formula, we find the monthly benefit would be 10($80) = $800 per month.

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(c) Following the formula, we find the monthly benefit would be 40($80) = $3,200 per month, except that this is more than the maximum. So the benefit would be the maximum $2,000 per month.

The benefit formula used in Example 7.1.1 does not take into account how much the retiree was earning before retirement. Often benefit formulas take earnings into account as well. The next example is illustrative of that type of formula.

Example 7.1.2 ***The XYZ Corporation Pension Plan provides a lifetime income to its employees on retirement at age 65. The formula provides 2% for each year of service of the average of the employee's earnings for the last 3 years on the job, up to a maximum of 70%. Jelena retired at 65 with 28 years of service. Her earnings for her last 3 years were $37,650, $39,525, and $40,187. What is her pension benefit?***

Since she had 28 years of service, she is entitled to 28(2%) = 56% of her final 3-year average income.

Her 3-year average income is ($37,650 + $39,525 + $40,187)/3 =$39,120.67.

So her annual pension benefit is (56%)($39,120.67) = $21,907.53 per year.

In both of these examples, the formula gave an income on the assumption that the employee retired at age 65. Of course, some people may choose to take an early retirement or keep working beyond age 65. Since a pension plan provides benefits for life, someone retiring at 60 would be expected to receive benefits for longer than if he retired at 65. Similarly, if someone keeps working past age 65, she would expect to receive benefits for fewer years. Thus, plans often include formulas that allow someone to start receiving a reduced benefit at an earlier date, or an enhanced benefit starting at a later date.

Example 7.1.3 ***The XYZ Corportation (from Example 7.1.2) offers employees the choice to retire as early as age 60 or as late as age 72. Those retiring before age 65 have their calculated benefit reduced by 2.5% for each year they retire prior to age 65; those retiring later have their benefit increased by 1.9% for each year beyond age 65 that they work.***

> ***(a) Brooke plans to retire this year at age 63. She has 19 years of service to the company, and her last 3 years' earnings were $48,000, $52,000, and $54,000.***
>
> ***(b) Kurt plans to retire this year at age 69. He has 37 years of service, and his last 3 year's earnings were $41,500, $43,750, and $46,300.***

Find each employee's pension benefit.

(a) Brooke's 19 years of service entitle her to 38%. Her final 3-year average salary is $51,333.33. If she were 65, her benefit would be (38%)($51,333.33) = $19,506.67. Because she is retiring 2 years early, she will give up 2(2.5%) = 5% of this benefit, leaving her with 95%. So she would receive (95%)($19,506.67) = $18,531.34 per year.

(b) Kurt's 37 years of service would give him 74%, except that the maximum percent is 70%. His final 3-year average salary is $43,850. If he were 65, his benefit would be (70%)($43,850) = $30,695. Retiring 4 years late means he will get an extra 4(1.9%) = 7.6%, so he will get 107.6% of this, or (107.6%)($30,695) = $33,027.82 per year.

In each of these examples, the calculated benefits will be paid for the life of the retiree. This is a very desirable feature; as long as you live, your pension income continues. However, if you have a spouse or other dependent who relies on your income, this can be a problem. If you pass away first, your dependent would be left without the pension payments that end with your death.

Pension plans usually offer the option of a benefit that is guaranteed for the lives of both people, sometimes with a reduction in the benefit paid when one person dies. The formulas for these benefits, though, can be very complicated, depending on the age (and hence the life expectancy) of the spouse. We won't deal with these more complicated formulas in this text, other than noting that since the benefits have the potential to be paid for a longer period of time, the amount of the benefit offered with a two-life guarantee should be expected to be lower than for a benefit guaranteed for only one life.

Defined Contribution Plans

Unlike defined benefit plans, ***defined contribution (DC) plans*** do not provide a fixed, guaranteed income in retirement. Instead, the employer contributes money to a retirement account for each employee on the basis of some set formula. The money is then invested and (hopefully) grows in value during the employee's working years. In retirement, the employee has the assets in this account to live on.

Sometimes the contributions to the plan are made entirely by the employer. In most plans of this type, though, the employee is either permitted or required to contribute money as well. It is not uncommon for the contribution formula to base the amount the employer contributes on the amount the employee does. The following examples represent typical defined contribution formulas.

Example 7.1.4 The company Jesse works for contributes 8% of each employee's annual earnings to a defined contribution plan, provided that the employee contributes at least 3%. Jesse makes $32,500 per year. How much will go into his account this year if he contributes (a) nothing, (b) 3%, and (c) 10% of his income?

(a) If Jesse does not contribute anything himself, neither does his employer. Nothing goes into his account.

(b) If Jesse contributes 3%, his employer will contribute 8%. So Jesse contributes (3%)($32,500) = $975, and the company contributes (8%)($32,500) = $2,600, for a total of $975 + $2,600 = $3,575.

(c) If Jesse contributes 10%, that amounts to (10%)($32,500) = $3,250. The company still contributes 8%, or $2,600. The total going into his account will be $3,250 + $2,600 = $5,850.

With another type of common formula, the employer contribution is a percent of the employee's. This sort of formula is often referred to as ***matching***.

Example 7.1.5 Daxxilon Digital Devices has a defined contribution plan. The company matches 75% of each employee's contributions up to a maximum of 10%. Hamid earns $60,000 at this company. How much will be deposited to his account this year if he contributes (a) nothing, (b) 5%, (c) 15% of his earnings?

(a) If Hamid does not contribute anything himself, neither does the company. Nothing will be deposited.

(b) If Hamid contributes 5%, this means he will deposit (5%)($60,000) = $3,000. The company matches 75% of this, or (75%)($3,000) = $2,250. The total deposited will be $3,000 + $2,250 = $5,250.

(c) If he contributes 15%, he will deposit (15%)($60,000) = $9,000. The company will match only up to 10%, though, so it will deposit (75%)(10%)($60,000) = $4,500. The total going into his account will be $9,000 + $4,500 = $13,500.

Vesting

What happens if you leave your job, for whatever reason, before you reach retirement age? Or, what if your employer decides to change or even discontinue its retirement plan before you reach retirement age? What happens to your retirement benefits? After you have accumulated a certain number of years of service, you become ***vested*** in the plan, meaning that any benefits earned will not be lost, regardless of whether or not you continue working or even if the plan is discontinued.

How quickly someone attains this vesting is determined by a ***vesting schedule***, which is determined when the plan is put into place. Vesting may be an all-or-nothing deal, or the

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schedule may allow for partial vesting that grows with years of service. With ***cliff vesting***, it is an all-or-nothing proposition. After a certain number of years of service, you become fully vested in the plan; prior to that point you have no vested benefit and will receive no benefits from the plan if you depart before that point. A cliff vesting schedule might look like this one:

5-YEAR CLIFF VESTING

Years of Service Completed	*Vesting Percent*
<5 years	0%
5+ years	100%

Cliff vesting is simple enough, but its all-or-nothing approach can be a bit severe. If your company's pension plan uses this schedule, leaving your job even 1 day before your fifth year of service is completed would mean that all of your earned benefits from the plan are forfeited. If you hang on just one single more day, all of your earned benefits are yours forever.

An alternative approach gradually increases the amount of your earned benefits that are vested as you accumulate additional years of service. With ***step vesting***, you may be entitled to a certain percent of your benefits, depending on your years of service when you leave. A step vesting schedule could look like this one:

7-YEAR 20% STEP VESTING

Years of Service Completed	*Vesting Percent*
<3 years	0%
3–4 years	20%
4–5 years	40%
5–6 years	60%
6–7 years	80%
7+ years	100%

Government regulations generally require that the vesting schedule be at least as favorable to the employee as either 5-year cliff vesting or the 7-year step vesting shown in the sample schedule above. (The schedule can be *more* favorable to the employee, but not less so.)

It is also generally the case that any contributions made by the employee himself, and any earnings on those contributions, are 100% vested at all times. Vesting schedules typically apply only to benefits funded by and contributions made by the employer.

Example 7.1.6 Dave has just been laid off from his job. He was covered by the company's defined benefit pension plan, which provides a benefit of 2% final year's salary for each completed year of service, beginning at age 65. Dave earned $43,600 at this job in the last year, and had completed 6 years of service. The pension plan uses the 7-year 20% vesting schedule shown above. What is Dave's vested benefit?

According to the benefit formula, Dave is entitled to 6(2%) = 12% of salary as a benefit. This works out to be (12%)($43,600) = $5,232 per year. According to the vesting schedule, though, he is only vested in 80% of this benefit. Thus, Dave's vested benefit is (80%)($5,232) = $4,185.60 per year, beginning at age 65.

Example 7.1.7 Kelly is leaving her job where she has been working for 3½ years. The company uses 7-year 20% vesting for its defined contribution plan. The money Kelly has contributed to the plan herself has accumulated to $5,722.16; the company contributions have accumulated a total of $4,810.33. Find her vested balance in this plan.

The amount accumulated from the company's contributions is 20% vested, so Kelly gets to keep (20%)($4,810.33) = $962.07. The accumulation from her own contributions is always fully vested. So in total, she will keep $5,722.16 + $962.07 = $6,684.23.

Defined Benefit versus Defined Contribution Plans

There are advantages and disadvantages to each type of plan, whether we are looking at things from the employer's or employee's perspective.

Defined benefit plans offer an employee a guaranteed level of income, and it is up to the employer to make sure that the funds are available to support that income. The employee does not have to worry about where the funding will come from, or how money set aside to provide retirement plans will be invested. The employer bears the cost of funding the plan, and, if the investment returns earned by the plan's assets fall short of expectations, is responsible for making up the difference. Furthermore, to ensure that the employer will be able to meet its pension obligations, these plans are regulated, and the employer must meet requirements for funding, managing, and financially reporting the results of this type of plan. (Unfortunately, despite these regulatory efforts, pension plans do occasionally fail to meet their promised obligations.) For an employer, complying with these regulations can be a significant burden of both cost and effort.

While defined contribution plans are also subject to regulation, the employer's issues of adequate funding and investment results largely—though not completely—disappear. Since the plan does not guarantee any particular income to its retirees, the employer does not have to worry about maintaining adequate funding levels or investment returns. The investment decisions are made by the employee, and if the investment returns disappoint, that is not, in general, the employer's problem.[1] These reasons provide a very strong incentive for employers to switch over to defined contribution plans, and that has certainly been the trend over the last few decades. Employers generally prefer the fewer risks and fewer headaches that DC plans offer.

Defined contribution plans offer advantages to the employee as well, however. The employee has control over how her funds are invested, and as we have seen mathematically, small amounts of money invested well can produce astonishingly large returns in the long run. An employee who makes good investment decisions has the potential to build up a large nest egg in a defined contribution plan. (Though conversely, if her investment choices do not work out so well, the end result will not work out so well either.)

Portability is a major advantage to the employee of defined contribution plans. The benefits offered with defined benefit plans are almost always based on years of service. For someone who works for many years for the same employer, this works out well, but for someone who changes jobs several times over the course of his career it doesn't. If you work for 10 years at one employer, 8 at another, 12 at another, and then 15 at yet another you may have a vested defined benefit at each company, but each of these will be comparatively small since it is based on your years of service. And in the modern working world, even this track record may be remarkably stable.

Defined benefit plans work best for someone who can accumulate a large benefit by the combination of accumulating many years of service with an increasing salary on which the benefit will be based. In Example 7.1.6, Dave was vested in a $4,185.60 per year benefit, but that benefit will never get any bigger, even if he is many, many years away from retirement. In distinction, when you leave an employer with a defined contribution plan, the vested funds in your plan can be ***rolled over*** into the new employer's plan, or into a personal retirement account. Unlike a vested defined benefit, a vested defined contribution balance can continue to grow over time.

This portability of plans can be a disadvantage to employers. A skilled and experienced worker at a company with a DB plan is likely to want to stay working for that company, if for no other reason than to keep building on his pension benefit. Without those "golden handcuffs," this employee would be easier to lure away to work for a competitor.

[1]Since the employer does decide what sorts of investments will be available in the plan, there can still be some risk of the employer being held responsible for bad investment results. For example, if the investment options are very limited or if employees are required to invest their money in company stock, the employer can still be liable for bad investment performance. However, as long as there is a wide enough range of investment options, the employer is not usually responsible for investment performance, and most employers make sure that their plans offer enough choices so that this issue is avoided.

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Some workers prefer DC plans simply because they like the feeling of having control over their retirement plan. Even if there are no mathematically measurable benefits to this, it is nonetheless true that many people prefer a sense of control and ownership. On the other hand, though, many others dislike DC plans for precisely the same reason. A DC plan requires workers to make choices they may not want to have to make, and to keep up on things they may not want to have to keep up on. Someone whose retirement is dependent on a DC plan will probably be watching the stock market's performance much more closely than someone relying on a DB plan.

Social Security Privatization

Much has been made over the past decade about proposals to "privatize" Social Security. This has been particularly (though not exclusively) advocated by conservative commentators and Republican politicians. As it is presently constructed, Social Security operates like a DB plan, providing retired workers with income for life on reaching retirement. (The formula used to determine someone's benefit is extremely complicated, but it follows the same general principles that the benefit increases with the number of years worked and amount earned.) Advocates of privatization argue that it would be better to replace this system with something similar to a defined contribution plan, where instead of paying taxes into "the system" for years, workers would have some or all of their taxes put into investment accounts, which they could manage for themselves.

There are many different competing proposals for how such a system would work, but by far the largest challenge to any sort of change comes from the way that the present system is funded. An employer with a DB plan is required to set money aside and invest it over the course of each employee's career so that when the worker retires there will be adequate assets in place to provide the promised benefits. Social Security, though, is funded on a pay-as-you-go basis. Most of the Social Security taxes that you pay today are used right away to pay benefits to current retirees. While at present the system does have a "trust fund" of assets set aside to provide future benefits, it is not nearly adequate to pay for all of the future benefits promised under the present system.

If Social Security were to be privatized, the most difficult question to answer is where the money would come from to pay benefits to current retirees. If the taxes that you now pay to fund the benefits of current retirees are instead directed to your own account, where will the money come from to pay those benefits? There are many other challenges to privatization, of course, and while we may eventually see such a change to the system, these challenges make it hard to see how it could happen in the near term at least.

EXERCISES 7.1

A. Defined Benefit Plans

Interglobal Consolidated Useful Products Corporation has a defined benefit pension plan for its workers. On retirement at age 65, the plan provides 1.75% of the 3-year final average salary for each year of service, up to a maximum of 60%. Exercises 1 to 4 are based on this DB plan.

1. Calculate the pension benefit that would be paid for the following years of service and last 3 years' salaries.
    a. 20 years; $42,045; $43,188; $45,657
    b. 30 years; $42,045; $43,188; $45,657

2. Calculate the pension benefit that would be paid for the following years of service and last 3 years' salaries.
   a. 40 years; $37,101; $36,544; $38,101
   b. 7 years; $103,548; $107,626; $112,055

3. Suppose that this pension plan allows its workers to begin receiving pension payouts as early as age 60. For each year below age 65, benefits are reduced by 2%. Calculate the pension benefit that would be paid for the following years or service, last 3 years' earnings, and retirement age: 28 years; $37,500; $38,425; $39,901; age 62.

4. Suppose that this pension plan allows its workers to begin receiving pension payouts as late as age 72. For each year beyond age 65, benefits are increased by 1½%. Calculate the pension benefit that would be paid for the following years or service, last 3 years' earnings, and retirement age: 39 years; $52,600; $52,825; $53,011; age 69.

B. Defined Contribution Plans

5. Mom and Pop's Big Box Shoppes offers its employees a defined contribution plan. The company matches employee contributions at 60% up to 8% of salary.

   Randy earns $28,795 working for this company. How much will be deposited to his account per year if he contributes (a) nothing, (b) 5% of his salary, (c) 10% of his salary?

6. A company offers its workers a defined contribution retirement plan, with 45% matching up to 9% of salary. If you take a job with this company, making a $39,400 annual salary, how much will be deposited to your account in your first year if you contribute (a) nothing, (b) 6% of salary, (c) 12% of salary?

7. Catskill University offers its faculty and staff a defined contribution retirement plan. If the employee contributes 4% of salary, the college will contribute 10% of salary to the plan. If an employee does not contribute anything, neither does the college. An employee may contribute more than 4%, but this will not increase the college's contribution.

   Suppose that you work at Catskill U. and earn $34,750 this year. How much will be deposited to your account this year if your contributions to this plan total (a) $1,000, (b) $1,800 and (c) $4,000.

C. Vesting

To answer these questions, use the vesting schedules given in this section.

8. For the 7-year step vesting schedule given in this section, what percent of the balance that comes from the employer's contributions are vested if someone has completed service totaling (a) 3 years and 4 months, (b) 2 years and 11 months, or (c) 6 years and 8 months? What percent of the balance that comes from the employee's contributions are vested for each of these lengths of service?

9. Bob has $10,350 in his defined contribution plan at work. Of that balance $7,573.25 comes from his own contributions and $2,776.75 comes from his employer's. The plan uses the 7-year step vesting schedule given in this section of the text. How much of his plan balance would Bob keep if he leaves his job now, assuming he has 5 years and 3 months of service?

10. Rita has $12,759.17 in her defined contribution retirement account at work. Of this amount, $4,387.15 comes from her contributions; the rest comes from her employer's contributions. Her company uses the 7-year, 20% step vesting schedule.

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How much would Rita be able to keep if she leaves her job with (a) 3 years, 5 months, (b) 5 years and 11 months, or (c) 11 years, 4 months of service?

11. Suppose that Bob's company (from Exercise 9) instead used 5-year cliff vesting. Rework Exercise 9 with this vesting schedule.

12. Suppose that instead of step vesting, Rita's company (from Exercise 10) used 5-year cliff vesting. Rework Exercise 5, using this vesting schedule.

D. Grab Bag

13. Suppose your employer offers you a defined contribution plan with 50% matching up to 12% of salary. What total percent of your salary would be deposited to this plan if you choose to contribute (a) 4%, (b) 8%, (c) 12%, or (d) 16%?

14. Ethan works for a company that offers both a defined contribution plan and a defined benefit plan. Ethan is 22 years old. Which plan is probably of most value to him? Why?

15. Jerianne is debating job offers from two different companies. The two jobs are very similar, and offer comparable salary and benefits, except that one has a DB plan and the other has a DC plan. Jerianne is 54 years old. Which plan is probably more attractive to her?

Most-Lee Quality Products offers its employees a defined contribution plan. The vesting schedule is shown below.

Years of Service Completed	***Vesting Percent***
<1 year	0%
1–2 years	25%
2–3 years	50%
3–4 years	75%
4+ years	100%

Use this plan in answering Exercises 16 to 17.

16. Ahmad has worked for the company for 3½ years. Right now, his plan balance is $8,105.45, of which $3,903.15 comes from the company's contributions. If he quits his job today, how much of this balance will he be able to keep?

17. Jennifer has worked for the company for 12 years. Right now, her plan balance is $19,505.99, of which $8,101.55 comes from the company's contributions. If she is laid off from her job today, how much of this balance will she be able to keep?

18. Cattarauqua Ginseng Enterprises offers its employees a DB pension plan. The plan pays 1.8% of the 2-final-year average salary for each year of service completed. If I made $49,756 last year and $51,845 this year and retire now with 18 years of service, what will my annual pension benefit be?

E. Additional Exercises

19. Brad and Tracy have both started working, earning $37,500 per year. Both will receive salary increases of 4% each year over the course of their careers.

    Brad will continue working for the same company for 35 years and then retire. His company offers a defined benefit pension that pays 2% of final salary for each year of service.

    Tracy will work for five different companies, changing jobs every 7 years. At each of his jobs, there will be a defined pension plan offering the same 2% of final salary for each year of service that Brad's company offers. Each of Tracy's employers will use 5-year cliff vesting.

    a. Calculate Brad's retirement benefit.
    b. Calculate the total of all the retirement benefits Tracy will receive from his five jobs.
    c. How does the total of Tracy's retirement benefits compare to Brad's?
    d. Suppose that instead of defined benefit plans, Brad and Tracy had had defined contribution plans. Assuming that all of the employers in question offered the same percent contributions, and assuming that both Brad and Tracy earned the same rate or return on their plan investments, how would Brad's total account balance compare to Tracy's?

20. Visit the Social Security website (www.socialsecurity.gov). On this site, you can find an interactive tool that will allow you to project Social Security benefits for someone on the basis of age and lifetime earnings.

    a. Find the projected Social Security benefit for Tracy, a female, presently 35 years of age, with no previous earnings, assuming a $30,000 annual salary this year increasing by 4% each year until age 65.
    b. Suppose that instead of this Social Security benefit you instead were credited 6.5% of your annual salary into an investment account. Using reasonable assumptions about how Tracy might invest this money and the rate of return that might be expected for that asset allocation, calculate the future value that this investment account might reach at age 65.
    c. Suppose that once she reaches retirement, Tracy invests her money mainly in fixed income investments and withdraws her money at a rate that would allow the account to last 20 years. How does the payment from this plan compare to the Social Security projection?
    d. Aside from the size of the payments, what other significant differences are there between traditional Social Security and a privatized plan?
    e. Suppose that instead of a 35-year old female, you did this exercise for someone of a different age and/or sex. How would this affect the comparison between traditional and privatized options?

7.2 Details of Retirement Plans

In Section 7.1 we discussed the two main types of employer-sponsored retirement plans: defined benefit and defined contribution plans. With defined benefit plans, the calculations that go into the funding and management of the plan that provides the promised benefits are not the concern of the individual employee. For the employer they are so complex that specialized professionals (called ***pension actuaries***) must be hired to handle these details, and so, not surprisingly, those details lie far beyond the scope of this book. Defined contribution plans, though, are another matter entirely, and in this section we will cover some of the main mathematical issues with these types of plans. At the same time, retirement plans that are not tied to employment are also available, and we will also consider those here.

To begin with, it should be stated that there does not have to be anything special about a retirement account. Any money that someone saves and invests with the intent to use it to fund their retirement can be considered a retirement account. Therefore, any of the future

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value calculations we have done in prior chapters can be thought of as retirement account calculations if we the consider them earmarked for retirement savings. However, there are a number of special types of retirement accounts available both to individuals on their own and to workers through their employment that offer special advantages. In this section, we will take a look at some of the more common types of these accounts.

Individual Retirement Accounts (IRAs)

An individual retirement account (IRA) is a special type of account that can be set up by an individual through almost any bank, brokerage firm, credit union, or other financial institution. The purpose of an IRA is to allow an individual to save for retirement, and the tax laws offer significant tax advantages to encourage people to set up and contribute to these sorts of accounts.

An IRA is not a type of investment in itself; it is a type of account that can contain almost any sort of investment within it. An IRA is a container that can hold a wide range of different contents. IRAs can contain ordinary bank accounts, certificates of deposit, stocks, bonds, mutual funds, as well some other more exotic options. Since the money invested in an IRA is intended for retirement, it should be invested with that long-term time frame in mind.

Depending on your income level and a few other factors, you may be able to take an income tax deduction for contributions made to an IRA. More significantly, the earnings on money invested in an IRA are not taxed until they are withdrawn from the account. This ***tax deferral*** can be a huge advantage. Gains on investments in an ordinary account are subject to federal, state, and local income taxes. In order to pay those taxes, money has to be taken from those accounts, or from some other source. For IRA investments, the taxes are deferred, so even though taxes will *eventually* be paid on the investment gains, along the way funds that would have had to be used to pay taxes are able to continue compound growth. The earnings on those earnings can provide a larger overall account value than could otherwise be achieved.

These tax advantages come with strings attached, though. There are limits to how much an individual can contribute to an IRA in any one year. (The purpose of an IRA is to enable ordinary people to save for retirement, not to allow multimillionaires to shield huge amounts of money from taxes.) For 2007, this limit is $4,000 under most circumstances. The limit is scheduled to increase in future years, though this may change with future changes in the tax laws. Also, money invested in an IRA can not usually be withdrawn until its owner reaches age 59½.[2] If you take money out of your IRA sooner, you must pay any income taxes due, and in addition will normally be subject to a significant additional financial penalty. (There are some special circumstances where the money can be withdrawn without penalty.) Also, if you have not already started withdrawing money from your IRA by age 70½, you must start to take withdrawals at that time, or face steep financial penalties. The amount you withdraw each year in retirement must be larger than a certain minimum, calculated on the basis of your account's value and your life expectancy—you cannot just leave unlimited amounts in the account to grow tax-deferred forever.

Roth IRAs are a special type of IRA. They get their name from the late Senator William Roth of Delaware, who championed their creation. Contributions to a Roth IRA are not tax deductible, but like ordinary IRAs their investment growth is not taxed while the money is in the account. Roth IRAs have a unique advantage over ordinary IRAs, though, in that *their investment growth is not taxed when money is withdrawn from the account either*. In other words, all of the investment growth from a Roth IRA is income tax free, not just income tax deferred. This is a potentially enormous advantage, and recognizing it, Roth IRAs have become much more popular than ordinary IRAs. The rules for making withdrawals from a Roth IRA are different (generally more liberal) than the rules for traditional IRAs as well.

[2] I am not making this up. You can begin to withdrawn money from your IRA on your 59½ birthday. Right after you blow out the 59½ candles on your cake.

Example 7.2.1 ***Sarah has $3,000 that she wants to invest in an IRA. She expects that this money will earn 9%, and does not expect to withdraw the money from her IRA for another 40 years. Assuming that she will pay a 30% rate for combined state and federal income taxes, how much will she have after taxes if she invests in (a) a Roth IRA or (b) a traditional IRA.***

(a) Compounded at 9% for 40 years, her $3,000 would grow to:

$$FV = PV(1 + i)^n$$
$$FV = \$3{,}000(1.09)^{40}$$
$$FV = \$94{,}288$$

No taxes will be owed on this money when she withdraws it from a Roth IRA.

(b) Her money would grow to the same amount in a traditional IRA, but then she would have to pay taxes on it when withdrawn. Paying 30% of this in income taxes when it is withdrawn would leave her with 70% of this after taxes, or (70%)($94,288) = $65,960. (With the traditional IRA she might be able to deduct her contribution on her income taxes, resulting in a (30%)($3,000) = $900 savings on her taxes now.)

401(k)s

401(k)s are a type of retirement account offered as a benefit by employers. Workers contribute to their 401(k) accounts by payroll deductions; money contributed to a 401(k) is not included in taxable income, and so putting money into a 401(k) can lower your overall income tax bill.[3] As in a traditional IRA, the money in a 401(k) account grows tax-deferred, and is subject to income tax when it is withdrawn from the plan. Withdrawals from a 401(k) are subject to restrictions similar to the restrictions on traditional IRAs. There are also limits on how much money can be contributed to a 401(k) plan per year, but these limits are high enough that they don't really affect most workers.

When a company sets up a 401(k) plan for its employees, it generally selects a ***plan administrator*** such as a bank, insurance company, or mutual fund company. Employees control how their money is invested by choosing from a selection of mutual funds offered by the administrator for the company's plan. In many, though not all cases, the employer offers some ***matching*** of the employee's contributions, up to some maximum. For example, a company might offer to match contributions 50% on the dollar up to 8% of salary. (The limit refers to the amount that can be matched, not to the amount of the employer contribution. In fact, the most this company will contribute would be 50% of 8%, or 4% of salary.) The money contributed by the employer, and the investment earnings on that money, may be subject to a vesting schedule.

Example 7.2.2 ***Lisette makes $26,735 annually working for Shender Chemical Corp. The company offers a 401(k) plan with 75% matching up to 6% of salary. How much in total would Lisette have deposited to her 401(k) each year if she decides to contribute (a) nothing, (b) 4%, or (c) 10%.***

(a) If Lisette doesn't contribute anything to the plan, neither will her company. The answer here is $0. She misses out entirely on both on the tax savings and the "free money" that the plan offers.

(b) 5% of her annual salary is (5%)($26,735) = $1,336.75. The company will match 75% of this amount, or (75%)($1,336.75) = $1,002.56. In total, this makes $1,336.75 + $1,002.56 = $2,339.51.

(c) 10% of her annual salary is (10%)($26,375) = $2,637.50. The company will not match this entire amount, though. The company match will only apply to the first 6% of her salary, so the match will be (75%)(6%)($26,375) = $1,186.88. This totals to $2,637.50 + $1,186.88 = $3,824.38.

Lisette will have to decide for herself how much to contribute to her 401(k), based on her present income needs and long-term financial planning goals. A case can easily be made,

[3] As with anything involving tax laws, there are some exceptions.

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though, that she should at least contribute the full 6% that her company will match, since the matching provides a bonus of 75 cents on each dollar.

The history of the 401(k) plan is interesting. IRAs were intentionally created to offer people encouragement and incentives to invest for their own retirement. On the other hand, 401(k)s came about when someone noticed a loophole in the tax laws that would allow for this sort of account. The strange name 401(k) comes from the section of the tax code where this loophole was found. From this humble and accidental origin, though, 401(k) plans have grown to become a major part of many companies' benefit plan offerings, and a cornerstone of retirement planning.

Example 7.2.3 ***Curt is 24 years old, and he has just started a new job, which offers a 401(k) plan. He will be making $31,000 per year. Looking to the future, Curt is wondering what percent of his salary he should contribute to the 401(k) if he wants to have $750,000 in this account when he reaches age 65. He is paid biweekly, his company does not offer any matching, and he expects that his investments in the plan can earn 10%.***

We can look at Curt's 401(k) contributions as an annuity designed to build up the desired $750,000 future value. Recall that biweekly works out to 26 times per year.

$$FV = PMTs_{\overline{n}|i}$$
$$\$750{,}000 = PMTs_{\overline{1066}|0.10/26}$$
$$\$750{,}000 = PMT(15{,}305.57734)$$
$$PMT = \$49.00$$

Curt's biweekly pay will be $31,000/26 = $1,192.31. A $49.00 contribution amounts to $49.00/$1,192.31 = 4.11%.

Of course, these sorts of calculations can't be taken too literally. Using an annuity assumes that Curt's contributions remain constant over the next 41 years, which may not be realistic. The $49.00 is 4.11% of his current salary, but one would hope that Curt would get a raise at some point over the next 41 years! The 10% rate of return assumed is a (hopefully, educated) guess, and the $750,000 figure may or may not be enough to meet Curt's needs in retirement. These figures ignore the impact of inflation as well. Section 7.3 will address some of these concerns mathematically, but even with them, this calculation does allow Curt to get a ball-park sense of just how much is reasonable to be contributing. It would not be reasonable for Curt to conclude that he should sign up to contribute exactly 4.11% of his salary to the 401(k)—taking this figure exactly and literally ignores all of the factors we've mentioned. It would, though, be reasonable for him to conclude that, to achieve his goal, signing up to contribute 4 or 5% would be a good idea. (Of course, more wouldn't hurt either.)

These sorts of calculations are more complicated when an employer match is involved, as the following two examples will show.

Example 7.2.4 ***Nancy earns $26,000 per year and is paid biweekly. She wants to have $250,000 in her 401(k) 30 years from now, and thinks that her investments can earn 8½%. Her employer matches her contributions 50% up to 8% of salary. How much of her salary should she contribute?***

Following the approach of the prior example gives:

$$FV = PMTs_{\overline{n}|i}$$
$$\$250{,}000 = PMTs_{\overline{780}|0.085/26}$$
$$\$250{,}000 = PMT(3{,}595.324968)$$
$$PMT = \$69.53$$

This total includes both her and her employer's contributions. How much does she need to contribute herself? The employer match means that the funds going into the 401(k) will be 150% of her own contributions, so:

$$(150\%)(\text{Nancy's contribution}) = \$69.53$$
$$\text{Nancy's contribution} = \$46.35$$

Her biweekly pay will be \$26,000/26 = \$1,000.00. A \$46.35 contribution amounts to \$46.35/\$1,000 = 4.64%.

In this example, the limit on employer contributions was not an issue. The next example will show how that limit can come into play:

Example 7.2.5 ***Suppose that Nancy, from the prior example, decides that \$250,000 will not be nearly enough. She instead decides to aim for \$750,000. What percent should she contribute then?***

$$FV = PMTs_{\overline{n}|i}$$
$$\$750{,}000 = PMTs_{\overline{780}|0.085/26}$$
$$\$750{,}000 = PMT(3{,}595.324968)$$
$$PMT = \$208.60$$

$$(150\%)(\text{Nancy's contribution}) = \$208.60$$
$$\text{Nancy's contribution} = \$139.07$$

This works out to 13.91% of her salary. Oops! We assumed that her employer would match her full contribution, but this percent is more than the 8% limit. So 13.91% is not correct.

The maximum her employer will contribute is (50%)(8%)(\$1,000) = \$40. So to get to \$208.60, Nancy needs to contribute \$208.60 – \$40 = \$168.60. This works out to \$168.60/\$1,000 = 16.86%, which is the correct answer.

Similar to 401(k)s are **403(b)** plans, which are offered by nonprofit employers, such as schools, hospitals, charitable organizations, and so on. There are also other types of similar plans. These plans differ in some of their technical details and are governed by different regulations, but in most respects operate in the same way as a 401(k).

Roth 401(k)s are a fairly recent addition to the retirement plan landscape. Roth 401(k)s operate much like Roth IRAs with respect to income taxes. While the money you contribute to a regular 401(k) is tax-deductible now, but fully taxable when it is withdrawn, money contributed to a Roth 401(k) is not tax-deductible today, but withdrawals are exempt from income tax.

Annuities

We have already discussed annuities in Chapters 4 and 5, defining them as any stream of equal payments made at regular time intervals. The term ***annuity*** is also commonly applied to a special type of investment plan offered by insurance companies. Money deposited into these insurance annuities will normally grow tax-deferred, as in an IRA or 401(k), though neither contributions to nor payments from an insurance annuity are tax free. In fact, in some states annuity contributions are subject to a state ***premium tax***.

With some annuities, funds are invested and credited with interest by the issuing insurance company, similar to a bank account. Other types, called ***variable annuities***, offer a range of investment choices like mutual funds, allowing the owner to choose among the investment options. Annuities often offer special features not available with IRAs and similar accounts. Though annuity contracts are available with low fees and expenses, some annuity contracts come with substantial loads and/or high expenses.

The original intent of insurance annuities was to provide income for the owner's or someone else's lifetime, or over some other period of time. An ***immediate annuity*** is an insurance contract for which the income begins right away on purchase of the contract; a ***deferred annuity*** is one where the funds deposited are intended to be invested for a period of time before the income stream begins. If you retire having accumulated a large sum of money in retirement accounts, you need to manage this money to provide yourself with

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income throughout your retirement, despite the fact that none of us know how long we have left on this earth. If you use this money to purchase an annuity, the insurer can provide you with an income guaranteed for as long as you live. Purchasing an annuity contract is one way to avoid the risk of outliving your money. Given the prevalence of 401(k)s as part of retirement planning, it is likely that immediate annuities will be an increasingly important financial product in the future, as retirees look to convert 401(k) balances into sources of guaranteed lifetime income.

Other Retirement Accounts

IRAs and 401(k)s are by no means the only forms of retirement accounts in common use. U.S. tax laws provide for a whole host of other accounts with similar purposes, designed to meet the needs of self-employed people, smaller businesses, and other situations. These types of plans include ***Keogh plans***, ***SIMPLE plans***, and many others. While these plans share many features with IRAs and 401(k)s, including tax advantages, the details of these features and the limitations and regulations that apply to each type of plan vary. While politicians always advocate simplifying things, and everyone agrees that simplification is a good thing, the fact of the matter is that we can expect new types of accounts to be created in the future, and the rules for existing accounts to be tinkered with. Fortunately, retirement savings is big business, and major banks, brokers, mutual fund companies, and other financial institutions that offer these types of account are usually well-informed about the latest rules and regulations, and are prepared to assist individuals and business owners in setting up and managing these accounts. A business owner may also be able to obtain advice on the different options available from his accountant.

EXERCISES 7.2

A. Traditional and Roth IRAs

1. Suppose Andrew deposits $5,000 into an IRA, which earns an average rate of 8.25% for the next 40 years. Suppose that Andrew's income tax rate bracket would require him to pay 35% in income taxes, whether paid today or in the future.

   a. What is the future value of this deposit?

   b. Suppose Andrew has a traditional IRA. Assuming that he is eligible to take this tax deduction, how much would he save on his income taxes this year because of this deposit? Forty years from now, how much would this deposit be worth after taxes are taken into account?

   c. Suppose Andrew has a Roth IRA. How much would he be able to save on his income taxes this year because of this deposit? Forty years from now, how much would this deposit be worth after taxes are taken into account?

2. Bonnie is 28 years old. She has $3,500, which she is considering either depositing in a regular IRA or a Roth IRA. Suppose that her account will earn 8.4% on average from now until she is 65, at which point she will withdraw the money from her account. Her income level is such that she will pay 28% in income taxes, whether paid today or in the future.

   a. If she puts this money into a traditional IRA, how much will she save in income taxes today? How much will she pay when the money is withdrawn?

   b. If she puts this money into a Roth IRA, how much will she save in income taxes today? How much will she pay in income taxes when the money is withdrawn?

B. 401(k) Plan Projections

Assume that all employees in these examples are paid biweekly (26 times per year).

3. LaNetta is 28 years old. She has just taken a new job at a company that offers a 401(k) plan with a 50% employer match up to 9% of salary. She will be earning $28,500 per year.

   If her 401(k) investments earn 7%, and she contributes 5% of her income, how much would her 401(k) balance grow to in 35 years. (Assume that her income remains constant at $28,500 per year.)

4. Kris's company offers a 401(k) plan with no employer match. Kris would like to have $500,000 in his account in 40 years. He makes $33,000 per year. He believes his investments can earn 6.5%. (a) How much should he be depositing with each paycheck? (b) To achieve this goal, what percent of his salary should he be contributing to this plan? (Assume his annual income does not change.)

The Winterburg Central School District offers its employees a 401(k) plan with a 40% employer match up to 10%. Walter has just taken a job with the district, earning $35,700 per year. Exercises 4 to 7 are based on Walter and this 401(k) plan.

5. Walter would like to have $250,000 in his 401(k) plan when he retires in 37 years. If his investments can earn 7.25%, what percent of his salary would he want to contribute?

6. Suppose that Walter decides that he should aim higher, and sets his target to be $500,000. What percent of salary would he want to contribute then?

7. Suppose that Walter sets his sights even higher and decides to aim for $750,000. What percent of his salary would he want to contribute then?

8. Suppose that Walter starts thinking that he wants to set an even higher goal, in hopes of either retiring early and/or being able to afford a higher standard of living when he does. What percent of his salary would Walter want to contribute if he sets a goal of $1,500,000.

C. Grab Bag

9. Eriko earns $45,755 per year. She contributes 8% of her income to her 401(k) plan at work. In her tax bracket, she would pay 37% of her income in state and federal income taxes.
   a. How much is she saving this year on her taxes by making these 401(k) plan contributions?
   b. Suppose that instead of contributing to her 401(k) plan, she decides instead to deposit the same amount of money to a Roth IRA. What would her tax savings be this year if the money went into the Roth instead?
   c. Eriko has projected that her retirement account deposits will have grown to $318,902 by the time she retires. Assuming the same 37% tax rate, what would the after-tax value of her account be if the money is in a 401(k)? In a Roth IRA?
   d. Suppose that Eriko's company offers a 75% match up to 10% of salary, how would this affect her decision of whether to stick with the 401(k) or instead switch to a Roth IRA?

D. Additional Exercises

10. Suppose that Walter (from Exercises 4 to 7) set his future value goal at $1,000,000. What percent of his salary would he want to contribute to achieve this goal?

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11. Dan has just retired and he has $743,500 in his 401(k) account. He plans to take equal monthly payments from this account, with the idea of making his money last for 25 years. Assuming that he earns 4.8% on his account over this time period, how much can he withdraw each month?

12. Suppose Jenny will retire in 19 years. She presently has $175,936 in her 401(k) account. When she retires, she plans to take equal monthly withdrawals of $3,500 from her 401(k) and have her money last for 20 years. If her account will earn 7.25% compounded monthly until she retires and 3.75% while she is retired, calculate (a) the amount she needs to have in her account at retirement and (b) the amount she needs to deposit to her 401(k) each month in order to reach this goal.

7.3 Assessing the Effect of Inflation

Many of the financial calculations we have done have dealt with long-term goals, such as figuring out how much you need to set aside each month to be a millionaire in retirement 40 years from now. While our work has been mathematically correct, we have ignored a very important practical concern: the impact of inflation.

Loosely speaking, inflation is the tendency of prices to rise over time. While there is nothing that *requires* this to happen, it is common knowledge that prices do tend by and large to go up over time. Though no one likes higher prices, inflation is not entirely a bad thing, at least when the rate of inflation is moderate or low. As long as prices do not rise too far or too fast we hardly even notice inflation in our daily lives.

Of course, when prices rise a lot, inflation is much harder to ignore and can be devastating to both individuals' personal finances and to the economy as a whole. An inflation rate of, say, 25% would mean that prices would double roughly every three years. Your earnings would have to rise by 25% annually, and your savings would have to grow at that rate, just to keep pace with the rise of prices. Yet even though this would be a very high rate, at least compared to the inflation rates historically seen in the United States, inflation rates can, and have, gone much higher in other places. Germany experienced devastating inflation rates after the First World War, and the economic insecurities they caused are often cited as a contributing factor in the rise of Adolf Hitler. Similarly awful inflation has been experienced in many different times and places, often in the aftermath of war or economic or political upheavals.[4]

While hopefully we will never have the pleasure of experiencing inflation on anywhere near that scale, even tame, barely-noticeable inflation rates can be significant over long periods of time. In modern history the inflation rate in the United States has varied quite a bit, but has averaged out to somewhere in the neighborhood of 3.5%.[5] This has not been a steady rate from year to year; in the 1970s and early 1980s the rate of inflation ran much higher, while in the 1990s inflation was nearly nonexistent, and some economists even worried about the possibility of ***deflation***, prices dropping overall rather than rising.

In the near term, inflation running at a 3.5% average is hardly even noticeable. But over the long term, the cumulative effect of compounding inflation can really add up. For example, assuming that a 3.5% rate of inflation continued for the next 40 years, a candy bar that costs 65 cents today would cost a whopping $2.57! So if you set aside 65 cents in your

[4]What is often regarded as the all-time world record for inflation occurred in Hungary in the wake of the Second World War. During this period, by some estimates the inflation rate reached a staggering 41.9 quadrillion percent per year, or 41,900,000,000,000,000%. With this inflation rate, prices would double on average every 15 hours.

[5]There is some controversy about how to measure the real rate of inflation, mainly because prices for all things do not rise at a uniform rate. The "inflation rate" is an attempt to measure how fast prices are rising *overall*, but your overall may not be the same as mine. If the prices of things that you buy are rising more quickly (or slowly) than the prices of things you don't, the inflation rate as far as you are concerned will be higher (or lower) than the overall rate.

dresser drawer, planning to treat yourself to a candy bar in 40 years, when the day arrived you would be sadly disappointed to find that you come up almost $2 short!

Of course, that example is obviously unrealistic. No one actually sets aside loose change and then patiently waits 40 years to buy a candy bar. Yet actually this is not all that far off from what we did in some of the examples in the previous section. We tend to base our future financial goals on what money is worth *today*, ignoring the fact that what looks like a lot of money today may not seem so impressive after years of inflationary compounding.

In problems where we set a goal of $1,000,000 in a retirement account, we failed to take into account that in the future, $1,000,000 will almost certainly not carry the same buying power as it does today. The mathematics was correct, yet when we set the million dollar goal we were thinking about how much money a million dollars is on the basis of how much that much money is worth at the prices we see today. In other words, we were thinking about $1,000,000 ***in today's dollars***, an expression that means that we really are thinking about the *buying power* that $1,000,000 represents today. Given the inflationary facts of life, 40 years from now $1,000,000 will almost certainly not have as much buying power as it does today. So if we set our goal to be one million *actual* dollars, when retirement comes we will find ourselves just as disappointed as the person trying to buy a $2.57 candy bar for 65 cents.

When making long-term financial planning decisions, it is easy, but foolish, to overlook the impact of inflation. This is especially important in retirement planning, since the goal is often a very long-term one, but it also can be important in long term business planning as well. Yet, it is not at all uncommon to see retirement or long-term business plans that nonetheless ignore inflation altogether. In this section, we will explore ways to take inflation into account mathematically.

Long-Term Predictions about Inflation

One huge problem that will confront us here is that we don't really have any idea of what will happen to prices in the future. Some economic prognosticators will tell you that free market competition paired with advances in productivity and technology will make inflation a non-issue for the foreseeable future. They can back up that claim with very convincing arguments and compelling evidence. Other forecasters predict that growing populations and rising living standards will combine with shortages of commodities and/or labor to drive prices through the roof. They can make an equally compelling case for this view. There is some truth in the old joke that God created economists to make weather forecasters look reliable.

The fact of the matter is that no one knows what the future holds, for inflation rates or anything else. While we can't pretend inflation does not exist, we also can't know in advance what will happen, either. Any projections we make will have to rest on assumptions, and so our predictions are just educated guesses calculated from other educated guesses. We can never know for certain that things will work out according to what we've planned, but if our assumptions are reasonable, we at least have a reasonable expectation that our projections will give us a good enough basis on which to make sound financial decisions.

The following example will illustrate the mathematics of making these sorts of projections.

Example 7.3.1 ***Suppose that a lawnmower costs $189.95 today, and that you expect lawnmower prices to rise at a 4% effective rate in the future. If your assumption is correct, how much would this mower cost 20 years from now?***

Prices rising at an annual inflation rate are mathematically equivalent to money growing at en effective compound interest rate. So:

$$FV = PV(1 + i)^n$$
$$FV = \$189.95(1 + 0.04)^{20}$$
$$FV = \$416.20$$

On the basis of these assumptions, we can predict that the lawnmower would cost $416.20 twenty years from now.

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The inflation rate we used in this example is slightly higher than the long-term average, so this $416.20 predicted price assumes inflation over the next 20 years that runs a bit above the long-term rate. However, 4% is close enough to the long-term average that we can say that, while this prediction may prove inaccurate, it is not completely unreasonable either.

Let's take another look at the example mentioned at the start of this section: how much does someone need to set aside each week in order to *really* have $1,000,000 in 40 years?

Example 7.3.2 ***Suppose you want to have $1,000,000 in your 401(k) account in 40 years. How much do you need to deposit into this account each week to achieve your goal? For the types of investments you plan to make, you expect to earn 9% on your investments. Also assume that your goal is not $1,000,000 in actual dollars, but instead is $1,000,000 in today's dollars.***

Our first step is to figure out what the actual goal is. Of course, since we can't know what inflation will be over the next 40 years, there is no way to know this for sure. However, just as with predictions about investment returns, we can reasonably approach the question by using an inflation rate based on what has happened in the past. So we will assume a 3.5% inflation rate in this problem, and use it to predict the cost of $1,000,000 (at today's prices) worth of goods and services in 40 years.

$$FV = PV(1 + i)^n$$
$$FV = \$1{,}000{,}000(1 + 0.035)^{40}$$
$$FV = \$3{,}959{,}260$$

So your goal is actually $3,959,260.

Now, using this as the future value of a sinking fund, we can calculate the required weekly deposits:

$$FV = PMTs_{\overline{n}|i}$$
$$\$3{,}959{,}260 = PMTs_{\overline{2080}|.09/52}$$
$$\$3{,}959{,}260 = PMT(20502.17)$$
$$PMT = \$193.11$$

So it turns out you would need to deposit $193.11 each week to reach this million dollar goal.

This is a deeply disappointing result. In Chapter 4, we saw that fairly modest deposits could grow into enormous future values, and the goal of becoming a millionaire seemed attainable, even easy. If we had used a $1,000,000 future value without adjusting for inflation, we would calculate that the weekly deposits need to be only $48.78. Weekly deposits of $48.78 amount to a bit over $2,500 a year—not pocket change, but not completely unattainable either. But $193.11 a week amounts to over $10,000 a year, an investment that most people would find challenging to say the least.

There is more to this story, though. We overlooked something in the prior example. We adjusted the targeted future value for inflation, but we did not make any adjustments to the payments. Our solution to the problem assumed that you will deposit $193.11 in the first week, and every week in between, so over the course of 40 years your weekly deposits never increase. If that is the goal—to find the fixed and never-changing weekly payment to reach the target—then our solution is correct. But as prices rise over the course of the next 40 years, hopefully your income will too. Probably a more realistic scenario would be to assume that you start making deposits at a certain rate today, but that over time as prices and your income rise, you will increase your deposits at a similar rate. This assumption changes things.

Projections in Today's Dollars

First of all, we have to make some assumption about the rate at which you will increase your payments. The simplest approach is to assume that you want to maintain a contribution rate that stays the same *in today's dollars*, and so assume that your deposits increase at the same rate as inflation. If you contribute \$50.00 per week this year, and inflation runs at 4%, you will increase your deposit next year by 4% to \$52.00 a week. But the \$52 in *actual* dollars next year is really the same as \$50 in *today's* dollars, since our 4% inflation rate assumes that goods and services that could be bought for \$50 today would cost \$52 next year.

Of course, once we assume that your deposits change, we no longer have an annuity, and so our annuity formulas won't work. We could deal with this by forgetting about using annuity formulas and instead switching over to the spreadsheet methods developed in Chapter 5. That will work, but there is another way.

Let's take a different approach than what we did in Example 7.3.2. In that example, we changed our future value to be the *actual* dollars you would need, and the payment we calculated was the *actual* dollars that you would deposit. Since annuities require equal payments, our solution had to require the same actual dollar deposits.

We can, though, work out the problem assuming that *all dollar amounts are understood as being in today's dollars*. Our equal deposits, then, would be equal in today's dollars, but not in actual dollars. When we calculate the payment in today's dollars, we understand that in reality the actual dollar deposits are increasing at the assumed inflation rate. Taking this approach, the future value would be \$1,000,000, understood as 1 million *today's* dollars, which we understand will probably mean a much larger number of *actual* dollars 40 years from now.

We will also need to adjust the rate of return that we assume. If you earn 9.1% on your account balance, \$100 will turn into \$109.10 in one year. But, if prices rise by 3.5%, you haven't really gained a full 9.1%, because in that year prices rose as well. What \$100 buys today will cost \$103.50 in a year. So in fact, you've actually only earned \$109.10 – \$103.50 = \$5.60, or 5.6% on your money. In other words, of the 9.1% we assume you earn, 3.5% is lost to inflation, so your ***real rate of return*** is actually 9.1% – 3.5% = 5.6%.[6] If we are going to work the problem out in today's dollars, we need to express our rate of return as the return on today's dollars, using 5.6% instead of 9.1%.

Example 7.3.3 ***Working entirely in today's dollars, what amount would you need to deposit each week to reach \$1,000,000 in 40 years, assuming that your account earns 9.1% and inflation averages 3.5%?***

Since we are working in today's dollars, the future value remains \$1,000,000. As explained above, we will use a rate of 9.1% – 3.5% for the sinking fund to find the required payment:

$$FV = PMTs_{\overline{n}|i}$$
$$\$1{,}000{,}000 = PMTs_{\overline{2080}|0.056/52}$$
$$\$1{,}000{,}000 = PMT(7{,}783.300973)$$
$$PMT = \$128.48$$

So the required payment is \$128.48 in today's dollars.

Don't forget that this must be understood as today's dollars. This year, \$128.48 would be the payment, but that amount needs to be adjusted upward each year to keep up with inflation. It also must be understood that if all works according to plan, the value of your account will be \$1,000,000 in today's dollars; the actual dollar amount in the account 40 years from now would actually be much larger, though whatever the amount is, we are projecting that it will have the same purchasing power as \$1,000,000 does today.

[6]Mathematically, this isn't entirely clear cut. Subtracting treats the gain of \$5.60 as a percent of the original \$100, when in fact you don't have that \$5.60 until the prices have risen to \$103.50. Your buying power after 1 year is better stated as \$109.10/\$103.50 = \$105.41, and it might be more accurate to say that your real rate of return is 5.41%. That said, it is conventional to find real rates using subtraction. Unless the rates of return and/or inflation are very high, the difference is minimal, and the rates we are using are approximations anyway. Because of this, though, if you work the problem out in actual dollars with increasing payments using a spreadsheet, the results will not exactly agree with the answer to Example 7.3.3.

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Projections Assuming Payments Change at a Different Rate than Inflation

We assumed above that your earnings, and with them your ability to fund your retirement account, will rise at a rate that keeps pace with inflation. Hopefully that assumption isn't too optimistic; you would certainly hope that your earnings will at least rise fast enough so that you don't lose ground to rising prices. However, you probably hope for more, that your earnings will rise at a faster rate than inflation, and that your buying power actually increases as your career advances. If that happens, hopefully you will be able to also increase your deposits at a similar rate. That would be helpful, because it would allow smaller deposits now to be made up for by increases later on.

How do we handle this? Whether we work in actual dollars or in today's dollars, the payments will be changing, so the annuity formulas don't work. The only realistic way to handle this is to work things out on a spreadsheet. We can do this by using either actual or today's dollars; since using today's dollars will not help us here the way it did in Example 7.3.3 it is probably more straightforward to go back to working in actual dollars.

Example 7.3.4 ***Rework the question from Examples 7.3.2 and 7.3.3, this time assuming that your payments increase at a 5% annual rate.***

In Example 7.3.2 we found that the future value you will actually need is $3,959,260. We set up a spreadsheet just as we did in Chapter 5. Since we don't know what the payment needs to be, we will just enter an educated guess, such as $100.00. Since the payments are weekly, we'll also need to change the header from Year to Week, and the interest earned column formula needs to be adjusted to divide by 52, since things are weekly. This new formula for the interest earned cell needs to be copied on down. For the moment, let's copy that cell down as far as row 106 (the end of the second year).

*In row 55, the first row of the second year, the payment cell should be changed to =Round(1.05*D54,2). This will increase the payment in that month and every subsequent month to $105.00.*

Since the 52 rows of the second year are set up the way we want all the remaining years to be, we will highlight the cells from A55 through E106, and copy those. Then highlight A107 through A2082, and click on Paste, which will duplicate the second year's cells into the cells for the remaining 38 years.

We then check to see the balance shown at the end of year 40. To be able to see this easily without having to scroll through a couple of thousand rows, you may want to hide the rows between the first and last.

	A	B	C	D	E
1	**Rate:**	9.10%			
2	**Week**	**Starting Balance**	**Interest Earned**	**Payment**	**Ending Balance**
3	1	$0.00	$0.00	$100.00	$100.00
4	2	$100.00	$0.18	$100.00	$200.18
5	3	$200.18	$0.35	$100.00	$300.53
	Rows Omitted				
55	53	$5,439.00	$9.52	$105.00	$5,553.52
	Rows Omitted				
2082	2080	$3,716,491.87	$6,503.86	$670.54	$3,723,666.27

This is actually close to the target future value, but a bit short. Trying different initial payments, through trial and error we find that the closest we can get to our target is by using an initial payment of $106.35 per week.

This problem could also be done in today's dollars. In that case, if we assume a 3.5% inflation rate, we would have the payments increasing by 5% – 3.5% = 1.5% per year, we would use a $1,000,000 future value, and would use a growth rate of 5.6%. It is equally reasonable to do the problem either way; what matters, though, is that, when you do one of these problems, you do not mix up today's dollars with actual dollars, since that inconsistency will lead to unreliable results. Whichever approach you take, always make sure to be consistent! If you are working a given problem in today's dollars, consistently use today's dollars and be clear about that fact when interpreting your answer. If you are working in actual dollars, consistently use actual dollars and be clear about that fact when interpreting your answer.

EXERCISES 7.3

A. Projecting Future Dollar Amounts with Inflation

1. Suppose that housing prices rise at a 2.75% annual rate for the next 20 years. Project the future price of a house that sells for $183,500 today.

2. Suppose that the rate of inflation averages 3.75% in the future. Garret wants to have an investment account worth the equivalent of $500,000 in today's dollars in 40 years. What is the actual dollar amount of his goal?

3. Suppose that food prices rise at an average 4.5% rate for the next 15 years. Based on this rate, predict the price of a box of cereal that currently costs $3.49.

4. In 2005, Gena was quoted a price of $24,583 for a large array of solar panels for her home. An industry expert was predicting that prices for solar arrays would drop at a roughly 10% annual rate for the next 10 years. If this prediction is correct, how much would this array cost in 2015?

B. Financial Projections with Inflation

5. Suppose that I believe that my investment portfolio can earn 8.5%. If the inflation rate averages 3.5%, what is my real rate of return?

6. In Section 6.4 we presented a table of average annual historical rates of return for different asset classes. These rates were not adjusted for inflation. Fill in the table below to include a column for the real rate of return, assuming inflation runs at a 3.5% average rate.

Asset Class	*Degree of Risk*	*Historical AverageRate of Return*	*Historical Average Real Rate of Return*
Cash	Minimal	2–5%	
Fixed Income	Moderate	4–7%	
Equities	High	8–12%	

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7. Suppose that I want to have $700,000 in today's dollars in my newly opened IRA when I retire in 36 years. Assume that my investments earn 9.3% and that inflation averages 3.5%.
   a. What is the actual future value I need in order to achieve my goal?
   b. How much should I deposit each year in order to reach this goal? Assume that I want to contribute the same amount of actual dollars each year; I do not intend to increase the contributions to keep pace with inflation.

C. Financial Projections in Today's Dollars

8. Suppose that I want to have $700,000 in today's dollars in my newly opened IRA when I retire in 36 years. Assume that my investments earn 9.3% and that inflation averages 3.5%
   a. What is the real rate of return I am expecting on my investments.
   b. How much should I deposit each year in order to reach this goal? Assume that I want the answer in today's dollars; I intend to increase my contributions each year to keep pace with inflation.

D. Projections with Payments Changing at a Different Rate than Inflation

9. Suppose that I want to have $700,000 in today's dollars in my newly opened IRA when I retire in 36 years. Assume that my investments earn 9.3%, that inflation averages 3.5%, and that I expect to increase my deposits by 5% each year. According to this plan, how much should I deposit this year.

E. Grab Bag

10. What is the real rate of return on my investment account if inflation runs at 3.15% and my account grows at a 7.25% rate.

Questions 11–14 deal with Emily's retirement savings plans.

Emily figures she needs $650,000 in her 401(k) account to retire comfortably 32 years from now. She realizes that it is not $650,000 actual dollars that she wants, but rather $650,000 in today's dollars.

Suppose that inflation averages 3.25% annually in the future, and that Emily's investments will earn an average 9.25% annual rate of return. She has nothing in her 401(k) account today.

11. Calculate the amount that Emily should deposit from each of her biweekly paychecks to reach this goal. Assume that she plans to contribute the exact same dollar amount from each paycheck from now until retirement.

12. Calculate the amount that Emily should deposit from each of her biweekly paychecks to reach this goal. Assume that she plans to increase her payments by 4.5% each year.

13. Calculate the amount that Emily should deposit from each of her biweekly paychecks to reach this goal. Assume that she plans to increase her payments over the years to keep up with inflation.

14. What real rate of return are we assuming for Emily's retirement account?

15. Using what she learned from studying this book, Vanessa made a projection of what future value she would need to retire comfortably and what rate of return she might reasonably project for her investments. She used the techniques of this section to take the impact of inflation into account, and determined how much she should be depositing into her retirement accounts. True or false: If she makes all of her deposits as planned, she will definitely have the amount she needs to retire comfortably.

16. In the summer of 2006, according to one method of calculating the inflation rate, the inflation rate was 4.8%. My savings account was paying 2½%. What was the real rate of return for my savings account?

17. Which is a better investment result: to earn 20% when the inflation rate is 24%, or to earn 2% when the inflation rate is 0%?

18. Linus's company offers a defined benefit pension. On a benefits summary statement he received from his company, based on a reasonable projection of his earnings into the future, he could expect to receive an annual pension of $43,560, assuming he does not leave his job until he retires 20 years from now.

    Linus realizes, though, that with inflation $43,560 is not as large a benefit as it seems. If inflation runs at 3.5% for the next 20 years, what is $43,560 worth in today's dollars?

19. Ambria and Todd figure that they will need the equivalent of $4,000 monthly income (in today's dollars) to retire comfortably. If they will retire in 18 years and inflation runs at 3.7%, what is the actual monthly income they would need upon retirement to meet this goal?

F. Additional Exercise

20. Glynnis received a statement from the U.S. Social Security Administration that indicated that, on the basis of her current earnings, she could expect to receive a monthly Social Security benefit of $2,558.56 when she reaches age 65 in 24 years. The statement also said that the value of this benefit in today's dollars is $1,187.50. What inflation rate is being assumed?

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CHAPTER 7 SUMMARY

Topic	Key Ideas, Formulas, and Techniques	Examples
Defined Benefit Plans, p. 307	• The benefits paid under a defined benefit plan are determined by a formula specified in the plan documents • Benefit formulas typically take into account years of service, average salary over some time period, and age at retirement.	A pension plan provides 2% of final 3-year average salary for each year of service on retirement at age 65. Jelena retired at 65 with 28 years of service. Her last 3 years' earnings were \$37,650, \$39,525 and \$40,187. Calculate her pension benefit. (Example 7.1.2)
Defined Contribution Plans, p. 309	• Defined contribution plans do not guarantee any specific income amount. • The employer makes contributions to each employee's account based on a set formula. • Employer contributions may be based on matching a percent of the employee's contribution.	Daxxilon Digital Devices matches 75% of each employee's plan contributions up to a 10% maximum. Hamid earns \$60,000. How much will be deposited to his account if he contributes 5% of his earnings? 15%? (Example 7.1.5)
Vesting, p. 309	• If you leave a job before retirement, the amount of benefits you keep is determined by a vesting schedule. • The schedule specifies the percent kept on the basis of the years of service completed. • Account values that come from your own contributions are always yours to keep.	Kelly is leaving her job where she has been working for 3½ years. The company uses 7-year step vesting. Her contributions have grown to \$5,622.16 and the company's have grown to \$4,810.33. Find her vested balance. (Example 7.1.7)
IRAs and Roth IRAs, p. 316	• Money deposited to a traditional IRA may be tax-deductible and grows tax-deferred. It is taxed when withdrawn. • Money deposited to a Roth IRA is not tax-deductible. Withdrawals are not subject to income tax.	Sarah has \$3,000 to invest in an IRA. She expects this money will earn 9% and will not withdraw it for 40 years. Assuming she will pay a 30% rate for state and federal taxes, how much will she have after tax if she invests in (a) a Roth IRA or (b) a traditional IRA? (Example 7.2.1)
401(k)s, p. 317	• 401(k) plans are retirement savings accounts offered as an employee benefit. • They are a type of defined contribution plan • 401(k)s often include an employer match of contributions.	Nancy earns \$26,000 per year and is paid biweekly. She wants to have \$250,000 in her 401(k) 30 years from now and thinks her investments will earn 8½%. Her employer matches contributions 50% up to 8%. How much of her salary should she contribute to reach this goal? (Example 7.2.4)

(Continued)

Topic	Key Ideas, Formulas, and Techniques	Examples
Projections with Inflation, p. 323	• Inflation has historically averaged 3 to 3.5% over the long term in the United States • Projections about inflation use the same mathematical formula as compound interest.	A lawnmower costs $189.95 today and you expect prices to rise at a 4% effective rate in the future. If your assumption is correct, predict the mower's price 20 years from now. (Example 7.3.1)
Retirement Projections in Today's Dollars, p. 324	• Long-term financial projections often overlook the impact of inflation • This may be addressed by doing calculations in today's dollars. • The real rate of return from an investment is found by subtracting the inflation rate	Working entirely in today's dollars, what amount would you need to deposit each week to reach $1,000,000 in 40 years, assuming your account earns 9.1% and inflation averages 3.5%? (Example 7.3.3)
Annuities Whose Payments Change at a Rate Different from Inflation, p. 326	• Calculations in today's dollars require the payments to change at the same rate as inflation. • If payments change at a different rate, spreadsheets are required to work out projections and other calculations.	What amount would you need to deposit each week to reach the equivalent of $1,000,000 worth of today's dollars, assuming your account earns 9.1%, inflation averages 3.5%, and your payments increase at a 5% annual rate? (Example 7.3.4)

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CHAPTER 8

Mathematics of Pricing

"This planet has—or rather had—a problem, which was this: most of the people living on it were unhappy for pretty much of the time. Many solutions were suggested for this problem, but most of these were largely concerned with the movements of small green pieces of paper, which is odd because on the whole it wasn't the small green pieces of paper that were unhappy."

—Douglas Adams, "So Long and Thanks for all the Fish"

Learning Objectives

LO 1 Determine retail prices by using markup from cost, markdown, and markup based on selling price.

LO 2 Calculate the percent markup or markdown based on cost and selling prices.

LO 3 Determine and interpret gross and net profit margins.

LO 4 Calculate invoice costs based on list price and trade discount.

LO 5 Understand the terminology surrounding cash discounts and be able to calculate them.

LO 6 Calculate the single discount percent equivalent to a series of discount percents.

LO 7 Distinguish between percent, straight-line, and MACRS depreciation and calculate depreciation rates and amounts, using these methods.

Chapter Outline

8.1 Markup and Markdown

A very large part of the business conducted in this world is a matter of buying things and then turning around and selling them to someone else at a profit. When we are setting prices, or when we are determining how successful those prices will be in meeting a business's profit goals, mathematics necessarily must be involved. In this section we will take a look at some of the most commonly used mathematical measurements and tools used to set and evaluate prices.

In this chapter, we will follow a convention that the price a business pays for an item is called the ***wholesale price*** or ***cost***. The price a business sells the item for will be called the ***retail*** price. We will use these terms even if the business purchases the item from a seller that might not formally call itself a "wholesaler;" likewise, we will use these terms even if the individual or business that sells the item is not strictly speaking a "retailer."

Markup Based on Cost

One common method used for setting the selling price for an item is ***markup based on cost***. This method is straightforward and agrees with the way most people usually think about setting prices. To determine a price with this method, we simply take the cost of the item, and add on a predetermined percent of the item's cost. For example, suppose that Eddie's Bike World can buy a particular model of bicycle for $255.00, and uses a mark up of 50% based on cost. Fifty percent of $255 is (0.50)($255) = $127.50, and so adding this on would mean a selling price of $225 + $127.50 = $352.50.

Finding the selling price in this way doesn't require too much effort, but we can make things even a bit simpler. A 50% markup means that every $1.00 of the cost turns into $1.00 + $0.50 = $1.50 of selling price. So a $255 cost turns into 225($1.50) = $352.50 in selling price. Using this logic allows us to find the selling price with a bit less effort, and will also pay off more richly in some of the problems that follow. We can sum this up in a formula:

FORMULA 8.1.1
Markup Based on Cost

$$P = C(1 + r)$$

where
P represents the SELLING PRICE,
C represents the COST
and
r represents the PERCENT MARKUP

Example 8.1.1 ***An auto mechanic charges a 40% markup based on cost for parts. What would the price be for an air filter that cost him $14.95? What is the dollar amount of his markup on this item?***

Since the markup is 40%, we multiply the cost by 1.40:

$$P = C(1 + r)$$
$$P = \$14.95(1.40)$$
$$P = \$20.93$$

The dollar amount of the markup can be determined in either of two ways. We can multiply 40% by the cost, to get (0.40)($14.95) = $5.98. Or we could subtract the cost from the selling price to get $20.93 − $14.95 = $5.98. Whichever way we find more convenient, the result is the same.

We can also work backward to find cost if we know the selling price and the markup percent. The following example will illustrate this.

Example 8.1.2 ***Hegel's Bagels and Vienna Coffeehouse sells souvenir coffee mugs for $7.95. The markup based on cost is 65%. Find (a) the cost of each mug, and (b) the dollar amount of the markup.***

(a) Working from our formula, we get:

$$P = C(1 + r)$$
$$\$7.95 = C(1.65)$$
$$C = \$4.82$$

(b) To find the dollar amount of the markup we can subtract $7.95 − $4.82 to get $3.13. We could also have got this by multiplying (0.65)($4.82) = $3.13.

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A word of caution is in order here, because it would have been very easy to have made a big mistake in Example 8.1.2. When using percents, we have to be careful what the percent is *of*. The 65% markup used in this example was a percent *of the cost*, not of the selling price. It is easy to overlook this and find the markup by multiplying by the selling price to get (0.65)($7.95) = $5.18, which is incorrect. In fact this would overstate the markup by more than $2! It is vitally important to make sure that you are applying the percent to the right thing when doing these sorts of calculations. These sorts of mistake not only lead to lower marks on exams, but when used in the real world they have the potential to lead to disastrously misguided business decisions.

If we know the cost and selling price, we can determine the percent markup from these, as the next example will demonstrate.

Example 8.1.3 ***An electronics retailer offers a computer for sale for $1,000. The retailer's cost is $700. What is the markup percent?***

Working from the formula, we get:

$$P = C(1 + r)$$
$$\$1{,}000 = \$700(1 + r)$$

Dividing both sides by $700 gives:

$$1.4285714 = 1 + r$$

We then subtract 1 from both sides and rewrite the result as a percent:

$$r = 0.4285714$$
$$r = 42.86\%$$

You may observe that we could also have calculated this by noting that there is a $300 markup, and so as a percent of cost the markup is $300/$700 = 42.86%. This is also a correct solution method.

The same danger we saw in Example 8.1.2 is in play here. It is tempting to reason that the dollar amount of the markup here is $300, and $300 is 30% of $1,000, and so wrongly conclude that the markup is 30%. The same issue is at play: *markup is a percent of cost, not of selling price*. Even though it is easy to see how someone could make this mistake (especially when the selling price is a nice round number), doing so leads to a result that misses the mark by quite a lot.

In the exercises you will have plenty of opportunity to work problems similar to these. You may find that you are able to find the correct answers without relying too heavily on the formula; if so, there is no need to rely on it too heavily. If you find, though, that you are making mistakes or getting confused, you may find it helpful to slow down and work things through carefully, and step by step, using the formula. When in doubt, slow down and use the formula!

Markdown in action! © PhotoLink/Getty Images/DIL

Markdown

From ordinary life, we are all familiar with the idea of prices being ***marked down***, for example, as part of a sale or some other promotion. Mathematically, markdown is quite similar to markup. To calculate a marked-down price, we simply apply the percent the price is to be marked down to the original price, and then subtract. For example, suppose Eddie's Bike World has a "10% off" sale on a bike that normally sells for $352.50. 10% of $352.50 is (0.10)($352.50) = $35.25, and so subtracting off this discount gives us a sale price of $352.50 − $35.25 = $317.25.

We can make things even a bit simpler, just as we did with markup based on cost. A 10% markdown means that every \$1.00 of the cost turns into \$1.00 − \$0.10 = \$0.90 of selling price. So a \$352.50 price turns into \$352.50(0.90) = \$317.25. Just as with markup, using this logic allows us to find the marked down price with a bit less effort, and will also pay off more richly in some of the problems that follow. We can sum this up in a formula:

FORMULA 8.1.2
Markdown

$$MP = OP(1 - d)$$

where
MP represents the MARKED-DOWN PRICE,
OP represents the ORIGINAL PRICE
and
d represents the PERCENT MARKDOWN

Note that this formula for discount very closely parallels the formula for markup. Note also, though, that while the markup percent is a percent of the cost, the markdown percent is a percent of the marked-up retail price.

Example 8.1.4 ***At its Presidents' Day Sale, a furniture store is offering 15% off everything in the store. What would the sale price be for a sofa that normally sells for \$1,279.95? What is the dollar amount of the markdown?***

Since the markdown is 15%, we multiply the original price by 0.85:

$$MP = OP(1 - d)$$
$$MP = \$1{,}279.95(0.85)$$
$$MP = \$1{,}087.96$$

The dollar amount of the markdown can be determined in either of two ways. We can multiply 15% by the original price to get (0.15)(\$1,279.95) = \$191.99. Or we could subtract the cost from the selling price to get \$1,279.95 − \$1,087.96 = \$191.99. Whichever way we find more convenient, the result is the same.

Again, as we did with markup, we can turn things around and find the original price based on the marked down price and markdown percent.

Example 8.1.5 ***Hal's Hardware Haven is having a going–out-of-business sale. According to its ad, everything in the store is marked down 40%. If a set of patio lights is offered at a marked-down price of \$29.97, what was the original price? How much of a dollar savings is this versus the original price?***

Working from our formula, we get:

$$MP = OP(1 - d)$$
$$\$29.97 = OP(0.60)$$
$$OP = \$49.95$$

To find the dollar amount of the markdown we can subtract \$49.95 − \$29.97 to get \$19.98. We could also have gotten this by multiplying (0.40)(\$49.95) = \$19.98.

The same cautions are in order with markdown as with markup. We have to be very careful to make sure we are clear just what the percent is *of*. In the case of markdown, the percent is a percent of the original price. It is not a percent of the marked down price. It would be incorrect to apply the 40% markdown rate to the marked-down price to find the amount of the markdown.

Once again, parallel to what we did with markup, we can find the percent markdown if we know the original and marked down price.

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Example 8.1.6 ***At the end of the summer, a backyard play set that usually sells for \$599.95 is marked down to \$450. What is the markdown percent?***

Working from the formula, we get:

$$MP = OP(1 - d)$$
$$\$450 = \$599.95(1 - d)$$

Dividing both sides by \$599.95 gives:

$$0.7500625 = 1 - d$$

Since the d is subtracted on the right side, while we want a positive d, we add d to both sides to get:

$$0.7500625 + d = 1$$

Then we subtract 0.7500625 from both sides and rewrite d as a percent to get:

$$d = 0.2499374$$
$$= 24.99\%$$

An alternative approach to the solution is the following. The markdown is \$599.95 − \$450 = \$149.95. As a percent of the original price this is \$149.95/\$599.95 = 0.2499374 = 24.99%.

Again we need to repeat a caution here. The markdown is 24.99% of the original \$599.95 price. It is not 24.99% of the marked-down price of \$450.

The amount, or percent, that a price is marked down is also commonly called ***discount***. The retailer selling the play set in Example 8.1.6 might describe this offer as a 24.99% (or, more likely, rounded to 25%), or \$149.95 discount. This is quite common phrasing, and probably phrasing that you are already quite familiar with. Be careful, though, not to confuse this with simple discount as discussed in Chapter 2. While the two ideas are similar, they are not exactly the same thing.

Comparing Markup Based on Cost with Markdown

There are many cases where markup and markdown are both involved, and it is important to exercise care with the calculations. When both markup and markdown are involved, sometimes "obvious" things are not nearly as obvious as they seem. The following example will illustrate.

Example 8.1.7 ***Gemma's Gemstone Gewelry bought a necklace for \$375. In the store, Gemma marked up this price by 20%. Several months later, when the necklace still had not sold, she decided to mark down the price by 20%. What was the marked-down price?***

The "obvious" answer is \$375; she marked it up by 20% and then down by 20%, so it seems to be common sense that the markup and markdown would cancel each other out. However, when we work it through, we see that this is not correct.

Markup:

$$P = C(1 + r)$$
$$P = \$375(1.20)$$
$$P = \$450.00$$

Markdown:

$$MP = OP(1 - d)$$
$$MP = \$450(0.80)$$
$$MP = \$360.00$$

The marked-down price was \$360, not \$375!

Why didn't things work out here as "common sense" would have suggested? The markup is a percent of the cost, the markdown is a percent of the marked-up selling price. Since those two 20%'s are not 20% of the same thing, they are not really the same at all. This can be confusing. It is important to not make assumptions off the cuff when comparing markup and markdown percents. Be careful and work things through!

Suppose we know a markup percent, and we want to mark things back down to cost. We can determine what the percent should be, as the next example will show.

Example 8.1.8 ***If prices are calculated with a 35% markup based on cost, what is the percent that those prices should be marked down to get back to their original cost?***

We don't know what sort of things we are pricing here, much less what the dollar amount of those prices would be. Fortunately, though, since we are working with percents the actual dollar amounts don't matter. We can work the problem out with whatever dollar amounts we like; the percent answer will be the same regardless of the price we assume. Following what we did to calculate effective interest rates back in Chapter 3, we choose a convenient cost of $100.

$$P = C(1 + r)$$
$$P = \$100(1.35)$$
$$P = \$135.00$$

$$MP = OP(1 - d)$$
$$\$100.00 = \$135.00(1 - d)$$
$$0.7407407 = 1 - d$$
$$d = 25.93\%$$

So a 25.93% discount "undoes" a 35% markup.

When "Prices" Aren't Really Prices

Even though we've approached this chapter from the point of view of retail prices, the mathematics we've developed here can be applied to other related situations. Whenever we have something that increases as a percent of the original amount, we can mathematically look at this as "markup." Likewise, whenever something decreases as a percent of some starting value, we can look at this as "markdown." The exercises provide several examples of this; the following example is just one of many possibilities.

Example 8.1.9 ***In 2004, 184 students graduated from Central City Business College. The graduating class of 2005 increased by 10.3% over 2004, but then the 2006 class decreased 9.8% versus 2005. How many students graduated in 2006?***

The size of the 2005 class can be thought of as a "markup" over the size of the 2004 class. Using this idea we get:

$$P = C(1 + r)$$
$$P = 184(1.103)$$
$$P = 203$$

(We round to the nearest whole number, since we are talking about a number of people.)

The 2006 class is given as a percent decrease from the 2005 class, so we can look at this decrease as a "markdown" from the 2005 class size.

$$MP = OP(1 - d)$$
$$MP = 203(0.902)$$
$$MP = 183$$

So there were 183 students in the 2006 graduating class.

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EXERCISES 8.1

When the final answer is a percent, round to two decimal places if necessary.

A. Markup Based on Cost

1. Complete the following table by filling in the retail prices based on the given costs and markup percents:

Item	Cost	Markup Percent	Retail Price
A	$39.03	12.5%	
B	$1.47	65%	
C	$69.22	5.9%	
D	$184.03	22.5%	
E	$23.36	85%	

2. Complete the following table by filling in the retail prices and the amount of markup based on the given costs and markup percents:

Item	Cost	Markup Percent	Markup Amount	Retail Price
A	$3.55	50%		
B	$459.03	24%		
C	$108.85	2.58%		
D	$18.32	26.5%		
E	$62.41	37.5%		

3. A hardware store buys a shipment of screwdriver sets of $8.37 each. If the store uses a 37.5% markup, what would the retail price be?

4. A coin dealer buys gold coins for $735 and then adds on an 8% markup. What is his selling price for each coin? What is the dealer's markup on each coin?

5. An electronics retailer charges $49.31 each for a particular model of emergency shortwave radio. Find the cost if the markup is 25%. Also find the amount of the markup.

6. A college bookstore uses an 18% markup for new textbooks.
   a. If a textbook costs the store $107.45, what would the selling price be for the book? What is the amount of the markup?
   b. If a textbook is offered for sale for $89.45, what did the bookstore pay for it? How much markup is included in the bookstore price?

7. Complete the following table by filling in the markup percent based on the given costs and retail prices.

Item	Cost	Retail Price	Markup Percent
A	$31.55	$89.99	
B	$0.82	$1.29	
C	$15,750	$16,995	

8. Complete the following table by filling in the markup amount and percent based on the given costs and retail prices.

Item	*Cost*	*Retail Price*	*Markup Amount*	*Markup Percent*
A	\$74.27	\$99.99		
B	\$4.22	\$12.99		
C	\$834.17	\$949.90		

9. A travel agency applies a 10% markup to airline tickets that they book. If you paid \$399.73 for a plane ticket to Las Vegas, what was the travel agency's cost? How much was the agency's markup?

10. An appliance store offers a new washing machine for \$799.95. The store's cost is \$597.03 for this machine. Determine the amount of markup, and determine the markup percent.

11. An employment agency pays a temp computer programmer \$18 per hour, and bills the company where she works \$22.50 an hour. Find the agency's markup percent.

B. Markdown

12. Complete the following table by filling in the markdown amounts and marked-down prices based on the given original prices and markdown percents:

Item	*Cost*	*Markdown Percent*	*Markdown Amount*	*Marked-Down Price*
A	\$249.97	35%		
B	\$19.95	10%		
C	\$75.00	16%		

13. Complete the following table by filling in the marked-down prices based on the given original prices and markdown percents:

Item	*Cost*	*Markdown Percent*	*Marked-Down Price*
A	\$79.95	12.5%	
B	\$9.50	30%	
C	\$29.29	85%	

14. A laptop computer that normally sells for \$799.95 is marked down by 12%. Find the sale price.

15. A pet supply store normally charges \$49.95 for a 37-pound bag of dog kibble. The store is having a 20% off sale. Find the sale price for this bag of dog food.

16. The list price of a new Jiangxi Motors Model J car is \$29,925. A dealer is advertising an end-of-model-year sale, with prices marked down 9.5% from list. Find the sale price of this car, and also the amount of the markdown.

17. A restaurant offered a "15% off your total bill" coupon. If you and some friends went out to this restaurant for dinner and the total bill came to \$79.46, how much would this coupon save you? What would your bill be after using the coupon?

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18. Complete the following table by filling in the markdown amount and percent based on the given original and marked-down prices.

Item	*Original Price*	*Marked-Down Price*	*Markdown Amount*	*Markdown Percent*
A	$74.99	$49.99		
B	$249.95	$239.99		
C	$8.00	$5.00		

19. Complete the following table by filling in the markdown percent based on the given original and marked-down prices.

Item	*Original Price*	*Marked-Down Price*	*Markdown Percent*
A	$37.99	$27.99	
B	$11,036	$9,999	
C	$19.99	$3.99	

20. A home theater system that normally sells for $299.95 is offered on sale for $189.99. Find the percent markdown.

21. A women's sweatshirt that normally sells for $19.95 is put on sale for $12.95. What is the markdown percent?

22. A mechanic charges $629.99 for a complete brake job. You have a coupon from the phone book for $50 off any service over $250. What percent markdown would using this coupon represent?

C. "Prices" That Aren't Really Prices

23. Zarofire Systems presently employs 1,875 people in its Allegany City headquarters. A story in the local paper quotes the company's president as saying that Zarofire intends to increase its headquarters workforce by 18% next year. How many new jobs are being created?

24. A newspaper advertising executive claims that her paper had 175,000 readers every day last year, and the number will increase by 8% this year. How many daily readers is she claiming for this year?

25. The Marchand Valley Wind Farm had 315 megawatts of installed electric generation capacity in 2003. By 2007, installed capacity had grown by 120%. What was the capacity in 2007?

26. An electronics store sold 18,547 videocassettes last year. This year, it expects sales to decline by 17.5%. How many does it expect to sell this year?

27. The property tax rate in Appaloossa Falls was $56.05 per thousand dollars of assessed value in 2005. In his reelection campaign, the mayor proudly claimed an 8.5% reduction in the property tax rate in 2006. What was the 2006 rate?

28. Full time enrollment at Devonshire Community College was 4,250 full-time students this year, a 23% increase over 5 years ago. What was the full time enrollment 5 years ago?

29. Thanks to conservation measures, Cattarauqua Ginseng Enterprises was able to reduce their office electric usage this year to 935,036 kilowatt-hours. This was a drop of 12.7% from last year's usage. What was the usage last year? How many kilowatt-hours did they save?

30. Attendance at the Campbelton Avengers' minor league hockey games averaged 5,988 this year. This was an increase of 8.4% over last year. What was the average attendance last year?

D. Comparing Markup with Markdown

31. A drugstore marked up a bottle of shampoo 40% over cost. The store manager wants to put the shampoo on sale at cost. What percent markdown should he use to bring the price back to cost?

32. If prices are marked up by 22.5%, by what percent should they be marked down to bring the price back down to cost?

33. A grocery store has been selling potato chips for 15% below the price printed on the bag. The store decides to discontinue this practice and sell the chips for the printed price. What percent of a price increase does this change represent?

34. A photocopier enlarges a copy by 25%. By what percent would you need to reduce a new copy to shrink the image back to its original size?

35. A farmer harvested 18,340 bushels of soybeans this year. This was an 8% drop from last year's harvest. By what percent does his harvest need to grow next year in order to reach last year's levels?

36. The volume control on an amplifier decreases the volume (in decibels) by 20% for each click down. By what percent does it increase the volume with each click up?

E. Grab Bag

37. a. An appliance store paid $193.05 each for a shipment of sewing machines, and marked the price up by 42%. What was the marked-up price?
    b. Three months later, the store decided to try to sell off the remaining machines by marking the price down by 25%. What was the sale price of the machines?

38. A music store offers its employees a 30% discount. If the retail price of a CD is $14.99, how much would an employee pay?

39. A department store uses a 30% markup on men's casual wear. At the end of the summer season, the department manager was told to mark everything in the store down to cost, so he marked down all of the prices by 30%.
    a. What would the marked-down price be for a pair of shorts that originally cost the store $12.53?

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b. What markdown percent should he have used?

40. Dishonest Dave's Discount Den is planning on advertising that "everything in the store is being marked down by 15%!". In reality, though, Dave plans to first increase his prices so that after the 15% markdown the prices are actually the same as they were all along. What percent will he first mark up his prices?

41. A retailer uses a 75% markup when setting the price of a piece of costume jewelry. The retail price is $20.86. What is the cost?

42. A pushcart snack vendor buys 12-ounce sodas for 25 cents each, and sells them for $1. What is the percent markup?

43. An amusement park had 74,006 visitors in the summer of 2005. In 2006, the number of visitors increased by 8.4%. In 2007, though, the number fell by 7.9%. How many visitors were there in 2007?

44. A furniture store marks up its "list" prices by 62.5% over cost, but then marks the list prices down by 20%. What is the markup percent that it is actually using?

45. An Internet service provider normally charges $59.95 per month for its service. It is offering a promotion where, for the first year, you can subscribe for $39.95 per month. What percent discount does this represent?

46. Vladimir makes $23,075 per year. He got a 6% raise. What is his new salary?

47. Last year, Howard's tree service business had gross sales totaling $67,525. This year, sales totaled $82,388. What was the percent increase in the business's sales?

48. A cable company offers high-speed Internet service for $49.95 per month, its most popular cable package for $64.99 a month, and unlimited local and long-distance phone service for $39.49 per month. The company offers a discount package as well; customers who order all three services pay $112.50 per month. What percent of a markdown does this represent?

F. Additional Exercises

49. Automobiles are usually listed for sale at a price suggested by the manufacturer (the **manufacturer's suggested retail price (MSRP)**), also called the **sticker price.** The **invoice price** is the cost of the vehicle listed on the invoice (i.e., the bill) sent to the manufacturer. While most people think of the invoice price as the dealer's cost, in reality most car manufacturers lower the price to the dealer by a **holdback,** a further discount below the invoice price.

    Suppose that a new car has a $29,995 list price, and a $26,792 invoice price. The holdback is 3% of the invoice price.

    a. Calculate the dealer's actual cost for this car.

    b. Suppose that a car dealer offers to sell this car for "1% over invoice." Calculate the amount of markup over invoice and the price at which the dealer is offering the car for sale.

c. At "1% over invoice," what is the percent markup over the dealer's actual cost?

50. When items are sold at auction, the auction house may charge a **buyer's premium,** a **seller's premium,** or both. A buyer's premium is an amount, usually given as a percent of the selling price, that the buyer pays to the auction house above and beyond the price of the item bought. A seller's premium is an amount, again usually a percent of the selling price, of the item's price that is kept by the auction house.

    Suppose that Auberschein Philatelic Auctions conducts a collectible stamp auction, at which a collection of rare U.S. stamps is sold for $14,750. The auction house charges a 3% buyer's premium, and a 5% seller's premium.

    a. How much will the buyer pay, including the premium, for this collection?

    b. How much will the seller receive, after the premium?

    c. How much does the auction house receive on this sale?

    d. What percent of the $14,750 selling price does the auction house receive?

    e. What is the house's percent markup over the cost of the stamps?

8.2 Profit Margin

In the previous section we looked at setting prices with a markup based on cost, and also with markdown from an initial selling price. These are both common business practices, but there is far more to the story than just buying an item for one price and selling it for another. If a clothing store buys a dress for $45, marks it up to $75, and sells it at a marked down price of $65, at first glance it may look like a profitable transaction. At first blush, the dress seems to have produced a $20 profit. The ***gross profit*** on an item is the difference between what the item cost and what it sold for. So in this example, we would say that the dress produced a $20 gross profit.

But of course it is not that simple. In the process of buying and selling the dress, the store had to take on plenty of other ***overhead*** expenses: rent, utilities, salaries, finance costs, advertising expenses, and all the many other costs involved in running a business. When calculating how much prices should be marked up, or how much room the business has to mark them down without losing money, management can't afford to overlook these other costs. Gross profit does not take overhead costs into account. The $20 gross profit is not what the store *really* made by selling the dress, since it overlooks the very real overhead expenses. The ***net profit***, on the other hand, is the profit made after taking into account all of the expenses of doing business. Net profit is a much better representation of the actual amount earned.

We will begin by looking at gross profit, since it is the simpler of the two. Finding the net profit on an individual item is a much more challenging question, since it is not at all obvious how much of the overhead can be attributed to any one specific item.

Gross Profit Margin

The ***profit margin*** is the profit expressed as a percent of the selling price. This can also be simply called the ***margin***; the word *profit* is sometimes omitted. As we would expect, the ***gross (profit) margin*** is based on the gross profit and the ***net (profit) margin*** is based on the net profit. The term *profit margin* is sometimes also used loosely to mean the profit itself—not as a percent—but when we use the term in this book we will always take it to mean a percent.

If we know both the cost and the selling price of an item, we can use this to calculate the gross profit margin on that item. Let's consider the gross profit margin for the example from the opening of this section.

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Example 8.2.1 ***Sally's Fashion Paradise sells a dress that cost $45 for $65. Find the gross profit margin from this sale.***

The gross profit from the sale of the dress is $65 − $45 = $20. Since gross profit is a percent of the selling price, the gross profit margin from this sale is $20/$65 = 0.3077 = 30.77%.

It should be apparent that, whenever we know both the selling price and cost, we can find the gross profit margin in this way. Likewise, if we know the gross profit margin and the selling price, we can determine the gross profit by multiplying.

Example 8.2.2 ***Sally's Fashion Paradise sells women's purses, pricing them with a 35% gross profit margin. If a purse is priced at $72, what is the gross profit in that price?***

The gross profit is (35%)($72) = $25.20. The question did not ask for it, but we can also find that the purse's cost is $72.00 − $25.20 = $46.80.

While a gross profit margin can be calculated for a single item, it can also be calculated for a business as a whole:

Example 8.2.3 ***Last year, sales at Sally's Fashion Paradise totaled $219,540. The cost of the items sold was $147,470. What was the business's overall gross profit margin for the year?***

Total gross profit is $219,540 − $147,470 = $72,070. As a percent of sales, $72,070/$219,540 = 32.83%, so that is the gross profit margin.

Note that this is an overall percent. It does not mean that every single item sold was priced with a 32.83% gross profit margin. Some items undoubtedly would have been sold at a higher gross profit margin, others a lower one. The 32.83% overall margin can be thought of as an average of the gross margins on all the store's sales for the year.

Net Profit Margin

How good is a 32.83% gross profit margin? The answer to this question really depends on how much overhead the dress shop has to cover. If expenses run at 20% of overall sales, a 32.83% gross margin is more than enough to cover that and provide a healthy net profit. On the other hand, if expenses run at 40% of sales, this would not be enough to cover costs.

As discussed earlier in this section, the net profit margin, which takes expenses into account, is a better measure of the business's actual profitability.

Example 8.2.4 ***Last year, Sally's Fashion Paradise had overhead expenses totaling $63,073. Find (a) the expenses as a percent of sales, and (b) the net profit margin.***

(a) $63,073/$219,540 = 28.73%

(b) The gross profit was $72,070. Subtracting expenses from this leaves a net profit of $72,070 − $63,073 = $8,997. As a percent of sales $8,997/$219,540 = 4.10%, so that is the net profit margin.

Notice that we could also have gotten the net profit margin by taking the gross margin and subtracting the expenses percent: 32.83% − 28.73% = 4.10%.

Before moving on, let's work through another example.

Example 8.2.5 ***Two years ago, Sally's shop had sales totaling $153,670. The cost of the goods sold was $118,945, and her expenses totaled $57,950. Find her overall (a) gross profit margin and (b) net profit margin for that year.***

(a) Gross profit = $153,670 − $118,945 = $34,725.

Then $34,725/$153,670 = 22.60%; that is the gross profit margin.

(b) We can calculate the net profit margin in either of two ways.

First approach: As a percent of sales, the expenses were $57,950/$153,670 = 37.71%.

$$22.60\% - 37.71\% = -15.11\%$$

Alternative approach: Net profits were $34,725 − $57,950 = −$23,225.

$$-\$23{,}225/\$153{,}670 = -15.11\%$$

A negative net profit margin indicates that the business lost money, unfortunately not an uncommon occurrence for many businesses. Gross profit margins can also be negative. A negative gross margin would indicate that things were being sold for less than cost.

In both of these examples of net profit margin, we have calculated the margin for the business as a whole. Calculating a net profit margin for an individual item is far less straightforward. The problem here is that there is no absolute way of determining just how much of a business's overhead expenses should be attributed to any given item. How much of Sally's electric bill, or rent, or insurance should be attributed to the dress she sold for $65?

One simple method of doing this is ***proportionate allocation***. This assumes that expenses should be attributed to each item sold in proportion to the overall sales. The following example will illustrate how this can be done.

Example 8.2.6 ***In the year in which the dress sold for $65, the total sales were $219,540 and expenses were $63,073. If expenses are allocated in proportion to sales, how much of the store's expenses is attributable to that dress?***

We first look at this dress in relation to the overall sales for the year. $65/$219,540 = 0.02961%. (We take this out to more than two decimal places because it is such a small percent.) Since the dress was 0.02961% of sales, we attribute to it 0.02961% of the expenses. Then (0.0002961)($63,073) = $18.67.

We could avoid the trouble of the small percentage by avoiding explicitly writing it out. We obtained this percent by dividing the dress sale by total sales; then we multiplied it by the overall expenses. Rather than write out the percent itself, we could simply have eliminated that intermediate step and gone directly to the multiplication:

$$(\$65/\$219{,}540)(\$63{,}073) = \$18.67.$$

A third option may be the most appealing. We previously calculated that the shop's overall expense percent was 28.73%. Proportionate allocation assumes that this same percent applies to all sales, so (28.73%)($65) = $18.67.

While this sort of allocation is attractively straightforward, it is probably overly simplistic for most business situations. Some items account for more overhead than others. A grocery store might think it entirely appropriate to attribute more of its utility bills to refrigerated and frozen foods than canned goods, for example. A department store might find it appropriate to attribute a larger percentage of its rent to appliances, which take up a lot of floor space, than to jewelry. And so on and so forth. Realistically speaking, the determination of net profit margin on any given item mainly depends on how the business's management deems it appropriate to allocate expenses. While this is an interesting subject to investigate, it is primarily a matter of accounting rather than mathematics and so we will not address it further here.

Markup Based on Selling Price

However a business determines that expenses should allocated, once the expenses as a percent of sales are known management can determine a gross margin which will be adequate to meet expenses and profitability goals. Suppose, for example, that Sally determines that she needs to set her dress prices by using a 35% gross margin. Of course, it is too late to fit that target on the dress from our first example, but we can look ahead and ask what price she would need to charge for a dress costing $45 to reach this target.

Determining a price by using a target gross margin is called ***markup based on selling price***, in contrast to markup based on cost. It may seem at first as though this will work

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in pretty much the same was as markup based on cost. There is a mathematical challenge here, though. If we know the item's cost, and if we know our markup percent based on cost, calculating the selling price is fairly straightforward. Profit margin, though, is a percent *of the selling price*. Obviously we don't know the selling price before we know the selling price!

We'll need to be a bit sneaky to get around this, but we can. Here is the idea: if 35% of the selling price is gross profit, we can conclude that the remaining 100% − 35% = 65% must be the cost of the item. Before taking this any further, we'll generalize this to a formula:

FORMULA 8.2.1
Markup Based on Selling Price

$$C = SP(1 - r)$$

where
C represents the item's COST
SP represents the SELLING PRICE
and
r represents the gross PROFIT MARGIN

We can now put this formula to work to solve the question at hand.

Example 8.2.7 ***Determine the selling price of an item costing $45 in order to have a 35% gross profit margin.***

Applying the formula:

$$C = SP(1 - r)$$
$$\$45 = SP(1 - 0.35)$$
$$\$45 = SP(0.65)$$

To find the selling price, we divide both sides by 0.65 to get:

$$SP = \$69.23$$

Notice that the 0.65 that we wound up with on the right side of this equation is the 65% that we said must be the percent cost of the item. Remember that expressed as a decimal 100% is 1, and so the (1 − r) in this formula serves the purpose of subtracting the gross profit margin from 100% to get the percent cost.

Markup based on selling price may also be used even when the idea of gross profit margin is not explicitly what is driving the pricing. Another example will serve to illustrate this.

Example 8.2.8 ***A cooperative market allows its members to place special orders for items they want to buy in bulk. The price the member pays is based on an 8% markup on selling price. Lynne ordered a case of protein bars, for which the market's cost was $24.17. How much will she pay for this order?***

$$C = SP(1 - r)$$
$$\$24.17 = SP(0.92)$$
$$SP = \$26.27$$

A Dose of Reality

Before getting too carried away with this method of pricing, a few cautions are in order. You should absolutely not draw the conclusion from this section that in the real world prices are set by determining a desired gross profit margin and then automatically applying that margin. Whether we think of prices as based on a markup based on cost or based on a markup based on selling price, the fact of the matter is that a retailer has to deal with the fact that the prices that they charge must be prices that their customers will pay. Based on your costs and profit targets you may calculate that your hardware store should sell a particular model lawnmower for $375, but if other stores are selling the same model for $300 you probably won't have too much luck selling it for this price. On the other hand, if you've calculated that you should sell a chain saw for $99 but your competitors are selling it for $129, you might want to seize the opportunity to raise your price and claim a higher percentage profit.

It is common practice to offer some items at very low margins because of strong competition, or even as "loss leaders" sold below cost to attract customers (who hopefully will also buy other, higher margin items). Other items, for which competition is less fierce and customers are less price sensitive, may be offered for sale with very high profit margins. Successful pricing is a matter of balancing low and high profit margin items to successfully compete for business while earning a good overall profit margin. Figuring out how to manage that balancing act, though, requires good judgment and a solid understanding of the business and its competitive environment.

EXERCISES 8.2

Percents in final answers should be rounded to two decimal places.

A. Gross Profit Margin

1. Complete the following table by filling in the gross profit margin based on the given costs and selling prices:

Item	*Cost*	*Price*	*Gross Profit Margin*
A	$39.03	$49.99	
B	$2.44	$4.88	
C	$73.00	$125.00	
D	$32,591	$50,000	

2. Complete the following table by filling in the gross profit and gross profit margin based on the given costs and selling prices:

Item	*Cost*	*Price*	*Gross Profit*	*Gross Profit Margin*
A	$7.62	$10.95		
B	$31.43	$57.50		
C	$108.23	$204.99		
D	$0.52	$0.75		

3. Eddie's Bike Shop sold a bike for $749.99, for which Eddie's cost was $493.57. Calculate the gross profit and gross profit margin for this item.

4. A grocer sells milk for $2.49 a gallon. The grocer's cost is $2.37 per gallon. Find the gross profit margin.

5. Ludd's Electronics had sales totaling $1,547,919 last year. The cost of the goods sold was $993,225. Find the gross profit margin.

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6. A used car dealer had sales of $3,588,034 in 2006. The cars they sold cost them $2,803,111. Calculate the dealer's gross profit margin for 2006.

7. StuffCo Stores uses a 18% gross profit margin for the small home appliances. A toaster is priced at $17.95. Find the gross profit and the cost for this toaster.

8. A consignment shop has a 60% gross profit margin. A children's sweatshirt is offered for sale for $3.75. Find the shop's gross profit for this sweatshirt and also the shop's cost.

9. International Consolidated Homestyle Industries reports that its gross profit margin was 7.39% last year, on sales totaling $85,873,002. Calculate the company's gross profits for the year.

B. Net Profit Margin and Proportionate Allocation

10. A used car dealer had sales of $3,588,034 in 2006. The cars the dealer sold cost it $2,803,111. Expenses totaled $518,046. Calculate the dealer's net profit margin for 2006.

11. Ludd's Electronics had sales totaling $1,547,919 last year. The cost of the goods sold was $993,225. Expenses totaled $385,997. Find the net profit margin.

12. In 2007, Eddie's Bike Shop had sales totaling $835,001. The cost of the goods sold was $624,023 and expenses totaled $287,002. Calculate the shop's (a) gross profit margin and (b) net profit margin for 2007.

13. OldeTyme Digital Sales had expenses of $87,559 in 2006, and sales totaled $204,013. The cost of the items sold totaled $177,402. Find the 2006 (a) gross profit margin and (b) net profit margin.

14. In 2007 StuffCo Stores had an overall gross profit margin of 23.9%, and expenses amounted to 17.4% of sales. What was the net profit margin?

15. Canalside Tapas had a gross profit margin of 37.9% last year, and expenses were 42.9%. Determine the net profit margin.

16. In 2006, Eddie's Bike Shop's expenses were 31.2% of sales. Assuming that the management allocates expenses in proportion to sales, what would be the expenses attributed to a bike that sold for $439.95?

17. Ludd's Electronics had sales totaling $1,547,919 last year. Expenses totaled $385,997. Assuming that management allocates overhead expenses in proportion to sales, calculate the overhead for a DVD player that sold for $89.99.

C. Markup Based on Selling Price

18. Complete the following table by filling in the retail price based on the given costs and gross profit margins:

Item	*Cost*	*Gross Profit Margin*	*Retail Price*
A	$8.45	50%	
B	$34.99	11.25%	
C	$97.43	25%	
D	$5.92	2.5%	

19. Complete the following table by filling in the retail price and markup amount based on the given costs and gross profit margins:

Item	*Cost*	*Gross Profit Margin*	*Retail Price*	*Markup Amount*
A	$5.55	62.5%		
B	$64.02	21%		
C	$17.03	22.95%		
D	$101.01	31.52%		

20. Complete the following table by filling in the retail price, markup amount, and markup percent *on cost*, given each item's cost and gross profit margin.

Item	*Cost*	*Gross Profit Margin*	*Retail Price*	*Markup Amount*	*Markup Percent*
A	$7.01	40%			
B	$3.24	2.75%			
C	$91.07	33.33%			

21. Complete the following table by filling in the retail price, gross profit amount, and markup percent *on cost*, given each item's cost and gross profit margin.

Item	*Cost*	*Gross Profit Margin*	*Retail Price*	*Markup Amount*	*Markup Percent*
A	$62.50	11%			
B	$15.44	35%			
C	$439.02	41.07%			

22. StuffCo Stores uses a 24% gross profit margin on men's clothing. What would the retail price be for a polo shirt that cost $8.42?

23. StuffCo Stores uses an 8% gross profit margin on grocery items. Determine its retail price for a box of crackers for which Stuffco's cost is $1.73.

24. Find the retail price of an item for which the cost is $583.02, assuming a gross profit margin of 22.5%.

25. A consignment shop uses a 55% gross profit margin. I place a men's suit on consignment there, telling the owner that I will not accept less than $20 for it. (The amount I receive is the shop's cost for the item.) What is the lowest price the shop can sell this item for?

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26. OldTyme Digital Sales prices laptop computers with a 18% markup based on selling price. Determine the retail price and gross profit for a laptop that cost OldTyme $449.38.

27. Kelly's plumbing supply business uses a 35% markup based on selling price. Determine the selling price and gross profit on piping that cost $47.03.

D. Grab Bag

28. Find the retail price of an item whose wholesale cost is $41.06 if the markup based on selling price is 32.5%.

29. In 2007 AnyCorp had sales totaling $179,303,925. Its total cost for the goods sold was $122,094,008. Expenses totaled $48,033,033. Determine the gross profit margin and net profit margin.

30. A rare stamp dealer had a gross profit margin of 19.3% this year. His net profit margin was 7.1%. Sales totaled $408,055. Determine the gross profit and net profit.

31. I own a greenhouse business. The gross profit margin last year was 41.5%. Expenses as a percent of sales came to 43.1%. What was my net profit margin for this business?

32. A hardware store sold a ratchet set for $19.97. Its cost was $12.43. Determine the gross profit margin on this sale.

33. A cell phone store sold a headset for $12.99 that cost the store $3.17. What was the gross profit? What was the gross profit margin?

34. Find the net profit margin for a business if the gross profit margin is 17.5% and expenses are 9.2% of sales.

35. A grocer uses a 9.5% markup based on selling price for produce. The grocer's cost for organic carrots is $1.18 per pound. How much will he charge for 1 pound of organic carrots?

36. Canalside Tapas' manager is projecting that sales next year will total $575,000 and the cost of the goods sold will amount to $280,000. What is the largest amount that expenses can total and still give the business a positive net profit margin?

37. An appliance dealer had expenses of $117,502 on sales of $782,035 last year. If she allocates overhead expenses proportionately based on selling price, what would be the overhead attributed to a freezer that she sold for $449.99?

38. Suppose that a pharmacy uses a 27% markup based on cost for all prescription medications. A 30-day supply of a cholesterol drug costs it $82.03 wholesale. Calculate the pharmacy's gross profit from a 30-day supply of this drug.

39. Company A has a gross profit margin of 24.81%. Company B has a gross profit margin of 31.92%. Both companies had $1,000,000 in sales last year. Which company was more profitable?

40. A travel agent keeps 7½% of the retail price of cruises that she sells as a commission. If she sells a cruise for $2,995, calculate her gross profit.

E. Additional Exercises

41. Determine expenses as a percent of sales if the gross profit margin is 22.1% and the net profit margin is −4.7%.

The terms used in Exercises 42 to 43 are explained in Exercise 49 of Section 8.1.

42. A car dealer offers to sell cars for 1% over invoice. The holdback is 3%. What is the dealer's gross profit margin?

43. A car dealer believes that expenses total 3.4% of sales. Holdback is 2.85%. Determine the minimum percent markup over invoice needed to break even.

8.3 Series and Trade Discounts

Trade Discounts

Merchants are normally free to set prices as they see fit, basing their pricing decisions on costs, profit targets, and competition.[1] Still, many manufacturers do set suggested prices for their products. The suggested price for an item often called its ***list price***, or ***manufacturer's suggested retail price (MSRP)***. When a product has a list price, it is not uncommon for the item to be sold to a merchant on the basis of a discount to the list price. This is known as a ***trade discount***.

The mathematics involved here is quite similar to work we have already done. In fact, though the context may be different, mathematically a trade discount is essentially the same thing as a markdown.

Example 8.3.1 Ampersand Computers bought 12 computers from the manufacturer. The list price for the computers is $895.00, and the manufacturer offered a 25% trade discount. How much did Ampersand pay for the computers?

As with markdown, we can either take 25% of the price and subtract, or instead just multiply the price by 75% (found by subtracting 25% from 100%). The latter approach is a bit simpler: (75%)($895.00) = $671.25 per computer. The total price for all 12 computers would be (12)($671.25) = $8,055.

[1]There are exceptions to this, though. In some states, it is not legal to sell wine or liquor at a discount, for example. Also, some manufacturers do not allow their products to be sold at a discount.

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Even though it is more mathematically convenient to multiply by 75%, there are sometimes reasons to work things out the longer way. When the manufacturer bills Ampersand for this purchase, it would not be unusual for it to show the amount of this discount as a separate item. (The bill is called an ***invoice***, and the net cost for an item is therefore sometimes called the ***invoice price***.) In addition, the manufacturer may add charges for shipping or other fees on top of the cost of the items purchased (after the discount is applied). The invoice might look something like this:

International Difference Engines Invoice No. 1207
Box 404
Marbleburg, North Carolina 20252

INVOICE

Sold To:

Ampersand Computers
4539 North Henley Street
Olean, NY 14760

Date: May 28, 2007
Order #: 90125
Shipped: May 17, 2007

Quantity	Product #	Description	MSRP	Total
12	87435-G	IDE-Model G Laptop	$895.00	$10,740.00
			Subtotal	$10,740.00
			LESS: 25% discount	($2,685.00)
			Net	$8,055.00
			PLUS: Freight	$350.00
			Total due	$8,405.00

The discount may sometimes be written in parentheses (as it is in the example above) because this is a commonly used way of indicating a negative or subtracted number in accounting. Other times, it may be written without the parentheses, it being understood that a discount should be subtracted.

It may be especially convenient to calculate the discount separately as shown in the following example where multiple different items are sold.

Example 8.3.2 ***Samir's House of Gadgets placed an order for 500 thingmies (list price $4.95), 350 jimmamathings (list price $8.95), and 800 hoozamawhatzits (list price $17.99). The manufacturer offers a 27½% trade discount, and includes shipping in its prices. Find the total due on the invoice for this order.***

For convenience, we will set this up as a table, similar to the invoice shown above:

Quantity	Product #	Description	MSRP	Total
500		Thingmies	$4.95	$2,475.00
350		Jimmamathings	$8.95	$3,132.50
800		Hoozamawhatzits	$17.99	$14,392.00
			Subtotal	$19,999.50
			LESS: 27.5% discount	($5,499.86)
			Net	$14,499.64
			PLUS: Freight	$00.00
			Total due	$14,499.64

The disadvantage of doing things in this way is that we do not directly see the net cost of each item. However, if we need to know this, we can readily calculate it.

Example 8.3.3 ***What is the net cost for each jimmamathing in the previous example?***

The net cost is 100% − 27.5% = 72.5% of the list price; (72.5%)($8.95) = $6.49.

A formula may not really be necessary, but if desired we can generalize what we have been doing to a formula:

FORMULA 8.3.1
Trade Disounts

$$NP = LP(1 - d)$$

where
NP represents the NET (DISCOUNTED) PRICE
LP represents the LIST PRICE
and
d represents the PERCENT DISCOUNT

Example 8.3.4 ***Rework Example 8.3.3, using Formula 8.3.1.***

$$NP = LP(1 - d)$$
$$NP = \$8.95(1 - 0.275)$$
$$NP = \$6.49$$

As we observed earlier, trade discounts are very similar to markdown, and in fact this formula is likewise very similar to the markdown formula. While it is probably not necessary to use this formula in every case, it may be helpful to have one available in some situations.

Example 8.3.5 ***Samir realized the he forgot to order 400 doohickeys. He called the manufacturer and was given a price of $2,652 for them. He did not ask the list price, but realizes that he needs to know it now. What is the list price for a doohickey?***

The net price for each doohickey is $2,652/400 = $6.63. To find the list price, we can use the formula with a bit of algebra:

$$NP = LP(1 - d)$$
$$\$6.63 = LP(1 - 0.275)$$

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$$\$6.63 = LP(0.725)$$
$$LP = \$9.14$$

Markdown and trade discounts are not, however, entirely the same thing. It is worthwhile to compare trade discounts to gross profit margins and markup based on cost.

Example 8.3.6 ***A manufacturer offers a 30% trade discount. If the merchant sells items at a 10% discount to list, what is the gross profit margin? What is the markup based on cost?***

Since we are working with percents, we can assume any list price that we want. We will use $100 in this case for convenience as we have in the past.

The cost would be (70%)($100) = $70.00
The selling price would be (90%)($100) = $90.00.
To find the gross profit margin:

$$C = SP(1 - r)$$
$$\$70 = \$90(1 - r)$$
$$0.7777777 = 1 - r$$
$$0.7777777 + r = 1$$
$$r = 0.2222223 = 22.22\%$$

To find the markup based on cost:

$$\$90 = \$70(1 + r)$$
$$1.2857143 = 1 + r$$
$$r = 0.2857143 = 28.57\%$$

Note that if the items were sold for list price, the trade discount would be the same as the gross profit margin. It is not unusual, though, for a manufacturer to set list prices that are higher than most retailers will actually charge; this practice enables retailers to make their usual prices sound more attractive by describing them as discounted below the list price.

Series Discounts

Series discounts are multiple discounts applied to a price in succession. Sometimes, a manufacturer may offer multiple trade discounts. For example, a company might normally offer a 25% trade discount, but, during a special promotion or to match a competitor's pricing, might offer an additional 15% discount. This series discount might be denoted as "25%, 15%," or sometimes as "25/15" (though that second notation can be confusing since the "/" looks like division).

As we have seen before, despite appearances it is incorrect to conclude that successive discounts of 25% and 15% are equivalent to a single discount of 40%. In fact, they are not.

Example 8.3.7 ***The list price for a herbal weight loss supplement is $39.95. The manufacturer normally offers a 25% trade discount, but during a special promotion it offers an additional 15% discount. Find the net price for this item.***

The first, 25% discount, reduces the price to (75%)($39.95) = $29.96. The second discount further reduces the price to (85%)($29.96) = $25.47.

This could also be calculated more simply as (75%)(85%)($39.95) = $25.47.

The single discount equivalent to a series of discounts is referred to as the ***single equivalent discount***.

Example 8.3.8 ***Find the single equivalent discount for successive 25% and 15% discounts.***

For convenience, we will work from an assumed price of $100. These discounts would reduce that price to (75%)(85%)($100) = $63.75. This is a total discount of $100 − $63.75 = $36.25. As a percent of the list price, this works out to $36.25/$100 = 36.25%.

Once we have calculated the discounted price, we could also find the single equivalent discount, using the formula:

$$NP = LP(1 - d)$$
$$\$63.75 = \$100(1 - d)$$
$$0.6375 = 1 - d$$
$$-0.3625 = -d$$
$$d = 36.25\%$$

As mentioned above, we cannot just add the percentages and conclude that this equates to a 40% discount. The reason is that the first 25% discount applies to the original price, while the second 15% of a percent of the reduced price. Since they are not percents of the same thing, they cannot be added.

Nonetheless, discounts are sometimes presented in this incorrect way anyway. If a store advertises "take an additional 20% off of prices that have already been discounted by 50%" they *may* mean successive discounts of 50% and 20% (equivalent to a single discount of 60%), or they may mean discounts totaling 50% + 20% = 70% off the original prices.[2]

Additional discounts are also often offered in response to competition.

Example 8.3.9 ***You work for a company that manufactures photovoltaic (solar) panels. Your company's 70-watt panel lists for $295, and you offer an 18% trade discount. A competing company lists its 70-watt panel for $305 and offers a 25% trade discount. Ingraham Solar Systems, one of your main customers, tells you that it is considering switching to your competitor's panels because of their lower cost. What additional discount do you need to offer to match your competitor's price?***

You are selling your panels for (82%)($295) = $241.90. Your competitor is selling for (75%)($305) = $228.75. To further discount your panels to match this price, you would need to drop your current net price by $241.90 − $228.75 = $13.15. As a percent of $241.90, this works out to $13.15/$241.90 = 5.44%.

Alternatively, we can work this out by using the formula treating the current $241.90 as the list price and calculating the discount to get the price to $228.75.

$$NP = LP(1 - d)$$
$$\$228.75 = \$241.90(1 - d)$$
$$0.9456387 = 1 - d$$
$$d = 5.44\%$$

Cash Discounts

A merchant may require customers to pay for their orders in advance, before the merchandise is shipped. It is not unusual, though, for a seller to sell items ***on credit***, especially to customers with whom the seller has (or hopes to build) a good relationship. When merchandise is sold on credit, the seller sends the buyer an invoice for the goods, but does not require the invoice be paid prior to shipment.

Even when credit is extended for a sale, though, the seller naturally still wants to be paid promptly. Late payments may incur interest charges. To encourage prompt payment, many sellers offer the carrot of a discount in addition to the stick of interest charges. A ***cash discount*** is a discount offered for prompt payment. This discount usually would usually apply only to the price of the goods themselves, not to shipping or other added-on charges.

The period of time during which the cash discount is available to the buyer is called the ***discount period***. The cash discount offered will often be denoted on an invoice with the percent discount offered followed by a slash and then the length of the discount period. For example, "3/10" would indicate a cash discount of 3% is offered for a 10-day period. If a seller offers a more complicated deal, this can be indicated similarly. "3/10, 1/30" indicates that a 3% discount if offered for payment within 10 days, and a 1% discount if the bill is paid within 30 days (from the start of the discount period; in this case, it would mean between the 11th and 30th day).

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[2]While the latter is technically incorrect, I have personally never seen fit to stand on principle and demand a smaller discount.

The period of time allowed before the payment is considered late may also be indicated. "2/15, n/30" indicates that a 2% cash discount is offered for payment within 15 days, and that between the 16th and 30th day the payment is considered on time, but no discount is offered. The n stands for "net." After the 30th day, payment would be considered late and late payment penalties and/or interest charges may be applied.

There are several different methods that are commonly used to determine the discount period. Some of the more commonly used ones are summarized in the table below:

Dating Method	Discount Period Begins:	Abbreviation	Example
Ordinary	On the invoice date	None. Assumed if not otherwise indicated.	"2/10, n/30"
End of month	On the end of the month following the invoice date	EOM	"2/10, n/30, EOM"
Proximo	On the end of the month following the invoice date	PROX	"2/10, n/30, PROX"
Receipt of goods	When the buyer receives the merchandise	ROG	"2/10, n/30, ROG"
Postdated	As of date indicated on the invoice	AS OF	"2/10, n/30, AS OF 3/7/08"

"End of month" and "proximo" dating are essentially the same. Proximo is a Latin term that has been used traditionally to indicate this dating method. EOM is probably the more modern usage, though both terms are still in use.

Example 8.3.10 ***Ned's Furniture Galaxy ordered a shipment of dining room sets. The invoice was dated March 14, 2007, and the order arrived on March 28, 2007. The manufacturer offers a 2½% cash discount for payment within 10 days. To take advantage of this discount, when must Ned make payment if the dating method is (a) ordinary dating, (b) end of month, (c) receipt of goods, or (d) postdated as of May 1, 2007?***

(a) With ordinary dating, the clock starts ticking on the invoice date, March 14. Answer: March 24, 2007.

(b) With EOM dating, the clock starts ticking at the end of March. Answer: April 10, 2007. (Note: Even though the discount period clock doesn't start ticking until the end of March, if Ned pays before the end of the month he still gets the cash discount.)

(c) The furniture arrived on March 28, so that's when the period begins. Answer: April 7, 2007.

(d) The discount period begins on May 1, 2007. Answer: May 11, 2007. (Just as with EOM, Ned receives the discount even if he pays before the "as of" date.)

Payments sent by mail are customarily considered to have been made on their postmark date, though this can not always be assumed. Some sellers consider the payment to have been made only when it is actually received. There may be other special rules used by a seller that deviate from the general methods explained above. If a seller's terms are unclear, naturally the best course of action is to request clarification.

Sellers sometimes also use another variation on these methods, known as ***extra*** dating. Extra dating is really nothing more than an extension of the discount period. An invoice marked "2/15–45X" would indicate that ordinary dating is being used, and that the usual 15-day discount period is being extended by an additional 45 days, for a total of 60 days. Such an offering would be equivalent to "2/60"; the distinction is really just a matter of emphasis that the 45 days are "extra," above and beyond what would normally be offered. Extra dating may be used, for example, by a seller trying to clear out excess inventory by encouraging merchants to place orders sooner than they otherwise might.

Example 8.3.11 ***Roy's Appliance Hut placed an order for 10 refrigerators that list for $775, and 8 front loader washing machines listing for $995. The manufacturer offers trade discounts of 20% and 20%. The delivery (i.e., shipping) cost is $185. The invoice was dated April 7, 2007, and stated "terms: 2/15, EOM." If Roy makes payment on April 30, 2007, how much must he pay?***

We first calculate the net cost of the items before considering any cash discount.

Quantity	Product #	Description	MSRP	Total
10		Refrigerators	$775	$7,750.00
8		Washers	$995	$7,960.00
			Subtotal	$15,710.00
			LESS: 20% discount	($3,142.00)
			Net	$12,568.00
			Additional 20% discount	($2,513.60)
			Subtotal	$10,054.40
			Shipping	$185.00
			Total due	$10,239.40

According to the terms offered, Roy is entitled to a 2% discount if he pays before May 15, 2007. He does, so the discount applies, and he will pay 98% of the calculated net cost. (98%)($10,054.40) = $9,853.31. The shipping cost is then added on top of this to arrive at the total of $9,853.31 + $185.00 = $10,038.31.

What if an invoice is partially paid, but not paid in full? Different companies treat partial payments differently. Some may offer the discount on the portion of the invoice paid within the period, others may offer the discount only if payment is made in full. (See the Additional Exercises at the end of this section for a partial payment example.)

EXERCISES 8.3

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Except where stated otherwise, assume that freight or shipping charges are not part of your calculation.

A. Trade Discounts

1. Complete the following table by calculating the total net amount for each order, given the total list price, shipping/freight charge, and trade discount percent.

Order	Total List Price	Trade Discount %	Freight	Net
A	$173,095	24%	$750	
B	$84,913	16.25%	$500	
C	$22,916	35%	$200	
D	$1,039,035	7.25%	included	

2. Complete the following table by calculating the total net amount for each order, given the total list price, shipping/freight charge, and trade discount percent.

Order	Total List Price	Trade Discount	Freight	Net
A	$22,500	8½%	Included	
B	$717,035	45%	$3,500	
C	$301,025	15.5%	$1,475	
D	$9,036	62.75%	Included	

3. The list price for a new computer system is $1,295.99. The manufacturer offers a trade discount of 32%. Ampersand Computers ordered 20 of these systems. Find the net price for each system and the total amount for this order.

4. The list price for a college biochemistry textbook is $117.50. The publisher offers college bookstores a 22.5% trade discount. If the University of Erie bookstore ordered 182 copies of the book for the fall semester, find the net price per book and the total for this order.

5. A fitness equipment store ordered 40 exercise bikes (list price $479.99), 65 elliptical trainers (list $549.99), and 23 rowing machines (list $375.99). The manufacturer offers a trade discount of 35%, and charges $250 for freight on all orders. Determine the total amount for this order.

6. An electronics retailer has just placed an order for a shipment of DVDs. Each DVD in the Hot New Classics line lists for $23.99, the Perennial Favorites line lists for $18.99 each, the Old-E-Good-E line lists for $12.99 each, and the Alan Smithee Selections line lists for $6.99 each. The manufacturer offers an 18% trade discount, and charges $0.25 shipping for each DVD. If the order includes 400 Hot New Classics, 575 Perennial Favorites, 815 Old-E-Good-E, and 1,750 Alan Smithee Selections, find the total amount of the invoice.

7. A dehumidifier manufacturer offers Hal's Hardware Hut a 35% trade discount. If the net price is $90.32 for a dehumidifier, what is the list price?

8. YummieNuttie Candy bars offers a 42% trade discount for their candy bars. Shipping is free. Jed's Snack Hut ordered 2,000 YummieNuttie bars for a total of $1,496.40. What is the list price for each candy bar?

9. StuffCo Stores sells greeting cards for 15% below list price. The greeting card manufacturers give StuffCo a 45% trade discount.
   a. Determine the retail price for a card that lists for $3.99.
   b. What is StuffCo's gross profit margin on greeting cards?
   c. What is StuffCo's markup percent based on cost?

10. A grocery store chain receives a 7.5% trade discount from Ejner's Authentic Amish Salsa Company. The store discounts the salsa 3.5% below list price.
    a. What is the grocer's gross profit margin on this product?

b. What is the grocer's markup based on cost for this product?

11. A new car lists for $29,995. The invoice price that the manufacturer charges the dealer is $26,348. What is the percent trade discount?

B. Series Discounts

12. A leather belt that was originally priced at $24.95 is put on sale for 15% off. Two weeks later, the store cuts the price by an additional 25%. Find the price of the belt.

13. A solar-powered radio normally lists for $79.95. A retailer normally sells this radio for 10% discount to list. During a clearance sale, the price is reduced by an additional 25%. Find the clearance sale price for the radio.

14. Find the single discount equivalent to successive 20% and 30% discounts.

15. a. Find the single discount equivalent to successive 15% and 35% discounts.
    b. Find the single discount equivalent to successive 35% and 15% discounts.

16. Eddies' Bike Shop sells all bikes for 12% below list. In September, the shop cut its prices by an additional 15%. In October, it has an end-of-season clearance where it cuts prices on all bikes in stock by an additional 25%. Suppose that a bike lists for $949.95. What would the price be for this bike at the end of season clearance? What is the single equivalent discount?

17. As mentioned in this section, successive discounts are sometimes stated in a way that is not mathematically correct. Suppose that Hausdorff's Department Stores puts out a flyer advertising that "we've cut all our prices by 40%, and *this Saturday only, take an additional 25% off!*"
    a. A sweater normally sells for $89.95. What two different prices might you be charged if you buy this sweater at Hausdorff's Saturday sale?
    b. Which price is the mathematically correct one?

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C. Cash Discounts

18. An invoice is dated July 21, 2007, and the order arrived on August 2, 2007. The manufacturer offers a discount for payment within 20 days. To take advantage of this discount, when must you make payment if the dating method is (a) ordinary dating, (b) end of month, (c) receipt of goods, or (d) postdated as of September 1, 2007.

19. Charlie's Anvil Sales offers its customers a 3% discount for payment made within 10 days. An invoice is dated January 19, 2008, and the order arrived on January 22, 2008. To take advantage of this discount, by when must you make payment if the dating method is (a) ordinary dating, (b) end of month, (c) receipt of goods, or (d) postdated as of February 6, 2008.

20. An invoice totaling $939.06 (excluding shipping charges) dated September 30, 2008, is marked "4/10, 2/15, n/45 ROG." Shipping charges are $24.00. The shipment arrived on October 16, 2008.

    a. If the bill is paid on October 18, 2008, find the total due.

    b. If the bill is paid on November 7, 2008, find the total due.

    c. If the bill is paid on November 30, 2008, find the total due.

    d. What is the last day the bill can be paid before it is considered late?

21. A seed packet wholesaler sells flower packets that list for $1.79 and vegetable packets that list for $1.39. They offer a 24% trade discount and charge $45 shipping per order.

    A garden store placed an order for 850 flower seed packets and 2,400 vegetable seed packets. The invoice was dated February 17, 2007, and the shipment arrived February 24, 2007, and was marked "2/15, n/30 EOM. 5% surcharge for late payment."

    a. Calculate the total due, assuming payment is made on March 1, 2007.

    b. Calculate the total due, assuming payment is made on March 17, 2007.

    c. Calculate the total due, assuming payment is made on March 29, 2007.

D. Grab Bag

22. Find the single discount equivalent to successive discounts of 30%, 20%, and 15%.

23. Find the gross profit margin and the percent markup based on cost for a product sold at list price, assuming a trade discount of 25%.

24. Northbrand 20-inch flat panel TVs list for $349.95, and the company offers a 17% trade discount. Southbrand, a competing company, offers a comparable television with the same list price, but with a 24% trade discount. Bigg Retail Stores tells Northbrand that it needs to match Southbrand's price, or else it risks losing the retailer's business. What additional discount does Northbrand need to offer to do this?

25. A jar of gourmet organic applesauce was originally priced at $6.99. This product did not sell well, so the grocer marked it down by 20%. Still, it did not sell well, so the grocer put it on clearance and marked the price down an additional 70%. Find the clearance price.

26. A cheese company sells three different types of brick cheese. Sharp cheddar costs $4.79 per pound, mild cheddar costs $3.99 per pound, and pepper jack costs $4.35 per pound. They do not charge shipping for orders over 500 pounds.

    A specialty food market placed an order for 700 pounds of sharp cheddar, 500 pounds of mild cheddar, and 350 pounds of pepper jack. The invoice was dated August 3, 2007, and the shipment arrived August 5, 2007, marked "1/10, n/30. 2% per month surcharge for late payment."

    a. Calculate the total due, assuming payment is made on August 10, 2007.

    b. Calculate the total due, assuming payment is made on August 29, 2007.

27. Using the information included, calculate the missing amounts from the total column in the invoice shown below.

International Difference Engines
Box 404
Marbleburg, North Carolina 20252

Invoice No. 1338

INVOICE

Sold To:

Ampersand Computers
4539 North Henley Street
Olean, NY 14760

Date: August 8, 2007
Order #: 271828
Shipped: August 9, 2007

Quantity	Product #	Description	MSRP	Total
8	82135-H	IDE-Model H Laptop	$695.00	(a)
15	81101-K	IDE-Model K Laptop	$1,175.00	(b)
			Subtotal	(c)
			LESS: 19% discount	(d)
			Net	(e)
			PLUS: Freight	$450.00
			Total due	(f)

28. A sofa lists for $1,999.00. The manufacturer offers a 37.5% trade discount. A furniture store offers a 10% discount off list, and during a special promotion discounts prices by an additional 10%.
    a. Find the sale price of the sofa.
    b. Find the gross profit on the sofa based on the sale price.
    c. Find the gross profit margin on the sofa based on the sale price.

29. Find the single discount equivalent to successive discounts of 18% and 22%.

30. A calculator manufacturer offers a 16% trade discount. The invoice for a shipment of 200 calculators totals $13,431.60. What is the list price for each calculator?

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31. A pharmacy receives a 27.5% trade discount on synthetic insulin, and sells it for 8.5% below list price. The pharmacy's retail price for a bottle of synthetic insulin is $67.43.
    a. Find the list price.
    b. Find the pharmacy's cost per bottle.
    c. Find the markup percent based on cost.
    d. Find the gross profit margin.

E. Additional Exercises

32. In Exercise 15, you were asked to find the single equivalent discount for successive discounts of 15% and 35%. Then the order was reversed, but the change did not affect the answer. Does the order of the discounts ever matter? Either give an example where the single equivalent discount comes out differently when the order is switched, or provide a convincing explanation of why order will never matter.

33. In the text of this section, we noted that sometimes series discounts are presented incorrectly. An advertisement that says "take an additional 20% off our prices already cut by 50%" may mean a total 70% discount, even though successive discounts of 50% and 20% are equivalent to a 60% discount.

    From the customer's point of view, this isn't anything to complain about; you are getting a larger discount than you would otherwise. However, the store is understanding the second discount. If the mathematically correct value for the second discount is used, it would be a larger number.

    Suppose that for a sale, prices being charged are actually discounted by 70%. The prices were already discounted by 50%. What is the correct percent for the second discount?

34. Suppose that an invoice totals $12,350, excluding shipping costs. If the bill is paid within 15 days, there is a 2% discount. If the payment is not made in full within 15 days, the seller will still apply the discount to the portion of the invoice that is. If you pay $7,000 within the discount period, how much will you owe for the remaining bill?

35. Suppose that you have an invoice for $40,000. The bill is due in 60 days. If you pay within 15 days there is a 2% discount. What is the simple interest rate equivalent to taking advantage of that discount offer?

8.4 Depreciation

The dollar value of anything that can be owned will change as time goes by. Some things, like real estate and collectibles, are expected (or at least hoped) to go up in value over time. We call that increase in something's dollar value price ***appreciation***. Other things, though, become less valuable with use and the passing of time. Computers and electronics become obsolete, used cars command lower prices than new ones (except for collectible cars), and business equipment becomes less valuable as it ages and wears out while improved equipment to do the same work comes on the market. The decline in something's dollar value is called ***depreciation***.

Of course, the prices for things in a free economy are determined by the market—what a willing buyer will pay a willing seller. The dollar value of a house, car, photocopier, computer, baseball card, tractor, hovercraft, or anything else for that matter will be whatever price it can command in the open market, not the value computed by any mathematical formula.

Nonetheless, mathematical formulas can be useful tools for prediction, for estimating the dollar value of something even if we don't actually want to offer it up for sale, and in other cases where we need to place a reasonable dollar value on something. In this chapter we will examine some of these situations and the mathematical tools that can be useful in dealing with them.

Calculating Price Appreciation

Let's take on the question of appreciation first. The following example will illustrate how this works mathematically, and will also illustrate that we already know how to do it!

***Example 8.4.1** According to a wine expert, the market price for a very rare bottle of Chateau la Plonque wine is $3,650. In an interview in a wine trade publication, he states that he expects this particular bottle to appreciate at a 7% annual rate for the next 10 years. If his prediction turns out to be correct, what will the price be 10 years from now?*

This problem is yet another example of a case where even though the 7% appreciation rate is not compound interest, it is a percentage rate of growth and is mathematically equivalent to compound interest. Therefore, we can use the compound interest formula:

$$FV = PV(1 + i)^n$$
$$FV = \$3,650(1.07)^{10}$$
$$FV = \$7,180.10$$

It's worth also noting that we could have estimated that the bottle would a little less than double in value, since Rule of 72 says that the doubling time at 7% is approximately 72/7 = 10.28 years.

Price appreciation is very often stated as a percent growth rate, and so using the compound interest formula as we did in Example 8.3.1 will handle price appreciation questions quite effectively.

Depreciation as a Percent

Just as price appreciation is often expressed in terms of a percent rate, price depreciation is also often expressed as a percent rate. The difference, of course, is that whereas the appreciation rate is a rate of growth, a depreciation rate is a rate of decline (sometimes also called ***decay***). However, since in both cases the dollar value is assumed to be changing at a percent rate, the same mathematical formulas can be used. The difference, though, is that since depreciation means a price is going down, we treat this as a *negative* growth rate.

The following example will illustrate how this works.

***Example 8.4.2** Todd just bought a new car for $23,407. According to an online used car pricing service, the value of this car will decline at a 15% annual rate. Assuming this is correct, what will the car's value be in 5 years?*

We take on this problem in just the same way as the previous example, except that here the rate is understood to be −15%.

$$FV = PV(1 + i)^n$$
$$FV = \$23,407[1 + (-0.15)]^5$$

Adding a negative number is the same as subtracting a positive one, so this becomes:

$$FV = \$23,407(1 - 0.15)^5$$
$$FV = \$23,407(0.85)^5$$
$$FV = \$10,386$$

So depreciation works in basically the same way as appreciation, which itself is essentially the same way as compound interest, even using the same formula as compound interest does. There is, however, one very significant difference. With appreciation, over time we

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are increasing the price by a percent of a growing value; so with each passing year, the price not only rises, but does so at an accelerating rate. If we look at the price of the collectible bottle of wine from Example 8.4.1 year by year, we can see this:

Year	Starting Price	Increase	Ending Price
1	$3,650.00	$255.50	$3,905.50
2	$3,905.50	$273.39	$4,178.89
3	$4,178.89	$292.52	$4,471.41
4	$4,471.41	$313.00	$4,784.41
5	$4,784.41	$334.91	$5,119.32
6	$5,119.32	$358.35	$5,477.67
7	$5,477.67	$383.44	$5,861.11
8	$5,861.11	$410.28	$6,271.39
9	$6,271.39	$439.00	$6,710.39
10	$6,710.39	$469.73	$7,180.12

This is exactly the same thing we saw with compound interest.[3]

With depreciation, though, matters are a little different. The price is not increasing, it is decreasing, and so, as we calculate the percent of a decreasing price, we get a decreasing amount. So the price is declining at a decelerating (slowing down) rate. If we look at the price of the car from Example 8.4.2 year by year, we get:

Year	Starting Price	Decrease	Ending Price
1	$23,407.00	$3,511.05	$19,895.95
2	$19,895.95	$2,984.39	$16,911.56
3	$16,911.56	$2,536.73	$14,374.83
4	$14,374.83	$2,156.22	$12,218.61
5	$12,218.61	$1,832.79	$10,385.82

Even though this is a big change from what we are used to seeing from the compound interest formula, this slowing down of price decreases is pretty much what we would expect for the value of a used car. As a car ages through its first year, it goes from being a new car to a used one. There is a big difference in what people are willing to pay for a new car versus what they are willing to pay for a used one. (There is truth in the old saying that the biggest drop in the car's value occurs "when you drive it off the lot.") On the other hand, there is usually less difference in the value people will assign to a 4-year-old car than to a 5-year-old one. This is reflected in the pricing of the table.

If we carry this further, we would expect to see even smaller differences in the car's value as years go by; we would not expect people to see much of a price difference between a 9-year-old car versus a 10-year-old one. Either way, it's a pretty old car. If we extend our car price projection out to year 10, we see this is exactly what happens:

6	$10,385.82	$1,557.87	$8,827.95
7	$8,827.95	$1,324.19	$7,503.76
8	$7,503.76	$1,125.56	$6,378.20
9	$6,378.20	$956.73	$5,421.47
10	$5,421.47	$813.22	$4,608.25

The car is predicted to lose just $813.22 in value between years 9 and 10, versus $3,511.05 in the first year. This agrees with common sense.

Just how low will the value of Todd's car go? The car's value each year is 85% of its value in the prior year. For this reason, even though the value will continue to decline with

[3]The 2 cent difference between the ending value in this table and the ending value we calculated in Example 8.4.1 is due to rounding the price change each year.

each passing year, the price according to our formula will never actually reach zero. This does not entirely agree with common sense. At a certain point, the car will either become worthless, or it will be considered a collectible and increase in value. This is not a problem, though. We did not claim that the price will actually drop by exactly 15% each year. The 15% a year figure is an estimate and a prediction, based on what has happened to the prices of similar cars in the past. If it is based on reliable data, we have reason to expect it to agree reasonably well with what will actually happen, but we should not take it too literally. It would be no surprise at all if the car's value in 5 years actually turns out to be an even \$10,000, or \$10,995, or some other value in the same general neighborhood instead of the predicted \$10,386. Likewise, we should not be too surprised if the car's value after 20 years is \$500 or nothing instead of the $\$20{,}387(0.85)^{20} = \$907.24$ that the formula predicts.

When we calculate depreciation in this way, assuming that each year's depreciation is a set percent of a decreasing value, the amount of annual depreciation decreases with each passing year. For this reason, this is often referred to as ***declining balance*** depreciation. As we will see next, this is not the only way to do it.

Straight-Line Depreciation

As we have suggested in the previous discussion, declining balance depreciation is often a very reasonable way to project the actual market price of something. There is another very commonly used method for calculating depreciation, though. With ***straight-line depreciation*** we assume that the price declines by *the same dollar amount* (not the same percent) each year.

In order to determine the amount of this annual price decline, we first must determine the period of time over which we expect the depreciation to occur. This is often termed the item's ***useful life***. We then determine the total amount that the price is to decline over the course of the item's useful life. In some cases, that amount is the entire original value of the item; once it reaches the end of its useful life, we consider its value to be zero. For example, a computer that has reached the end of its usefulness has little if any value, and in fact old computers are of so little value that they can be hard to get rid of. In other cases, though, we assume that at the end of its useful life the item still has some salvage value, called its ***residual value*** or ***salvage value***. A 15-year old car may be basically worthless as a vehicle, but it may still have some value for parts or scrap metal, and can probably still be sold to a salvage yard for a few hundred dollars even if it doesn't run.

Once we have determined the useful life and salvage value, we then calculate the total amount of depreciation to be taken and divide this by the useful life. This gives the amount the price will decrease each year. We call this amount the ***rate of (straight-line) depreciation***, or the ***(straight-line) depreciation rate***. The value of the item after a period of time calculated from this rate is called the item's ***depreciated value***.

Before trying to make a formula out of this, let's illustrate the idea with an example.

Example 8.4.3 ***The Cotswold Real Estate Agency purchased a computer for \$2,000. The useful life of the computer is 5 years. The computer is assumed to have no salvage value. Find (a) the straight-line depreciation rate, (b) the depreciated value of the computer after 3 years, and (c) the depreciated value after 7 years.***

The computer will lose its full \$2,000 initial value in 5 years, and since it loses the same amount each year, the depreciation rate is \$2,000/5 = \$400 per year.

If the computer's value drops by \$400 per year for 3 years, that means it will be worth \$2,000 − 3(\$400) = \$2,000 − \$1,200 = \$800 at the end of the 3 years.

Since the computer's useful life is 5 years, from that time on it has a value of zero. So at 7 years, the depreciated value would be \$0.

It may not be necessary to have a formula for straight-line depreciation, since any other problem of this type will be done in pretty much the same way. Nonetheless, we can sum this up with formulas:

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FORMULA 8.4.1
Straight-Line Depreciation Rate

$$D = \frac{IV - SV}{UL}$$

where
D represents the ANNUAL DEPRECIATION AMOUNT
IV represents the INITIAL VALUE OF THE ITEM
SV represents the SALVAGE (RESIDUAL) VALUE
and
UL represents the USEFUL LIFE of the item

FORMULA 8.4.2
Straight-Line Depreciation

$$DV = IV - D(n) \text{ when } n < UL$$
$$DV = SV \text{ when } n \geq UL$$

where
DV represents the DEPRECIATED VALUE
and
n represents the NUMBER OF YEARS that have passed

If you were comfortable with how we worked Example 8.4.3 without having formal formulas, it is not really necessary to use them explicitly. However, the following example will illustrate their use.

Example 8.4.4 ***Rework Example 8.4.3 using Formulas 8.4.1 and 8.4.2.***

First we find the annual depreciation amount.

$$D = \frac{IV - SV}{UL} = \frac{\$2{,}000 - \$0}{5} = \$400$$

Then, we find the depreciated value after 3 years:

$$DV = IV - D(n) = \$2{,}000 - \$400(3) = \$2{,}000 - \$1{,}200 = \$800$$

Since $n = 7$ and $UL = 5$, and $7 > 5$, we conclude that the value after 7 years is \$0.

Comparing Percent to Straight-Line Depreciation

Both of the types of depreciation that we have considered are useful in different situations. Having seen how each works, it is worthwhile to compare the two methods both mathematically and in how they are most commonly used.

As a first step to comparison, let's reconsider the car from Example 8.4.2, this time with straight-line depreciation. Suppose we assume a useful life of 10 years, and assume that at the end of 10 years the car has a residual value of \$4,608.25 (the same value we predicted by using percent depreciation). The depreciation rate would then be (\$23,407 − \$4,608.25)/10 = \$1,879.88 per year. Showing this in a table similar to the one for this example with 15% declining balance depreciation (see "Depreciation as a Percent," above), we get:

Year	*Starting Price*	*Decrease*	*Ending Price*
1	\$23,407.00	\$1,879.88	\$21,527.12
2	\$21,527.12	\$1,879.88	\$19,647.24
3	\$19,647.24	\$1,879.88	\$17,767.36
4	\$17,767.36	\$1,879.88	\$15,887.48
5	\$15,887.48	\$1,879.88	\$14,007.60
6	\$14,007.60	\$1,879.88	\$12,127.72
7	\$12,127.72	\$1,879.88	\$10,247.84
8	\$10,247.84	\$1,879.88	\$8,367.96
9	\$8,367.96	\$1,879.88	\$6,488.08
10	\$6,488.08	\$1,879.88	\$4,608.20

(The slight difference between the 10th year value in the table and the residual value we stated is due to rounding of the straight-line depreciation rate.)

This steady decrease is clearly different from what we saw with percent depreciation. To make it easier to compare the two, let's look at the car's values under the two different depreciation methods side by side:

Year	*Percent*	*Straight Line*
Start	$23,407.00	$23,407.00
1	$19,895.95	$21,527.12
2	$16,911.56	$19,647.24
3	$14,374.83	$17,767.36
4	$12,218.61	$15,887.48
5	$10,385.82	$14,007.60
6	$8,827.95	$12,127.72
7	$7,503.76	$10,247.84
8	$6,378.20	$8,367.96
9	$5,421.47	$6,488.08
10	$4,608.25	$4,608.20

We can also illustrate this comparison with a graph (which also illustrates where the name *straight-line* depreciation comes from), as seen in Figure 8.1.

While both methods reflect the same reality—that the value of the car is declining with time—there are some significant differences in the values that they predict. The most significant difference between the two lies in how the value declines with each method. With percent (declining balance) depreciation, the value takes heavier hits in the early years, and then the rate of decline slows down. In contrast, the graph shows where straight-line depreciation gets its name; the depreciated value declines by the same steady amount each year, reflected by the steady, straight-line graph. Given that we concluded in our prior discussion that it is realistic to expect a faster drop in value in the early years, this should raise the question of why we bother with straight-line depreciation as an alternative.

One advantage of straight-line depreciation is that it is easier to calculate with pencil and paper. That may not be such a big deal today, but in the past this mattered much more, giving the straight-line method a foothold. A second reason is that even though the percent method gives a more realistic estimate of actual market value for most items, this is not necessarily always true. For some items the straight-line method may be the more realistic choice. Straight-line depreciation might give a more realistic estimate of the value of a copper mine that contains $500 million worth of copper and can produce a steady $25 million

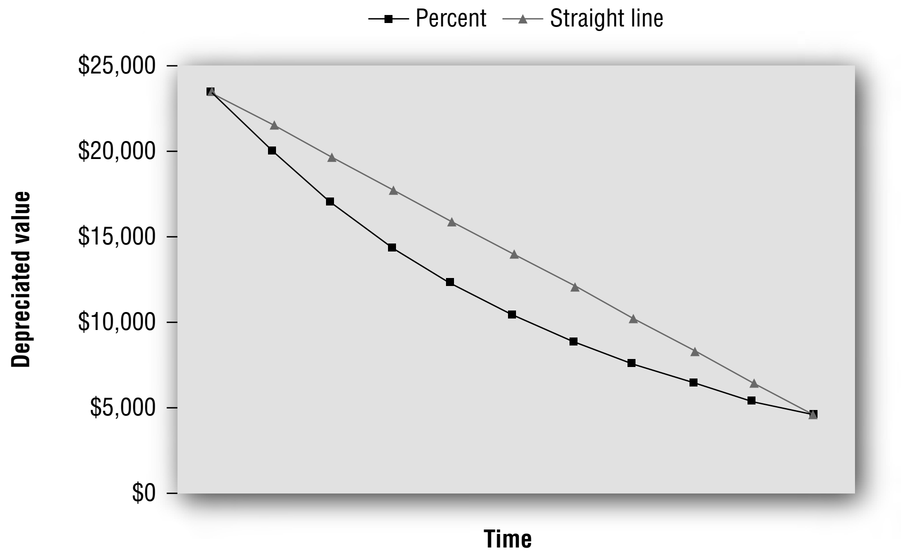

FIGURE 8.1
Straight-Line Versus Percent Depreciation

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worth per year. After 1 year, $475 million worth remains; after 2 years, $450 million is left, and so on. (This of course assumes no change in the market price for copper and no significant changes in operating costs.)

By far the greatest reason to consider the straight-line method, though, is this: *depreciation is used not only as a way of assessing the actual fair market value of an item*. In business, depreciation is also used for accounting purposes, and often those purposes call for straight-line depreciation. Accountants use depreciation to match a business's cost of purchasing an item with the revenues that are produced by using that item. A thorough explanation of this lies beyond the scope of this book, but a simple example will illustrate the basic idea.

The Cotswold Real Estate Agency in Example 8.4.3 paid $2,000 for a computer. This is a business expense, and so when the agency sets out to calculate its profit for the year, the cost of this computer should be taken into account, just as the agency would take into account its office rent or electric bill. But there is a very important difference between the computer and the office rent or electric bill. The office rent and electric bill are expenses for business costs incurred entirely for that year, but the expense of the computer is for a piece of equipment that will be useful for years to come. There is something not quite right about attributing the entire cost of the computer to this year's business expenses. If, as we assumed, the computer has a useful life of five years, then only 1/5 of its cost really is attributable to the current year (assuming that the computer was in use for the full year; if it was only in use for part of the year, the fraction would be even smaller).

Example 8.4.5 ***Suppose that Cotswold Real Estate put its computer into use 7 months before the end of 2007. For straight-line depreciation, how much depreciation for the computer should be attributed to 2007? How much for 2008?***

The depreciation amount is $400 per year. In 2007, the computer was in use for 7/12 of the year, so the company should count (7/12)($400) = $233.33 worth of the computer's cost as an expense for 2007.

In 2008, the computer will be in use for the full year, so the depreciation would be $400.

As far as accounting for business expenses is concerned, then, $233.33 of the purchase price is attributable to 2007, and the remaining $1,788.67 remains an expense for future years. Looked at this way, *the depreciated value of the computer is not intended to reflect its actual value if sold*. Instead, the depreciated value represents the amount of money spent for it that is yet to be counted as an expense against the business's income.

Before moving on, let's try one more example of this type.

Example 8.4.6 ***Suppose that a retailer buys fixtures for $14,835. The useful life is 12 years, and the residual value is $1,500. If the fixtures are put into use 5 months before the end of 2007, determine the depreciated value of the fixtures at the end of 2007 and at the end of 2008.***

First, we determine the annual depreciation amount:

$$D = \frac{IV - SV}{UL} = \frac{\$14,835 - \$1,500}{12} = \$1,111.25$$

In the first year, the retailer would take 5/12 of a full year's depreciation, or (5/12)($1,111.25) = $463.02.

So at the end of 2007, the depreciated value is $14,835 − $463.02 = $14,371.98. In 2008, a full year's depreciation would be taken, so the end-of-year depreciated value is $14,371.98 − $1,111.25 = $13,260.73.

The use of depreciation in accounting has some interesting side effects. It may be to a company's benefit to make its expenses seem as large as possible to minimize the amount of income taxes it will have to pay. The useful life of an item may be estimated very aggressively low to allow the business to use as much depreciation as possible as soon as possible. Even if Cotswold Realty really expects its computer to be useful for 5 or more years, they may prefer to use a 2 or 3 year "useful life" in its accounting to be able to count more money as expenses against earnings this year (and thus to keep its income tax liability lower.)

On the other hand, if the company is trying to make itself look as profitable as possible to convince outside investors or potential buyers of the business to place a higher value on the business, the company may want to minimize depreciation to keep the expenses shown on their books as low as possible. A 10- or 15-year or even longer useful life might be desirable in that case so as to make the depreciation expense in each year small and make profits seem larger.

It does not seem quite right that a company could game its annual profits by choosing a useful life for a depreciating asset based entirely on how it wants its profits to look. Fortunately, professional accounting standards put reasonable limits on how quickly items can be depreciated, as do Internal Revenue Service regulations (though the two do not always agree, and this may require a business to have two different calculations of its annual profits). While a business's accountants do not have free rein in how long to set the useful life for an asset, it is nonetheless true that the useful life for depreciation purposes may be different (usually shorter) than the length of time that the business can realistically expect the item in question to be useful.

More surprisingly, accounting principles may require a business to depreciate for accounting purposes an item that is not even expected to actually lose value. People who own rental real estate, for example, in their accounting often depreciate the real estate that they own, treating the purchase price of the real estate as an expense to count against the income that they get in rent. This flies in the face of the fact that in reality the real estate will hopefully actually *grow* in value over time. This may seem strange, but remember that in such cases depreciation is a matter of accounting for a business's costs, not an attempt to assign a realistic market value to the property.

Before moving on, we should note one apparent difference between straight-line and percent depreciation. It may appear from this example that salvage values are used only with straight-line depreciation, not with percent depreciation. While it is true that salvage values are more commonly encountered with straight line, it is possible to have a salvage value with percent depreciation. We will not, however, deal with that situation in this text.

MACRS and Other Depreciation Models

Straight-line and percent depreciation are commonly used methods of calculating depreciation, but there are others. Probably the best known is the ***Modified Accelerated Cost Recovery System (MACRS)***. MACRS (the acronym is pronounced like "mackers") was put into use in the United States under the Tax Reform Act of 1986. Generally speaking, its purpose was to allow businesses to claim larger depreciation expenses early on than would have been permitted by straight-line depreciation. MACRS is widely used for the purpose of computing taxable income in the United States.

The formulas on which MACRS is based are seldom seen in practice. Instead, accountants use tables. Each type of depreciable item is assigned a MACRS useful life (which may or may not match up with its realistic useful life), and then a certain percent of depreciation is allowed each year. This percent normally applies to the entire acquisition price of the item; the residual value is taken to be zero, regardless of whether or not the item really could be expected to have no actual residual value.

For example, under MACRS computers are usually depreciated over 5 years. The percents used for this equipment under MACRS each year are:

Year	*Percent*
1	20%
2	32%
3	19.2%
4	11.52%
5	11.52%
6	5.76%

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Hang on—why does something depreciated over 5 years have depreciation percents for 6? Under MACRS, we assume that all items are put into service halfway through the year, regardless of when that actually happens. So the first year of accounting for the computer's expense is assumed to only be the first half-year that the computer was in use. The 20% depreciation is not considered a full year's worth of depreciation; it is the first half-year's worth. That is the reason why the percentage for year 2 is higher than the percentage for year 1; MACRS assumes we are taking a full 12 months' worth of depreciation in year 2. At the end of the fifth year, we have only taken 4½ years of depreciation, leaving another half year to run into year 6.

Fortunately, the calculation is easier than the explanation!

Example 8.4.7 ***Calculate the depreciation each year for the Cotswold Real Estate computer, using MACRS.***

In the first year, the depreciation is (20%)($2,000) = $400.
In the second, the depreciation is (32%)($2,000) = $640.
In the third, it amounts to (19.2%)($2,000) = $384.
In both the fourth and fifth years, it is (11.52%)($2,000) = $230.40.
Finally, in the sixth year the depreciation is (5.76%)($2,000) = $115.20.

The $400 depreciation in the first year is based on the assumption that the computer was in use for 6 months. However, the company takes this same $400 depreciation regardless of how much of the year the computer was actually in use.

This simple example illustrates how MACRS works mathematically. This is a simplified example; as with many things involving the tax code, the specific details, exceptions, and other vagaries of MACRS can be quite involved.

EXERCISES 8.4

A. Appreciation

1. The housing market in Watsonville Falls is hot, and a local real estate agent predicted at a meeting of the chamber of commerce that prices should rise by an average of 6.3% per year for the foreseeable future. If the average single-family home costs $273,592 today, what would the agent predict it will be in 5 years?

2. In August 2006 the price of an ounce of gold was $640. A commentator on a late-night radio program predicted that the price of gold would rise by an average of 15% per year for the next 5 years. What is he projecting the price will be in August 2011.

B. Percent (Declining Balance) Depreciation

3. Gary bought a boat for $32,750. If the market value declines at a steady 8% annual rate, what will the boat's value be in 5 years? In 10 years?

4. Renee paid $29,409 for a new truck. She expects that the market value will decline by 11% annually. How much does she expect to be able to sell it for in 3 years?

5. A car that cost $21,450 new is worth $11,050 5 years later. What percent depreciation rate would lead to this value?[4]

[4]NOTE: Chapter 6 is not formally a prerequisite for this chapter; however, a precise answer to this question requires use of a formula from that chapter. Students who have not yet covered Chapter 6 should skip this question.

C. Straight-Line Depreciation

6. A new refrigerator for a restaurant costs $6,000. The useful life is 9 years, and the salvage value is $1,500.
   a. Determine the annual depreciation amount using straight-line depreciation.
   b. Determine the depreciated value of the refrigerator after 4 years of use.

7. Zarofire Systems bought new filing cabinets for the home office. The total cost was $47,500. The useful life is 8 years, and the salvage value is $7,500. Find the amount of straight-line depreciation per year, and determine the depreciated value of these cabinets after 5 years of use.

8. A gas station owner purchased new gas pumps for $192,300. The useful life is 6 years, and the salvage value is zero. Find the per-year straight-line depreciation amount, and use it to determine the depreciated value after 4 years of use.

9. Suppose that a retail store owner buys fixtures for $20,000. These fixtures have an 8-year useful life and no salvage value.
   a. Determine the per-year straight-line depreciation amount.
   b. Suppose these fixtures are put into use 7 months before the end of 2008. How much depreciation should be taken for them in 2008?
   c. Find the depreciated value of the fixtures as of the end of 2011.

10. Suppose that Bradenford State College bought new computers for a student lab for $64,500. The useful life is 5 years and the computers have no salvage value.
    a. Determine the per-year straight-line depreciation amount.
    b. Suppose the computers were put into use 4 months before the end of 2007. How much depreciation should the college take for these computers for 2007?
    c. Determine the depreciation amount for the remainder of the computers' useful life by completing the table below:

Year	***Depreciation Amount***	***Depreciated Value at End of Year***
2007		
2008		
2009		
2010		
2011		
2012		

D. MACRS

11. Bradenford National Corporation bought new computers for its offices. The company uses MACRS depreciation. The total cost of the computers was $64,500. (The MACRS table for 5-year assets is given in the text of this section.)
    a. Determine the MACRS depreciation for these computers for the first year.
    b. Suppose the computers were put into use 4 months before the end of 2007. How much depreciation will the company take for these computers for 2007?

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c. Determine the depreciation amount for the remainder of the computers' useful life by completing the table below:

Year	Depreciation Amount	Depreciated Value at End of Year
2007		
2008		
2009		
2010		
2011		
2012		

E. Grab Bag

12. Zmarek Associates purchased laboratory equipment for $450,000. Determine the depreciated value of this equipment after 4 years using:

   a. 10.5% annual depreciation

   b. Straight-line depreciation with a $25,000 salvage value and a 9-year useful life

13. Boldhurst Conservatory purchased new pianos for $1.2 million. The useful life of the pianos is 20 years and the salvage value is $95,000. Using straight-line depreciation, determine the depreciated value of the pianos after 7 years.

14. Zarofire Systems bought new computers for its offices for $250,000. Using 5-year MACRS depreciation, determine the total depreciation the company will take for these computers in the first 3 years and their depreciated value after 3 years.

15. Stonehurst Trucking bought a new truck for $111,925. Find the depreciated value of this truck after 5 years, using:

   a. 12% annual depreciation

   b. Straight-line depreciation with a $7,300 salvage value and a 10-year useful life

16. Ralydex Pharmacorp bought office equipment for a total of $162,500. The useful life is 7 years, and the salvage value is $3,500. The equipment was put in use 214 days before the end of 2007. Determine the amount of straight-line depreciation that the company could claim for this equipment for 2007, 2008, and 2009.

17. Segulon Corp. bought new soundproofing equipment for its audio testing labs at a cost of $319,500. The useful life is 12 years and there is no salvage value. Determine the per-year straight-line depreciation rate.

18. Suppose that a piece of office equipment is purchased for $7,200. The useful life is 5 years and the salvage value is $1,400. Create a table showing the depreciated value of the office equipment after each year of its use.

19. Suppose that a piece of office equipment is purchased for $7,200. The useful life is 5 years and the salvage value is $1,400. The equipment is placed in use at the end of September 2006. Create a table showing the depreciated value of this equipment at the end of each calendar year, beginning with 2006 and ending when it reaches its salvage value.

20. Suppose that a company's stock sells for $37.50. An investment analyst predicts that the stock price will rise at a 15% annual rate for the next 7 years. If this is correct, what will the price be in 7 years?

21. A mechanic purchased new equipment for $32,000, with a useful life of 5 years and a salvage value of $4,000. Complete the table below to show the straight-line depreciated value of the equipment each year:

After the equipment has been in use:	***The depreciated value is***
1 year	
2 years	
3 years	
4 years	
5 years	

22. A mechanic purchased new equipment for $32,000, with a useful life of 5 years and a salvage value of $4,000. The equipment was put into use 9 months before the end of 2006. Complete the table below to show the straight-line depreciated value of the equipment each year:

At the end of year:	***The depreciated value is***
2006	
2007	
2008	
2009	
2010	
2011	

23. A dairy farmer purchased a piece of equipment for $37,500. The useful life is 12 years and the salvage value is $3,775. If he sells this equipment 7 years after buying it, how much will he get for it?

F. Additional Exercise

24. In the Book of Leviticus in the Jewish and Christian Bible, an unusual economic system is proposed. Every fiftieth year is to be a "Year of Jubilee," in which many types of property are to return to their original owners. When the jubilee year is far away, the price paid for an item would therefore be higher than when the jubilee year is close at hand.

    Suppose that you owned a house that would be worth 1,000 shekels immediately after a jubilee year. How much would I pay you for this house 17 years before the next jubilee year?

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CHAPTER 8 SUMMARY

Topic	Key Ideas, Formulas and Techniques	Example(s)
Markup Based on Cost, p. 333	• Markup based on cost formula: $P = C(1 + r)$ Markup amount is the difference between retail price and cost.	An auto mechanic charges 40% markup based on cost for parts. Determine the retail price and amount of markup for an air filter that cost \$14.95. (Example 8.1.1)
Determining Markup Percent, p. 334	• Use the markup based on cost formula and solve for r. • Or calculate the markup amount and divide by the cost.	An electronics retailer offers a computer for sale for \$1,000. The cost is \$700. Determine the markup percent. (Example 8.1.3)
Markdown, p. 334	• Markdown formula: $MP = OP(1 - d)$. • Markdown amount is difference between original and marked-down price.	A furniture store offers a 15%-off sale. A sofa normally sells for \$1,279.95. What is the sale price? (Example 8.1.4)
Markup versus Markdown, p. 336	• Despite appearances, if a price is marked up by a certain percent and then marked down by the same percent, the result is not the original cost. • To determine the markdown to get back to cost, calculate the percent markdown, using the cost and retail prices. • If price is unknown, use any amount for the cost (such as \$100).	If prices are calculated with a 35% markup based on cost, what is the percentage that those prices should be marked down to get back to original cost? (Example 8.1.8)
Gross Profit Margin, p. 343	• Gross profit is the difference between cost and selling price. • Gross profit margin is gross profit as a percent of selling price.	Last year, sales at Sally's Fashion Paradise totaled \$219,540. The cost of the items sold was \$147,470. What was the business's gross profit margin? (Example 8.2.3)
Net Profit Margin, p. 344	• Net profit is the profit left after expenses are subtracted from gross profit. • Net profit margin is net profit as a percent of selling price.	Last year, Sally's Fashion Paradise had overhead expenses totaling \$63,073. Find the net profit margin. (Example 8.2.4)
Proportionate Allocation of Expenses, p. 345	• The net profit margin for an individual item is difficult to determine since expenses cannot usually be directly attributed to each item. • Proportionate allocation applies the business's overall expense percent to each item.	A dress sold for \$65. The business's overall sales were \$219,540 and overall expenses were \$63,073. Using proportionate allocation, find how much expense is attributable to this dress. (Example 8.2.6)
Markup Based on Selling Price, p. 345	• Markup based on selling price formula: $C = SP(1 - r)$.	Determine the selling price of an item costing \$45 in order to have a 35% gross margin. (Example 8.2.7)

(Continued)

Topic	Key Ideas, Formulas and Techniques	Example(s)
Trade Discounts, p. 351	• Items that have a list price or MSRP may be sold to retailers at a discount to that price. • Calculation of trade discounts is done in the same way as markdown • Trade discount formula: $NP = LP(1 - d)$.	Ampersand Computers bought 12 computers from the manufacturer. The list price is $895.00 each and the manufacturer offers a 25% trade discount. How much did Ampersand pay? (Example 8.3.1)
Series Discounts, p. 354	• Series discounts are multiple discounts applied to a price in succession. • A single discount can be calculated that is equivalent to a series of discounts.	Find the single discount equivalent to successive 25% and 15% discounts.
Cash Discounts, p. 355	• Cash discounts are additional discounts offered for prompt payment. • Cash discounts are calculated like other discounts. • Some specialized terminology may be used to indicate what cash discounts are offered.	Roy's Appliance Hut placed an order for 10 refrigerators (list $775) and 8 washers (list $995). The manufacturer offered trade discounts of 20% and 20%, and the invoice was dated April 7, 2007, with "terms 2/15, EOM." If Roy makes his payment on April 30, 2007, how much must he pay? (Example 8.3.11)
Price Appreciation, p. 363	• Projected future prices based on an assumed percent appreciation rate can be calculated by using the compound interest formula.	A wine expert predicts that a rare bottle of wine currently worth $3,650 will appreciate at a 7% annual rate. Predict the price in 10 years. (Example 8.4.1)
Percent (Declining Balance) Depreciation, p. 363	• Depreciation based on an assumed percent can be calculated by using the compound interest formula. • The depreciation percent rate is a negative percent.	Todd just bought a new car for $23,407. If the value declines at a 15% annual rate, what will the value be in 5 years? (Example 8.4.2)
Straight-Line Depreciation, p. 365	• Straight-line depreciation assumes the same dollar amount of depreciation each year. • Annual depreciation amount: $DR = \frac{IV - SV}{UL}$ • Depreciated value: $DV = IV - DR(n)$	A company purchased a computer for $2,000. The useful life is 5 years and the salvage value is $0. Find the depreciated value of the computer after 3 years. (Example 8.4.3)
Partial Year Depreciation, p. 368	• If an item is not in use for an entire year, it may be depreciated for only the percent of the year for which it was in use.	If the computer from the above example was put into use 7 months before the end of 2007, how much depreciation should be attributed to 2007? (Example 8.4.5)
MACRS, p. 369	• MACRS depreciation is used by businesses mainly for income tax accounting. • The percent depreciation for each year is given by a table.	Calculate the 5-year depreciation for a $2,000 computer, using MACRS. (Example 8.4.7)

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CHAPTER 9

Taxes

"Giving money and power to government is like giving whiskey and car keys to teenage boys."

—P. J. O'Rourke, "Parliament of Whores"

Learning Objectives

LO 1 Calculate the sales tax due on purchases.

LO 2 Determine the price before sales tax, given the price including tax.

LO 3 Understand the structure of U.S. income taxes, and calculate total income taxes owed.

LO 4 Calculate federal income tax withholding and FICA for payroll deductions.

LO 5 Calculate the tax due on a property, given the assessed value and tax rate.

LO 6 Understand the role of assessment in determining the tax due on a property, and be able to compare tax levels in jurisdictions using different assessment rates.

LO 7 Calculate tax rates based on total tax levy and total assessment.

LO 8 Be familiar with and able to compute various types of excise taxes.

LO 9 Be familiar with the current structure and recent changes in the federal estate tax, and be able to calculate the tax due on a given estate.

Chapter Outline

9.1 Sales Taxes

Sales taxes are taxes imposed on retail sales. In the United States, sales taxes are imposed by state and/or local governments. Businesses are generally required to calculate and collect these taxes on the sales that they make, and then forward what they collect to the appropriate government tax collection agency. This places some mathematical and accounting responsibilities on businesses; a business needs to make sure that sales taxes are calculated and collected correctly at point of sale, and that the business's overall tax collections are reported and paid correctly to the government.

As of this writing, state sales taxes are in place in 45 states, plus the District of Columbia. Only Alaska, Delaware, Montana, New Hampshire, and Oregon do not impose state sales taxes. There is a great deal of diversity, though, in both tax rates and in the types of sales on which these taxes are collected. In some areas, purchases of things like food, clothing, or prescription and/or over the counter drugs may be exempt from sales taxes, or may be taxed at a lower rate than other purchases. For example, in much of North Carolina the general sales tax rate is 7.0%, but a 2.5% rate applies for groceries. Different rates may also be charged on other types of sales, such as utility bills, gasoline, hotel rooms, hybrid cars, home improvement services, or any other sort of sale that lawmakers deem fit for a different rate. Furthermore, regardless of the type of item sold, sales made to government agencies and nonprofit organizations may be exempt from sales taxes, regardless of the item purchased.

Each state sets its own sales tax rates, and so naturally rates differ from one state to another. Matters can be further complicated by sales taxes imposed by a county or city, meaning that the sales tax rate differ from one location to another within the same state. For example, the general state sales tax rate in New York is 4%, but the actual rate in effect in that state ranges from 7% to 9⅜%, depending on the additional sales tax imposed by local governments.

When different levels of government impose taxes on the same purchases, each rate is a percent of the price before taxes. This differs from series discounts (or series markups). We saw in Chapter 8 that a 5% discount followed by a 3% discount is not the same as an 8% discount. This is because the first discount is a percent of the original price, and the second is a percent of the already discounted price. The situation is different with sales tax rates: since both rates are a percent of the same thing, the overall sales tax rate *can* be calculated by adding the two rates.

Example 9.1.1 ***If the state sales tax rate is 4.5% and the local sales tax rate is 3.75%, what is the overall sales tax rate?***

The overall rate is 4.5% + 3.75% = 8.25%.

At this time, there is no national sales tax in the United States, though the idea of a national sales tax has often been proposed as an alternative to the current income tax system. Advocates of a national sales tax often use the term ***consumption tax*** to emphasize the claim that collecting taxes on purchases rather than income would encourage savings over spending. Many other countries have a national ***value-added tax***, which functions in much the same way as a sales tax. While we are unlikely to see a national sales tax in the near term in the United States, it is possible that such a tax may eventually come into place.[1]

Because sales taxes are a state or local tax, matters become interesting with mail order or Internet sales. If a Pennsylvania resident buys merchandise over the Internet from a company in California, which state collects the sales tax? In general a merchant is not required to collect tax on sales made to an out-of-state customer, unless the merchant has a business presence (such as a retail store) of some sort in that state. If the merchant in

[1]This would have an interesting effect on retirement plans. Withdrawals from Roth IRAs (discussed in Chapter 7) are not subject to *income* tax. Things that you buy with money from a Roth IRA, though, are not immune from *sales* taxes.

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this example is a nationwide chain that has stores in Pennsylvania, it would probably be required to collect Pennsylvania sales tax on the purchase. If, though, the merchant has no business operations based in Pennsylvania, it would most likely not collect any sales tax on this purchase. This does not mean that the purchase is tax-free, only that the merchant does not collect it. The purchaser may be legally required to pay the tax on the sale to his state of residence. In those cases, the tax is referred to as a ***use tax***, payable to the state in which the item purchased is intended to be used.

It will come as no surprise that, in practice, use taxes are often ignored for small purchases, with many people not bothering with, or not even being aware of, their obligation to pay them. For larger purchases, use taxes generally will be collected, however. One common example is automobiles, when a car is purchased in one state but registered in another. Auto dealers often handle the use tax collection for the state in which the vehicle will be registered.

Calculating Sales Taxes

Sales taxes are normally set as a percent of the amount of the sale. Calculating the sales tax on a purchase is a matter of converting the percent to a decimal, and multiplying by the sales amount. The following example will illustrate.

Example 9.1.2 ***A sweater costs $42.95 and the sales tax rate is 7⅜%. Calculate (a) the amount of sales tax and (b) the total price including tax.***

(a) 7⅜% is 0.07375 as a decimal. Multiplying this by the price we get (0.07375)($42.95) = $3.18.

(b) $42.95 + $3.18 = $46.13

Sales tax is usually calculated this way because the amount of sales tax is usually shown separately on the receipt. If we are mainly interested in the total, though, we can also calculate sales tax in a slightly different way. A 7⅜% tax rate means that $1 of purchase price becomes $1.00 + $0.07375 = $1.07375, and so we can just multiply the actual price by this figure to get the price including tax. We can express this as a formula:

FORMULA 9.1.1
Sales Tax Formula

$$T = P(1 + r)$$

where
T represents the TOTAL PRICE including tax
P represents the PRICE BEFORE TAX
and
r represents the SALES TAX RATE (as a decimal)

Example 9.1.3 ***Repeat Example 9.1.2, using Formula 9.1.1***

Since the formula gives the total price, it is easier to do part (b) first.

$$T = P(1 + r)$$
$$T = \$42.95(1 + .07375)$$
$$T = \$42.95(1.07375)$$
$$T = \$46.13$$

To answer (a), the amount of sales tax, we can subtract:

$$\$46.13 - \$42.95 = \$3.18$$

If a sale consists of multiple different items, it does not matter (in theory at least) whether we calculate the tax on each item individually and total the results, or total the items' prices first and then calculate the tax on the total.

Example 9.1.4 ***On a trip to his favorite local discount store, Jack bought a DVD player costing $79.95, a toaster for $29.95, and a case of cranberry juice for $12.75. The sales tax rate on all purchases is 6¼%. Find the total cost of his purchases including tax.***

If we calculate the total price of each item individually we get:
DVD player: $T = P(1 + r) = \$79.95(1.0625) = \$84.95$
Toaster: $T = P(1 + r) = \$29.95(1.0625) = \$31.82$
Cranberry juice: $T = P(1 + r) = \$12.75(1.0625) = \$13.55$
Adding these up we get a total of $\$84.95 + \$31.82 + 13.55 = \$130.32$

If we total the items first and then calculate the price with tax, we get:
Total before tax = $\$79.95 + \$29.95 + \$12.75 = \$122.65$
With tax: $T = P(1 + r) = \$122.65(1.0625) = \$130.32$

Occasionally these two different methods may give very slightly different results because of rounding. If we did not need to round, the two methods would always give exactly the same answer, but the need to round can cause minor discrepancies. With the first method, we rounded three times, with the second we rounded only once, and that sometimes may create a difference of a penny or two. This never amounts to enough money to be much of a business concern (although state or local regulations may require a business to calculate the tax charged to a customer in one way or another). At most it may be a matter of a few cents difference. The answers given in this book are calculated by calculating sales tax on totals, not on individual items separately, since this is the method most commonly used in actual practice.

Of course, it does matter if there are different tax rates on some items than on others. In that case, the items must be grouped together based on their tax rate.

Example 9.1.5 ***Rework Example 9.1.4 if the sales tax rate on food is 2½%.***

While we can combine the prices of items that will be charged the same rate, items with differing rates must be kept separate.

The DVD player and toaster cost a total of $\$79.95 + \$29.95 = \$109.90$.
Nonfood items: $T = P(1 + r) = \$109.90(1.0625) = \$116.77$
Food item: $T = P(1 + r) = \$12.75(1.025) = \$13.07$
Total with tax: $\$116.77 + \$13.07 = \$129.84$.

Students who have covered Chapter 8.1 should note the similarities between sales tax and markup calculations. They are essentially the same thing, mathematically speaking. You may also notice that the markup formula given in Section 8.1 is essentially the same as the sales tax formula given in this chapter, the two differing only in the letters used.

Finding a Price before Tax

Sometimes, we may need to work things in the opposite direction. We may know the price including tax and need to figure out the price before tax. For example, a business owner may know her business's total sales receipts for the day, but want to know how much of those receipts actually represent income for the business as opposed to taxes collected for the government. The following example will illustrate another such situation, as well as the mathematics needed to answer the question.

Example 9.1.6 ***Ardana is shopping for a new car. She figures she can afford a $325 monthly payment, and on the basis of this payment and the rates she can get on a car loan from her credit union, she has determined she can borrow $21,900. With no down payment or trade-in, she started shopping, figuring she could afford to pay $21,900 for her new car.***

Unfortunately, she forgot that in the county where she lives the sales tax rate is 8.75%. She now realizes that this total must include the sales tax, and so the price she can

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afford to pay for the car is actually lower. What price for the car can she actually afford? How much sales tax would she then pay?

We start with Formula 9.1.1, plugging in the values that we know:

$$T = P(1 + r)$$
$$\$21,900 = P(1.0875)$$

To find P, we divide both sides by 1.0875:

$$P = \$20,137.93.$$

In fact, Ardana can only afford a car costing a little over $20,000. To find the amount of sales tax on this amount, we subtract $21,900 − $20,137.92 = $1,762.07.

A word of warning is in order here, because it is easy to fall into a common mistake on problems like this one. It is tempting to use the following incorrect reasoning: Since the tax rate is 8.75%, just take 8.75% of the $21,900 and subtract. The problem with this approach is that the tax rate is a percent of the before-tax price, not a percent of the after-tax total. Just as with simple interest versus simple discount or markup vs. markdown, it is important to be careful that, when using percents, we are use them as a percent *of the right thing*. Since Ardana didn't know the before-tax price, she can't just take 8.75% of it, and so the formula approach we used in this example is the only reasonable way to approach her problem.

Another similar situation sometimes arises for a business. Usually, businesses quote their prices before tax, and then add the tax on. Sometimes, though, prices are quoted including sales tax. This might be done as part of a sales promotion, or by a business where most purchases are made in cash, and where, to avoid having to make so much change, it is more convenient for final prices to be nice round numbers.

Example 9.1.7 ***A souvenir shop sets all of its prices to include 6.75% sales tax. For the month of June, sales totaled $16,739. How much sales tax is due on these sales?***

$$T = P(1 + r)$$
$$\$16,379 = P(1.0675)$$
$$P = \$15,343.33$$

Since the P is the total price of the items sold before tax, the difference between this and the total sales must be the tax. Therefore, the tax is $16,379 − $15,343.33 = $1,035.67.

Sales Tax Tables (Optional)

You may occasionally see tables for calculating sales taxes. In the past, retailers would often keep a chart next to the cash register so that completing a sale would require adding a looked-up tax amount, which made for easier arithmetic than multiplication by a decimal. An example is below:

5% SALES TAX TABLE

AMOUNTS LESS THAN $1.00		AMOUNTS OVER $1.00 ADD TO AMOUNT FROM THE LEFT FOR THE <$1.00 AMOUNT			
If the Amount is:	***The Tax is:***	***Dollars:***	***Tax:***	***Dollars:***	***Tax:***
Less than $0.10	$0.00	$1	$0.05	$6	$0.30
$0.10 to $0.29	$0.01	$2	$0.10	$7	$0.35
$0.30 to $0.49	$0.02	$3	$0.15	$8	$0.40
$0.50 to $0.69	$0.03	$4	$0.20	$9	$0.45
$0.70 to $0.89	$0.04	$5	$0.25		
$0.90 to $0.99	$0.05	*For each $10, add $0.50 in tax.*			

Example 9.1.8 ***Use the 5% sales tax table shown above to calculate the tax on a $34.74 purchase.***

For the 74 cents, the tax from this table is $0.04.
For the $4.00, the tax from this table is $0.20.
For the $30.00, the tax from this table is 3($0.50) = $1.50.
So the total tax is $1.50 + $0.20 + $0.04 = $1.74.

While these tables offered somewhat easier arithmetic when it had to be done with pencil and paper or simple adding machines, in the present day there is not much reason to use these charts. Most businesses are equipped with cash registers that can calculate sales tax automatically, and even when a price quote or sale is made without a cash register, calculators are in such common use that the arithmetic is not much of an issue any more. Nonetheless, sales tax tables are still sometimes used.

EXERCISES 9.1

A. Calculating Sales Tax

1. Find the sales tax on a purchase of $147.35 if the sales tax rate is 8%.

2. Find the sales tax on an $87.29 purchase if the sales tax rate is 6.5%.

3. Find the sales tax on a $25.02 purchase if the sales tax rate is $8\frac{3}{8}$%.

4. Find the sales tax on a $41.04 purchase if the sales tax rate is $4\frac{3}{4}$%.

5. Bill bought a DVD player for $49.99, a TV for $149.99, and a rice cooker for $39.95. The sales tax rate is $6\frac{3}{4}$%. Find the total amount of his purchase, including tax.

6. Shenara bought a blouse for $79.88 and a bracelet for $49.95. The sales tax rate is 7.25%. Find the total amount of her purchase, including tax.

7. At a discount store, Ed bought a tool kit for $17.59, a file drawer for $21.35, and a case of granola bars for $9.35. The general sales tax rate is 9%, but the sales tax on food is $2\frac{3}{8}$%. Calculate (a) the total sales tax Ed will pay on this purchase and (b) the total amount of the purchase including tax.

8. Chelsea bought two music CDs for $12.99 each, a pair of jeans for $17.99, and a swim suit for $21.99. The general sales tax rate is 8.25%, but the sales tax on clothing is just 4%. Calculate the total amount of her purchase, including sales tax.

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9. Deon's pizza shop had sales of $5,039.45 before tax in April. The sales tax rate in his area is 5.75%. Find the total amount of sales tax he should have collected in that month.

10. Calculate the missing subtotal, sales tax, and overall total for the receipt shown below:

JOE'S DISCOUNT BOOKSTORE

ORDER 32		APRIL 7, 2007
DINOSAUR BARBECUE COOKBOOK		$27.95
2007 UNIVERSAL GUIDEBOOK		$9.95
QUANTUM MECHANICS MADE FUND AND EASY!		$34.95
	SUBTOTAL	(A)
	7.35% SALES TAX	(B)
	TOTAL	(C)

11. Calculate the missing subtotal, sales tax, and overall total for the receipt shown below:

STASSLER'S HARDWARE

ORDER 34		MARCH 30, 2008
2 GALLONS PAINT@$23.95		$47.90
STEP LADDER		$27.99
BATTERIES SALE VAL-U-PACK		$17.59
KEY DUPLICATE		$3.95
	SUBTOTAL	(A)
	6.25% SALES TAX	(B)
	TOTAL	(C)

B. Finding a Price before Tax

12. Sandy has budgeted $450 for back-to-school shopping this year for her son. The sales tax rate where she lives is 6.75%. Assuming this tax rate applies to everything she buys, how much can she spend before tax and still fall within her budget?

13. Abe figures that he can afford to spend up to $12,000 for a car. The sales tax rate in his area is 7½%. Since that $12,000 figure includes the amount he would pay for sales tax, what is the maximum price he can actually afford to pay for the car?

14. Devoura Pizza offers a lunch special: two slices and a medium soda for $4.00 including tax. If the sales tax rate is 7%, what is the before-tax price?

15. Roberto's Tree Service quotes a price of $475 to remove a tree from a property, including 4.25% sales tax. What is the price before tax?

16. A shoe store advertises a special "We pay the sales tax!" promotion. During this promotion, sales (including tax) totaled $9,504.95. If the sales tax on shoes is 5.85%, calculate (a) the total sales before tax and (b) the amount of sales tax that will be due on these sales.

17. Brad's Furniture Village had a weekend "no sales tax" sale. Of course, sales tax actually must be paid; since Brad did not charge it to his customers, the business will have to pay it. The total sales for the weekend amounted to $39,558.03 (including tax). If the sales tax rate is 7¾%, how much will Brad have to pay from this weekend sale?

C. Sales Tax Tables (Optional)

18. Use the sales tax table given in the text of this section to find the sales tax on purchases of:
    a. $88.35
    b. $12.02
    c. $35.73
    d. $108.50
    e. $17.91

D. Grab Bag

19. If the sales tax rate is 12.25%, find the total price including tax of a toaster oven priced at $19.95.

20. Rene bought a self-powered radio over the Internet for $49.95. Since the merchant was out of state, no sales tax was charged. The use tax in Rene's area is 8%. How much use tax is Rene supposed to pay on this purchase?

21. Jervais went to an appliance store to buy a dishwasher. The sales tax where he lives is 8.25%, but the law in his area exempts "capital improvements" to a home from sales tax. If he buys the dishwasher and takes it home himself, it is not considered a "capital improvement" to his house and he will have to pay sales tax. If he has it installed by the dealer, it is a "capital improvement" and no sales tax will be charged.

    The price of the dishwasher is $739.95. The dealer charges $50 for installation. How much will he pay if he installs it himself? How much will he pay if the dealer installs it for him?

22. Kyle runs a concession stand at a local baseball stadium. To make things simpler and keep sales moving quickly, he includes sales tax in all the prices at his stand. Last Saturday, the total receipts at his stand were $845.13. The sales tax rate is 5.25%. How much of his sales does Kyle need to pay as sales tax?

23. The sales tax rate where I live is 7.75%. I bought a pair of dress slacks last week and lost the receipt. I want to know the price of the slacks. My credit card statement shows a charge of $75.37 for this purchase, but that amount includes the sales tax. What was the price of the slacks?

24. Suppose that the sales tax on prescription drugs is 3% and on other purchases it is 7%. Lana went to her local pharmacy and filled two prescriptions costing $39.45 and $81.02. She also bought two bottles of contact lens solution for $4.49 each, a paperback novel for $7.95, and a bottle of shampoo for $4.59. Find the total of her purchases including sales tax.

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25. Suppose that the state where Clay lives charges a 4% sales tax rate. Clay lives in Huron County, which charges a 3% sales tax in addition to the state tax. Five miles away in Champlain County, the county sales tax rate is also 3%, but clothing is specifically excluded from county sales tax.

    Clay's wardrobe consists mostly of ripped jeans and T-shirts with witty but off-color cartoons on them. He has just taken a new job where he will need to dress business casual, and so he desperately needs to do some clothes shopping. If he spends $650 before taxes, how much would he save by doing his shopping in Champlain County instead of Huron County?

26. Calculate the missing subtotal, sales tax, and overall total for the receipt shown below. The sales tax on clothing is 4.5% and the sales tax on other items is 8.25%.

HAUSDORFF'S DEPARTMENT STORE

ORDER 41	SEPTEMBER 19, 2008
MEN'S DRESS SHIRT	$47.90
WINDBREAKER END OF SEASON CLEARANCE	$27.99
BREAD MACHINE SALE	$57.59
SILK NECKTIE	$17.49
EMERGENCY RADIO KIT	$33.95
4.5% TAX RATE SUBTOTAL	(A)
8.25% TAX RATE SUBTOTAL	(B)
4.5% SALES TAX	(C)
8.25% SALES TAX	(D)
TOTAL	(E)

27. Dan has budgeted $3,000 for materials to refinish his basement. The sales tax rate in his area is 5%. How much can be afford to pay for materials before taxes and still fall within his budget?

28. Gino's Hots menu is shown here. Calculate the menu prices before tax.

Gino's Hots

Egg and Cheese Sandwich	$3.75
Red Hot Hot	$2.50
Bacon Cheese White Hot	$5.25
Veggie Hot	$4.00
Dumpster Plate (2 hots, 2 burgers, mac salad, fries, hot sauce)	$7.50
Sodas	$1.25

All prices include 6.75% sales tax!

29. Find the sales tax due on a $148.95 purchase if the tax rate is 6.95%.

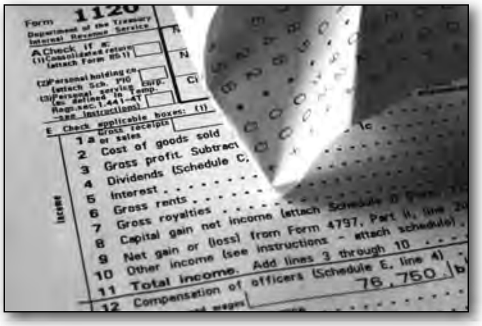
30. Calculate the use tax on a $749.95 computer if the tax rate is 9.15%.

E. Additional Exercises

31. Leo bought a new car. The price before tax was $20,357. The price after tax was $21,832.88. What was the sales tax rate?

32. Create a sales tax table similar to the one shown at the end of this section for a sales tax rate of 4%.

9.2 Income and Payroll Taxes



Income taxes are an unpleasant, but unavoidable, fact of financial life. © C. Sherburne/PhotoLink/Getty Images/DIL

The 16th Amendment to the U.S. Constitution, ratified in 1913, provides that "The Congress shall have power to lay and collect taxes on incomes, from whatever source derived. . . ." This constitutional amendment authorized what has become the modern U.S. income tax system.[2] The income tax has become an enormous source of revenue to the federal government, and the tax code has become enormously complex, with numerous special exceptions and rules. The income tax landscape is further complicated by the fact that most states have some sort of personal (i.e., paid by individuals) income tax (the only exceptions as of this writing are Alaska, Florida, Nevada, South Dakota, Texas, Washington, and Wyoming). Some counties and cities do as well. Corporate income taxes differ from personal income taxes, and also vary by jurisdiction. Of course no one likes to pay taxes, and income taxes are often singled out for particular disdain, but however one feels about income taxes they are a major financial issue for both individuals and businesses.

Because the tax laws are so complex, in this chapter we will cover only the basic principles and the mathematics needed to put those principles into action. You should understand that

[2]The 16th Amendment is a popular topic with conspiracy theorists and "tax resistors" who claim on various grounds that it was never properly ratified and that the entire income tax system is therefore illegal. It is not hard to find websites that promote this idea. There have been many, many court cases where individuals have tried to use this as a defense for not having paid their income taxes. Unsuccessfully. As appealing as not having to pay your taxes might sound, courts have repeatedly ruled that the income tax rests on solid legal ground.

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the discussion and examples included in this chapter are intended to illustrate the income tax system in general terms, but there are so many exceptions, details, and special cases that it would be impossible to cover them all in detail here. It should be understood that, in any particular situation, there may be facts that would cause taxes to be determined in a way that differs—possibly significantly—from what is discussed here. In addition, it should be noted that income tax rates are rules that can and do change from one year to the next, and so the rates that are in effect when you read this will probably not be exactly the same as the ones given here.

Calculating Personal Income Taxes

The first step to calculate the amount of income tax to be paid is to determine just how much income is subject to tax! While income from almost any source is potentially taxable, not every dollar of income is actually taxed. An individual's (or married couple's) ***taxable income***, the income upon which income tax is actually paid, is found by making certain subtractions from actual income.

Tax exemptions and ***deductions*** are subtracted from income before the tax is calculated. Taxpayers are generally allowed to ***exempt*** a certain amount of income ($3,300 in 2006, though the amount generally increases from year to year) for themselves and for each person they can claim as a dependent. You may not take an exemption for yourself if someone else (typically a parent) claims you as a dependent; your ability to claim exemptions may also be limited if your income is high.

In addition, many expenses are ***tax deductible***, meaning that they can be subtracted from income before calculation of the amount of tax owed. Tax deductible expenses include mortgage and some student loan interest, many state and local taxes, charitable contributions, medical expenses (above a certain threshold), some educational expenses, and many, many others, though each of these categories is subject to certain limitations. Taxpayers with significant tax deductible expenses file a form with their tax returns where they ***itemize*** these deductions, listing the expenses of each type so that they can be deducted. Taxpayers who choose not to itemize usually may take a ***standard deduction*** instead. The standard deduction amount in 2006 was $5,150 for a single taxpayer; a married couple filing their taxes jointly could take a $10,300 standard deduction (again, these amounts generally increase from one year to the next).

There are other adjustments that may be made to someone's taxable income, though since these other adjustments would apply only in special situations we will not address them here.

Example 9.2.1 ***Luisa and Alfonso are a married couple who had a joint income of $68,579 in 2006. They have two dependent children. They paid $3,450 in state and local taxes in 2006, and gave $3,700 to charity. They have no other tax deductible expenses, and no other special adjustments to their taxable income. What was their taxable income for 2006?***

Luisa and Alfonso can claim four exemptions: two for themselves and two for their dependent children. At $3,300 that works out to 4($3,300) = $13,200. Also, they could deduct $3,450 + $3,700 = $7,150 for their state and local taxes and charitable contributions; however, since the standard deduction for a married couple is $10,300 they are better off taking the standard deduction instead of itemizing. Their total taxable income therefore was $68,579 − $13,200 − $10,300 = $45,079.

The rates used to calculate income taxes vary, depending on the taxpayer's income level. Lower incomes pay lower rates; higher incomes are subject to higher rates. Tax rates set up in this way are known as ***progressive tax rates***. (While there are many advocates for a single ***flat tax*** rate that would apply to all levels of income, progressive rates are likely to remain in use for the foreseeable future.) The specific rates used can be, and often are, changed by Congress, and do in fact vary over time. Nonetheless, the 2006 tax rate schedules for a married couple filing their taxes jointly, given below, are typical.

2006 INCOME TAX RATES—MARRIED FILING JOINTLY OR QUALIFYING WIDOW(ER)[3]

If Taxable Income is Over	*But not Over*	*The Tax is:*
$0	$15,100	10% of the amount over $0
$15,100	$61,300	$1,510.00 plus 15% of the amount over $15,100
$61,300	$123,700	$8,440.00 plus 25% of the amount over $61,300
$123,700	$188,450	$24,040.00 plus 28% of the amount over $123,700
$188,450	$336,550	$42,170.00 plus 33% of the amount over $188,450
$336,550	No limit	$91,043.00 plus 35% of the amount over $336,550

Note that different tax rates are used, depending on whether the taxes are being filed for a single individual or for a married couple filing together.

Example 9.2.2 ***Calculate the 2006 income tax for Luisa and Alfonso (from Example 9.2.1).***

Their taxable income was $45,079. Since this is more than $15,100 but less than $61,300, according to the table their taxes should be $1,510.00 plus 15% of the amount over $15,100. Then $45,079 − $15,100 = $29,979, so applying a 15% rate to this would give (0.15)($29,979) = $4,496.85. So their total tax would be $1,510.00 + $4,496.85 = $6,006.85.

Alternatively, we could have written the calculation in this way:

$$\begin{aligned} \text{Tax} &= \$1{,}510.00 + (0.15)(\$45{,}079 - \$15{,}100) \\ &= \$1{,}510.00 + (0.15)(\$29{,}979) \\ &= \$1{,}510.00 + \$4{,}496.85 = \$6{,}006.85 \end{aligned}$$

The income levels at which the different tax rates apply depend on the taxpayer's ***filing status***. The table we used for this example was for married couples filing their taxes jointly. Most taxpayers file as either "single" or "married filing jointly." Taxpayers are also allowed to file as "head of household" or "married filing separately." The 2006 tax rate tables for all four of these filing categories are given below:[4]

2006 INCOME TAX RATES—SINGLE TAXPAYERS

If Taxable Income is Over	*But not Over*	*The Tax is:*
$0	$7,550	10% of the amount over $0
$7,550	$30,650	$755 plus 15% of the amount over $7,550
$30,650	$74,200	$4,220.00 plus 25% of the amount over $30,650
$74,200	$154,800	$15,107.50 plus 28% of the amount over $74,200
$154,800	$336,550	$37,675.50 plus 33% of the amount over $154,800
$336,550	No limit	$97,653.00 plus 35% of the amount over $336,550

2006 INCOME TAX RATES—MARRIED FILING JOINTLY OR QUALIFYING WIDOW(ER)

If Taxable Income is Over	*But not Over*	*The Tax is:*
$0	$15,100	10% of the amount over $0
$15,100	$61,300	$1,510.00 plus 15% of the amount over $15,100
$61,300	$123,700	$8,440.00 plus 25% of the amount over $61,300
$123,700	$188,450	$24,040.00 plus 28% of the amount over $123,700
$188,450	$336,550	$42,170.00 plus 33% of the amount over $188,450
$336,550	No limit	$91,043.00 plus 35% of the amount over $336,550

[3]Source: U.S. Internal Revenue Service, www.irs.gov. Reformatted.

[4]Source: U.S. Internal Revenue Service, www.irs.gov. Reformatted.

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2006 INCOME TAX RATES—MARRIED FILING SEPARATELY

If Taxable Income is Over	*But not Over*	*The Tax is:*
$0	$7,550	10% of the amount over $0
$7,550	$30,650	$755.00 plus 15% of the amount over $7,550
$30,650	$61,850	$4,220.00 plus 25% of the amount over $30,650
$61,850	$94,225	$12,020.00 plus 28% of the amount over $61,850
$94,225	$168,275	$21,085.00 plus 33% of the amount over $94,225
$168,275	No limit	$45,521.50 plus 35% of the amount over $168,275

2006 INCOME TAX RATES—HEAD OF HOUSEHOLD

If Taxable Income is Over	*But not Over*	*The Tax is:*
$0	$10,750	10% of the amount over $0
$10,750	$41,050	$1,075.00 plus 15% of the amount over $10,750
$41,050	$106,000	$5,620.00 plus 25% of the amount over $41,050
$106,000	$171,650	$21,857.50 plus 28% of the amount over $106,000
$171,650	$336,550	$40,239.50 plus 33% of the amount over $171,650
$336,550	No limit	$94,656.50 plus 35% of the amount over $336,550

While different tables are used for each filing status, the tax calculation can be done in essentially the same way.

Example 9.2.3 ***Zoe's taxable income for 2006 was $43,000. Her filing status is single. Based on the tax table given above, how much did she have to pay in federal income tax?***

According to the table, Zoe's tax would be $4,220.00 plus 25% of the amount over $30,650. Then $43,000 − $30,650 = $12,350. Applying the 25% rate to this gives (0.25)($12,350) = $3,087.50. So her total income tax would be $4,220.00 + $3,087.50 = $7,307.50.

The term ***tax bracket*** is sometimes used to indicate which of these rate levels a taxpayer falls into. For example, Zoe from the previous example would fall into the 25% tax bracket. This wording sometimes causes a misunderstanding about how income taxes are calculated. People often mistakenly assume that someone in the 25% bracket pays 25% of her taxable income in income taxes. This is not correct. Zoe's taxes are *not* 25% of her taxable income (which would have amounted to $10,750).

Example 9.2.4 ***Zoe (from Example 9.2.3) is in the 25% tax bracket. What percent of her taxable income does she actually pay in federal income taxes? If her actual income (before deductions) was $58,375, what percent of her actual income does she actually pay in federal income taxes?***

Her income tax was $7,307.50 on $43,000 of taxable income. As a percent, $7,307.50/$43,000 = 16.99%.

As a percent of her actual income, her tax would be $7,307.50/$58,375 = 12.52%.

The first $7,550 of an individual's taxable income is taxed at the lowest 10% rate, regardless of how much total income she has. Your first $7,550 of income would be taxed at the same rate as Zoe's, or LeBron James's, or Bill Gates's for that matter. Likewise, the amount of Zoe's income between $7,550 and $30,650 is taxed at the same 15% rate as anyone else's. Actually her taxes are calculated by breaking her income up by level:

Tax Rate	*Zoe's Income Subject to Rate*	*Zoe's Tax on This Income*
10% (up to $7,550)	$7,550	$755.00
15% (from $7,550 to $30,650)	$23,100	$3,465.00
25% (from $30,650 to $74,200)	$12,350	$3,087.50
Total	$43,000	$7,307.50

Notice that the total tax on the first two rows adds up to $4,090.00, the amount from the tax table. This is where the "$4,090.00 plus 25%" in the tax table comes from. Since anyone in the 25% bracket will have these same tax rates applied to his first $30,650 of income, instead of calculating the tax in this detail over and over again, the table just gives the "$4,220.00 plus 25%" formula. This also helps avoid any confusion over whether the 25% rate applies to all income of just some of it.

Notice that even though Luisa and Alfonso's taxable income was larger than Zoe's, they are in a lower tax bracket and pay less in tax than Zoe did as a single taxpayer. This is because their income is combined, and the tax rates that apply to married couples have higher cutoffs for each tax bracket to reflect this fact. Nonetheless, it is often the case that, overall, a married couple winds up paying more in taxes filing jointly than they would have as individuals, particularly if both spouses have income from employment. This is sometimes referred to as the ***marriage penalty***. However, this is not always the case. If Luisa (or Alfonso) does not work outside the home and most or all of their combined income is earned by one person, they may end up paying quite a bit *less* in tax that they would have as individuals.

There are some other details that may affect the overall amount of tax owed. The rates given above apply to income from most sources, referred to as ***ordinary income***. Certain types of income, however, may be taxed at different rates. For example, income from ***capital gains*** (the profit from selling stocks or other investments at a higher price than paid) may be taxed at a lower rate than ordinary income; there is also currently a lower tax rate for dividends paid to stockholders. Also, once the tax is calculated, the amount of tax may be reduced by special ***tax credits***. Tax credits are offered for families with children, for child care expenses, for adoption expenses, for buying a hybrid car, and many other things. The specific items for which tax credits are offered, and the requirements for being able to claim those credits, change frequently.

There has been much discussion in the news over the last few years of the ***alternative minimum tax***, or ***AMT***. The AMT is an alternative way of calculating the income tax due, one that uses different rates and does not allow for as many tax deductions as the usual formula. If the amount a taxpayer would owe under the AMT is larger, that is the amount of tax that must be paid. The AMT was originally intended to make sure that very high income taxpayers did not avoid paying income taxes by taking unusually large tax deductions. Unfortunately, the dollar amounts specified by the law were not adjusted for inflation, and as prices have risen over time incomes that were once considered very high and deductions that were once unusually large are no longer so high or unusual. The AMT now affects many more taxpayers than it was originally intended to. Because of the complexity involved, and because the rules for the AMT are likely to change over the next few years, our discussion of this topic will be limited to mentioning its existence.

State income taxes are generally calculated in much the same way as federal taxes, except that the various states have different rates, and may have different rules for dependent exemptions and what items are and are not deductible. Some states mirror the federal rules quite closely, while others may differ in significant ways. Also, while most states use progressive tax rates, there are several states (including, as of this writing, Colorado, Indiana, Illinois, Massachusetts, Michigan, and Pennsylvania) that employ a flat state income tax rate.

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Income Tax Withholding

While income taxes are calculated and owed on the basis of the taxpayer's annual income, the government is not content to sit around and wait until the end of the year to collect the tax owed. The income tax you owe for your income must be paid when you earn it. To make sure that this happens, employers are responsible for ***withholding*** money for taxes from their employees' paychecks and forwarding this money to the Internal Revenue Service (for federal taxes) and state (and/or city or county) tax agencies.

An employer is responsible for withholding taxes on the basis of withholding rates published by the IRS (and state/local tax agencies for state/local taxes). The employer then forwards the taxes withheld to the appropriate government tax collection agencies. The 2006 withholding rates for an employer using a biweekly pay schedule are given as an example below:[5]

SINGLE PERSON (INCLUDING HEAD OF HOUSEHOLD)

IF THE AMOUNT OF WAGES (AFTER SUBTRACTING WITHHOLDING ALLOWANCES) IS:		THE AMOUNT OF INCOME TAX TO WITHHOLD IS:	
Over	***But not Over***		***of Excess Over***
$0	$102	$0	
$102	$385	10%	$102
$385	$1,240	$28.30 plus 15%	$385
$1,240	$2,817	$156.55 plus 25%	$1,240
$2,817	$6,025	$550.80 plus 28%	$2,817
$6,025	$13,015	$1,449.04 plus 33%	$6,025
$13,015		$3,755.74 plus 35%	$13,015

MARRIED PERSON

IF THE AMOUNT OF WAGES (AFTER SUBTRACTING WITHHOLDING ALLOWANCES) IS:		THE AMOUNT OF INCOME TAX TO WITHHOLD IS:	
Over	***But not Over***		***of Excess Over***
$0	$308	$0	
$308	$881	10%	$308
$881	$2,617	$57.30 plus 15%	$881
$2,617	$4,881	$317.70 plus 25%	$2,617
$4,881	$7,517	$883.70 plus 28%	$4,881
$7,517	$13,213	$1,621.78 plus 33%	$7,517
$13,213		$3,501.46 plus 35%	$13,213

Note that different rates apply depending on filing status, and these rates apply to the amount of wages after subtracting ***withholding allowances*** (withholding allowances are roughly equivalent to the deductions for dependents). This reflects the effects of different filing statuses and numbers of dependents. Each employee is required to file a ***W-4 form*** with his employer to indicate his dependent exemptions and filing status; the employer then bases his tax withholding on the information supplied on this form.

The amount of each withholding allowance is the annual allowance divided by the number of pay periods in the year. For example, the annual allowance for 2006 is $3,300, so for someone paid biweekly the withholding allowance would be $3,300/26 = $126.92 for each exemption claimed. Also, the amount an employee pays toward the cost of benefits like health insurance or contributes to a retirement account like a 401(k) is normally not taxable and hence is not included when calculating the amount of withholding.

[5]Source: U.S. Internal Revenue Service, www.irs.gov. Reformatted.

Example 9.2.5 ***Erik is paid biweekly. He is married, and claims three exemptions on his tax return (one for himself, one for his wife, and one for a dependent child). After he takes out his contributions for health insurance coverage and his 401(k), his gross taxable pay is $1,535.06. How much should his employer withhold for federal income taxes?***

Each exemption is worth $126.92, so his three exemptions amount to (3)($126.92) = $380.76. Subtracting this from his income leaves $1,535.06 − $380.76 = $1,154.30. For this amount, his employer should withhold $57.30 plus 15% of the excess over $881, so the withholding is $57.30 + (0.15)($1,154.30 − $881) = $57.30 + (0.15)($273.30) = $57.30 + $41.00 = $98.30.

The tax withholding tables are intended to make sure that enough money is withheld from Erik's paychecks to cover his income tax liability for the year. Unfortunately, there is much that could affect the amount of taxes he owes in the end that these tables don't—and couldn't reasonably be expected to—take into account. They are based on *reasonable* assumptions for a *typical* taxpayer, but one size does not fit all. If Erik's wife does not earn any income from work outside of the home, they may be in a different tax bracket than if she earns a large income. Erik may have income from work he does on the side, or income from investments. Or he may have made large contributions to charity, or paid a lot in mortgage interest or medical expenses, and therefore claim much larger itemized deductions that could make his ultimate tax total much lower than would be usual. None of these possibilities are factored into his paycheck withholding. Regardless of the amount withheld, though, he is still responsible for making sure that his tax withholding is adequate to cover his overall tax bill.

If Erik expects that his withholding will not be enough to cover his ultimate tax bill, he may be required to make ***estimated tax*** payments. Self-employed taxpayers, those who receive much of their income from investments, or those whose tax withholding falls short of their overall tax liability for whatever reason must make quarterly estimated tax payments, based on their earnings for that quarter. If, however, he has just a moderate amount of additional income and doesn't want to be bothered with estimated tax payments, he can request that his employer base withholding taxes on fewer exemptions or the higher single rate.

Example 9.2.6 ***Suppose that Erik files a new W-4 form claiming 0 exemptions and requesting withholding at the higher single rate. How much would be withheld from his paycheck in that case?***

Using the single table and the full $1,535.06, we calculate that his withholding will be $156.55 plus 25% of the amount over $1,240. Thus the withholding is $156.55 + (0.25)($1,535.06 − $1240) = $156.55 + (0.25)($295.06) = $156.55 + $73.77 = $230.32. Note that this is much higher than the withholding with three exemptions and the married rate.

Failing to have enough money withheld means facing a large tax bill later on, and also may result in significant penalties. To avoid these unpleasant possibilities, claiming fewer than the allowed number of exemptions and a single rather than married filing status is quite commonly done.

Even though the withholding calculations presented here are indicative of how things work in general, there are a number of other details that can affect the withholding rate. (The IRS document that spells out all the details of withholding calculations is 68 pages in length, which should give you a pretty good sense of the complexity involved.)

Tax Filing

Remember that the taxes withheld from your paycheck and/or estimated tax payments that you make are not your actual income taxes. The income taxes that you owe for a year can only be determined after the end of that year, once your total income and deductible expenses are known. To determine your actual income taxes, each year you as a taxpayer must file an ***income tax return*** with the Internal Revenue Service.

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After the year has passed, each taxpayer files a tax return with the appropriate tax agencies, listing income, deductions, etc. and calculating the total tax owed based on this income. If the amount withheld and/or paid in estimated taxes is more than the actual amount owed, the taxpayer receives a refund of the difference. If the amount withheld or paid in estimated taxes is less than the amount owed, the taxpayer must pay the difference, and may be charged penalties and interest for the underpayment.[6]

Example 9.2.7 ***Suppose that Erik's actual income taxes for the year are $4,795.16. Determine the amount he would owe or would receive as a refund after he files his taxes if he had taxes withheld for the entire year at (a) the rate calculated in Example 9.2.5 and (b) the rate calculated in Example 9.2.6.***

(a) In Example 9.2.5 we calculated Erik's withholding to be $98.30 for each of his 26 paychecks, and so his total withholding for the year would be 26($98.30) = $2,555.80. This is less than his taxes, so he must pay the $4,795.16 − $2,555.80 = $2,239.36. He may also have to pay penalties and interest as well.

(b) In Example 9.2.6 we calculated Erik's withholding to be $230.32 for each paycheck, so his total withholding would be 26($230.32) = $5,988.32. This is more than he owes, so he will receive a $5,988.32 − $4,795.16 = $1,193.16 refund.

FICA

Income tax withholding is not the only tax collection for which an employer is responsible. The taxes that fund the U.S. Social Security system and Medicare are also collected by withholding from an employee's paycheck. The law that requires these payments is known as the ***Federal Insurance Contributions Act***, and hence these taxes are commonly referred to by the acronym ***FICA***.

For 2006, the Social Security portion of the FICA tax was 6.2% up to $94,200 of gross wages. This percent is paid by both the employee and the employer, so in actuality this totals 12.4%. The Medicare portion of FICA was 1.45%, also paid by both employer and employee, though without any upper limit.

Example 9.2.8 ***Suppose that Lesley has a gross (i.e., before deductions) salary of $43,500 per year and is paid biweekly. She pays $150 with each paycheck for deductions for health insurance or other benefits. How much will be deducted from her biweekly paycheck for FICA?***

Lesley's gross biweekly pay is $43,500/26 = $1,673.08. Subtracting out her deductions leaves $1,673.08 − $150 = $1,523.08. The Social Security tax will be (6.2%)($1,523.08) = $94.43. The Medicare tax is (1.45%)($1,523.08) = $22.08. So her total FICA deduction will be $94.43 + $22.08 = $116.51. Note that her employer will also pay an equal amount of tax.

One of the objections to the FICA tax is that unlike the progressive income tax rates, the Social Security tax is actually in some ways ***regressive***: higher incomes pay a *lower* rate (as a percent of total income). The following example will illustrate this objection.

Example 9.2.9 ***Ramon has a gross annual salary of $148,900. Calculate his total annual FICA taxes.***

The Social Security tax applies only to the first $94,200 of income: (6.2%)($94,200) = $5,840.40. The Medicare tax does not have an upper limit, so it amounts to (1.45%)($148,900) = $2,159.05. The total FICA tax is thus $5,840.40 + $2,159.05 = $7,999.45. Once again, this amount will need to be matched by his employer.

[6]Penalties do not apply in every situation; there are a number of rules that exempt you from paying a penalty, particularly if the amount owed is small or due to a special situation. However, whenever the amount of tax withheld is less than the amount owed, there is at least the possibility of penalties being imposed.

In practice, Ramon will probably have the full 6.2% taken from his paycheck until the $5,840.40 maximum is reached, at which point Social Security taxes will stop being withheld from his check until the beginning of the next year.

Because deductions, credits, and so on do not apply to FICA taxes, there is no need to file a tax return for them at the end of the year. The sole exception would be the rare situation where someone earns more than the Social Security maximum, but has that income coming from more than one employer. For example, if you earn $89,000 from one job and $37,000 from another job on the side, both employers will withhold the full 6.2% for Social Security (since you are not reaching the maximum at either of the two jobs, and one employer cannot be expected to keep track of how much you have been paid by another). Since the tax actually applies to only $94,200 of income, though, you would be eligible to claim a refund of any over payment at the end of the year.

For the self-employed, FICA taxes must be handled a bit differently, since you are your own employer in that case. For the self-employed, FICA taxes become the so-called ***self-employment tax***. A self-employed worker must pay both the employer and employee share of FICA. This tax is included in the quarterly estimated tax payments that would be made, and a final calculation of the total owed for the year is done as part of the usual annual income tax filing.

Example 9.2.10 ***Heather is a self-employed plumber who earned $47,600 in 2006. Calculate her self-employment tax.***

Her Social Security tax is (2)(6.2%)($47,600) = $5,902.40. Her Medicare tax is (2)(1.45%)($47,600) = $1,380.40. Note that we multiplied both of these by 2 since she is paying both employer and employee shares of the tax. The total is $5,902.40 + $1,380.40 = $7,282.80.

Sometimes, other payroll taxes for things such as state disability insurance taxes or other similar taxes must be paid. Generally, these are small in comparison to Social Security and Medicaid; mathematically they usually work in much the same way.

Note that FICA taxes are paid on income from employment only. Income from interest, investments, and so on is not subject to this tax.

Business Income Taxes

Just as the incomes of individuals are subject to tax, so are the incomes (i.e., profits) of businesses. A business owner must make sure to collect the appropriate taxes from her employees, pay her share of the FICA payroll taxes, and must also make sure to pay the appropriate taxes on the business's income as well.

When the business is set up as a sole proprietorship, its owner normally would treat income from the business just like any other income, reporting it on her personal income tax return and paying income tax. Business income generally requires additional forms to be filed with the tax return, and may be subject to somewhat different rates and rules, but is handled as just another type of income.

When the business is set up as a corporation, the corporation itself is a taxpayer, and may be required to file a tax return and pay income taxes on its own account. The tax rates and laws that apply to corporations are conceptually similar to those for individuals, but differ in specific details. Some corporations, known as ***subchapter S corporations***, do not pay income taxes themselves but instead pass their earnings directly to the stockholders. This may be done to simplify the tax filings, and also to avoid the ***double taxation*** of earnings. The earnings of a corporation are subject to income tax for the corporation, and then taxed again as income to the stockholders when paid out in dividends. Subchapter S corporations avoid this issue. Subchapter S corporations are popular among small businesses, but subchapter S status is subject to significant limitations and restrictions. It is not an option for large businesses.

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EXERCISES 9.2

In the exercises for this section, use the 2006 tax year rates given in this section. Specifically:

- *The personal exemption allowance is $3,300 per person.*
- *The standard deduction is $5,150 for a single person and $10,300 for married filing jointly.*
- *The Social Security payroll tax rate is 6.2% up to $94,200.*
- *The Medicare payroll tax rate is 1.45% with no income limit.*
- *The income tax and withholding rates are given in the tables on pages 387–388 and 390.*

A. Calculating Taxable Income

1. Suppose that Dave and Jenni are married and had gross earnings of $78,953 last year. They have two dependent children, and did not have enough deductible expenses to bother itemizing their deductions. What is their taxable income for the year? How many personal exemptions can they claim?

2. Ken earned $31,440 last year. He is single and has one dependent child. He paid $2,075 in state and local taxes last year, and gave $775 to charity. Should he itemize or take the standard deduction? Calculate his taxable income. How many personal exemptions can he claim?

3. Sal earned $61,995 last year. He is single and has no dependents. He paid $5,075 in state and local taxes, gave $3,000 to charity, paid $5,804 in mortgage interest, and had $1,200 in deductible medical expenses. Calculate Sal's taxable income. How many personal exemptions can he claim?

4. Steve and Nancy are married and earned $43,004 last year. They have five dependent children. They gave $4,000 to charity, paid $6,305 in mortgage interest, paid $2,084 in state and local taxes, and had $3,500 in deductible medical expenses. Find their taxable income for the year. How many personal exemptions can they claim?

B. Calculating Income Taxes

5. a. Calculate Dave and Jenni's federal income tax based on the information given in Exercise 1.
   b. What tax bracket are they in?
   c. What percent of their taxable income do they pay in federal income taxes?
   d. What percent of their gross income do they pay in federal income taxes?

6. a. Ken claims head-of-household status when filing his income taxes. Calculate his federal income tax based on the information given in Exercise 2.
   b. What tax bracket is Ken in?
   c. What percent of his taxable income does he pay in federal income taxes?
   d. What percent of his gross income does he pay in federal income taxes?

7. Calculate Sal's federal income tax based on the information given in Exercise 3.

8. Calculate Steve and Nancy's federal income tax based on the information given in Exercise 4.

9. Suppose Britt earned $38,995 last year. She is single and has two dependent children. Her state and local income taxes were $2,075 last year, she paid $3,073 in mortgage interest, and she gave $575 to charity. Her filing status is head of household. She has no other deductible expenses. Using this information, calculate her total federal income tax for the year.

10. Suppose Gene earned $37,450 last year and his wife Kelly made $32,095. They do not have any dependent children, and their deductible expenses were not large enough to itemize.
    a. If Gene were single, how much would he owe for federal income taxes for the year?
    b. If Kelly were single, how much would she owe for federal income taxes for the year?
    c. As a married couple filing their taxes jointly, how much do Gene and Kelly owe for federal income taxes for the year?

11. Suppose Jean earned $54,545 last year and her husband Kelvin earned $15,000. They have two dependent children.
    a. If Jean were single, how much would she owe for federal income taxes? (Assume that she claims herself and the two children as dependents.)
    b. If Kelvin were single, how much would he owe for federal income taxes? (Assume he claimed only himself as a dependent.)
    c. As a married couple filing their taxes jointly, how much do Jean and Kevlin owe for federal income taxes for the year?

12. What do Exercises 10 and 11 tell you about the "marriage penalty"?

C. Calculating Tax Withholding

13. Mike earns a $38,905 salary and is paid biweekly. His biweekly deductions for health insurance and other benefits amount to $177.92. Mike is single and claims three exemptions on his W-4 form. His filing status is single.
    a. Calculate Mike's biweekly gross salary.
    b. Calculate the amount of Mike's withholding allowances.
    c. Calculate Mike's biweekly income subject to tax withholding.
    d. How much will his employer withhold from his check for federal income taxes?

14. Lita makes $18,035 annual salary. She is paid biweekly. She has no deductions for health insurance or other benefits. She claims married filing status with four dependents on her W-4.
    a. Calculate Lita's biweekly gross salary.
    b. Calculate the amount of Lita's withholding allowances.
    c. Calculate Lita's biweekly income subject to tax withholding.
    d. How much will her employer withhold from her check for federal income taxes?

15. Surima earns $111,925 in salary. She is paid biweekly. Her biweekly deduction for health insurance and other benefits is $286.35. She is married and claims 0 exemptions on her W-4 form. How much should her employer withhold from each paycheck for federal income taxes? If she instead files her W-4 requesting taxes be withheld at the higher single rate, what would her withholding amount be?

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16. Tris has a part-time job, where his hours and hence his income vary from one biweekly pay period to the next. He is single, and claims one exemption on his W-4. His company does not offer him any benefits, and so he has no deductions for them. What would his federal tax withholding be for a pay period in which his gross pay was:

 a. $205.92

 b. $525.25

 c. $846.15

D. Annual Tax Filings

17. In 2006 Alan and Cathy earn a combined income of $75,372. Their taxable income was $50,037. Alan had $4,265.09 withheld from his paychecks for federal income taxes, and Cathy had $2,777.33 withheld.

 a. How much did they have withheld for federal taxes in total?

 b. How much do they actually owe in federal income taxes for the year?

 c. Will they have to pay extra when they file their taxes, or will they get a refund? How much will they owe or how much will their refund be?

18. Suppose that Lita's (from Exercise 14) husband earned $31,035 last year. He also filed his W-4 based on married status with four exemptions.

 a. Calculate the federal tax withholding from each of his biweekly paychecks.

 b. Calculate the total tax withhold for the year for both Lita and her husband.

 c. When they complete their tax return, they find that their total federal income taxes for the year are $3,109.62. Will they receive a refund or owe extra taxes? How much?

19. Suppose that Surima's (from Exercise 15) husband earned $52,045 last year and had $5,892.37 withheld from his paycheck for federal income taxes. When they file their taxes, they find that their total federal income taxes for the year are $27,500.

 a. If Surima filed her W-4 form for married withholding, how much will they owe or receive as a refund?

 b. If Surima filed her W-4 form for single withholding, how much will they owe or receive as a refund.

20. Suppose that Brian has a part-time job at a golf-course and his hours vary a lot over the course of the year. In the summer he works lots of hours, but in the winter months he usually only puts in a few hours each week. When he files his annual income tax return, would you expect that Brian would owe money or get a refund? Why?

E. FICA and Self-Employment Tax

21. How much total FICA taxes will I pay for the year if I earn a $39,456 gross salary with no deductions for benefits?

22. Calculate the total employee share of FICA taxes for the year for someone who earns $8,035 gross income. Assume no deductions for benefits.

23. How much total Social Security tax will you pay for the year if you earn $125,769 in gross salary? How much total Medicare tax will you pay for the year? Assume no deductions for benefits.

24. How much total self-employment tax would be paid by someone who had $80,543 in income from self-employment income?

25. Calculate the total self-employment tax for someone who made $27,300 as a self-employed software engineer.

26. Wally earned $32,453 in self-employment income last year. How much did he pay in self-employment taxes for Social Security? For Medicare?

F. Grab Bag

27. Ralph earns a $36,000 salary. He is paid biweekly. Ralph is married and has three dependent children, but he files his W-4 claiming single status and no exemptions. His biweekly deductions for health insurance and other benefits total $37.05.

    a. Calculate the amount of withholding from each of Ralph's paychecks for federal income taxes.

    b. Calculate the amount of FICA taxes for each paycheck.

28. *This exercise is a continuation of Exercise 27.* Ralph's wife Shara earns a $38,450 salary, and is also paid biweekly. She filed her W-4 claiming married status and two exemptions. Her biweekly benefit deductions are $87.55.

    a. Calculate her withholding from each paycheck for federal income taxes.

    b. Calculate the amount of FICA taxes for each paycheck.

29. *This exercise is a continuation of Exercises 27 and 28.* Suppose that Ralph and Shara file their federal income taxes, taking the standard deduction instead of itemizing. How much total federal income tax will they owe?

30. *This exercise is a continuation of Exercises 27 to 29.* Will Ralph and Shara receive a tax refund, or will they owe additional taxes? How much?

31. *This exercise is a continuation of Exercises 27 to 30.* What tax bracket are Ralph and Shara in? What percent of their gross income do they pay in federal income taxes?

Suppose that Shimura-Taniyama Design Associates has eight employees. The company uses a biweekly pay period. For Exercises 32 to 39, complete the table below by calculating the federal income tax withholding and FICA tax for each employee and each employee's net pay. (The business is located in Texas, which has no state income tax, so there is no state tax withholding.)

Exercise	Employee	Gross Pay	Deductions	Status	Exemptions	Withholding	FICA	Net Pay
32	Duplessis	$755.38	$0.00	Single	0			
33	Watson	$1,865.13	$265.04	Married	1			
34	Hsu	$1,808.19	$303.65	Single	0			
35	Gonsalves	$2,165.03	$108.45	Single	3			
36	Tansini	$1,303.55	$208.16	Married	5			
37	Rauschman	$2,545.00	$382.05	Married	4			
38	Lee	$4,089.05	$399.94	Single	0			
39	Becker	$1,776.76	$255.04	Married	5			

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G. Additional Exercises

40. What is the maximum percent of taxable income that a single person in the 25% tax bracket could actually have to pay in federal income taxes? What is the minimum percent?

41. A star baseball player signs a contract under which he will be paid $17.5 million next year. How much will he pay in Social Security payroll taxes? What percent of his income does this represent?

9.3 Property Taxes

Property taxes are, as the name suggests, taxes that must be paid on certain types of property. ***Real estate taxes*** are the most widespread and financially significant form of property taxes. In some places, there may also be a ***personal property tax*** in effect, which must be paid on certain types of personal property such as cars or other goods.

In the United States, real estate taxes are state or local taxes, often assessed by towns, counties, cities, and/or school districts. Since they are primarily local taxes, the amount of tax charged on property therefore will vary from one locality to the next. Two identical houses that sit directly across the street from each other may be charged significantly different real estate taxes if the street is a border between two different towns or school districts. While the rate of taxation may vary dramatically from one place to another, real estate taxes are a fact of life virtually everywhere.

Despite local variation, the overall level of real estate taxation tends to vary most dramatically from state to state. Even though these taxes are not necessarily state taxes, the overall tax structure in a state will affect how much of their revenues local governments expect to collect from real estate taxes. In states where much revenue is obtained from sales and income taxes and then shared by the state with local governments, the need for real estate taxes may be less than when this does not occur. In a state like New Hampshire, for example, where there is no sales tax and personal income taxes are quite limited, property taxes might be expected to be relatively high; the money to run the government has to come from somewhere. In states that collect more from sales and/or income taxes, the need for revenue from property taxes may be less, and hence the property tax rates may be lower. Of course, the overall level of government spending is often a much more significant factor. Sales and income taxes provide plenty of tax revenue in New York for example, yet property taxes are not known for being especially low in that state.

In this section, we will focus on real estate taxes. The mathematics underlying personal property taxes is similar, though the specific details of what types of property are taxed and how the tax is assessed and collected will vary from one jurisdiction to another.

Assessed Value

The amount of tax owed on a given property is determined by applying a tax rate to the property's ***assessed value***. The assessed value is a value assigned to a property for tax purposes. It is natural to assume that the assessed value of a property would be its actual market value. This is not necessarily the case.

First of all, it is not realistically possible to determine the actual market value of a given piece of property exactly. The market value of a property varies over time, and depends on a wide range of factors. The ***tax assessor***, the government official responsible for determining assessments, may take a reasonable appraisal of market value into account when determining a property's assessment, but will generally base the assessment on factors such as square footage, age of the structure, the number of bedrooms and bathrooms, lot size, and features like garages and pools. A tax assessor would not normally take into account things like the attractiveness of the landscaping, interior decoration, or age of the roof or furnace, even though those factors might well affect the property's market value.

Assessed values often differ from actual market value in another important respect. In some jurisdictions, the assessed values of properties may fall far below the realistic market value across the board. It is not uncommon for assessed values to be as low as half of a reasonable appraisal of market value. In some areas the assessed value of a property may be even lower; assessments that run 5% or 10% of a reasonable appraisal of market value are not unheard of. If Josh's house is assessed for $75,372, that figure may be in line with its actual market value. Or, depending on how appraised values are set where he lives, that assessed value might indicate a house worth many times that much.

While the assessed value need not be in line with actual market value, tax assessors must follow certain fairness guidelines when setting assessed values. If Josh's house is actually worth $250,000, he might feel as though he is getting a bargain being assessed at only $75,372. But if his neighbor has a similar house assessed at only $50,000, he is probably is not getting a fair deal. Fairness requires that similar properties should receive similar assessments. So even though the assessed value may differ dramatically from market value, we would expect that different properties in the same jurisdiction would be assessed at roughly the same percent of their market value. This percentage may be referred to as the ***uniform assessment percent***.

Example 9.3.1 ***Josh's house, assessed at $75,372, has a fair market value of around $250,000. Find the uniform assessment percent based on Josh's assessment, and use it to approximate the assessed value of a house worth $350,000 in this same tax jurisdiction.***

Josh's assessment is $75,372/$250,000 = 30.15% of market value. This is our best available estimate of the uniform assessment percent. Applying this to a $350,000 house, we arrive at a $105,525 assessed value.

It might also be reasonable to assume from these numbers that the uniform assessment percent is actually intended to be 30%. In that case, we arrive at a (30%)($350,000) = $105,000 assessed value.

In either case the calculation is an approximation, since Josh's house is probably not worth exactly $250,000, nor is the house in question likely to be worth exactly $350,000.

As much as we might want to have a precise percentage to apply uniformly to the exact fair market value of each property, the best we can reasonably hope for is consistency, not perfect uniformity, since this is all based on estimates. As discussed above, even in areas where properties are supposed to be assessed on their fair market value (called ***full assessment***), the assessed values will not exactly agree with actual market prices.

Still, even though perfect fairness is impossible, taxpayers have a right to expect assessments to be reasonably consistent in relation to actual property values. A taxpayer who believes that his assessment is unfair may file a ***tax grievance*** or ***appeal*** requesting that his assessment be lowered. For example, if Josh could demonstrate that residential properties in his area are assessed at an average of 20% of market value, he would probably have a strong case to appeal his assessment. (On the other hand, if Josh just doesn't like how high his tax bill is, but can't demonstrate that he is being treated worse than anyone else, he is unlikely to have much success.) Each jurisdiction has its own procedures for filing assessment appeals.

If we know the uniform assessment rate and a property's assessment, we can use these facts to find the fair market value being assumed, though that requires a bit of algebra.

Example 9.3.2 ***Suppose that the uniform assessment rate is supposed to be 25% in Boltonboro township. What market value does a $53,502 assessment suggest?***

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We should have:

$$(25\%)(\text{Market Value}) = \$53{,}502$$

To find the market value, we can divide both sides by 25% (i.e., by 0.25):

$$\frac{(0.25)(\text{Market Value})}{0.25} = \frac{\$53{,}502}{0.25}$$

$$\text{Market Value} = \$214{,}008$$

Calculating Real Estate Taxes on a Property

The tax rate may be expressed in a number of different equivalent ways. It may be expressed as a ***rate per thousand***. As the name suggests, a rate per thousand is simply the dollar amount of tax per thousand dollars of assessed value. For example, if the real estate tax rate in the Waterfield Central School District is $22 per thousand, the tax on a property assessed at $100,000 would be ($22 per thousand)(100 thousands) = $2,200.

Mills are an equivalent way of expressing a rate per thousand. A mill is 1/1,000 of a dollar, or 1/10 of a cent. (The name comes from the Latin word *mille* meaning thousand.) Rather than saying that the Waterfield Central School District (CSD) tax rate is $22 per thousand, we could equivalently say it is 22 mills. (It is understood that these mills are a rate per dollar, so we do not normally say "22 mills per dollar.") Either way, the amount of tax is expressed in a way that is based on 1/1,000 of the assessed value.

In theory, using a 22-mill rate, we would calculate the tax on a $100,000 assessed property value as ($0.022 per dollar)($100,000) = $2,200. In practice, however, we can just read "mills" as "dollars per thousand" While *technically* speaking this is not exactly the same as calculating the rates per thousand, in practice the result is the same. Therefore, we could also have done this in the same way as we did when it was stated as a rate per thousand: ($22 per thousand)(100 thousands) = $2,200.

A tax rate may also be expressed as a ***rate per hundred***. This is a bit less convenient than a rate per thousand, but it works in much the same way. Waterfield CSD could equally well express its tax rate as $2.20 per hundred—1/10 the per thousand rate since there are 10 hundreds in each thousand. Using this rate, we would find that an assessed value of $100,000 is $100,000/100 = 1,000 hundreds of assessed value [($2.20 per hundred)(1,000 hundreds) = $2,200].

Using a rate per hundred may seem awkward, but there is an equivalent way of expressing a rate per hundred that may seem much more familiar. The Latin word for hundred is *centum*—which is also the origin of the English word ***percent***. The $2.20 per hundred can be equivalently expressed as 2.2%. Using this form of the rate, we would calculate the tax as (2.2%)($100,000) = $2,200. Just as with mills versus rate per thousand, even though rate per hundred is not technically exactly the same as percent, in practice "rate per hundred" can be read as "percent."

Before moving on to an example, let's summarize this discussion:

If the Tax Rate is Expressed as:	***Calculate the Property Tax by:***
Rate per thousand	• Divide assessed value by 1,000 • Multiply by the rate per thousand
Mills	• Treat mills the same as a rate per thousand • Divide assessed value by 1,000 • Multiply by the rate per thousand
Percent	• Multiply the percent by the assessed value
Rate per hundred	• Treat the rate per hundred as a percent • Multiply the percent by the assessed value

Example 9.3.3 ***Josh's house has an assessed value of $75,372. He pays real estate taxes to his county, town, and school district. The county rate is 14 mills, the town rate is $8.57 per thousand, and the school district rate is $1.35 per hundred. How much does he pay to each of these, and how much does he pay in total?***

The assessed value of the property is \$75,372, this is 75,372/1,000 = 75.372 thousands. We will consider the county rate, expressed in mills, as a rate per thousand.

County tax: (\$14 per thousand)(75.372 thousands) = \$1,055.21.

Town tax: (\$8.57 per thousand)(75.372 thousands) = \$645.94

For the school district, it is most convenient to read \$1.35 per hundred as 1.35%. So:

School tax: (1.35%)(\$75,372) = \$1,017.52.

Overall, Josh's real estate taxes total \$1,055.21 + \$645.94 + \$1,017.52 = \$2,718.67.

Setting Property Tax Rates

Property tax rates are normally set by first determining the total amount of tax to be collected (called the ***tax levy***) and then dividing it by the total assessed value of taxable property. This result is then put in terms of a rate per thousand, or percent, or whatever the desired form to express the rate may be. The following example will illustrate this.

Example 9.3.4 ***The Bloome County legislature has set the overall property tax levy for 2008 to be \$39,600,000. The total assessed value of taxable property in the county is \$4,873,595,000. Determine the 2008 real estate tax rate for the county. Express the result as (a) a percent (to four decimal places), (b) a rate per hundred (to three decimal places), (c) a rate per thousand (to two decimal places), and (d) mills (to two decimal places).***

Dividing this tax levy by the taxable property gives a percent rate of:

$$\text{Rate} = \frac{\$39{,}600{,}000}{\$4{,}873{,}595{,}000} = 0.0081254354$$

(a) Moving the decimal place, we can express this as a 0.8125% rate.

(b) A rate per hundred is essentially the same as a percent, except expressed in dollars. Therefore the rate is \$0.813 per hundred.

(c) To find the rate per \$1,000, we can apply our original percent to a \$1,000 assessed value. Then (0.0081254354)(\$1,000) = \$8.13 per thousand.

(d) Mills are essentially the same as a rate per thousand. So the rate is 8.13 mills.

The table below summarizes the ideas used in this example:

If the Rate is Expressed as:	*Divide the Total Levy by the Assessed Value and:*
Rate per thousand	• Multiply the result by \$1,000
Mills	• Multiply the result by 1,000
Percent	• Move the decimal two places to the left (as usual)
Rate per hundred	• Rewrite the percent rate as a dollar amount

Rounding can be an issue with tax rates. To get the *exact* \$39,600,000 tax levy, the rate would have to be applied without any rounding, or at least carried out to quite a few decimal places. Any reasonable degree of rounding may cause the actual taxes charged to work out to more, or less, than the intended levy. This is why the instructions to Example 9.3.4 asked for the percent to more than the two decimal places that we have been customarily using with percents; the more decimal places, the less rounding, and hence the less difference due to rounding.

Example 9.3.5 ***Suppose that the county sets the tax rate to be \$8.13 per thousand. How much will the actual taxes total?***

(4,873,595 thousands)(\$8.13 per thousand) = \$39,622,327.35

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In this example we can see that rounding provides a bit more than the tax levy would have called for. In proportion to the overall tax levy, the difference is not all that large; however, if this is too large of an error, the only recourse is to carry the rate out to more decimal places.

Comparing Tax Rates

Comparing tax rates can be a tricky matter, because two different localities may use different assessment practices. For example, suppose that the real estate tax rate in the City of Eastville is $31.75 per thousand, while in the City of Westville the real estate tax rate is only $18.34 per thousand. Obviously, real estate taxes are lower in Westville, right? Despite appearances, we can't immediately tell from this information which city's taxes are lower. Even though Westville's rate is lower, it may be that properties in Westville are assessed at a higher rate.

Example 9.3.6 ***The real estate tax rate in Eastville is $31.75 per thousand, and properties are assessed at 50%. In Westville, the tax rate is $18.34 per thousand, and properties are fully assessed. Which city has the higher property taxes?***

To be able to make an "apples to apples" comparison, we will calculate the total tax owed on properties of equal actual value in each community. For convenience, let's use a $100,000 market value (any value could be used equally well).

In Eastville, we would expect a $100,000 property to be assessed at (50%)($100,000) = $50,000. The tax rate would then be (50 thousands)($31.75 per thousand) = $1,587.50.

In Westville, we would expect a $100,000 property to be assessed at $100,000. The tax rate would then be (100 thousands)($18.34 per thousand) = $1,834.00.

Realistically speaking, the tax rate in Westville is actually higher than in Eastville.

Tax rates can be misleading in comparing property taxes from year to year as well. Suppose that next year Westville lowers the tax rate to $18.00 per thousand. Good news for Westville's taxpayers, right?

Example 9.3.7 ***In 2007 the tax rate in Westville was $18.34 per thousand. In 2008, thanks to a strong real estate market, properties in Westville increase in value by 10% on average. The tax assessor kept up on this, and the assessed values also grew at the same rate, allowing the city to lower the tax rate to $18.00 per thousand. How much did property taxes drop between 2007 and 2008?***

In Example 9.3.5 we calculated that the 2007 property tax on a $100,000 house in Westville would have been $1,834.00.

On the basis of the average increase in assessed value, that $100,000 house would have increased in assessed value by 10%, so its 2008 assessed value would be $100,000(1.10) = $110,000. The 2008 property tax would be (110 thousands)($18.00 per thousand) = $1,980.00.

How much did the property tax drop? It didn't! In fact, we can divide $,1980/$1,834 = 1.0796 and see that, realistically speaking, property taxes actually ***increased*** *by 7.96%!*

Naturally, local politicians running for re-election will still want to talk about the decreasing tax rate.

Special Property Tax Rates

In some areas different types of property may be assessed at different rates. Residential property may be taxed at a higher, or lower, rate than commercial or agricultural property. Businesses may be offered special breaks on property taxes as an incentive to locate in a community, or reduced tax rates may be offered for veterans or senior citizens. Other properties (such as churches, schools, or hospitals) may be exempted from them altogether.

The legal language that surrounds these special property tax rates can often involve some highly specialized terminology, since the methods employed in assessing property taxes must follow procedures and regulations that are intended to make sure that things are done properly. Tax assessors are not allowed to cut special deals whenever they see fit! Naturally the laws and procedures and hence the terminology can vary from one jurisdiction to another.

Special property taxes may be handled by having different rates for different classes of property. Alternatively, a single tax rate can be used, but adjustments may be made to the assessed value.

Example 9.3.8 ***The commercial property tax rate in Incentiveburg is $14.99 per thousand. Globalco International Inc. built a factory there 5 years ago, and as part of an incentive package was granted a 35% reduction in property tax on this factory for 20 years. The assessed value of the factory is $10,375,000. What will be the company's real estate tax this year?***

Without the reduction, the company's tax would have been (10,375 thousands)($14.99) = $155,521.25. After a 35% reduction, the company will pay 100% − 35% = 65% of this tax. Therefore the taxes are (65%)($155,521.25) = $101,088.81.

Note that the same result could have been accomplished by leaving the tax rate just where it is and lowering Globalco's assessment.

Example 9.3.9 ***Suppose that the deal between Incentiveburg and Globalco specified a 35% reduction in assessment instead of a 35% reduction in property tax. What would the assessment have been? How would this have affected the taxes owed?***

Globalco's assessment would thus be reduced to 65% of what it otherwise would have been, or (65%)(10,375 thousands) = 6,743.75 thousands. Applying the usual tax rate to this assessment would give ($14.99 per thousand)(6,743.75 thousands) = $101,088.81. The actual tax is the same.

Another alternative would be to leave the assessment as is, but lower the tax rate by 35%.

Example 9.3.10 ***Suppose that the deal between Incentiveburg instead left the assessment where it otherwise would have been, but offered a 35% lower tax rate. What would the new rate have been? How would this have affected the taxes owed?***

The special tax rate would be (65%)($14.99) = $9.74 (assuming we round to two decimal places).

Applying this rate to the assessed value gives ($9.74 per thousand)(10,375 thousands) = $101,052.50. This is trivially less than in the other calculations, and the difference is due entirely to rounding. If the special tax rate were carried out to more decimal places, this difference would disappear.

Of course, if some taxpayers get a lower rate, naturally that means that more of the tax levy must come from others. This may seem unfair—and in some ways it is—but the simple truth is not always truly simple. If Incentiveburg had not made this offer of reduced taxes, Globalco might not have built the factory at all, in which case the town would not have received even the reduced taxes, and the community also would not have benefited from the jobs offered by the factory. On the other hand, Globalco might have built the factory anyway.

Selling the idea that giving a corporation a more favorable tax rate, though, can be a difficult sale, especially to taxpayers unhappy about high property taxes. Structuring tax incentives as an overall reduction, as reduction in assessment, or a reduction in rate is often a game of structuring a deal to conform to legal requirements and public demand, without actually changing the end result of the deal itself.

Exercise 25 provides an opportunity to explore how tax rates are affected by these incentives.

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EXERCISES 9.3

A. Calculating Real Estate Taxes

1. My house has an assessed value of $15,375. The property tax rate is $48.82 per thousand. Determine the property tax.

2. Bob's house has an assessed value of $113,045. The property tax rate is $2.17 per hundred. Calculate the property tax.

3. Rachel and Bruce own a house in the Northville Central School District. The school tax rate is $28.35 per thousand, and the house's assessed value if $285,000. Calculate the school district property tax on their house.

4. Suppose you own a house which has a $350,000 assessed value. The county property tax rate is 4.75 mills. Calculate your county property tax.

5. The city property tax rate in Chester Rock is 4.72%. Suppose you own a house in this city that is assessed at $68,305; find the city property tax.

B. Assessed Value

6. Felix owns a house that is assessed at $35,043. He believes that the fair market value of the house is $175,000. For this value, what would you expect the uniform assessment percent to be?

7. Sammi owns a property that is assessed at $78,345. She believes the fair market value of the property is around $315,000. For this value, what would you expect the uniform assessment percent to be?

8. My town uses full assessment. If the market value of my property is $184,000, approximately what should my assessment be?

9. The town of Lenoxburg uses a 15% uniform assessment percent.
   a. If the fair market value of Shayne's condo there is $457,000, what would you expect the assessment to be?
   b. If his condo is assessed at $69,225, would Shayne have a justification for filing a tax grievance?

10. a. If the uniform assessment percent is 20% and your property is assessed at $48,945, approximately what fair market value does this assume?
    b. If you believe that your property's actual market value is $300,000. Would you have a justification for filing a tax grievance?

11. a. If the uniform assessment percent is 40% and your property is assessed at $83,509, what fair market value does this assessment assume?

    b. Suppose that you believe that your property is actually worth $175,000. Would you have a justification for filing a tax grievance?

C. Setting Tax Rates

Use the same rounding conventions as used in the text of this section.

12. The total tax levy for the Hopewell Corners Union School District is $25,005,295. The total assessed value of property in the district is $475,925,000. Determine the property tax rate expressed as (a) a rate per thousand and (b) a rate per hundred.

13. The total tax levy for Stuyvesant County is $184,925,037, and the total assessed value of property in the county is $9,045,913,025. Find the property tax rate, expressed as (a) a rate per thousand, (b) a rate per hundred, (c) a percent, and (d) in mills.

D. Comparing Tax Rates

14. The City of Betaburg has a property tax rate of 15 mills and uses full assessment. The City of Alphaville has a property tax rate of $7.25 per thousand, and a uniform assessment percent of 35%. Which city has the higher property tax rate?

15. Last year, my town's tax rate was $40.35 per thousand. This year my assessment went up by 8.5%, but I was happy to hear that the tax rate had been lowered to $39.08 per thousand. What percent increase or decrease will I see in my town property tax bill this year?

E. Special Tax Rates

16. Zarofire Systems has been offered property tax incentives to locate its headquarters in Olentangy County. The normal property tax rate is $28.35 per thousand, and the property the company would use as its headquarters would be assessed at $12,500,000. The county has offered a 40% reduction in property taxes for 10 years. What would the company's property taxes be for the first full year if it takes the deal?

17. Zarofire Systems has been offered property tax incentives to locate its headquarters in Blakeboro County. The usual property tax rate is $17.45 per thousand, and the property the company would buy for a headquarters would be assessed at $6,800,000. The county offered the company a 45% reduction in property taxes for 20 years.

    a. What would the company's county property taxes be for the first full year under this arrangement.

    b. Suppose that, in response to public objections to giving a blanket tax reduction to the company, the county restructures its offer and instead offers a 45% reduction in the tax rate, but no change to the assessment. What would the tax rate be under this arrangement? How would it change the company's county taxes?

    c. Suppose that public objections rang just as strong to "giving a lower tax rate to big corporations." So the county once again restructured the deal, this time keeping the tax rate the same, but offering a 45% lower assessment. What would the assessment be in that case? How would it change the company's county taxes?

F. Grab Bag

18. The total tax levy is $30,000,000 and the total assessed value is $475,000,000. What is the tax rate, expressed as (a) a rate per thousand and (b) as a percent?

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19. Suppose that the Ischua Public Library District tax rate is $0.42 per hundred. The total assessed value of property in the district is $18,725,095. Calculate the total tax levy.

20. Jose's house has a market value of $320,000. The uniform assessment percent is 17.5%. What would you expect his assessment to be?

21. North Falls has a tax rate of $31.72 per thousand, while South Falls has a tax rate of $58.43 per thousand. North Falls uses full assessment, while South Falls has a 40% uniform assessment percent. Which municipality has the lower actual tax rate? Justify your answer.

22. Adrian's condo is assessed at $22,745. The uniform assessment percent is 20%. What fair market value does this assessment imply?

23. Dan owns a house in a community where the property tax rate is $67.43 per thousand. The market value of his property is $128,900 and the uniform assessment percent is 40%. On the basis of this information, what would you expect his property tax to be?

G. Additional Exercises

24. The tax rate in a town is $132.75 per thousand, based on a 25% uniform assessment percent. If the town switches to full assessment but keeps the total tax levy the same, what would the next tax rate be as a rate per thousand?

25. A county has a total tax levy of $22,503,000 and total assessed value of $1,579,045,000, of which $302,045,000 is classified as commercial property. If the county decides to use a tax rate for commercial property that is 20% lower than the rate for other property (making no changes to the assessments), calculate the tax rates for commercial and noncommercial property as a rate per thousand.

9.4 Other Taxes

Income, property, and sales taxes are the most significant taxes faced by most individuals and businesses, but they are by no means the only ones. In this section, we will discuss some other types of taxes that must be paid.

Excise Taxes

Excise taxes are taxes levied on specific products or services. A quick glance at the break-out of charges on your telephone bill will probably provide several examples of excise taxes. Excise taxes may be charged to connect the cost of a government service to activities that use that service, such as a tax on airplane tickets to pay for airport security or a tax on gasoline to pay for road maintenance. They may be charged to discourage the purchase of certain products, such as taxes on cigarettes, alcohol, and firearms. And of course they may be charged simply as a way to raise revenue without any grander rationale, such as taxes on mortgages or business licenses.

Excise taxes are often used in what the government considers "special situations." ***Luxury taxes*** are an example of this; an extra sales tax (above and beyond ordinary taxes) may be levied on the sale of luxury items such as sports cars, yachts, furs, jewelry, and so on. Interestingly, some taxes on home telephone service were originally considered luxury taxes; when they were first instituted, having a telephone at home was considered a luxury. These taxes remain in force even though having a home telephone is no longer considered to be living large. Luxury taxes were once quite common (witness the space of luxury tax on the Monopoly game board), but in recent years they have fallen out of favor.

In other cases, an excise tax may represent a lowering of the tax that would be otherwise charged. For example, in May 2006 the New York State legislature responded to high gas prices by eliminating the 4% sales tax on gasoline and replacing it with a new tax of 4% up to a maximum of $0.08 per gallon.

Example 9.4.1 For the New York gasoline tax of 4% up to $0.08 per gallon, what would the tax be on a gallon of gas if the price (before taxes) is (a) $1.79 per gallon, (b) $2.00 per gallon, (c) $3.75 per gallon.

(a) (4%)($1.79) = $0.0716 = 7 cents per gallon. (We assume here that the tax is rounded to the nearest cent.)

(b) (4%)($2.00) = $0.08 = 8 cents per gallon.

(c) (4%)($3.75) = $0.15 = 15 cents per gallon. However, the maximum applies, so the actual tax is 8 cents per gallon.

Tariffs and Duties

Items imported into the United States from overseas may be subject to ***duties***, ***import fees***, or ***tariffs***. Likewise, American goods exported to foreign countries may be subject to tariffs, import fees, or duties imposed by those countries. While there may be some technical differences between these terms, for our purposes they are essentially the same thing: a tax on bringing things into the country. (Both the process of managing this and the taxes themselves are referred to by the term ***customs***.)

In theory, these taxes are intended to protect companies producing products domestically from unfair foreign price competition. Foreign governments are often accused of unfairly subsidizing certain businesses. What really constitutes "unfair" competition is a matter of debate, though, and disputes between countries over these sorts of taxes are frequent and sometimes fairly nasty. One country may accuse another of "dumping" cheap products on the world market to gain market share for its domestic companies, while another may be accused of putting up "protectionist" barriers to keep out foreign companies and their products.

In an increasingly global marketplace, tariffs and duties can become a significant factor for businesses. The import taxes charged by the United States on goods from a foreign country will obviously have an effect on businesses that import goods from that country. Likewise, the import taxes charged by other countries on goods imported to those countries from the United States will have an impact on businesses looking to sell their products abroad. International agreements such as the North American Free Trade Agreement (NAFTA) between the United States, Canada, and Mexico can be of great significance to business, as they limit or eliminate many of the taxes charged on imports among the countries that are party to the agreement.

Customs and duties can be of significance for individuals when traveling. Items that you buy overseas while traveling on business or on vacation may be subject to duties when you bring them back to the United States. Each individual can bring goods back from a trip abroad up to a certain dollar value (the exact amount depends on how long you were there and may also depend on the country visited). Special exemptions may apply for certain goods such as tobacco or alcohol products, and certain products may be restricted regardless of their value (fruits and vegetables, for example, may be restricted because of concerns over the spread of plant diseases). It is a very good idea to check on current customs regulations before traveling abroad to avoid being hit with unexpected steep duties.

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Example 9.4.2 ***The Republic of Ithuvania charges an import duty of $1.25 per pack plus 18% of the market value on cigarettes imported from the United States. (The prices here are in Ithuvanian dollars.) On a business trip to Ithuvania you bring 12 packs of cigarettes, which would sell for $5 a pack there. How much of a duty will you have to pay?***

The market value of the cigarettes is 12($5) = $60. Taking 18% of this gives (18%)($60) = $10.80. In addition, you must pay $1.25 per pack, totaling ($1.25)(12) = $15.00. Altogether, this adds up to $15.00 + $10.80 = $25.80.

Example 9.4.3 ***Suppose that imports of wine are subject to an import duty of $8.00 per case. A case of wine is defined to be 9 liters. How much import duty would have to be paid on a shipment of 100 bottles, each of which contains 1.5 liters?***

The total amount imported is (100 bottles)(1.5 liters/bottle) = 150 liters.
Since there are 9 liters per case, this is (150 liters)/(9 liters/case) = 16.6667 cases.

The duty would be (16.6667 cases)($8.00/case) = $133.33.

Tourists are sometimes permitted to bring back a certain amount of an item without having to pay any duty. In those cases, duties would only be payable only on the amount which exceeds the exemption amount.

Example 9.4.4 ***Suppose that you are returning from a trip to Asia, and customs regulations permit you to bring back up to $500 worth of electronics duty-free. Beyond that exemption, there is an import duty of 6% of the price paid. You bring back two digital cameras, each of which cost $325. How much duty must be paid?***

The total purchase price is ($325)(2) = $650.
This exceeds the exemption by $650 – $500 = $150.
The duty would then be (6%)($150) = $9.00.

Estate Taxes

When someone dies, her property passes on to her survivors as specified in her will. (If there is no will, a court determines who is entitled to inherit the property according to state law.) This property is referred to as the deceased person's ***estate***, and ***estate taxes*** may be assessed against the value of the property in the estate. Estate taxes are levied by the federal government, and also by some states.

Especially in recent years, estate taxes have been a hot political issue. Critics call them "death taxes," and many politicians and political commentators have loudly demanded they be eliminated. Estate taxes make an easy target for outrage; it is not hard to take offense at the notion of a tax on dying! On the other side, supporters of the estate tax note that this tax only applies to quite large estates, and that in actuality only a very small percent of estates wind up facing any estate tax liability at all.

Estate taxes apply only to a person's ***taxable estate***. Generally speaking, the taxable estate is the fair market value of the deceased person's property, after subtracting out:

- The amount required to repay any debts
- Funeral expenses
- Gifts specified in the will for charities (in most cases)
- Amounts left to a surviving spouse (in most cases)
- Fees paid to the person selected to administer the estate (the ***executor*** or ***executrix***) and for legal expenses of administering the estate

Death benefits from life insurance policies may or may not be considered part of the taxable estate, depending on several particular details of how the policy was set up. The estate tax status of special retirement accounts such as IRAs and 401(k)'s can also be a complicated matter. Wealthy individual also sometimes take advantage of special types of accounts such as ***trusts***, which may circumvent estate taxes; the details of how these work though is a matter for specialized attorneys and accountants, and lies far beyond our scope here.

Example 9.4.5 ***Suppose that Anselm died, leaving assets worth $925,063. His will left $400,000 to his wife, $75,000 to his church, and $35,000 to other charities. At his death he had debts totaling $54,053 and his funeral expenses were $22,500. Executor's and other legal fees amounted to $30,000. What was his taxable estate?***

The money left to his wife, church, and charity are not taxable. In addition, all of the expenses listed would be subtracted as well. In total, these amounts add up to $616,553. The remaining taxable estate is $925,063 – $616,553 = $308,510.

Estate taxes are structured in such a way that every estate is allowed a fairly large exemption from the tax. The estate tax then applies to the value that exceeds this exemption. In recent years the amount of the exemption has been growing, and the tax rate has been reduced, in response to strong political opposition to the estate tax. The exemptions and rates for recent years are shown below.

Year of Death	Exemption	Rate
2003	$1,000,000	49%
2004	$1,500,000	48%
2005	$1,500,000	47%
2006	$2,000,000	46%
2007	$2,000,000	45%
2008	$2,000,000	45%
2009	$3,500,000	45%

As of this writing, under current tax law, the federal estate tax will be eliminated entirely in 2010. However, it will be eliminated for that year only; according to present law it will return in 2011 with a much lower exemption ($1,000,000) and higher rate (55%) than before. While it can't be said with any certainty what will happen in the future, it is very likely that some changes will be made before 2010.

Example 9.4.6 ***If Anselm (from Example 9.4.5) died in 2006, how much federal estate tax would have been owed on his estate?***

Anselm's taxable estate fell far below the exemption in 2006 (or in any of the years in the table for that matter). No federal estate tax would be paid.

Example 9.4.7 ***Suppose that an unmarried person dies, leaving an estate worth $2,800,000. After excludable expenses are subtracted out, her taxable estate came to $2,600,000. How much federal estate tax would be owed if her death occurred in 2008? In 2009?***

2008: The taxable estate is $2,800,000 – $200,000 = $2,600,000. Since the exemption for 2008 is $2,000,000, the tax rate applies to $2,600,000 – $2,000,000 = $600,000. The estate tax owed would then be (45%)($600,000) = $270,000.

2009: Since the exemption in 2009 is $3,500,000 and this is more than the taxable estate, no tax would be owed.

In this section we have been discussing only federal estate taxes. Some states also assess estate taxes, though there is a great deal of variation in estate taxes among the states. The amount of estate tax paid to the federal government may be affected by the amount of state estate tax. The examples above did not take this into account, and since the nature and amount of estate tax—if any—varies widely, we will consider only federal taxes here.

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There are many common misunderstandings about the estate tax. Many people do not realize how large the exemption is, and mistakenly assume that the entire estate is taxed. In fact, it should come as no surprise that most people's estates do not even approach the exemption amount.

People often also fail to take into account how inherited property is treated for income tax purposes. Under current law, if you sell property that you inherit, you pay capital gains (income) tax only on the amount that the property has grown in value since you inherited it, not since it was originally purchased. Under some proposals for eliminating the estate tax, this would be changed so that you would pay capital gains tax on the gain in the property's value since it was purchased. Suppose your rich uncle passes away, leaving you $100,000 worth of stock which he bought many years ago for $20,000 and you sell it soon afterward for $105,000. Presently, you would owe income tax on a $5,000 gain. If that law were changed as described here, you would pay income tax on an $85,000 gain. For those whose estates fall below the exemption amount, this sort of repeal of the "death tax" could turn into a tax increase, rather than a tax cut! (See Exercises 27 and 28 at the end of this section for an example of this.)

Estate taxes can be a major issue for business owners who want to leave a business "in the family." It does sometimes happen that the estate tax on the value of a farm or business can end up being large enough that the heirs have to sell the farm or business to be able to raise the cash to pay the estate tax. With the recent increases in the estate tax exemption this will likely be less of a concern, but even then it is important for business owners to keep this possibility in mind when planning for the future. A number of attorneys and financial professionals specialize in helping business owners plan for estate taxes (as well as other related issues), and there are many strategies that can be used to help financially plan for any future estate tax issues.

One idea to avoid estate taxes might be to simply give property to your intended heirs, especially when the donor is in failing health. The ***gift tax*** is intended to prevent people from avoiding estate taxes altogether in this way. Each individual is allowed to directly give only a limited amount of property to any other individual each year. As of 2006, this limit was $12,000 per recipient per year. A married couple may give $24,000 per recipient (i.e., $12,000 is considered to be from each spouse). Any gift over $12,000 must be reported to the Internal Revenue Service, and is counted against the estate tax exemption upon death. However, it is not quite that simple; up to a certain point, gifts that exceed this limit have no actual effect on the estate tax. (Property given to a spouse or charity is not subject to gift tax; property given to children or grandchildren or others is, however.)

Example 9.4.8 In 2006, Gus gave each of his four children $25,000. He also gave each of his seven grandchildren $40,000 to pay for their future educations. How much of these gifts is potentially subject to gift tax?

The gifts Gus made to his children and grandchildren exceeded the $12,000 limit and so the gift tax potentially applies. The amount subject to gift tax was $25,000 − $12,000 = $13,000 for each of his children, and $40,000 − $12,000 = $28,000 for each of his grandchildren. This works out to a total of 4($13,000) + 7($28,000) = $248,000.

The effect that this $248,000 will actually have on Gus's taxable estate when he passes away depends on many factors and complex formulas. For our purposes, it will suffice to note that these gifts have the *potential* to reduce Gus's estate tax exemption in the future, and we will leave the specific calculations to the accountants and attorneys who specialize in the vagaries of the estate tax codes.

Taxes: The Whole Story

No one likes to pay taxes. But government can't be run for free, and the services we expect from our government have to be paid for somehow. As long as there is a government, there will be taxes to pay.

Unfortunately, whenever large amounts of money and politics are involved, matters are never simple. Despite loud political calls to "simplify the tax code," the fact of the matter remains that the tax system is incredibly complex, and the prospects for things getting much simpler any time soon are dim. Given this, it is important for both individuals and for businesses to make sure to educate themselves about the specific tax issues they face, and to take advantage of competent, professional tax advice and assistance, particularly when special situations apply. While this chapter has provided an introduction to the basics of taxation in the United States, it is nowhere near a complete treatment of the subject. Until and unless the tax code is dramatically simplified, its complexity will continue to provide headaches to individual taxpayers and businesses, and a good living to accountants and attorneys who specialize in maneuvering through the tax codes for their clients.

EXERCISES 9.4

A. Excise Taxes

1. Suppose that a state charges an excise tax on liquor of $0.32 per 100 milliliters. What would the tax be on a 500-milliliter bottle of gin? A 750-milliliter bottle?

2. The excise tax on cigarettes is 7 cents per pack in South Carolina. The rate is $2.46 per pack in Rhode Island. If a pack of cigarettes would cost $2.39 before excise taxes, what would the price be in each of these two states?

3. The Republic of Ithuvania charges an import duty of $2,000 (Ithuvanian dollars) plus 12.5% of the market value on imported automobiles. If Zoltan wants to import a sports car worth $38,450 (Ithuvanian), how much would he pay in import duties?

4. Suppose that American citizens visiting Canada are allowed to bring home one case of beer duty free, but must pay a duty of $8.35 for each additional case. Ted and Sarah spend a weekend in Toronto, and bring home four cases of Alexander Keith's India Pale Ale. How much will they pay in import duties?

5. Suppose that your company imports digital cameras from overseas. The federal government passes a law imposing a 17.5% import duty on all digital camera imports from the Republic of Cascaria. If your company imports a shipment of $978,350 worth of these cameras from that nation, how much import duty would be owed?

6. Suppose that the state tax on gasoline is 5% of the wholesale price up to a maximum of 13.5 cents per gallon. Determine the tax on a 1,600-gallon shipment if the wholesale price is $1.79 per gallon.

7. Suppose that the state tax on gasoline is 4.75% of the wholesale price up to a maximum of 10 cents per gallon. Determine the tax on a 1,450-gallon shipment if the wholesale price is $2.39 per gallon.

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B. Estate Taxes

8. Rajinder passed away, leaving an estate worth a total of $839,075. His will left $150,000 to various charities, and the rest of his estate to his children. He had no debts, his funeral expenses amounted to $19,700, and the estate administrative expenses totaled $42,000. Calculate his taxable estate.

9. A farmer passed away leaving a farm worth $2,500,000 in total. His other assets amounted to $74,500 and he had $240,000 in debt. His funeral expenses and estate administrative expenses totaled $54,000.

   a. Calculate his taxable estate.

   b. Determine the federal estate tax on this estate, assuming he died in 2003.

   c. Determine the federal estate tax on this estate, assuming he died in 2008.

10. Determine (a) the taxable estate and (b) the federal estate tax owed, given:

    The overall value of the estate is $5,743,025.

    The deceased's will left $100,000 to charity and $2,000,000 to his spouse.

    Funeral expenses were $31,535 and estate administration expenses were $184,000.

    Death occurred in 2009.

11. Suppose that someone dies, leaving an estate of $4,546,902. His will leaves $300,000 to charity, with the rest passing on to his children and grandchildren. Funeral and administrative expenses total $195,000. Calculate the amount of the federal estate tax if this death were to occur in (a) 2008, (b) 2009, and (b) 2010.

12. Mike's father died in 2004, leaving an estate worth $172,405 in total. Calculate the estate tax on his father's estate.

13. Lisa's grandfather dies in 2006. His taxable estate was worth $582,043. How much federal estate tax would be paid on this estate?

14. Ken gave each of his four grandchildren $100,000. How much of this gift is potentially subject to gift tax?

15. Jarron and Lucy gave each of their three children $30,000, and each of their four grandchildren $25,000. They also gave $50,000 to their church. How much of this gift is potentially subject to gift tax?

16. Nguyen has a large investment account and wants to give as much of it as possible to his family this year. He plans to make gifts to his three children and five grandchildren. How much in total can he give away without exceeding the gift tax limit?

17. Under the 2007 estate tax rates, who would pay the most estate taxes?

    a. Someone who dies leaving a taxable estate of $1,750,000.

    b. Someone who dies leaving a taxable estate of $500,000.

c. Someone who dies leaving a taxable estate of $60,000.

d. Someone who dies with no assets at all.

C. Grab Bag

18. Suppose that a state imposes taxes of $2.45 per pack on cigarettes in addition to a sales tax of 7.5% of the retail price. If a pack of cigarettes costs $1.93 before taxes, what would the price be, including taxes?

19. On December 31, 2009, Junius is taken to the hospital with a severe and life-threatening illness. Despite the best efforts of his physicians, near the stroke of midnight he passes away, leaving a taxable estate of $12,700,000. (a) For the tax rates given in this section, what would the estate tax be on this amount if he died at 11:59 P.M.? (b) What would the estate tax be if he died 2 minutes later?

20. Calculate the estate tax due on a taxable estate of $237,000, assuming a 2008 date of death.

21. Suppose that the import duty on perfume is $1.25 for up to 5 milliliters, and $0.85 for each additional 5 milliliters or portion thereof. Calculate the duty on a 17-millilter bottle of perfume.

22. Sheila gave each of her three grandchildren $14,575 in 2006. How much of her gift is potentially subject to gift tax? How much did she have to pay in gift tax when she made these gifts?

23. Suppose that there is a state excise tax on long-distance service of 75 cents monthly plus $0.0025 per minute of use. Determine the excise tax that would be charged for someone whose long-distance use in the month of June was 817 minutes.

24. Suppose that there is an import duty in place for imported automobiles. The duty is $350 plus 7½% of the purchase price. If you import a high-performance car that cost $58,935, how much would the import duty amount to?

D. Additional Exercises

25. At the 2007 estate tax rates, how large of a taxable estate would you need to leave to owe $1,000,000 in estate taxes? What percent of your taxable estate would this tax represent?

26. The estate tax rate for 2008 is scheduled to be 45%. Of course the rate is not really 45% of the entire estate because of the exemption amount.

    How large of a taxable estate would someone have to leave in order for federal estate taxes to total:

    a. 25% of the total taxable estate

    b. 35% of the total taxable estate

    c. A full 45% of the total taxable estate

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27. Suppose that Rick dies leaving a taxable estate of $50,000, which consists entirely of a stock portfolio that cost him $3,000 when he bought the stocks years ago. Suppose that the capital gains tax rate is 18%. As mentioned in this section, under current law, when you sell inherited property, you pay capital gains tax only on the gain over the value at death.

    a. Suppose that Rick dies in 2007. Calculate the amount of estate tax that would be owed on his estate, and calculate the capital gains tax that his heirs would have to pay, assuming that they sold these stocks for $52,700.

    b. Suppose that the estate tax is abolished altogether, but that when it is, the law is changed so that capital gains tax would have to be paid on the entire gain in the value of property from the original cost. Calculate the amount of estate tax that would have been owed on Rick's estate under this law, and also the capital gains tax his heirs would have to pay, assuming they sold the stocks for $52,700.

    c. In total, how would the overall taxes on this inheritance change?

28. Suppose that Rick dies, leaving a taxable estate of $5,000,000, which consists entirely of a stock portfolio that cost him $300,000. Suppose that the capital gains tax rate is 18%. As mentioned in this section, under current law, when you sell inherited property, you pay capital gains tax only on the gain over the value at death.

    a. Suppose that Rick dies in 2007. Calculate the amount of estate tax that would be owed on his estate, and calculate the capital gains tax that his heirs would have to pay assuming that they sold these stocks for $5,270,000.

    b. Suppose that the estate tax is abolished altogether, but that, when it is, the law is changed so that capital gains tax would have to be paid on the entire gain in the value of property from the original cost. Calculate the amount of estate tax that would have been owed on Rick's estate under this law, and also the capital gains tax his heirs would have to pay, assuming they sold the stocks for $5,270,000.

    c. In total, how much would the tax on this inheritance change?

Topic	Key Ideas, Formulas, and Techniques	Examples
Calculating Sales Tax, p. 378	• Sales tax is a percent of the purchase price. • Multiply the percent by the price to get the sales tax. • Add calculated sales tax to the price to get total price including tax. • Alternatively, use the formula: $T = P(1 + r)$.	A sweater costs $42.95 and the sales tax rate is 7⅜%. Calculate (a) the amount of sales tax and (b) the total price including tax. (Example 9.1.2)
Finding a Price before Tax, p. 379	• Use the formula $T = P(1 + r)$. • Use algebra to solve for P.	A souvenir shop sets all of its prices to include 6.75% sales tax. For the month of June, sales totaled $16,739. How much sales tax is due on these sales? (Example 9.1.7)
Calculating Taxable Income, p. 386	• Subtract $3,300 for each dependent who can be claimed. • Subtract itemized tax deductible expenses, or subtract the standard deduction if larger.	Luisa and Alfonso are a married couple who had a joint income of $68,579 in 2006. They have two dependent children. They paid $3,450 in state and local taxes in 2006, and gave $3,700 to charity. They have no other tax-deductible expenses, and no other special adjustments to their taxable income. What was their taxable income for 2006? (Example 9.2.1)
Calculating Total Tax, p. 387	• Use the IRS income tax tables for the appropriate filing status.	Calculate the 2006 income tax for Luisa and Alfonso (from Example 9.2.1). (Example 9.2.2)
Calculating Tax Withholding, p. 390	• Income tax payments must be withheld from each paycheck. • Withholding rates apply to gross income after subtraction of nontaxable deductions and exemption allowances. • Use the IRS withholding tables for the appropriate filing status.	Erik is paid biweekly. He is married, and claims three exemptions on his tax return (one for himself, one for his wife, and one for a dependent child). After his contributions for health insurance coverage and his 401(k) are taken out, his gross taxable pay is $1,535.06. How much should his employer withhold for federal income taxes? (Example 9.2.5)
Calculating FICA and Self-Employment Taxes, p. 392	• Taxes for Social Security and Medicare are calculated as a percent of pay less nontaxable deductions. • Employer and employee pay equal amounts. • If self-employed, you pay both employer and employee shares.	Suppose that Lesley has a gross salary of $43,500 per year and is paid biweekly. She pays $150 with each paycheck for deductions for health insurance or other benefits. How much will be deducted from her biweekly paycheck for FICA? (Example 9.2.8)
Assessed Value and Uniform Assessment Percent, p. 398	• Property taxes are based on assessed value, not actual market value. • Assessed values should be approximately a consistent percent of market value. • Assessed Value = (Uniform Assessment Percent)(Market Value)	Suppose that the uniform assessment rate is supposed to be 25% in Boltonboro township. What market value does a $53,502 assessment suggest? (Example 9.3.2)

(Continued)

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Topic	Key Ideas, Formulas, and Techniques	Examples
Calculating Property Tax Due, p. 400	• Tax is assessed value multiplied by tax rate. • Tax rates may be expressed as a rate per thousand, percent, rate per hundred, or in mills.	Josh's house has an assessed value of $75,372. He pays real taxes to his county, town, and school district. The county rate is 14 mills, the town rate is $8.57 per thousand, and the school district rate is $1.35 per hundred. How much does he pay to each of these, and how much does he pay in total? (Example 9.3.3)
Setting Property Tax Rates, p. 401	• Tax rate is total tax levy divided by total assessed value. • Rates may be expressed as a rate per thousand, percent, rate per hundred, or in mills.	The Bloome County legislature has set the overall property tax levy for 2008 to be $39,600,000. The total assessed value of taxable property in the county is $4,873,595,000. Determine the 2008 real estate tax rate for the county. Express the result as (a) a percent, (b) a rate per hundred, (c) a rate per thousand, and (d) mills. (Example 9.3.4)
Excise Taxes, p. 406	• Taxes may be placed on specific activities or items. • These taxes may be expressed in a variety of different ways.	For the New York gasoline tax of 4% up to $0.08 per gallon, what would the tax be on a gallon of gas if the price (before taxes) is (a) $1.79 per gallon, (b) $2.00 per gallon, (c) $3.75 per gallon. (Example 9.4.1)
Estate Taxes, p. 408	• Taxes may be due on inherited property upon someone's death. • The taxable estate is the value of the deceased property after certain items have been subtracted. • Federal estate taxes are a percent of the amount of the taxable estate above a certain limit.	Suppose that Anselm died in 2006, leaving assets worth $925,063. His will left $400,000 to his wife, $75,000 to his church, and $35,000 to other charities. At his death he had debts totaling $54,053 and his funeral expenses were $22,500. Executor's and other legal fees amounted to $30,000. What was his taxable estate? How much estate tax would be owed on it? (Examples 9.4.5 and 9.4.6)

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CHAPTER 10

Consumer Mathematics

"I want it now!"

—Veruca Salt

Learning Objectives

LO 1 Calculate credit card interest using the average daily balance method.

LO 2 Determine the commission cost to the merchant of a credit card purchase.

LO 3 Compare different credit card offers for their annual fees and interest rates.

LO 4 Determine the equity and maximum loan amount for a mortgage loan.

LO 5 Calculate the total monthly payment (PITI) on a mortgage loan, including escrow and private mortgage insurance.

LO 6 Apply standard income qualification ratios for mortgage loans.

LO 7 Calculate the total cash needed to close on a mortgage loan.

LO 8 Determine the payments on an installment payment plan based on carrying charges or simple interest rates.

LO 9 Calculate the annual percentage rate (APR) for an installment plan by using amortization and/or approximation formulas.

LO 10 Calculate the monthly payment on an auto or other lease, given the residual value and interest rate.

Chapter Outline

10.1 Credit Cards

It's no secret that credit cards can be useful to both businesses and individual consumers. It's also no secret that letting things get out of control with credit cards can be disastrous to your financial well-being. Given these facts, it is worthwhile and important to have a solid understanding of the mathematics involved in their use.

The Basics: What Is a Credit Card *Really?*

Credit cards are a convenient and flexible way of paying for things *with borrowed money*. Swiping your credit card through the payment machine at the gas pump may not *feel* like taking out a loan, but that is what you are doing every time you pay with a credit card. The bank (or other financial institution) that issued your card pays the gas station on your behalf, and then you pay them back later. You are borrowing money.

People do use credit cards to buy things that they do not have the money to pay for up front. There are, though, many other reasons to use a credit card to pay for something beyond needing to borrow the money to pay. You might perfectly well be able to pay at the gas station with cash, but simply choose to use a credit card because it is more convenient to pay at the pump. Using a credit card to pay for Internet purchases is usually much easier than paying by other means. And for some payments (such as hotel rooms and car rentals) a credit card may avoid the need to leave a large deposit. There are other reasons to use a credit card as well, such as "cash back" or frequent flier mile awards offered by some card issuers. Whatever your motives may be, though, using a credit card means borrowing money, whether that is your reason for using the card or not.

Since using a credit card means borrowing money, it also means interest. One common complaint about credit cards is that their interest rates are often very high compared to the rates on other loans, such as car loans or home loans. With those sorts of loans, the lender has ***collateral***. This means that if you do not repay the loan as promised, the lender has the right to take the property for which you borrowed the money. With a car loan the collateral is the car; with a mortgage or home equity loan the collateral is the house. Loans that have collateral are called ***secured*** loans. Lenders would much rather have you pay back the loan, but if you don't, they at least have the ability to recover what you owe them by claiming the collateral.

Credit cards, on the other hand, are normally ***unsecured*** loans.[1] Usually, a credit card is issued without any collateral. The only guarantee that the lender has that you will pay back what you borrow is the fact that you promised you would when you applied for the card. If you fail to pay back what you borrow, the lender can—and will—take steps to collect what it is owed, but it does not have the right to seize any particular property from you as compensation. This makes the matter of collecting from someone who is unable or unwilling to pay far more complicated. Credit card debts often wind up being settled in bankruptcy court, where the lender typically will be able to collect only a fraction of what is actually owed.

The higher interest rates charged by credit cards are there to give the lender the opportunity to make up for these sorts of losses. This also means that applicants for credit cards who have strong credit histories may be able to get a card with a much lower rate, since the risk of someone with strong credit failing to pay is lower.

Debit Cards: The Same Except Different

As we already mentioned, there are reasons to want to pay for something with a credit card that have nothing to do with borrowing money. ***Debit cards*** present an alternative for those who really don't want to borrow. Debit cards have grown dramatically in their popularity in

[1] There are some exceptions to this. Credit cards are sometimes issued in connection with a home loan or with some other sort of collateral. There are also secured credit cards that require you to put money on deposit to serve as collateral for the card. These cards are not the norm, but they are sometimes used to get a lower interest rate, because of certain tax advantages, or in cases where someone wants a card but has a credit history that is not strong enough to qualify for an unsecured credit card.

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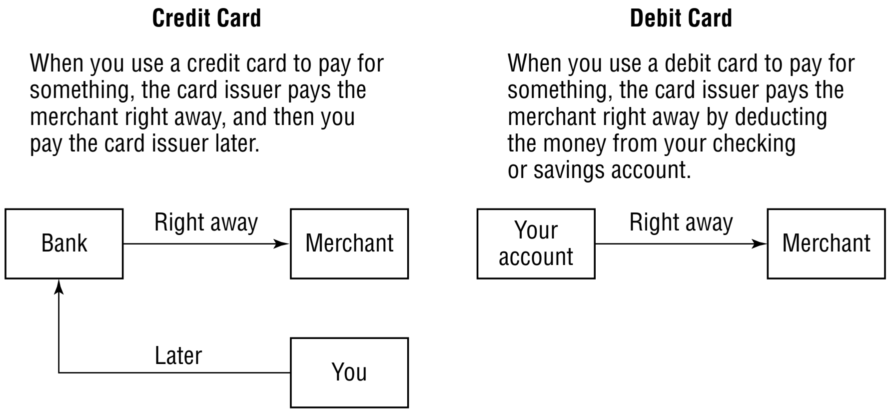
the last decade. A debit card may look like, and may be used in essentially the same ways as, a credit card, but using it does not involve any actual borrowing. Purchases made with a debit card are paid for out of a checking, savings, or similar account immediately. Since the money comes directly from your account, you are not borrowing when you make a payment with one of these cards.



While debit cards work just like credit cards at the point of use, they are really quite a different thing. Since no loans are involved, there is no interest to consider with debit cards (which makes them a lot less interesting mathematically). Also, a consumer should be aware that despite their many resemblances to credit cards, the regulations and consumer protection laws that apply to credit cards may not apply, or may apply differently, to debit cards. It is also important to remember that, while credit cards allow some flexibility as to when you actually pay for your purchases, with a debit card the money is withdrawn from your account right away. You need to have the money in your account at the moment that you use the card.

As a merchant you would also need to be aware that credit and debit card transactions may require different processing procedures, and the expenses of accepting payment by credit or debit cards may differ. While it is easy to confuse credit and debit cards because of their similarities, it is important not to forget the significant differences between them.

"Travel and Entertainment Cards"—Also the Same, and Also Different

Another similar type of card is sometimes known as a ***travel and entertainment (T&E) card***. The American Express card is probably the best known example. Payments for purchases made with these cards are handled in much the same way as with credit cards: the issuer pays the merchant, and you pay the issuer back later. However, while a credit card allows you flexibility in when you pay the money back, with a T&E card you are normally required to pay off any charges in full each month. Even though you are essentially borrowing money, albeit for a short time, there is usually no interest charged on these cards.

Calculating Credit Card Interest—Average Daily Balance

The calculation of credit card interest poses a bit of a challenge. On the one hand, since statements are produced and payments are due monthly, it makes sense that interest should be computed and charged to the account monthly. On the other hand, since the balance changes from day to day, it seems that interest should be calculated daily, to take care of the fact that the principal owed is not the same for the entire month.

Suppose, for example, that at the beginning of April I owe a $500 balance on my credit card. On April 5 I charge $300, on April 14 I make a $200 payment, and on April 30 I charge $1,600. A history of my balance from day to day in April would look something like this:

Effective Date	*Activity*	*Balance*
April 1	Start of Month	$500
April 5	Charged $300	$800
April 14	Paid $200	$600
April 30	Charged $1,600	$2,200

If we are going to calculate my interest monthly, then what should we use as the principal? At the end of the month the balance was $2,200. But it hardly seems fair to charge a full month's worth of interest on that amount, since except for the one day at the end of the month my balance was much lower. At the start of the month the balance was $500, but using that as the principal would be ignoring the higher balances carried through the rest of the month.

There is more than one way that interest can be calculated for a credit card, but the most common is known as the ***average daily balance (ADB) method***. With the ADB method, interest is computed and added to the account monthly, and the question of principal is answered by charging interest on the *average* of the daily balances through the month. This seems like a fair way to tackle the problem of the fluctuating balances.

But how do we calculate the average? If we were to take a straight average of the four balances shown in the table, we would get:

$$\text{Average} = \frac{\$500 + \$800 + \$600 + \$2{,}200}{4} = \$1025$$

This doesn't seem to be a fair representation of the "average" balance on the card for the month, though. For 29 of the 30 days in the month, the balance was $800 or less, and so an "average" of $1,025 isn't consistent with what really happened on the card during the month.

The problem with a straight average like this is that it doesn't take into account the amount of *time* that the card stayed at each of these balances. The $2,200 balance was only in effect for a single day, and so it should not carry the same weight in our average as the lower balances that were in effect for longer. The time that each balance was in effect should be factored into our average in some way.

Let's try looking at the average in a different way. Imagine that we take a calendar of the month of April, and write the account balance on each day. The result would be something like this:

APRIL						
1 $500	**2** $500	**3** $500	**4** $500	**5** $800	**6** $800	**7** $800
8 $800	**9** $800	**10** $800	**11** $800	**12** $800	**13** $800	**14** $600
15 $600	**16** $600	**17** $600	**18** $600	**19** $600	**20** $600	**21** $600
22 $600	**23** $600	**24** $600	**25** $600	**26** $600	**27** $600	**28** $600
29 $600	**30** $2,200					

A more reasonable and fairer way to find an average daily balance would be to average all of the balances written on this calendar. This would take into account the length of time

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that the account carried each balance; each balance amount would count once for each day that it was the card's balance. The $500 balance with which we started the month will be included 4 times, since that was the balance for 4 days, while the $2,200 at which the month ended will only count once, reflecting the fact that it was only the balance for a single day. Thus, we can calculate the ADB by adding up all the daily balances and dividing the result by 30 (since there are 30 days in the month). Thus:

$$\text{ADB} = \frac{\$500 + \$500 + \$500 + \$500 + \$800 + \cdots + \$2{,}200}{30} = \frac{\$21{,}000}{30} = \$700$$

which seems a much better representation of the "average" that the $1025 figure was.

The dots, indicating that "you get the idea so please don't make me write it all out," reflect the fact that, while this is a logical approach, it is also a tedious one. We can simplify this calculation simply by noting that our total included $500 four times, since it was the balance for 4 days, the $800 nine times, since it was the balance for 9 days, and so on. So we could more easily calculate the ADB as:

$$\text{ADB} = \frac{4(\$500) + 9(\$800) + 16(\$600) + 1(\$2{,}200)}{30} = \frac{\$21{,}000}{30} = \$700$$

This is an example of something known in mathematics as a ***weighted average***. In a weighted average, things being "averaged" are given different "weights," which gives some amounts more importance in the average than others. In this case, the weights are the days that the account carries each of the different balances.

Calculating Average Daily Balances Efficiently

Expanding on the table we originally created for the balance history can provide an efficient way to calculate the ADB. We repeat that table, this time adding columns for the number of days at each balance and for the result of multiplying each balance by the number of its days. We also total both of these new columns:

Effective Date	*Activity*	*Balance*	*Days at Balance*	*(Balance) (Days)*
April 1	Start of month	$500	4	$2,000
April 5	Charged $300	$800	9	$7,200
April 14	Paid $200	$600	16	$9,600
April 30	Charged $1,600	$2,200	1	$2,200
	Totals:		30	$21,000

Notice that the total of the (Balance)(Days) column is just the top of the ADB fraction, while the Days at Balance total is just the bottom. So we can just divide these column totals to get:

$$\text{ADB} = \frac{\text{Total of (Days)(Balance)column}}{\text{Total of Days column}} = \frac{\$21{,}000}{30} = \$700$$

In this calculation, the credit card's billing month coincided with a calendar month. This generally won't be the case, and the following example will illustrate how to deal with this:

> ***Example 10.1.1*** ***Joanna has a credit card whose billing period begins on the 17th day of each month. On July 17 her balance was $815.49. She made a $250 payment on July 28. She also made new charges of $27.55 on July 21, $129.99 on August 5, and $74.45 on August 8. Find her average daily balance.***
>
> *The first step is to organize the information in a table like the one we used before. We will leave the last two columns blank for the moment while we put the information in order. Also, to save space, we will indicate charges simply with a "+" and payments with a "−".*

Effective Date	*Activity*	*Balance*	*Days at Balance*	*(Balance)(Days)*
July 17	Start	$815.49		
July 21	+$27.55	$843.04		
July 28	−$250.00	$593.04		
August 5	+$129.99	$723.03		
August 8	+$74.45	$797.48		
	Totals:			

Once we have the information organized, we can focus on completing the last two columns. The days between July 17 and July 21 can be calculated easily by subtracting: 21 − 17 = 4 days. Similarly, we find that there are 7 days between July 21 and 28.

Finding the days between July 28 and August 5 is a bit trickier, since the crossover between months makes simply subtracting impossible. This can be done easily in either of two ways. One approach would be to count the days in July and August separately, and then add the total. This is essentially the approach we used in Chapter 1 when similar situations arose with the dates of notes.

A second, equally effective, approach would be to just pretend that the month of July continued. Since August 5 is 5 days beyond the end of July, we think of it as "July 36th". We can then just subtract 36 − 28 to get a total of 8 days.

The remaining dates all fall within the month of August, so we can go back to simply subtracting. 8 − 5 = 3 days between August 5 and 8, and 17 − 8 = 9 days between August 8 and the end of the billing month. Once we have calculated the days for each balance, we can also complete the last column by multiplying each row's balance by its days at that balance.

Completing the table, we get:

Effective Date	*Activity*	*Balance*	*Days at Balance*	*(Balance)(Days)*
July 17	Start	$815.49	4	$3,261.96
July 21	+$27.55	$843.04	7	$5,901.28
July 28	−$250.00	$593.04	8	$4,744.32
August 5	+$129.99	$723.03	3	$2,169.09
August 8	+$74.45	$797.48	9	$7,177.32
	Totals:		31	$23,253.97

Completing the ADB calculation by dividing the column totals, we get:

$$ADB = \frac{\$23{,}253.97}{31} = \$750.13$$

Calculating Credit Card Interest

Once we have the ADB, calculating the credit card interest is fairly straightforward. Since no compounding occurs within the billing month, we can use the simple interest formula, with the ADB as the principal. One potential sticky point is the time. The time period can be thought of either as 1 month (hence T = 1/12) or as the actual number of days in the billing period, in which case T = (number of days in the billing period)/(number of days in the year). In most cases, the interest with time is calculated by using the number of days in the billing period.[2]

Example 10.1.2 ***Suppose that Joanna's credit card (from Example 10.1.1) carries an interest rate of 15.99%. How much interest would she owe for the billing month from Example 10.1.1?***

[2]In all of the examples and problems in this section, we will use the simplified exact method and assume 365 days per year. During leap years, 366 days would be used, but we will assume that the year is not a leap year in the examples and exercises of this section.

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$$I = PRT$$
$$I = (\$750.13)(0.1599)(31/365)$$
$$I = \$10.19$$

So her interest for this billing month would be $10.19.

Be careful when adding interest onto the balance to determine the total on a monthly statement. Remember that the ADB is simply a tool used for calculating interest—it is not the amount that you actually owe. It is a very common mistake in answering these types of questions to add the interest onto the ADB. We put so much work into finding the ADB, it seems important enough that we want to use it wherever possible! But if you stop to think about things, it makes sense that you would need to repay the total of your charges, not their average. And so, once the interest is calculated, it should be added to the last balance, not onto the ADB.

Example 10.1.3 What will the total balance be on Joanna's monthly statement?

Interest will be added onto her balance at the end of this billing period; the total balance on her August 17 statement will be $797.48 + $10.19 = $807.67.

Credit Card Interest—The Grace Period

Most credit cards offer a feature known as a ***grace period***, which adds an interesting wrinkle to the matter of interest. The grace period is a period of time (typically 20 to 25 days), beginning on the card's billing date. If you pay the entire balance within this grace period, and if you paid your previous month's balance off in full (so that none of your balance is a carryover from the prior month), you pay no interest at all.[3]

For people who use a credit card primarily for convenience's sake, this can provide a tremendous savings. Many people have credit cards that they use regularly, yet pay no interest whatsoever. Such credit card holders are often referred to in the industry as ***convenience users***.[4] Taking advantage of the grace period is one of the few "free lunches" in the financial world; charges made to the card become a short-term interest-free loan. A convenience user benefits from both the convenience of her credit card and from the temporary use of someone else's money for free. However, grace periods are an all-or-nothing deal; the grace period will apply only if the balance is paid *in full*. Paying even one penny less than the full balance means that interest charges will still apply.

Example 10.1.4 The billing date for Nick's credit card falls on the 18th day of each month. His card carries an interest rate of 21.99%, and has a 20-day grace period. On October 18 his balance was $935.14. He made charges of $56.65 on October 20, $309.25 on October 29, $81.17 on November 9 and $101.42 on November 17. He paid $935.14 on November 3. If he pays the balance on his November 18 statement in full before the end of the grace period, how much interest will he owe?

Since he paid his October 18 balance in full within the grace period, if he pays the November balance in full the grace period applies and he will owe no interest. The answer is: $0!

In addition to the obvious financial advantage of knowing about this, as a student it is important to always check for the grace period in questions about credit card interest. Not only did Nick save himself from paying interest in this example, he also saved us from having to calculate his ADB!

You don't have to pay interest if you pay your balance in full within the grace period, but when the bank prints your statement, it obviously has no way of knowing in advance whether or not you are going to do this. So how does the bank know whether or not to place an interest

[3] This is the typical deal for the grace period on most cards. However, the exact provisions for any particular card may vary. For example, some cards have a grace period for purchases made to the card, but no grace period for cash advances (obtained by using your credit card in an ATM to get cash). For that matter, while an overwhelming majority of credit cards do offer a grace period, occasionally cards are offered with no grace period at all. You should make sure to carefully read the terms of any credit card agreement.

[4] At least that is the polite term.

charge on your monthly bill? Assuming that you paid your previous month's balance in full (and thus are eligible to take advantage of the grace period), most issuers will send a bill that shows no interest due. If you fail to pay your balance in full during the grace period, the interest charges for the month will show up on your following month's statement.

Other Fees and Expenses

Probably the main source of profit for credit card issuers is interest, but interest is not the issuer's only way of making money. Two others are annual fees and commissions.

An ***annual fee*** is a fee paid by the cardholder simply for having the credit card. Annual fees can range as high as $100 per year (or even higher), but typically are much more modest. Many credit cards are available that charge no annual fee at all. T&E cards often carry high annual fees (since the issuer does not make any money from interest charges).

Commissions are not paid by the cardholder, and in fact many credit card users are not even aware of their existence. When a merchant accepts a payment by a credit card, the merchant pays a fee to the credit card company. Commissions apply to debit card transactions as well. These commissions may be a percent of the amount charged, a flat amount per transaction, or a combination of the two.

Example 10.1.5 ***Travis bought a pair of shoes for $107.79 and charged them to his credit card. The credit card company charges the shoe store 45 cents for each transaction, plus 1.25% of the amount charged.*** **How much will the credit card company pay to the shoe store?**

The percent portion of the commission would be:

$$(0.0125)(\$107.79) = \$1.35.$$

To this, we add the 45 cent charge to arrive at a total commission of $1.35 + $0.45 = $1.80. Subtracting this from the amount of the charge, we can determine that the shoe store will receive $107.79 − $1.80 = $105.99.

As a consumer, these commissions may not matter much to you, but they can be a significant issue for businesses. The shoe store would of course just as soon not have to give up the $1.80 to the credit card company and instead get the full $107.79 for Travis's shoes. However, few merchants can afford to refuse credit card payments. While the shoe store would just as soon not give up the $1.80, it would probably prefer sacrificing $1.80 on the sale to losing the business to another store. Plus, with a credit card the store knows that it will receive the payment quickly, with no risk of a bounced check. And of course, the shoe store's owners have to keep in mind that credit cards are a very popular payment method among consumers, and the store's competitors probably do accept them. While some business owners choose not to accept credit cards, most simply choose to think of the costs incurred by accepting credit card payments as just another cost of doing business.

Commissions can, however, provide a particular challenge to small businesses. Larger businesses may be in a position to negotiate lower commission rates, whereas a "mom and pop" operation doesn't have much negotiating power. Different commission rates are often one of the key reasons why some merchants may accept one major credit card but not another, or why some merchants will accept credit but not debit cards (or vice versa).

Choosing the Best Deal

The credit card industry is highly competitive, with literally hundreds of different card issuers competing for each potential card holder. Many consumers will simply go with whatever credit card offer seems most convenient, but cards are offered in a wide range of interest rates and fees. People who just take whatever offer is first presented to them often overlook or miss out on opportunities to pay significantly less for their credit card use.

To some extent, the choice of credit card and issuer is not a mathematical matter at all. If you do all of your banking at Friendly Neighborhood National Bank and decide that you just want to keep your credit card account at the same place, then there are no

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calculations behind your decision. There is something to be said for simplicity—keeping all your financial dealings at one institution is more appealing that having accounts in a dozen different places. However, given that interest rates and fees can vary dramatically from one issuer to another, it is usually worth shopping around to find the best deal, or to at least make sure that you have a competitive one. The absolute lowest costs may not be the most important thing to you, but you probably want to make sure that the deal you are getting is at least reasonable compared to what's available elsewhere.

Ideally, as a consumer you would want to choose the card that has both the lowest interest rate and the lowest annual fee. If one of your options is the lowest cost for *both* of these, no calculations are necessary. The choice is obvious. What if, though, the card with the lowest annual fee carries a higher interest rate, while the card with the lowest interest rate has a high annual fee. How can we balance the interest rate against the annual fee to determine which is the best deal?

For example, suppose you have the choice of a Visa card issued by any of three different banks. The banks' offers are shown in the table below. (Note that in this table the abbreviation ***APR***, short for annual percentage rate, is used; it is common practice on credit card offers to label the interest rate in this way.)

Card Issuer	APR	Annual Fee
Bank A	9%	$80
Bank B	15%	$25
Bank C	23.99%	None

Which deal is best? The answer really depends on how you use the card. If you are a convenience user, paying your bill within the grace period each month, the interest rate is irrelevant. You're not going to pay any interest anyway, and so the interest rate doesn't matter to you in the slightest. In that case, your best choice would be Bank C, because with no annual fee you will be able to use this card for free. On the other hand, for someone who carries a very large balance, the savings from a low interest rate would more than make up for a high annual fee, and so Bank A would be the obvious winner.

The question is more challenging for someone in the middle, who may carry a balance, but not a large enough one that a lower interest rate would obviously compensate for a higher annual fee. In such a case, we must make a reasonable estimate of how much of a balance will be carried, and then crunch the numbers. The following example will illustrate:

Example 10.1.6 ***Jerome expects that he will normally carry a credit card balance of around $800. Which of the three options in the table given above would be the lowest cost option for him?***

We can calculate the total annual cost of each card. Since he will be carrying a balance of around $800, at Bank A his interest charges over the course of a year would be:

$$I = PRT$$
$$I = (\$800)(0.09)(1)$$
$$I = \$72$$

Combining this with the $80 annual fee means that his total annual cost at Bank A would be approximately $80 + $72 = $152.

For Bank B, the interest would be I = PRT = ($800)(0.15)(1) = $120. Added to the $25 annual fee would total $145.

For Bank C, interest would be I = PRT = ($800)(0.2399)(1) = $191.92. There is no annual fee, so this it total cost.

In Jerome's case, clearly it is worth paying an annual fee and going with either Bank A or B. Bank B has the lowest overall cost. If Jerome has some other reasons (convenience, existing banking relationship, etc.) to prefer Bank A, though, he might choose that option, since the difference is only a few dollars and is only an estimate anyway. Bank B, however, has the lowest projected cost, and so barring any other factors it is the best choice.

Jerome's $800 predicted balance is large enough to justify passing up the no annual fee offering at Bank C and instead pay his $25 annual fee at Bank B. But it's not large enough to justify the $80 annual fee at Bank A. We noted before, though, that someone with a "large" balance would have reason to choose Bank A. So, how large is "large"?

Example 10.1.7 How large would my balance need to be for it to be worth paying the annual fee at Bank A to get the lower interest rate?

First let's compare Bank A to Bank B. The difference between the two annual fees is $80 − $25 = $55. The difference between the interest rates is 15% − 9% = 6%. So, the balance would have to be large enough that a 6% rate applied to it for a year would be at least equal to $55. Solving, we get:

$$I = PRT$$
$$\$55 = P(0.06)(1)$$
$$P = \$916.67$$

So we can conclude that the cutoff for choosing Bank A over Bank B is $916.67.

Now let's compare Bank A to Bank C. Using the same logic as before, we see that the difference in fees is $80 and the difference in rates is 14.99%. Solving for interest gives:

$$I = PRT$$
$$\$80 = P(0.1499)(1)$$
$$P = \$533.69$$

Overall, Bank A beats out both of the other banks as long as the average balance is larger than either of these two amounts. So as long as the average balance is $916.67 or more, Bank A is the winner.

Note that in the example above it was not really necessary to compare Bank A to Bank C. We know that as the balance increases, a lower interest rate becomes more important. Therefore we really only needed to compare Bank A against the bank with the next lowest interest rate.

Choosing the Best Deal—"Reward Cards"

The choice of the best credit card offer can be especially complicated for so-called ***reward credit cards***. Many credit card deals are available that offer some sort of reward to their cardholders for using the card. When first launched, the Discover card made quite a splash in the market by offering the then-remarkable deal of paying cardholders back a portion of their annual charges in cash. The idea might have sounded crazy at first, but proved to be very successful. Today, there are a wide range of similar deals, ranging from cards that offer cash-back rewards, to airline frequent flier miles, to credits that can be applied toward buying specific products, and so on.

In choosing the "best" offer, these rewards can't be ignored, but it is often hard—if not impossible—to apply a mathematical analysis. If the reward offered is cash, then you can estimate the amount, and then subtract that from your total costs. But if a credit card offers frequent flyer miles based on the amount of your purchases, what sort of value should you place on them? If you enjoy traveling, the frequent flyer miles from your credit card may combine with other miles you earn from travel to get you a free trip. In that case the value of this reward to you may be great, though it is not entirely clear how to put a specific dollar value on it. On the other hand, if you hate traveling, this is obviously of no value to you. Likewise, if you are planning on buying a new Pontiac next year, a credit card that allows you to earn credits toward purchase of a GM car would be of great value. If you have no plans to buy a new car and like Toyotas better anyway, such a reward may not be worth anything.

While such rewards can't be ignored, judging their value is subjective. A convenience user with a reward card offering something she values is in the terrific position of actually getting paid to use her credit card! But someone paying a high annual fee and interest rate

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on a large balance in order to earn a half-penny back on each dollar spent might be better off giving up the "reward" and getting lower fees and/or interest. Where rewards are offered, you have to be careful to be realistic about their value, and just how much that reward might actually cost in fees and interest. Some of the Additional Exercises will illustrate this.

EXERCISES 10.1

A. Average Daily Balance

1. A credit card's billing period begins on the 12th day of each month. The activity on the card for the billing period beginning April 12 is summarized in the table below. Fill in the missing table components to calculate the ADB for this billing period.

Effective Date	*Activity*	*Balance*	*Days at Balance*	*(Balance)(Days)*
April 12	Start	$1,755.28		
April 14	Charged $128.53			
April 29	Paid $500.00			
May 7	Charged $62.45			
May 9	Charged $197.65			
	TOTALS:			

$$\text{ADB} = \frac{\text{TOTAL (Balance)(Days)}}{\text{TOTAL Days}} = \underline{\hspace{3cm}}$$

2. Jacinthe has a Visa credit card whose billing period begins on the third day of each month. On March 3 her balance was $707.45. She made a payment of $350.00 on March 19. She made charges of $105.00 on March 8, $75.00 on March 18, and $206.95 on April 2. Make a table similar to the one given in Exercise 1 to list out her monthly card balance, then use it to find her average daily balance for this billing period.

3. The billing period for Daeshawn's MasterCard begins on the 27th of each month. On November 27 his balance was $455.25. He made charges of $75.05 on November 29, $92.07 on December 11, $177.42 on December 12, and $875.50 on December 24. He paid $375.00 on December 9. Organize this information in a table, and then use it to find his average daily balance for this billing period.

4. Erik's credit card billing period begins on the first day of each month. On January 1 his balance was $4,039.88. He charged $303.75 on January 9, $611.05 on January 16, and made charges of $75.59 and $108.92 both on January 28. He paid $100 on January 17. What was his average daily balance for this billing period?

B. Calculating Credit Card Interest

5. Find the interest due for the credit card from Exercise 1 if the interest rate is 18.99%.

6. Find the interest Jacinthe will owe for the billing period in Exercise 2, assuming her interest rate is 21.99%.

7. a. If Daeshawn's credit card carries a 10.99% interest rate, how much interest will he pay for the billing period from Exercise 3?
   b. What will be his total balance on his December 27 statement?

8. a. Erik's credit card's interest rate is 17.95%. How much interest will he owe for the billing period in Exercise 4?
   b. What will the total balance be on his February 1 statement?

C. The Grace Period

*In each of the following situations, determine whether or not any interest would be owed for the billing period. It is not necessary to calculate the **amount** of interest due (if any). However, if interest will be owed, specify the reason (or reasons) why the grace period does not apply. Assume that the grace period is 20 days in each problem.*

9. Dave's credit card billing period ends on the sixth of the month. On October 6, his balance was $750.41. He made charges of $175.00 on October 9 and $308.92 on October 11, and paid $750.41 on October 25. Will he owe any interest on his November 6 statement?

10. The billing period for Ellie's MasterCard ends on the third of each month. On February 3 her balance was $205.19, which she paid in full on February 15. She made charges of $119.75 on February 28, $71.11 on March 1, and $938.75 on March 2. Will she owe any interest on her March 3 statement?

11. The billing period of Kent's Visa card ends on the 19th of each month. On December 19 his balance was $2,458.19. He made charges of $355.49 on December 27, $101.01 on January 5, and $317.52 on January 6, and he paid $800 on January 3. Will he owe any interest on his January 19 bill?

12. My credit card statement is produced on the 23rd of every month. On April 23 I owed $829.15, which I paid in full on May 15. The only charge I made during the next billing period was $89.95 on May 17. Will I owe any interest for my May 23 bill if I pay it in full on May 31?

D. Other Fees and Expenses

13. Jack's Solar Gadget Shoppe sold a radio for $78.35 and accepted a credit card payment. The commission was 1.75%. How much was the commission? How much did Jack actually get paid for the radio?

14. Paula paid $127.45 for groceries, using her debit card. The store paid a commission of 50 cents plus 1.35%. How much was the commission? How much did the grocery store actually receive?

15. Mom N Pop's Big Box Hyper Mart accepted a debit card payment of $199.59. The commission charged was 50 cents plus 1¼%. How much did Mom and Pop actually receive?

16. Mamiko charged $407.59 to her credit card for a new stereo system. The retailer paid a commission of 2¼%. How much did the retailer actually receive for the stereo?

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E. Choosing the Best Deal

17. Arturo is looking around for a Visa card, and has narrowed his choices to three different banks. Each offers a different annual fee and interest rate, as given in the table below:

Card Issuer	*APR*	*Annual Fee*
Turtlepoint Trust	19%	$20
Myrtle Mutual	15%	$35
Rew Regional	21.99%	None

Which card will be the least expensive for him if:

a. His balance varies from month to month, but is usually very high?

b. He carries an average balance of $2,000?

c. His balance averages $800?

d. His balance averages $250?

e. He is a convenience user?

18. Seema applied for a MasterCard from her credit union. Her application was approved, and she was offered the choice of three different plans:

Plan	*Interest Rate*	*Annual Fee*
Interest Super-Saver	8.99%	$75
Fee Freedom!	18.99%	None
Standard	13.5%	$25

Which plan should Seema choose if she plans to:

a. Carry a balance averaging around $500?

b. Transfer all of her other credit card balances (totaling $5,000) to this card?

c. Have an average balance of around $1,000?

d. Pay off her balance in full each month?

19. How large of a balance would Arturo (from Exercise 17) need to carry on average for him to:

a. Prefer Myrtle Mutual over Turtlepoint Trust?

b. Prefer Turtlepoint Trust over Rew Regional?

20. What sort of average balance would Seema (from Exercise 18) need to carry in order to:

a. Prefer the Interest Super-Saver over Standard?

b. Prefer Standard over Interest Super-Saver?

c. Prefer Fee Freedom! over Standard?

21. Suppose you are offered a MasterCard from each of four different banks, with fees and interest rates as shown in the table below. You would be equally happy with any of the four, and are just interested in finding the lowest-cost option.

Card Issuer	*Interest Rate*	*Annual Fee*
Northern National	15%	$20
Southern Mutual	19%	$35
Western Trust	11½%	$30
EastBancorp	9.99%	$75

Under what circumstances (be specific) would you:

a. Prefer Western Trust over Northern National?

b. Prefer EastBancorp over Western Trust?

c. Prefer Southern Mutual over EastBancorp?

d. Prefer Southern Mutual over Northern National?

F. Grab Bag

Assume a 20-day grace period in each of the following problems.

22. A clothing store accepted a debit card payment for $127.49 in merchandise. The commission charged was 75 cents plus 1.4%. How much did the store actually receive?

23. Brad's credit card billing period ends on the seventh of each month. On August 7, his balance was $915.79. The table below shows the activity on his account for the following month:

Date	*Activity*
August 12	Charged $204.15
August 14	Charged $9.75
August 23	Paid $915.79
August 30	Charged $54.08

The interest rate on his card is 18.99%. How much interest will he owe on his September 7 statement? What will his total balance be on that statement?

24. Karen's credit card billing period ends on the eighth day of each month. The interest rate is 17½%. On March 8 her balance was $454.19. She charged $70.22 on March 21, $98.76 on March 23, and $211.98 on April 6. She made a payment of $150 on March 20. Organize this information in a table, and then use it to complete the following calculations. For the billing period ending on April 8, find (a) her ADB, (b) the interest for this billing period, and (c) her total balance at the end of the billing period.

25. I am looking around for a Visa card, and have narrowed my choices to three different banks. Each offers a different annual fee and interest rate, as given in the table below:

Card Issuer	*APR*	*Annual Fee*
Kashong Trust	21.99%	None
Crooked Lake Savings and Loan	15¾%	$55
Dresden Credit Bank	17.25%	$20

a. Which card will be the least expensive for me if I am a convenience user?

b. Which card will be least expensive if I carry a balance of $725 on average?

c. Which card will be the least expensive if I carry a balance of $2,500 on average?

d. How large of an average balance would I need to carry to prefer Crooked Lake?

e. What sort of average balance would I need to carry to prefer Dresden Credit?

26. Rikki's credit card bill is produced on the 19th of each month. On January 19 her balance was $454.02. She made charges of: $175.96 on January 25, $34.44 on January 27, $303.12 on February 15, and $17.73 on February 18. She made a payment of $400.00 on February 3. Her card carries a 17.99% interest rate. Find the total balance for her February 19 bill.

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27. Rikki's credit card bill is produced on the 19th of each month. On January 19 her balance was $454.02. She made charges of: $175.96 on January 25, $34.44 on January 27, $303.12 on February 15, and $17.73 on February 18. She made a payment of $454.02 on February 3. Her card carries a 17.99% interest rate. Find the total balance for her February 19 bill.

28. Amos charges $95.09 to his credit card at a grocery store. The store pays a 1.5% commission on the purchase. How much does the grocery store actually receive?

29. Michael's credit card bill is produced on the 23rd of each month. On July 23 his balance was $775.19. He paid this balance in full on August 14. Is he eligible to take advantage of the grace period for the billing period ending on August 23? Why or why not?

30. A debit card charges a merchant 1.45% of the amount of each payment, subject to a minimum commission of $1.05. Calculate the commission that would be paid on a purchase of (a) $25.13, (b) $52.47, (c) $82.02, (d) $179.33.

31. CC's credit card billing period ends on the 20th of each month. On April 20 his balance was $500. He paid $200 on April 28, and made charges of $350 on April 30 and $675 on May 15. If his card carries a 23.99% interest rate, calculate the interest he will owe on his May 20 statement.

32. Shinglehouse National Bank offers a credit card with no annual fee and a 12.99% interest rate. Eldred Mutual Savings Bank offers a card with a $25 annual fee and a 15% interest rate. Under what circumstances would someone prefer the card offered by Eldred Mutual?

G. Additional Exercises

33. A credit card processor charges a florist a commission for credit card purchases of 1.6% or $1.75, whichever is larger. Because the florist doesn't want to have to pay $1.75 on a small amount, he decides to make a store policy that he will only accept credit card payment for purchases on which the 1.6% commission is at least $1.75. What will he make the minimum credit card purchase amount? Would doing this be a good business decision?

34. Neil is looking for a new credit card, and has shopped around for the best deal he can find. He doesn't like to carry much cash or write checks, and so he plans to use this card as much as possible. He has narrowed his choices down to four different cards, some of which offer a "reward":

Issuer	***Rate***	***Annual Fee***	***Reward***
Bank A	15%	$29	1% cash back on all purchases
Bank B	18%	None	½% cash back on all purchases
Bank C	9.99%	None	None
Bank D	12.9%	$59	1 frequent flier mile for each dollar charged

Suppose that Neil expects that he will make purchases totaling $8,500 on the card over the course of the year.

a. Calculate the total rewards that Neil would receive over the course of the year with each of these cards.

b. Which card would be the best deal for Neil if he is a convenience user?

c. Which card would be the best deal for Neil if he carries an average balance of $1,250?

d. What if he carries an average balance of $5,000?

35. Suppose that Neil's girlfriend Alecia was also looking for a credit card. Unlike Neil, though, she doesn't plan to use it much, but just wants to have it for occasional purchases. When she does use her card, she plans on paying the entire balance right away. Which of the four options Neil was considering would be best for her?

36. A credit card offers the following cash-back rewards: ¼% of the first $5,000 in purchases, ½% of the next $5,000 in purchases, and 1% of all purchases beyond that annually, subject to an annual maximum cash-back of $150. Find the total cash-back reward you would receive if your charges to the card for the year totaled:

    a. $2,000

    b. $5,000

    c. $8,000

    d. $12,500

    e. $20,000

    f. $50,000

37. At what point (in terms of the total amount charged to the card for the year) do you stop earning any cash back on the card described in Exercise 29?

10.2 Mortgages

Owning a home has long been the considered a key part of "the American dream," and home ownership can offer plenty of financial advantages. Real estate, however, is not cheap, and it is rare for someone to be able to buy a house[5] without having to borrow the money to do it. Borrowing and lending money to finance real estate ownership is an enormous business, both in the United States and around the world. Having a good understanding of the mathematics of these loans is important both because most of us at one point or another will be involved in such a loan of one type or another, and because of the size and importance of this lending to the overall economy and business world.

The Language of Mortgages

The entire process of buying real estate, including mortgage loans, can be awfully intimidating and confusing. The amounts of money involved are large, and so the stakes are high, and the terms and process involved are usually unfamiliar, especially for first-time buyers. Before proceeding, it is worth defining and explaining some basic terms and processes. Additional terms will be defined throughout this section as the need arises.

A ***mortgage*** is a loan that is secured by real estate. That is to say, when the loan is made the borrower pledges some piece of real estate as collateral. If the borrower fails to pay back the loan as promised, the lender has the right to take the real estate from the borrower. The legal process by which a lender does this is called ***foreclosure***. Another way of saying that the lender has the right to do this is to say that the lender has a ***lien*** on the property.

Mortgages can be taken out on all sorts of real estate. Probably the most familiar example to most people would be a mortgage that someone takes out to buy a single-family house, but mortgage loans are also commonly made for apartments, townhouses, condominiums, vacation homes, and so on. (Mobile homes, however, are not considered to be real estate since they can be moved, and so loans made for them are not considered mortgages.) A landlord often will have a mortgage on rental property, a farmer may have a mortgage on agricultural land, and a business may have a mortgage on properties it owns as well.

[5] Throughout this section, we will use the terms *house* and *home* loosely, and not exclusively to refer to a single-family home.

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While it offers many, many advantages, buying a home usually involves borrowing a large amount of money. © Photodisc

With few exceptions, the amount borrowed with a mortgage loan must be less than the value of the property. Mortgage lenders typically set a maximum percentage of a property's value that can be borrowed. This is sometimes called the maximum ***loan to value percentage*** (or simply ***loan to value*** or ***LTV***). For example, if a lender's maximum LTV is 95%, the largest amount that could be borrowed against a $200,000 property would be (0.95)($200,000) = $190,000.

The difference between the value of a property and the amount that is owed against it is called the ***homeowner's equity***, or, more simply, just the ***equity***. In the previous example, if the value of the property is $200,000 and the amount owed on a mortgage against it is $190,000, the homeowner's equity would be $200,000 − $190,000 = $10,000. Rather than state the maximum someone can borrower, lenders sometimes will state the minimum equity. You can easily relate these two ideas, though, since the amount borrowed and the equity must add up to 100%. So if the maximum loan to value is 95%, we could equivalently say that the minimum equity is 100% − 95% = 5%.

A property's owner can have more than one mortgage on a given property. A ***first mortgage***, as the name suggests, is the primary mortgage loan. A ***second mortgage*** is an additional mortgage loan made against a given property. In the event of foreclosure, the first mortgage lender has first claim against the property, and the second mortgage lender gets paid only if there is money left after the first mortgage is satisfied. For this reason, a second mortgage will usually have a higher interest rate than a first mortgage, because while the second mortgage is secured, the security is not as great as that of a first mortgage.

Example 10.2.1 ***Les and Rhonda own a house worth $194,825. The balance they owe on their first mortgage is $118,548. They want to take out a second mortgage, and all of the lenders they have spoken with require a minimum equity of 5%. Find (a) their equity now, (b) the maximum amount they can borrow with the second mortgage, and (c) their equity if they borrow the maximum.***

(a) The equity is the difference between the property's value and the amount owed against it. Since

$$\$194{,}825 - \$118{,}548 = \$76{,}277$$

their equity now is $76,277.

(b) Since the minimum equity is 5%, the maximum they can borrow in total is 100% − 5% = 95%. So the maximum they can owe in total is

$$(0.95)(\$194{,}825) = \$185{,}084$$

Since they already owe $118,548, the most they could borrow with a second loan is

$$\$185{,}084 - \$118{,}548 = \$66{,}536.$$

(c) If they borrow the maximum, they will owe a total of $184,084 and their home equity would be reduced to

$$\$194{,}825 - \$185{,}084 = \$9{,}741.$$

We could also have calculated this by noting that if they borrow the maximum, they will have 5% equity, and so their equity would be:

$$(0.05)(\$194{,}825) = \$9{,}741$$

If Les and Rhonda go ahead and take out the second mortgage, this will greatly reduce their homeowner's equity. A second mortgage can be thought of as borrowing against the equity in the property, and for this reason such loans are often called ***home equity loans***.

This can be easily confused with a similar financial product called a ***home equity line of credit***. The difference between the two is that, with a home equity line, instead of being lent a set amount up front, the borrowers are given a checkbook or debit card that they can use to borrow money as needed against their home equity up to a set limit.

Occasionally, some finance companies will offer mortgage loans for more than the value of the property. In those cases, it is actually possible to have ***negative equity*** in a property. (Owing more against something than its value is sometimes referred to as ***being upside-down*** with the loan.) The following example will illustrate:

Example 10.2.2 Suppose that Les and Rhonda from the previous example are offered a loan by a company that will allow them to borrow up to a maximum of 125%. How much could they borrow with this loan, and what would their equity be if they did?

Les and Rhonda could have a maximum mortgage debt of

$$(1.25)(\$194{,}825) = \$243{,}531$$

Since they already owe $118,548, the most they could borrow with a second loan is

$$\$243{,}531 - \$118{,}548 = \$124{,}983.$$

If they borrow the maximum, they will owe a total of $184,084 and their home equity would be reduced to

$$\$194{,}825 - \$243{,}531 = -\$48{,}706.$$

This loan would certainly allow them to borrow quite a bit more, but the idea of owing nearly $50,000 more on a property than it is worth may not be entirely appealing for obvious reasons.

Types of Mortgage Loans

The market for mortgages is enormous, and the variety of different types of mortgages available is likewise immense. It would be an exaggeration to say that mortgages are available with absolutely any features someone might want—but not as much of an exaggeration as you might think. Most needs, though, can be met with a mortgage fitting into one of these standard types.

A ***fixed*** (or ***fixed-rate***) mortgage has an interest rate that is set at the beginning and never changes over the entire term of the loan. All of the mortgage examples in Chapter 4 were fixed-rate loans. The most common term for a fixed rate loan is 30 years, with 15 years being the next most common. Terms of 10 and 20 years are also not unusual, and in recent years terms as long as 40 years have become available. However, fixed-rate loans can be found with just about any reasonable term (and, depending on your point of view, perhaps some unreasonable ones as well). Because the interest rate on a fixed-rate loan never changes, neither does the payment. While you usually can pay extra to save on interest and reduce the time required to pay off the loan (as discussed in Chapter 4), doing so does not normally allow you to lower your payment in later months.

If mortgage rates drop during the term of the loan, a borrower can often take advantage of this by refinancing (as discussed in Chapter 4). However, if mortgage rates go up, the lender cannot raise a fixed rate. This system obviously gives an advantage to the borrower. The rates on shorter-term fixed mortgages will often be lower than on longer terms, largely because with a longer term the risk that interest rates will work against the lender at some point is greater.

An ***adjustable-rate mortgage*** (or ***ARM***) is a more complicated type of loan. An adjustable-rate loan will usually have some initial period of time for which the rate is fixed. After that, the rate can change, and when the rate changes the payment may change as well (we say that the loan is ***reamortized*** at that point). The loan's provisions should specify exactly when the rate can be adjusted (often once a year), and usually set the new rate according to some independent benchmark. Common benchmarks used for this are the average yield on U.S. Treasury bonds or the London Interbank Offered Rate (the LIBOR). Using an index ensures that while the lender can raise the rate, it cannot do so arbitrarily. Changes in the rate must be in line with prevailing interest rates in the market.

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There will also normally be a cap on how much the rate can change at a time (such as not more than 2% at each adjustment), and an overall limit on the highest the rate can go under any circumstances. These limits are referred to as ***caps***.

The term of an adjustable rate loan will specify both the length of the initial fixed period and the overall term of the loan. A 7/30 adjustable rate, for example, would be a loan that has a rate that is fixed for the first 7 years, and then varies for the next 23 years.

Adjustable-rate loans often will carry better interest rates up front than fixed-rate loans, since if interest rates rise, the lender is not at nearly the same risk of being stuck with a loan at an uncompetitive rate. For the same reason, the shorter the fixed period, the better the initial rate is likely to be.

In recent years, there has been an increase in the availability of some very flexible mortgage products. One example of this is ***interest-only mortgages***. With such loans, the borrower may only be obligated to pay enough to cover the interest on the loan each month, and may not have to pay anything against the principal. This option to pay only the interest may be permitted for a short introductory period, or for a more extended period of time. While these loans are attractive because of their flexibility, they of course pose the danger that, if nothing is paid against the principal, the process of amortization never begins, and no progress is made toward paying off the loan.

Occasionally loans may be even be offered that allow payments that are even less than the amount of interest for a period of time, in which case instead of declining, the amount owed actually grows. This situation is sometimes called ***negative amortization***. These sorts of loans can be attractive because of their flexibility and because they can make payments on a large loan appear more digestible, but they can also be dangerous. As we have seen in our prior work, if no progress is made toward paying off the balance, the interest costs over time can add up to staggering amounts. These sorts of loans also can be dangerous in that keeping the initial payments low can tempt a borrower to take on more overall debt than he can really handle in the long term. In some cases the interest may not be allowed to compound, meaning that while interest accumulates on the balance, there is no "interest on interest" building up. The mathematics behind those sorts of loans is therefore quite a bit trickier than on standard sorts of loans, which assume that interest is paid on the entire balance each month.

It is generally believed that home ownership is a good thing both for individual families and for society as a whole. To make it easier for families to own their own homes, and to encourage them to do so, there are many government-sponsored programs, at both the state and federal levels (and sometimes even the local government level). Two of the more common programs are the ***Federal Housing Administration*** (FHA), and programs offered through the ***Department of Veterans Affairs*** (VA). These programs may offer help in paying some of the costs of obtaining a loan, may offer especially low interest rates, or may offer loans to people who would not otherwise qualify for a mortgage loan. Such programs usually have specific criteria for who can qualify (such as only people with incomes below a set level, or those who have served in the military, and so on).

A loan for which the borrower qualified without any such programs is called a ***conventional*** loan.

Calculating Monthly Mortgage Payments (Fixed Loans)

We discussed how to find monthly mortgage payments in Chapter 4. The following example is intended to refresh your memory about these calculations.

Example 10.2.3 Chantal took out a $142,000 mortgage with a 30-year, fixed-rate loan at 7.2%. Find her monthly mortgage payment.

$$\begin{aligned} PV &= PMT\, a_{\overline{n}|i} \\ \$142{,}000 &= PMT\, a_{\overline{360}|0.006} \\ \$142{,}000 &= PMT(147.321356802) \\ PMT &= \$963.88 \end{aligned}$$

You may want to review the material from Section 4.4 covering present values of annuities. In particular, you should make sure that you know how to calculate present value annuity factors, as we will be using them quite a bit in this section. You are encouraged to work through the calculations of the example above to make sure that you can find the annuity factor correctly.

Calculating Monthly Mortgage Payments (Adjustable-Rate Loans)

Payments for adjustable-rate loans are calculated in the same way as those for fixed rate loans. The difference is that periodically the payment will be recalculated for the remaining loan balance, term, and new interest rate.

Example 10.2.4 ***Suppose that instead of being a 30-year fixed loan, the mortgage in the prior example was a 5/30 adjustable rate loan, with a 7.2% interest rate. Find Chantal's monthly payment.***

Her mortgage payment would be calculated in exactly the same way as in Example 10.2.3, since the term is still 30 years and the interest rate is still 7.2%. The difference between the two examples is that with the fixed-rate loan her payment will remain the same for the entire 30-year term. With the 5/30 ARM, her payment will be $963.88 for the first 5 years. After that time, her payment will most likely change, depending on the interest rate at the remaining balance owed at that time (it may either increase or decrease).

APRs and Mortgage Loans

Mortgage lenders are required by law to disclose an APR for their mortgage loans. The intended purpose of this APR is the same as the APY for an interest-bearing account (as discussed in Chapter 3): to allow an "apples-to-apples" comparison between different loan offerings. One lender may offer a 7.25% rate for a 30-year fixed loan, while another may offer a 7% for a similar loan, but make up for this lower rate with much higher fees and other charges. The APR is intended to take into account many of the extra fees and expenses.

The calculation of the APR is a highly complicated matter. There is little need for us to get involved in the calculation of APRs here. As a consumer, though, it is important to recognize certain things about mortgage APRs:

- APRs are meant to help make an overall comparison between similar mortgage products offered by different lenders by taking into account both interest and some other expenses that may be charged
- APRs do not necessarily include every expense that might be charged
- APRs cannot reasonably be used to compare different mortgage products. A fixed-rate loan is by its nature significantly different from an ARM, and so simply comparing APRs between the two products overlooks the differences in the nature of the products
- APRs for adjustable rate loans must be calculated on the basis of assumptions about where the rates will be adjusted in the future, even though this cannot be known in advance

And, most importantly for our purposes:

- In calculating the monthly payment, *do not use the APR!* The APR is not the actual interest rate used to determine the payment, nor is it used in a loan's amortization schedule.

Some Additional Monthly Expenses

In Chapter 4, we saw how to find the monthly payment on a fixed mortgage loan. The amount borrowed is the present value of an annuity. However, there are other regular expenses of owning a home beyond just the mortgage payment.

Real property taxes (sometimes called ***real estate taxes*** or simply ***property taxes***) are taxes that you must pay on the basis of the value of real estate that you own. They may be levied by state, county, or local governments and/or school districts. Real property tax

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rates can vary dramatically from one state to another, and even from one community to another. The real property tax rate in a given community will depend on the overall level of government spending and taxation, the tax structure (in some areas these taxes are the main source of tax revenue, while in others the government may collect more revenue from sales or income taxes), and the overall tax base. However, whether high or low, these taxes cannot be ignored in considering the cost of owning property.

What happens if you don't pay your property taxes? The short answer to this question is this: the government takes your property. Of course, this won't normally happen immediately. If you happen to be a month or two late with a payment, you may be hit with some late fees, but matters usually have to get much further out of hand before you would face a tax foreclosure. In most cases the government will make numerous attempts to collect before resorting to seizing the property. The exact process and procedures for this vary from one jurisdiction to another, but whatever the details may be, in the long run, if you don't pay, it's taken away. (Usually once seized, the property is sold at a ***tax auction***.)

Of course, your mortgage lender does not want to see this happen. It is incredibly unlikely that you would keep on making your mortgage payments on a property that has been seized for taxes, and it is unlikely that the lender will be able recover the amount you owe from such a property after the government has recovered the taxes owed. Again, the details of how this all plays out would depend on the specific situation and jurisdiction. But while the specific details might differ, one thing is certain: it will be a mess that the lender would just as soon avoid.

To ensure that your taxes are paid as they should be, lenders often require an ***escrow*** account for real estate taxes. An escrow account is a type of bank account into which you make monthly payments to build up the amount needed to pay your real estate taxes when they come due. The money in the account is yours, and you may earn a bit of interest on it, but the account is controlled by the mortgage lender. Each month, when you make your mortgage payment, you also pay into your escrow account. The amount of each payment is typically 1/12 of the expected property tax for the year. So if, for example, the taxes on your property are expected to be $3,000 for the year, each month you would send $3,000/12 = $250 with your mortgage payment, as shown here.

Of course, the actual tax bill will seldom come out to be exactly the amount expected, and so when the actual bill arrives, you may have to fork over some extra money if the bill is higher (or get a refund if it is lower). However, any difference will likely be small, and so the escrow account will still have fulfilled its purpose of making sure that enough money is set aside so that paying the taxes isn't a problem.

Mortgage lenders do not always require an escrow account, and are usually more willing to waive one for borrowers who make a large down payment (someone who has a large amount of money invested in the property is probably less likely to let the tax situation get out of hand). However, even borrowers who could get out of having an escrow account often prefer to have one, finding monthly escrow payments easier to swallow than one large annual tax payment.

Escrow may also be required for ***homeowners' insurance premiums***. Suppose that your house is destroyed in a fire, and you have either chosen not to carry any insurance or neglected to pay your premiums. Insurance would have covered the loss and provided money to pay off the mortgage or rebuild the property, but without it you'd be stuck still owing the mortgage on a property that has been destroyed. Of course, being an *honest* person you would *no doubt* honor your obligations and keep on making the payments. Mortgage lenders are concerned, though, that *some* people might just be tempted to stop making their payments and leave the lender to foreclose on a smoldering pile of ashes. To avoid this possibility, lenders want to be sure that those insurance premiums are paid, and so will also usually require escrow for insurance. Insurance escrow works in essentially the same way as tax escrow.

One additional expense that may be included with your monthly payment is ***private mortgage insurance*** (usually just referred to as ***PMI***). PMI is an insurance that protects the mortgage lender financially in the event that you do not pay and they are forced to foreclose.

At first, this sounds completely unnecessary. Suppose you bought a $150,000 house. Even if you made a small down payment of $5,000 (just over 3%), the amount borrowed would be $145,000, less than the house is worth. Even if you never made a single payment, the lender has the right to foreclose on a house worth more than the amount they are owed. Why on earth would they need insurance to protect them from this situation?

This is an overly simple way of looking at things though. If the bank has to foreclose on you, it will incur all sorts of expenses, including:

- Legal fees incurred to foreclose
- Costs of transferring ownership of the property
- Lost interest on its money (its money is tied up in the loan, but you aren't paying interest on it any more)
- Costs of marketing and selling the property
- Costs of maintaining the property (someone has to mow the lawn, pay the electric bill, patch roof leaks, etc.)
- Costs for repairs to the property (people about to lose their homes to foreclosure are often not in the mood to keep the place ship-shape)
- Taxes and insurance (both delinquent from what you did not pay as well as the costs for the time the lender owns the property)

Furthermore, the mortgage lender is in the business of making loans, not owning real estate. Often, in order to sell the property quickly, it will have to accept a lower price than the property might command with a seller willing to hold out for the best offer. The "$150,000 house" may sell for quite a bit less, and the costs and expenses of foreclosure may add up to tens of thousands of dollars. The end result is that the lender will recover a good deal less than the $145,000 it is owed. All in all, the lender in this situation can be left with a substantial loss, even though the loan was secured by a house worth more than the amount owed. The purpose of PMI is to help cover the lender's losses from this sort of situation.

If your down payment is 20% or more, the lender will not require PMI. In that case, the gap between the property's value and the amount owed is wide enough so that the lender's risk is minimal, even if it does wind up having to foreclose. Some lenders also offer loans without PMI with down payments less than 20%, though you may pay for that with a higher interest rate. When your equity reaches 22% based on the original amortization schedule, PMI must be removed automatically. You can also apply to have it removed sooner whenever your equity reaches 20%, which will usually occur as a result of a combination of the balance declining from your payments and an increase in the value of the property.

Example 10.2.5 ***Suppose Ben bought a house for $146,000, taking out a $135,000 mortgage. Three years later, his mortgage balance has declined to $132,833.15 and the house's market value has grown to $168,000. Can Ben apply to have his PMI discontinued?***

Ben's equity is now $168,000 − $132,883.15 = $35,116.85. This is $35,116.85/$168,000 = 20.90% equity, so the answer is yes.

Total Monthly Payment (PITI)

Since escrow and PMI (if required) are additional monthly expenses, looking at the monthly mortgage payment by itself as your "monthly payment" is a bit misleading. In fact, in most cases the monthly payment made to the mortgage lender will include all of these costs

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lumped together. This total payment is referred to as the ***PITI*** (an acronym for principal, interest, taxes, and insurance).

Example 10.2.6 ***Let's revisit Chantal from Examples 10.2.3 and 10.2.4. Suppose that her annual property taxes are expected to be $3,300 and her homeowners' insurance premium is $750. PMI is $45 a month. Find her total PITI.***

We have already calculated her mortgage payment to be $963.88.

The annual total for her escrow accounts will be $3,300 + $750 = $4,050. Each month she will need to pay 1/12 of that, or ($4,050/12) = $337.50.

Her PMI is $45. Note that we do not divide that by 12, since it is already expressed as a monthly value.

Her total PITI would then be $963.88 + $337.50 + $45.00 = $1,346.38.

Qualifying for a Mortgage

The risk a lender takes on with a mortgage loan is limited by the fact that the loan is secured, but, as we've already discussed, there is still plenty of risk involved. Even if the lender is completely sure that it would come out ahead if it had to foreclose, the fact is that a mortgage lender would much prefer to have the payments. As with any other loan, a lender wants to make sure that the people to whom it makes a loan are likely to pay as promised.

How does a lender decide whether or not someone gets a mortgage? The simple answer is that you qualify for a mortgage if and only if a lender decides that you are a good enough risk. Naturally there are some limitations—antidiscrimination laws apply and government regulations may restrict a lender from endangering its financial stability by taking on too many high-risk loans—but lenders do have a fair amount of leeway.

Most lenders, though, will take into account certain common factors:

- **Your credit history.** Lenders want to see that you both have a track record with credit and have handled your obligations responsibly. Those with a history of borrowing money and paying it back on time are much more attractive to a lender than those who have a history of getting in over their heads and paying late (or not at all). Those who have little experience with credit are also less attractive, since the lack of a track record makes it hard to judge how they will handle their debts.
- **Employment stability.** A mortgage loan is a long-term proposition. A potential borrower who has a track record of stable employment will be more attractive than a borrower who has come and gone from a wide range of jobs, or who doesn't have much of an employment history. (Lenders are usually willing to consider time spent in school acquiring the educational credentials for a career as part of employment history.)
- **Income.** This is probably the single most important factor of all. Regardless of how good your credit and employment history may be, if your income is not large enough to make your payments you are not going to be a good risk for the loan.

How does a lender determine whether or not your income is adequate for a loan? Most lenders decide this by using ***ratio tests***, comparing the total monthly PITI for the loan to a percentage of your gross (before taxes and other deductions) monthly income.

There are two commonly used ratio tests. The first is the simplest:

The 28% Rule

Total PITI cannot exceed 28% of gross monthly income.

The purpose of this rule is simply to make sure that your income is adequate to cover the total monthly payment. The 28% figure is commonly used for this test, but different lenders may use different percents. However, even if a different percent is used, that percent is likely to be used in the same general way.

The following example will illustrate how the 28% rule is used.

Example 10.2.7 ***Antoine and Maria earn a combined annual income of $76,500. They are trying to qualify for a mortgage on a house for which the total monthly payment (PITI) would be $1,494.57. Do they pass the 28% test?***

Their gross monthly income is $76,500/12 = $6,375; 28% of this is (0.28)($6,375) = $1,785. Since the PITI is less than this, they PASS the 28% test.

The 28% rule is pretty simple, and there is a lot it does not take into account. In particular, it does not take into account any other financial obligations. If Antoine and Maria have no other debts, their situation is quite different than if they have two huge car loans, maxed out credit cards, and enormous student loans to boot. A second rule commonly used rule takes these other expenses into account:

The 36% Rule

Total PITI and all other long-term debt payments cannot exceed 36% of gross monthly income.

Long-term debt includes any debt with more than 1 year left in its term.

Just as with the 28% rule, not all lenders use the same percent for this type of test, and not all lenders count the same things as long-term debt payments. In particular, some lenders use 38% rather than 36% for this rule. However, the general idea is the same, even if the specific details may vary.

The following example will illustrate how this rule is used.

Example 10.2.8 ***Suppose that Antoine and Maria (from the previous example) have two car loans. The first carries a monthly payment of $309.15 and has 2 years and 7 months left to go; the second has 6 months left to go and the monthly payment is $175.14. They have student loan payments of $109.15 per month, and the minimum monthly payments on their credit cards total $75.00. Do they pass the 36% test?***

36% of their gross monthly income is (0.36)($6,375) = $2,295.00.

Their total debt payments would be $1,494.57 + $309.15 + $109.15 + $75 = $1,987.87. (The second car loan is not included in this because less than 1 year remains in its term.) This total is less than 36% of their monthly gross income, and so they PASS the 36% test as well.

Usually potential borrowers must pass both of these tests. If you pass one, but not the other, you would not normally qualify for the mortgage. Antoine and Maria pass the income qualification with no problem, and so assuming that everything else is in order they would qualify for this mortgage easily.

An additional word of caution is called for here. The tests used by the lender are designed to assess whether or not you are likely to be able to make the payments. They are not designed to assess whether or not you will personally find the payments comfortable. A PITI payment that you can afford may still eat up more of your income than you find comfortable. The lender's concern is: can you afford to make the payments? The lender is not concerned with whether or not the payments will eat up so much of your income that you don't have money left to do other things that you might want to and buy other things you might want to buy. Be careful not to confuse a lender's determination of what you can afford with your own assessment of what you can afford.

Up-Front Expenses

In addition to the monthly PITI payments, buying real estate usually involves a significant cash outlay up front. Some of the many expenses involved include:

- **Down payment.** As we have already discussed, you cannot normally borrow 100% of the property's value. Some down payment is required. Ordinarily, the minimum down payment will be at least 3 to 5% of the price.
- **Legal fees.** Home buyers often will hire an attorney to make sure that everything in the transaction is done correctly. Even if you decide not to use an attorney, the mortgage lender will have an attorney, and the borrower pays those fees.

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- **Appraisal.** The lender will require an appraisal of the property's value. Since the property is their security for the loan, the lender wants to make sure that you are not overpaying for the property.
- **Title search and insurance.** If the seller is not the true owner of the property, even if you buy it in good faith you may lose out when the actual owner seeks to reclaim the property. A title insurance company researches the ownership history to make sure that the seller has a clean title to the property, and provides an insurance policy to cover you and/or the lender in case a mistake is made that they don't catch. Problems with titles can arise from lost or improperly completed paperwork, inheritance, divorces, fraud, or simple errors.
- **Inspections.** A house that is structurally damaged by a termite infestation is worth less than one that isn't. The lender may require certain inspections to make sure that there are no such hidden problems.
- **Flood check.** Damage from floods is not covered by homeowners' insurance. The lender will require a check to make sure that the property does not lie in an area where flooding is a risk, and if it is, will want to know about it to make sure that a separate flood insurance policy is in place.
- **Recording fees.** Local governments maintain records of who owns what properties, and charge fees to record changes in these records.
- **Mortgage taxes.** The state, county, city, village, and/or town in which the property is located may assess taxes on mortgage loans, as well as other taxes on the sale of a property. Mortgage taxes are currently charged in Alabama, Florida, Georgia, Hawaii, Kansas, Maryland, Minnesota, New York, Oklahoma, Tennessee, and Virginia.
- **Application and origination fees.** The mortgage lender may charge various fees for processing a mortgage application, running credit checks, and so on.
- **Miscellaneous other fees and expenses.** This includes fees for notaries, overnight postage, and various other expenses.

All together, these up-front expenses can add up to a lot of money. These costs (excluding the down payment) are often referred to collectively as ***closing costs***. Closing costs can run anywhere from several hundred to several thousand dollars on a typical property.

Another demand for up-front money comes from what are commonly known as ***prepaids***. The best way to see what these are is by means of an example. Suppose that you close on the purchase of a house on October 1, 2006. The annual property taxes are \$3,600, and they are due on January 1. Property taxes are paid in advance, so the people you bought the house from paid the taxes for 2006. Since you will own the house for the last 3 months of 2006, you must reimburse them for 3 months' worth of the taxes.

You also have a problem with your escrow payments. You will be making payments of $\$3{,}600/12 = \$300$ into your escrow account with your November, December, and January payments, but that adds up to only \$900. The taxes due on January 1 are \$3,600. The problem is that you need a full year of escrow payments to accumulate the full year's taxes, and so when you close on the house, you need to make a payment to catch up for those 9 months' payments. And so, you will need to come up with 9 months' worth of taxes for your escrow account.

Putting these two figures together, you will need a total of $3 + 9 = 12$ months, or one full year, of taxes. With a moment's thought it should be clear that if you had bought the house on another date, the split between reimbursing the previous owner and escrow catch-up would be different, but the total would still be a full year of taxes. For example, if the closing had been on May 1, 2006, you would have had 8 months of taxes reimbursing the seller, and 4 months worth of catch up. But in any case, you will need a full year's worth.

Because insurance premiums must also be paid in advance, you will also need to come up with a full year of homeowners' insurance as well. The monthly payments to your escrow account will pay for *next* year's premium, but they don't help with the current year.

Putting all of these expenses together can add up to quite a large amount of money, as this example will illustrate:

Example 10.2.9 ***Drew and Joanne are buying a house for $128,550. They will make a minimum 3% down payment, and closing costs will total $2,100. Annual property taxes are $2,894 and homeowners' insurance is $757 annually. How much money will they need up front?***

Their down payment comes to (0.03)($128,550) = $3,856.50.

The closing costs are given, and they will also need $2,894 + $757 = $3,651 for prepaids. Putting this all together, we get a total of:

$3,856.50 + $2,100 + $3,651 = $9,607.50

Ouch! Even if Drew and Joanne can comfortably manage the monthly payment on this loan, the up-front costs present a difficult barrier to overcome. People who are selling one house and buying a new one often will have enough equity in the home they are selling to cover these costs easily, but first-time home buyers don't have that advantage.

Fortunately, special programs, both government-sponsored and others, will sometimes provide ways to get around this hurdle (for example, ***"no-closing cost" loans*** are sometimes offered, under which the lender shoulders the closing costs, often in exchange for a higher interest rate on the loan). Sometimes lenders will require less than a full year of taxes and insurance up front, and instead make up any escrow deficits by requiring higher escrow payments during the first year.

An Optional Up-Front Expense: Points

Lenders often offer the opportunity to "buy" a lower interest rate by paying ***points***. Points are a fee paid to the lender up front, in exchange for a lower interest rate. One point is equal to 1% of the amount of the loan.

For example, a lender might offer the following choices for a 30-year fixed loan:

Points	Interest Rate
None	7.5%
2.5	6.25%

A borrower who can afford to come up with the money to pay points up front can potentially end up saving quite a bit of money, as demonstrated in the following example.

Example 10.2.10 ***Suppose that Drew and Joanne (from Example 10.2.9) are offered the mortgage choices shown above.***

(a) How much more would they have to come up with if they chose to pay the points?

(b) What would their monthly mortgage payment be with each of the options offered?

(c) How much would they save over the 30-year life of the loan if they paid the points?

(a) The price of the house is $128,550. Drew and Joanne are making a $3856.50 down payment, and so they will be borrowing $128,550 − $3,856.50 = $124,693.50.

Since each point is 1% of the loan, 2.5 points is 2.5%, or (0.025)($124,693.50) = $3,117.34.

(b) Using the present value annuity formula with PV = $124,693.50 and a 7.5% interest rate for 30 years gives that the monthly payment for the loan with no points would be $871.88. Using the 6.25% rate that they would have if they paid points gives a monthly payment of $767.76.

(c) Looking at the total payments over the life of the loan, we see that:
No points: (360)($871.88) = $313,876.80
Points: (360)($767.76) = $276,393.60

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Subtracting to find the difference, we can see that, over 30 years, they would save \$37,483.20.

While the \$3,117.34 up front doesn't sound particularly appealing, saving \$37,483.20 certainly does. This figure can be deceptive though, because it is based on the assumption that Drew and Joanne will actually keep this loan for the full 30 years. If they decide to move, if interest rates drop and they seize the opportunity to refinance, or if they pay extra on the loan to kill it off sooner, they will not see the full savings. In fact, it is rare that a 30-year loan actually stays around for a full 30 years, because one of those events is very likely to happen over the course of 30 years.

One tool that is sometimes used to assess whether or not an up-front expense is worth it or not is the ***payback period***. To calculate the payback period, we simply look at how much Drew and Joanne would save on their monthly payment, and determine how many months it would take for those savings to add up to the cost of the points.

Example 10.2.11 ***Find the payback period for the decision to pay points in the previous example.***

If they pay the points, their monthly payment will be \$871.88 − \$767.76 = \$104.12. Then \$3,117.34/\$104.12 = 29.94 or just about 30 months.

So we conclude it will take about 30 months for Drew and Joanne to recoup their investment in the points.

Payback periods are a crude tool, and don't take everything into account perfectly (they ignore the time value of money for example), but they do provide a helpful way of looking at the points decision. Until 30 months have passed, the accumulated savings from the lower payment does not add up to the cost of the points. But from that point on, the savings from the points will exceed their cost. Knowing how long it will take for them to recoup their investment in those points may help them to evaluate whether or not they are worth the expense. (Payback periods as a tool for financial decision making are discussed in greater depth in Chapter 14.)

EXERCISES 10.2

A. The Language of Mortgages

1. Howard and Lita own a house worth \$128,000. They have a mortgage on the house, with an outstanding balance of \$89,537. How much home equity do they have?

2. A corporation owns the office building that houses its headquarters. The value of the property is \$1,535,000. The balance on the mortgage on the property is \$915,888. Find the corporation's equity in the building.

3. Sally owes \$87,309 on the mortgage on a condo that she owns, which is worth \$115,000. She wants to take out a home equity loan. The bank loan officer tells her that the maximum loan to value percent is 90%. How much could Sally borrow?

4. Bill owns an apartment building worth \$450,000. He has a mortgage on the property with a balance of \$338,919. If he takes out the largest home equity loan he can from a bank that requires him to have a minimum of 8% in equity in the property, how much can he borrow? If he does this, what will his equity be?

5. Liankung owns a house worth $218,500. He has two mortgages on the property. The balance on the first mortgage is $132,890 and on the second the balance is $52,675. A finance company offers him the opportunity to borrow more against the house, allowing a maximum loan to value of 115%. If he takes out the largest loan the bank will allow, how much can he borrow? What would his equity be then?

B. Calculating Monthly Mortgage Payments

Note that the problems in this section require calculation of the payment on the mortgage loan itself ***only.***

6. Find the monthly payment on a 30-year fixed mortgage loan of $108,525 if the interest rate is 6¾%.

7. Find the initial monthly payment on a 3/20 ARM of $178,802 if the interest rate is 5.35%. How long will it be before the interest rate (and thus the payment) can change?

C. Escrow Accounts

8. How much will my monthly escrow payment be if my property taxes are expected to be $2,895 annually and my homeowners' insurance premium is $745 per year?

9. Find the monthly escrow payment for someone whose annual property taxes are $3,900 and homeowners' insurance premium is $663.

10. Jack and Tania's homeowners' insurance premium is $1,252, and their property taxes are $5,254. How much will their monthly escrow payment be?

D. Total Monthly Payment (PITI)

11. Mike's monthly mortgage payment is $675.13. His property taxes are $1,392 and his homeowners' insurance premium is $885. He also pays $48.17 a month for PMI. Find his total monthly PITI payment.

12. Suppose that for the property in Exercise 6 the property taxes are $1,725 and homeowners' insurance costs $607. PMI of $57.25 is also required. Find the total monthly PITI.

13. Aaron and Cienna are considering buying a house for $175,000. They would make a $10,000 down payment, and borrow the rest with a 30-year fixed mortgage at 7.45%. Annual property taxes would be $5,700, and homeowners' insurance would be $935. They would also need to pay $84.00 monthly for PMI. What would their total monthly payment (PITI) be?

14. Russ is contemplating making an investment by buying a rental property. The cost of the building would be $399,000, and he can afford to make a 20% down payment. He plans to finance the rest with a 25-year mortgage at 6.98%. Annual property taxes are $10,384 and insurance would cost him $2,850 per year. What will his monthly payment be?

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E. Qualifying for a Mortgage

15. True or false: If you pass both the 28% and 36% tests, you qualify for the loan.

16. True of false: If you pass either the 28% or the 36% test and meet all other qualification requirements (credit, employment history, etc.), you qualify for the loan.

17. True or false: Every mortgage lender uses the 28% and 36% tests.

18. True or false: If you qualify for a mortgage loan, that means you can comfortably afford the house payments.

19. Lilianna makes $45,000 per year. She wants to buy a house on which the total monthly PITI would be $1,297.55. Her only other long-term debt payments are for a car loan on which the monthly payment is $255.19. Does she pass the 28% test? Does she pass the 36% test? Assuming that everything else with her application is acceptable, would she qualify for the loan?

20. The Sumners earn a combined annual income of $89,700. Would they pass the 28% test for a property for which the monthly PITI would be $1,799.23? If their total additional long-term debt payments total $855.92, would they pass the 36% test?

21. If Aaron and Cienna from Exercise 13 have a combined annual income of $82,500, would they pass the 28% test? If their long-term debt payments total $355.99, would they pass the 36% test? Assuming everything else is OK, would they qualify for the mortgage?

22. Bruce and Elena earn a combined $57,840 per year. They have two car loans, both of which have more than a year left to go, with payments totaling $464.25. They have student loan payments totaling $122.75, and they have 6 months left to pay on a loan for some furniture on which the payment is $125 monthly. The minimum monthly payments on their credit cards total $80.

    They want to buy a house for $117,300. They plan to make a 5% down payment, and finance the rest with a 30-year fixed-rate mortgage at 7.3%. Property taxes amount to $1,856 per year, and homeowners' insurance premiums are $762 annually. PMI would be $47 a month.

    They have applied for a mortgage with a lender that uses the 28% and 36% tests.

    a. Calculate their PITI for this loan.

    b. Calculate 28% of their monthly gross income. Do they pass the 28% test?

    c. Calculate their total monthly long-term debt payments, and calculate 36% of their monthly gross income. Do they pass the 36% test?

    d. Assuming everything else is OK, would they get the mortgage loan?

F. Up-Front Expenses

23. Herb wants to buy a house for $192,800. He is planning on making a 5% down payment, and closing costs will amount to $3,170. Annual property taxes are $5,155 and annual homeowners' insurance premiums will total $1,250. How much money will he need up front?

24. Will and Stacey are selling their house and moving to a new city. In the time that they've owned their current house, housing prices have risen a lot, and after paying their current mortgage off they will be left with $72,853. Their new house costs $279,800, and they want to make a 20% down payment. Property taxes come to $6,825 per year, and homeowners' insurance will cost them $1,809. Their new loan has no closing costs. Do they have enough from the sale of the old house to cover the up-front costs?

25. In the examples we have seen in this section, the up-front expenses often add up to a very large amount of money. True or false: In order to have any hope of buying a house, you must have a large amount of money in savings to pay for closing costs, prepaids, down payments, etc.

G. Up-Front Expenses—Points

26. Suppose you take out a mortgage for $219,836 with 2 points. How much do these points cost you?

27. Suppose you buy a townhouse for $109,000, and make a 5% down payment. If you take out a mortgage that requires paying 1¾ points, how much will you pay for the points?

28. Suppose that Will and Stacey's (from Exercise 24) lender suggests that they could save quite a bit on their loan if they paid 2½ points. Do they have enough from their old house to pay these points in addition to the up-front costs?

29. Heidi is looking for a mortgage to buy a condominium. The condo costs $122,950 and she will make a 3% down payment. She is considering two different choices for a 30-year fixed rate loan:

Points	***Interest Rate***
0	7¾%
1¾	6⅞%

a. Find the up-front cost of the points if she chooses that option.

b. Find her monthly mortgage payment with each option.

c. How much would she save over 30 years by paying points, assuming that she keeps the loan for the full 30 years.

d. Find the payback period for paying points.

30. You are considering two different options for a 15-year fixed-rate mortgage of $144,315:

Points	***Interest Rate***
None	6.85%
1.5	6.04%

a. How much could you save over the full 15-year term by paying points?

b. What would the payback period be for these points?

H. Grab Bag

31. Jack and Diane want to buy a house for $83,000. They are required to make a 5% down payment. Property taxes are $2,500 per year, and homeowners' insurance is $400 per year. Closing costs for their mortgage loan are $2,800. How much money do they need up front to buy the house?

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32. Ricky and Lucy bought an apartment for $110,000. They made a 25% down payment and financed the rest with a 15-year loan at 7.8% interest. Taxes are $1,900 per year, insurance is $400 and PMI (if required) would cost $60 per month. What is their total monthly payment?

33. Jerry bought a house for $182,000 with a $32,000 down payment. Is it likely that Jerry would need to pay PMI?

34. Suppose Wendy and Peter decided to buy a $297,000 house with an 8% down payment. Property taxes are $7,200 and homeowners' insurance is $1,550 per year. Closing costs are $3,300. Assuming their mortgage requires no points, how much money do they need up front to buy the house? If instead they took out a mortgage loan requiring 2 points, how much would they need up front then?

35. Jamie and Cameron own a house with a market value of $258,000. They presently owe $148,923.54 on their mortgage loan. They are considering taking out a home equity loan. A lender offers to make a home equity loan with up to a 105% LTV. What is the maximum they could borrow from this lender? What would their equity in the house be if they took out the maximum loan?

36. Glenys is buying a $130,000 house with a 10% down payment.
    a. If she finances the purchase with a 30-year loan at 8.5% and no points, how much will her monthly mortgage payment be?
    b. If she finances the purchase with a 30-year loan at 7.5% and 2 points, how much money will she need up front for the points and how much will her monthly mortgage payment be?
    c. What is Glenys' payback period for the points?

37. Travis and Lisa have a joint gross annual income of $47,035. If they apply for a mortgage with a lender that uses only the 28% income test, what would be the maximum monthly PITI for which they would qualify?

38. Tom is buying a $145,000 house with a 20% down payment. Closing costs are $4,200, property taxes are $3,800 annually, and homeowners' insurance is $520.
    a. If he finances the purchase with a 30-year loan at 8.2% and 0.5 point, how much money will he need up front and how much will his monthly PITI payment be?
    b. If he finances the purchase with a 30-year loan at 7.0% and 2.5 points, how much money will he need up front and how much will his monthly PITI payment be?

39. Ramesh and Mackenzie plan to buy a townhouse for $150,000. They would make a $25,000 down payment and finance the rest with a 30-year loan at 8.4% interest. Property taxes are $2,800 per year, homeowners' insurance is $620, and PMI would be $60 per month.
    a. What will their total monthly PITI payment be for the house?
    b. If they have a combined annual income of $61,500, do they pass the 28% test?

40. Janet's property taxes are $2,034.76 annually, and her homeowners' insurance premium is $585.02 annually. What is her monthly escrow payment?

41. Warren and Stella bought a house for $154,035. They made an $8,000 down payment. Their lender required them to pay PMI on the loan. Four years later, their mortgage balance has been reduced to $139,035, and the market value of their house had grown to $163,500. (a) Find their equity today. (b) Is their equity high enough that they will be able to eliminate their PMI?

42. Karam owes $172,835.19 on his mortgage. His house is worth $279,000. How much equity does he have in the house?

43. Bruce is taking out a 25-year fixed-rate mortgage for $119,500. He has a choice of a 7.9% rate with no points or a 7.5% rate with 1¼ points. What is the payback period for the points on this loan? If Bruce expects to sell this house in 5 years, is it financially worthwhile for him to pay the points? Explain.

44. Felipe's house is worth $183,500 and he owes $162,104. His bank offers him a home equity loan with a maximum LTV of 95%. How much could Felipe borrow?

I. Additional Exercise

45. Five years ago, Kris bought a house for $135,000. He made a 5% down payment, and financed the rest with a 30-year fixed-rate loan at 6.96%. He has made all of his monthly payments exactly as scheduled. In the past 5 years, the real estate market in his town has been hot, and the value of his house has risen at a 5.5% annual rate. How much equity does Kris have in the house today?

10.3 Installment Plans

Many major purchases are made with the aid of some sort of borrowing. The reasons for this are obvious; many buyers do not always have the ability to pay for expensive consumer goods they want or need in cash. For consumers who want to buy, and merchants who want to sell, goods such as refrigerators, computers, big screen TVs, and pianos, "a few dollars down and easy monthly payments" can make the expense of these purchases seem much easier to swallow.

Any purchase made with the assistance of borrowing is said to be ***on credit***. We have already explored a number of options for credit purchases. For large purchases like houses or cars, a secured loan (a mortgage or auto loan) is the most common approach.

For other, smaller scale but still expensive, items, the purchase may often be made with a credit card. Installment plans present another option that we will explore in this section.

A ***fixed-payment installment plan*** is essentially a loan, usually made to allow the purchase of a particular item, repaid by making a set series of payments. These plans may be offered by a merchant (though usually the merchant would arrange the actual financing through an outside finance company) or through a traditional bank. These installment plans normally (though not always) require some down payment to be made, and then the remaining balance is assessed a ***finance charge*** (also known as the ***carrying charge***), and the total of the remaining balance and its carrying charge is divided evenly among the number of payments. These plans are typically fairly short term arrangements, and often carry very high interest rates.

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Example 10.3.1 ***Bill is shopping for new dining room furniture. A furniture store offers a dining room set he likes for $1,600. The store offers to arrange financing for this purchase for $200 down, and 9 monthly payments on an installment plan with a 10% carrying charge. What would the monthly payment be?***

The amount being financed is $1,600 − $200 = $1,400. The 10% carrying charge amounts to (10%)($1,400) = $140. Adding this to the amount borrowed gives us $1,400 + $140 = $1,540 to be paid on the plan. (We could equivalently have multiplied $1,400 by 1.10). To compute the payment, we divide to get $1,540/9 = $171.11 monthly.

It is easy to misunderstand the 10% carrying charge used in this example, and think of it as an interest rate. It is not! For one thing, interest rates are understood to be rates per year, while this rate applied in full even though the term was only 9 months. Also, this 10% rate applied to the entire $1,400 initial balance, even though payments are being made that should be reducing the balance over time. With a little thought, it should be apparent that the interest rate for this loan is actually significantly higher than 10%.

This can be even more easily misunderstood with so-called ***simple interest loans***. This sort of loan is a type of installment plan where the carrying charge is expressed as a simple interest rate. However, the interest is calculated by applying the simple interest rate to the entire original principal of the loan, even though the payments are working to pay off the loan during its term. Interest calculated in this way is sometimes called ***add-on interest***.

Example 10.3.2 ***Tonya bought an electronic piano for $1,850. She financed this purchase with a $50 down payment and a 2-year simple interest loan at an 8% rate and monthly payments. Find her monthly payment amount.***

Subtracting her down payment leaves $1,850 − $50 = $1,800 to be financed. Her carrying charge is:

$$I = PRT$$
$$I = \$1{,}800(0.08)(2)$$
$$I = \$288$$

We add this on to arrive at a total of $1,800 + $288 = $2,088. Dividing this by the 24 monthly payments, we arrive at $2,088/24 = $87.00 per month.

Again, the rate quoted here is easily misunderstood. The actual interest rate that Tonya is paying would actually be higher than 8%. In this case, the rate quoted is per year, just as we would expect an interest rate to be, *but it applies to the entire initial principal* for the whole 2 years. The fact that payments are being made on the debt is completely ignored in the interest calculation.

The Rule of 78 (Optional)

Back in Chapter 4 when we studied loans whose payments were calculated using annuities, we found that we could use an amortization table to find the breakdown between principal and interest for each payment. We also could use either an amortization table or the present value of the remaining payments to determine the remaining balance owed at any point in time. This was all based on a certain way of looking at the interest calculation. For each monthly payment, interest was calculated on the balance, the payment was used to pay this interest, and the remainder of the payment went to reduce the balance. For the next month, the same process applied, using the new, lower balance, and so on until the balance was reduced to zero.

That method does not entirely seem to fit the situation we are describing here. While it still seems reasonable that there must be some way of splitting the payments between principal and interest or finding the amount owed midway through the loan's term, the amortization process is exactly reflective of how the loan works. One traditional way of handling these sorts of questions is the ***Rule of 78*** (not to be confused with the Rule of 72, despite the similarity of the names). In fact, fixed-term installment loans are sometimes referred to as ***Rule of 78 loans***.

The Rule of 78 reflects the idea (which we also saw with amortization) that interest should constitute a higher proportion of the payments on a loan early in the term. For a 12-month loan, we assume that 12/78 of the total interest is paid with the first payment, 11/78 is paid with the second, 10/78 is paid with the third, and so on until the last payment takes care of 1/78 of the total interest. The reason that we are dividing by 78 is that $12 + 11 + 10 + 9 + \cdots + 2 + 1 = 78$, and so in total over the 12 months, $12/78 + 11/78 + 10/78 + 9/78 + \cdots + 2/78 + 1/78 = 78/78$ (i.e., all) of the interest is paid.

The Rule of 78 gets its name from the fact that the total of the interest fractions for 12 months is 78/78, and 12 months is a very common term for an installment plan. However, if the term is different than 1 year, we use whatever the total of all the whole numbers from 1 to the number of payments adds up to. For Bill's dining room set in Example 10.3.1, the number to use would be $9 + 8 + 7 + \cdots + 2 + 1 = 45$, so his first payment would take care of 9/45 of the total interest, and so on.

It would be helpful to have a convenient formula to calculate this total of the whole numbers up to a given number. Fortunately, a surprisingly simple formula can do this.[6]

FORMULA 10.3.1
The Sum of Whole Numbers Formula

The sum of all the whole numbers from 1 to n is:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

We could have used this formula to find the total $9 + 8 + 7 + \cdots + 2 + 1$ above by using $n = 9$ in this formula to get $(9)(10)/2 = 90/2 = 45$. Having this formula is especially helpful with longer-term loans.

Example 10.3.3 ***How much interest does Tonya (from Example 10.3.2) pay with her first monthly payment? How much interest does she pay with her second payment? What about her last payment?***

Tonya's loan is for 24 months, so the denominator we need is $1 + 2 + 3 + \cdots + 24 = (24)(25)/2 = 300$. The total interest she is paying is \$288. So, with her first payment she is paying (24/300)(\$288) = \$23.04. With her second payment, she pays (23/300)(\$288) = \$22.08. With her last payment, she pays (1/300)(\$288) = \$0.96 in interest.

If Tonya decides to pay off her loan early, the Rule of 78 can be used to determine the payoff amount. To do this, we calculate the remaining interest that she would pay over the remaining term of the loan, and then subtract that from her remaining payments.

Example 10.3.4 ***Suppose that after 10 months, Tonya decides to pay off her piano in full. Assuming the Rule of 78 is used for his payment plan, how much would she need to pay it off at that point?***

After 10 months, she has $24 - 10 = 14$ payments left. At \$87.00 a month, this would mean a total of 14(\$87.00) = \$1,218. If she pays early, though, we would reduce this by the amount of the interest that she would have paid over these last 14 payments.

Over the course of those 14 remaining payments, Tonya would have paid $14/300 + 13/300 + 12/300 \cdots + 1/300$ of the interest. We can use our sum formula to find that $14 + 13 + 12 + \cdots + 1 = (14)(15)/2 = 105$. So in total her last 14 payments work out to 105/300 of the total interest, or (105/300)(\$288) = \$100.80. By paying early she avoids this interest, so her loan payoff would be \$1,218.00 − \$100.80 = \$1,117.20.

[6]This formula has a story behind it. According to legend, one day in school the boy who grew up to be the great mathematician Carl Friedrich Gauss was bored and causing trouble in class. The teacher, knowing that Gauss liked arithmetic, gave him the assignment of adding up all the whole numbers from 1 to 1,000, thinking it would keep him busy for a while. Moments later, Gauss supplied the correct total, having come up with the formula on his own.

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Installment Plan Interest Rates: Tables and Spreadsheets

As we noted early in this section, the realistic interest rate actually being paid on one of these loans may be easily misunderstood. The Truth in Lending Act requires lenders to disclose the “actual” interest rate for a loan, generally speaking meaning the interest rate that would result in the plans’ payments if the interest were calculated in the amortized way that we have previously seen. This “actual rate” is generally referred to as the ***APR***.

The regulations surrounding calculation of the APR are complex, but we can get a general idea from some of the mathematical tools we have already seen.

We noted that the actual rate for Tonya’s piano loan must be much higher than the 8% nominal simple interest rate. We know the term (2 years), the payments ($87/month) and the initial balance ($1,800), but calculating the APR for this poses a bit of a challenge.

As we saw in Chapter 5, a problem of this type can be solved by setting up an amortization table, and then using guess and check methods to find the interest rate that works. Realistically speaking this is probably the most efficient method to use today. We can find the rate she is actually paying by setting up an amortization table, using $1,800 as the present value and $87.00 as the payment, and then using either guess and check or goal seek to find the rate that would have the payment killing off an $1,800 debt in 24 months.

Example 10.3.5 ***Calculate the actual interest rate Tonya is paying on her piano installment plan, using a spreadsheet amortization table like the one here.***

	A	B	C	D	E
1	**Rate:**	14.65%	**Initial Balance:**	$1,800	
2	**Month**	**Payment**	**To Interest**	**To Principal**	**Ending Balance**
3	1	$87.00	$21.98	$65.02	$1,734.98
4	2	$87.00	$21.18	$65.82	$1,669.16
	Rows Omitted				
26	24	$87.00	$1.04	$85.96	-$0.66

When the Truth in Lending law was passed, though, personal computer technology was not what it is today, and that sort of solution was not a realistic option. One option is to find the rate using tables. The idea behind this table approach rate is to find the present value annuity factor implied by the present value and payments. For example, for Tonya’s loan, we would start with the formula:

$$PV = PMT a_{\overline{n}|i}$$

Then, we would plug in the known values of PV and PMT, and solve for $a_{\overline{n}|i}$.

$$\$1{,}800 = \$87 a_{\overline{n}|i}$$
$$a_{\overline{n}|i} = 20.68965517$$

The next step would be to go to a table of annuity factors, looking at the annuity factors for various different interest rates with n = 24. The interest rate that gives the closest match to the annuity factor we calculated is then determined to be the ***actuarial rate***, since using that rate with its annuity factor would produce the same payments.

While tables are not much used today, an excerpt from such a table is shown below as an illustration.

ANNUITY FACTORS BY RATE AND NUMBER OF PAYMENTS

	14.50%	***14.55%***	***14.60%***	***14.65%***	***14.70%***	***14.75%***
n = 24	20.72563428	20.71546249	20.70529779	20.69514016	20.68498961	20.67484613

Since the factor shown for the 14.65% rate is the closest we can get to the annuity factor we are looking for, we would conclude that this is the APR.

Installment Plan Interest Rates: The Approximation Formula

An alternative means of calculating the APR is an approximate calculation of the actual rate by using a formula such as the one given below.

FORMULA 10.3.2
APR Approximation Formula

$$\text{APR} \approx \frac{2nr}{n + 1}$$

where
n represents the NUMBER OF PAYMENTS
and
r represents the NOMINAL SIMPLE INTEREST RATE

The symbol ≈ used above indicates "is *approximately* equal to."

Example 10.3.6 ***Calculate the APR interest rate for Tonya's piano loan by the approximation formula.***

$$APR \approx \frac{2nr}{n+1} \approx \frac{2(24)(0.08)}{25} \approx 15.36\%$$

This is not the same as the 14.65% rate we found by using a spreadsheet or table. However, even though these two rates are not exactly the same, they are "in the same ballpark." The approximation formula is claimed to give only an approximation, not the exact value. Note also that the APR interest rate is considerably higher than 8%. Either of these rates makes that quite apparent!

We could also find the APR for Bill's dining room furniture by both of these methods as well. Since we did not have a simple interest rate for Bill's plan, though, we would first have to calculate it.

Example 10.3.7 ***Bill is paying off a $1,400 purchase with nine monthly payments of $171.11. Calculate the APR for this installment plan, using amortization tables and using the approximation formula.***

Using an amortization table with guess and check or solving for the annuity factor and finding it in an annuity factor table we come up with a rate of 23.40%.

By spreadsheet:

	A	B	C	D	E
1	Rate:	23.40%	Initial Balance:	$1,400	
2	Month	Payment	To Interest	To Principal	Ending Balance
3	1	$171.11	$27.30	$143.81	$1,256.19
4	2	$171.11	$24.50	$146.61	$1,109.58
5	3	$171.11	$21.64	$149.47	$960.11
	Rows Omitted				
11	9	$171.11	$3.27	$167.84	$0.02

By table:

We can solve for the annuity factor:

$$\$1,400 = \$171.11 a_{\overline{n}|i}$$
$$a_{\overline{n}|i} = 8.181871311$$

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We then can find the rate for this factor by using a table.

By the approximation formula:

We have to be careful about using the approximation formula. The 10% carrying charge used to calculate Bill's payment was ***not*** *a simple interest rate, which is what that formula requires. We first must figure out the simple interest rate for Bill's loan.*

Bill is paying \$140 interest for a 9-month loan with \$1,400 principle. So the simple interest rate for this would be:

$$I = PRT$$
$$\$140 = (\$1{,}400)R(9/12)$$
$$R = 13.33\%$$

Using this rate in the approximation formula gives us:

$$APR \approx \frac{2nr}{n+1} \approx \frac{2(9)(0.1333)}{10} \approx 23.99\%$$

Once again, we see that these two different methods of calculating the rate differ quite a bit, but still give the same general sense of the actual rates being charged.

Installment Plans Today

The sorts of installment plans described in this section, while once commonplace, have largely disappeared from the American consumer finance landscape. This has happened for a number of different reasons.

One reason is certainly a result of the Truth in Lending Law and other similar consumer financial protection laws and regulations. If Tonya is told that her installment plan carries an 8% simple interest rate, she might not think that is too bad a deal; a 14.68% rate sounds far less attractive and knowing this might well lead her to consider other options.

Another reason for a move away from these types of plans is mathematical. As we have seen, calculating annuity factors is a computational challenge, requiring either calculator or computer tools that have become widely available only in the last few decades, or cumbersome tables. In contrast, calculating the payments for a simple interest plan requires only a bit of basic arithmetic. The Rule of 78 offers similar advantages over building an amortization table. Because the technology for amortized loans is now widely available, though, these advantages are no longer so important.

Yet another reason is the increased use of credit cards. Not all that long ago, credit cards were not as widely accepted or used as they are today. Many readers of this book may be old enough to remember when grocery stores and most small businesses only grudgingly accepted credit card payments, if at all, and "pay at the pump" was a novel concept. It wasn't all that long ago! But in today's world it would not be at all unusual for Tonya and Bill to pay for their purchases on a credit card.

One last reason is the rise of ***rent-to-own*** retailers. These businesses offer an enormous array of consumer goods on plans that allow customers to make monthly rental payments until, after a certain number of payments have been made, the rented item becomes property of the renter. These are not really installment plans, since instead of buying the item with borrowed money the customer rents the item, but they function in much the same way and appeal to much the same consumer market as installment plans.

Because of their similarities to installment plans, it can be interesting and worthwhile to consider them, using the techniques of this section.

Example 10.3.8 ***Suppose that Bob's Super Rental World offers an LCD television that would normally sell for \$375 on a 12-month rent-to-own contract for \$35.00 a month. If this were an installment loan, what would the interest rate be?***

Using a spreadsheet amortization table with guess and check, we find that the rate works out to be 21.46%.

Using the approximation formula, we would first calculate the simple interest rate. The monthly payments total 12($35) = $420, and so the extra $45 can be thought of as interest.

$$I = PRT$$
$$\$45 = (\$420)R(1)$$
$$R = 10.71\%$$

Plugging this into the approximation formula gives:

$$APR \approx \frac{2nr}{n+1} \approx \frac{2(12)(0.1071)}{13} \approx 19.77\%$$

Whichever rate you use, it may be helpful to have a comparable interest rate in mind when you are considering a plan of this type. If the rates implied by a rent-to-own plan are comparable to the rates you can obtain elsewhere (such as with a credit card), then such a plan might be reasonable. On the other hand, rent-to-own companies are often accused of charging rental rates that equate to exorbitant interest rates. If you found, say, that a rent-to-own plan was equivalent to paying 40% interest, simply buying the item on a credit card might be a more financially attractive proposition.[7]

One significant difference between rent-to-own plans and borrowing of whatever type is that, until the full rental term has been completed, the renter does not own the item in question. There is no point in asking a question such as “after 10 months, how much do you still owe” because you don’t owe anything. While it is appealing to not have any debt for the item, it also means that if you stop paying on it, it goes back to its owner.

Land contracts are a means of buying real estate that work like rent-to-own plans. They are most attractive to buyers whose credit is insufficient to qualify for a mortgage; the downside is, of course, that the buyer does not actually own the property until all payments are made.

[7]Or, you could just save up and pay for it in cash. It’s not as crazy as it sounds. Really.

EXERCISES 10.3

Throughout the exercises for this section, payoff amounts and interest/principal calculations should be found by using the Rule of 78, unless explicitly stated otherwise.

A. Calculating Installment Plan Payments

1. Lennie’s Appliance Megamart has arranged to offer installment plans for major appliance purchases. The plans require $25 down and 12 monthly payments, and include a 10% carrying charge. If you buy a refrigerator for $850 under this plan, what will your monthly payments be?

2. Suppose I buy a lawnmower on an installment plan. The price of the lawnmower is $775.32, the carrying charge is 15%, there is a $50 down payment, and the rest is paid off in monthly installments over the next 2 years. Calculate the monthly installment payment.

3. Calculate the monthly payment for a 3-year simple interest loan if the initial balance is $2,750 and the simple interest rate is 8.25%.

4. Julian bought a sound system on a simple-interest installment plan. The initial balance for the loan was $835. The plan requires 18 monthly payments, and the simple interest rate is 12½%. Calculate his monthly payment.

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B. The Rule of 78 (Optional)

Use the Rule of 78 to determine interest/principle splits and payoff amounts.

5. Calculate $1 + 2 + 3 + \cdots + 400$.

6. Calculate the sum of the whole numbers from 1 up to (and including) 837.

7. Suppose that you buy the refrigerator under the installment plan described in Exercise 1. How much of your first payment counts toward interest? How much would count toward paying off the principal?

8. For the lawnmower purchase described in Exercise 2, how much interest will I pay with my first payment? How much total interest will I pay in the first year?

9. For the installment plan described in Exercise 3, determine:
   a. The total interest paid in the first 6 months.
   b. The amount required to pay off this loan after the first 6 monthly payments.

10. Suppose that Julian (from Exercise 4) decides to pay off the remaining balance he owes after making 12 monthly payments. How much will he need to do this?

11. What is the sum of the whole numbers starting with 7 and ending with 295? (*Hint:* First find the sum of the whole numbers from 1 to 295.)

12. Dario made a purchase with a 16-month installment plan. His payments are $50 a month, and the initial balance was $600. How much total interest will he pay over the course of this plan? How much interest does he pay with his first seven payments? After his seventh payment, what would the payoff amount for this installment plan be?

C. APR Calculations

13. Calculate the APR for the installment plan from Exercise 3, using:
   a. A spreadsheet amortization table
   b. The approximation formula

14. Calculate the APR for Julian's sound system purchase from Exercise 4, using:
   a. A spreadsheet amortization table
   b. The approximation formula

15. For the refrigerator installment plan from Exercise 1, find the APR by using:
   a. A spreadsheet amortization table

b. The approximation formula

16. For the lawnmower installment plan described in Exercise 2, find the APR by using:
    a. A spreadsheet amortization table
    b. The approximation formula

17. In each of Exercises 13 to 16, the APR is calculated in two different ways, and the two different methods give two different answers. Which rate, the one calculated from the spreadsheet or the one calculated from the formula, would you consider to be the more accurate one? Why?

D. Grab Bag

18. Joyce bought a sleeper sofa for $953 on an installment plan. The plan calls for $35 down and the rest to be paid off with 24 monthly payments. The simple interest rate is 7½%.
    a. Calculate Joyce's monthly payment.
    b. Calculate this plan's APR, using the approximation formula.

19. A farmer bought some equipment for $20,000. He and the seller agreed to a 5-year installment plan with monthly payments, and a 25% carrying charge. Find the monthly payments.

20. Use the approximation formula to calculate an APR for a 30-month installment plan with a 9% simple interest rate.

21. Mark bought a plot of land for $80,000. He and the seller agreed to a 10-year installment plan, with a 6.25% simple interest rate.
    a. Calculate Mark's monthly payment.
    b. Calculate the APR for this plan, using the approximation formula.
    c. If Mark made this purchase by taking out a mortgage, with an amortized loan, with the same monthly payments, what would the interest rate be?

22. Find the weekly payment on a 26-week installment plan with an initial balance of $500 and an 8% carrying charge.

23. Shelly is considering a rent-to-own plan for a new stove and range. The cash price to buy it would be $895. A rent-to-own plan would require 36 monthly payments of $28.75. Calculate the interest rate that would be equivalent to this plan using either an amortization spreadsheet or the approximation formula.

E. Additional Exercises

24. Visit a rent-to-own store in your area and see what the monthly payments would be for a few items (choose any items that interest you.) How does the rent-to-own payment plan compare to other options? What sorts of interest rates are built into the rental rates they offer?

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25. Suppose that I make \$100 monthly payments for 24 months on an installment plan where the initial balance was \$2,000.
    a. For the Rule of 78, what would my payoff amount be after the 15th payment?
    b. For an amortization table, what would my payoff amount be after the 15th payment?

26. Using the approximation formula, I calculated the APR for a 12-month installment loan to be 28.92%. What would the rate have come out to be if I had used an amortization table?

10.4 Leasing

Whether it is an individual looking for a car or apartment, or a business looking at office space or a photocopier, leasing is a widely used alternative to buying property. Some aspects of the choice between leasing and buying are a matter of personal preference, and so there really is not much to say about that mathematically. There are, though, interesting and important mathematical aspects to leasing, and understanding them can give valuable insights into this option.

Because car leasing is probably the most familiar use of leasing to most readers, we will initially approach the topic by looking specifically at car leasing. Leasing in other contexts will be discussed at the end of the section, and explored in some of the exercises as well.

Differences between Leasing and Buying

Suppose Jessie is looking for a new car. She has picked out the specific model and features she wants, but now is left to decide whether to lease the car or to buy it outright, taking out a loan to finance the purchase. Obviously, one important factor in her decision will be the monthly payment, but there are other significant differences between the two options.

Leasing is essentially the same as renting the car for a set period of time. If she leases, she will have the right to drive the car, will have the responsibility for maintaining it and insuring it, and so on, but she will not actually be the *owner* of the car. During the term of the lease, the car belongs to the leasing company, and at the end of the lease, Jessie's payments cease and the car must be returned to the leasing company.

Because the car never actually belongs to her and must be ultimately be returned, the leasing company has the right to place some conditions on how Jessie uses it. The leasing company will most likely require Jessie to take the car in for all scheduled maintenance, will require her to repair any damages, and also will limit the total miles that she can put on the car. Anything about the condition of the car (beyond the results of "normal" use) that lowers its value will be Jessie's responsibility to cover when the car is returned.

If Jessie buys the car, she becomes the car's actual owner. This is true even if she takes out a loan to buy it. While the lender will most likely have a ***lien*** on the car (a provision on the title that allows the lender to take possession of the car if the loan is not paid as promised), the car actually is Jessie's property, albeit with a string attached. At the end of the loan term, Jessie's payments cease, but the car remains her property. The lien is then taken off of the title and she owns it "free and clear."

Because the car belongs to her, Jessie is not obligated to perform scheduled maintenance. She doesn't have to change the oil (though she obviously would be foolish not to). If the paint is scratched or coffee spills stain the upholstery, she may want to clean these up, but no one will make her to do this. And she is not subject to any limitations on how many miles she can drive. The car is hers, and while it may be in her best interest to keep it up to maintain its value, she is not obligated to do so.

Calculating Lease Payments

Suppose that the total price (including all fees, taxes, and other expenses) of her car comes out to be $19,875. We've already (in Chapter 4) found how to calculate the monthly payments on a car loan; the following example will serve as a reminder.

Example 10.4.1 ***Suppose that Jessie decides to finance the entire cost of the car with a 5-year loan at an 8.4% interest rate. Calculate her monthly payment.***

Recall that her monthly payments form an annuity, and the amount borrowed is the present value. Therefore:

$$PV = PMTa_{\overline{n}|i}$$
$$\$19{,}875 = PMTa_{\overline{60}|.007}$$
$$\$19{,}875 = PMT(48.85587164)$$
$$PMT = \$406.81$$

For comparison, now, let's consider what would happen if she decided to lease the car. When she borrows the money to buy the car, she will actually be borrowing $19,875, which she then uses to buy the car. When she signs the lease, she doesn't *directly* borrow any money, and so a lease is not *exactly* the same as a loan. However, she does borrow something: the car itself. Looking at the mathematics of the situation, we can think of it as a kind of loan. Since the car is worth $19,875, she is, in effect, borrowing $19,875 after all, albeit in the form of the car.

There is a big difference though, in how this "loan" is repaid. With the actual loan, her monthly payments repay all of the principal borrowed together with all of the interest. The "loan" she takes out with the lease is only partly repaid with her monthly lease payments. A large part of the "loan" will be repaid at the end of the lease by returning the car to the leasing company.

Suppose that the leasing company has determined that, after 2 years of normal use and proper maintenance, the value of this car should be $14,055. This is referred to as its ***residual value***. Of the $19,875 that she borrowed in the form of the car, then, she will repay $14,055 by returning the car. So her lease payments need to cover the difference between the original and residual values of the car, or $19,875 − $14,055 = $5,820, together with the interest on this debt. In addition, even though $14,055 worth of principal will be repaid by returning the car, she still must pay interest on the $14,055 as well. The situation can be summed up in the following diagram:

Amount	***Interest***	***Principal***
$14,055 (residual value)	Paid by monthly payments	Paid by return of car
$5,820 (loss in value)	Paid by monthly payments	

Thus, her monthly lease payments are built of two parts: the principal and interest for the loss in value, plus the interest (but not the principal) on the residual value. The first part can be found in the same way as any other loan payment, using the present value annuity formula. Since the interest on the residual value will be covered in each month's payment, it won't ever compound, and so it can be found by using the simple interest formula. We can sum this up with the following "formula":

"FORMULA" 10.4.1
The (theoretical) monthly payment on a lease can be found by adding the following two parts:

1. Payment on Loss: Subtract the residual value from the original price, and calculate the annuity payment with this difference as the present value

PLUS

2. Interest on Residual: Use the residual value as principal and calculate the monthly interest, using I = PRT

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The following example will illustrate:

Example 10.4.2 ***Using the information provided so far about Jessie's car lease, determine the appropriate monthly lease payment. Assume an 8.4% interest rate.***

Payment on Loss: The loss in value is $5,820 . The monthly payment on this would be:

$$PV = PMTa_{\overline{n}|i}$$
$$\$5{,}820 = PMTa_{\overline{24}|0.007}$$
$$\$5{,}820 = PMT(22.02160857)$$
$$PMT = \$264.29$$

Interest on Residual: The residual value is $14,055. The monthly interest on this would come to:

$$I = PRT$$
$$I = (\$14{,}055)(0.084)(1/12)$$
$$I = \$98.39$$

The lease payment would then be the sum of these two values:

$$\text{Lease payment} = \$264.29 + \$98.39 = \$362.68$$

Note that, of the two components that go into the total lease payment, the payment on loss is the larger one, even though the loss in value itself is small in comparison to the residual value. This is because the payment on loss must include both principal and interest, instead of just interest. This can lead to some surprising results, though, as the following example will illustrate.

Example 10.4.3 ***Ajay is considering taking out a 2-year lease on a new car. The total price of the car would be $23,850, and the residual value is $18,350. He owns a car now, worth $4,800, which he will trade in when he gets his new vehicle. Calculate his lease payment, assuming an 8.7% interest rate.***

The idea here is pretty much the same as in Example 10.4.2, except for the existence of the trade-in. When he leases the car, he "borrows" $23,850. But his trade-in is worth $4,800, so in fact this is immediately reduced to $23,850 − $4,800 = $19,050. The loss in value, then, is only $19,050 − $18,350 = $700! Proceeding with our calculation:

Payment on loss:

$$PV = PMTa_{\overline{n}|i}$$
$$\$700 = PMTa_{\overline{24}|0.00725}$$
$$\$700 = PMT(21.95523688)$$
$$PMT = \$31.88$$

Interest on residual:

$$I = PRT$$
$$I = (\$18{,}350)(0.087)(1/12)$$
$$I = \$133.04$$

Total payment:

$$\text{Payment} = \$31.88 + \$133.04 = \$164.92$$

By comparison, if Ajay had bought the car and financed it with a 5-year loan at the same 8.7% rate, his monthly payment would have been $392.68. The trade-in dramatically cut the loss in value, and since that makes up the more significant component of the lease payment, the lease payment is shockingly low for a new car costing well over $20,000. If Ajay didn't have a car to trade in, he could have still got to this low monthly lease payment by putting $4,800 down (assuming, of course, that he can come up with that much money for his lease down payment). Sometimes you may see lease payments advertised that just look too good to be true; the payments may well be for real, but require a significant down payment, either in the form of cash or a trade-in.

Before moving on, we should note that the method for calculating lease payments that we are using here is a theoretical one. Just as with the rent on an apartment or anything else for that matter, the amount of a lease payment will be largely influenced by what the market will bear. In an earlier example, we calculated Jessie's lease payment to be $362.68. While this is a theoretically correct lease payment based on the prices and interest rates used, the rates a leasing company will actually charge will be strongly affected by rates accepted in the market. If the market would bear lease payments of $375, the leasing company will most likely move the rate up. On the other hand, if this rate is too high relative to what people are willing to pay, the rate may require a downward adjustment, say to $350.00. It should be clearly understood that lease payments as calculated in this section provide a guideline, but they are not written in stone.

Mileage Limits

In order for leasing to work out well financially for the leasing company, the residual value used in determining the payment is critical. The lease payments need to cover the interest on both the residual and the loss in value, so where the line is drawn between these two is not all that important for the portion of the payments that go to interest. However, the payments also must cover the loss in value itself. If the residual value is overestimated, the loss in value will be underestimated, and so the lease payments will not fully cover the loss in value.

There are several reasons why the residual value might be overestimated. If the leasing company misjudged the used car market and wrongly expected the car's value to be more than it really turns out to be, that is unfortunate for the leasing company, but not a problem for the lessee. The lease terms were set in advance, and it is not the lessee's responsibility to make sure that the leasing company is not being overly optimistic about the residual value.

If the residual value is lower than it should be at the end of the lease because things the lessee did with the car, though, that is another story. If the car is returned with serious paint scratches, upholstery stains, damage due to negligent maintenance, or other problems with the vehicle that go beyond what would be expected from normal wear and tear, the lessee is responsible for paying for those damages. The leasing contract will usually spell out the details of this responsibility.

Without doing any explicit damage, though, the lessee can also reduce the value of the car by putting too many miles on it. A 3-year-old car with 80,000 miles is naturally worth much less than one with 30,000 miles, even if both cars have been properly maintained. For this reason, leasing contracts almost always specify some limits on the number of miles the car can be driven. If the car is returned with mileage that exceeds this limit, the lessee must compensate the leasing company for this. This is usually based on a predetermined penalty per mile over the limit. The following example will illustrate how this works.

Example 10.4.4 ***The terms of Ajay's 2-year lease allow for 12,500 miles per year, with a 32 cents per mile charge for any overages. How much will Ajay have to pay as a penalty if at the end of the lease his mileage is (a) 16,500 miles, (b) 28,743 miles, or (c) 72,400 miles?***

The total miles allowed over the course of the lease is (2 years)(12,500 miles per year) = 25,000 miles.

(a) Since 16,500 is less than the allowed 25,000, Ajay does not owe any penalty. (Sorry, Ajay, there is no rebate for being below the limit.)

(b) 28,743 − 25,000 = 3,743 miles over. (3,743 miles)($0.32 per mile) = $1,197.76.

(c) 72,400 − 25,000 = 47,400 miles over. (47,400 miles)($0.32 per mile) = $15,168.

The fee in (c) above may seem outrageous, but on the other hand Ajay has dramatically reduced the car's resale value by putting such enormous mileage on it. Even if he agrees the fee is fair, though, it certainly poses a problem for him. Most leases contain a provision that allows the lessee to buy the car at the end of the lease for the assumed residual value. If you

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buy the car for the assumed residual value, the justification for these fees disappears. In the situation of (c) above, Ajay would probably choose to buy the car for its $18,350 residual value (maybe to keep, maybe to sell himself) rather than pay this overage fee.

The Lease versus Buy Decision

So, which is the better choice: leasing or buying?

Of course, there is no simple answer to that question. To some extent the answer depends on individual circumstances and personal preferences. While the choice is seldom black and white, there are certain things to consider in approaching that decision.

In Jessie's case, the lease payment was quite a bit lower than what she would have had to pay on her car loan. In Ajay's case the difference was enormous. When this is happens, and it very often does, the attraction of leasing is obvious.

However, the lower monthly payment can be deceptive. At the end of the lease's term, the car goes back to the leasing company. At the end of a car loan, you stop making payments but keep the car. Of course, at the end of the lease on one car you can always take out a new lease on a new car, but as long as you keep leasing, you keep paying. Buying the car may require a higher monthly payment, but those payments eventually end. The buyer has the opportunity to eventually own the car "free and clear." Of course, once the loan is paid off, the car is no longer new, and so whether or not this is worthwhile depends on whether it is more important to you to have a nearly new car or a paid-for car.

Leasing also comes with limitations, the most significant of which is mileage limits. If the mileage limits allowed for a lease are below what you would normally expect to drive in a year, the disadvantage of leasing should be apparent.

For businesses, the choice of leasing versus buying can take on some extra dimensions. Leased property is treated differently on the business's financial statements than owned property, and there may be tax consequences as well. A business owner may need to take advantage of the advice of an accountant to be able to sort out the pros and cons of leasing as opposed to buying property the business needs.

In addition, leases are often offered under which the leasing company takes responsibility for the maintenance and repair of the equipment leased. This can be an attractive feature for a business that does not want to have to train its employees to maintain and service equipment or pay for a separate maintenance service contract.

Leases for Other Types of Property

While we have used car leases as a familiar example of leasing, many other items are commonly leased as well. We will close this section with one such example.

Example 10.4.5 ***Cotswold Real Estate needs new computers for its offices. The computers would cost $10,543. The company is given the opportunity to lease the computers for 3 years instead of buying them. If the leasing company assumes a residual value of $845 and a 7.2% interest rate, what would the monthly lease payment be?***

The calculation is handled here in the same way as we did for car leases. The loss in value is $10,543 − $845 = $9,698.

Payment on loss:

$$PV = PMT a_{\overline{n}|i}$$
$$\$9{,}698 = PMT a_{\overline{36}|.006}$$
$$\$9{,}698 = PMT(32.29074882)$$
$$PMT = \$300.33$$

Interest on residual:

$$I = PRT$$
$$I = (\$845)(0.072)(1/12)$$
$$I = \$5.07$$

Total payment:

$$\text{Payment} = \$300.33 + \$5.07 = \$305.40$$

Note that in this case the lease payment is probably not much different from what the payment would be to buy the computers outright and finance them at the same interest rate. Nonetheless, as mentioned previously, the business may find leasing attractive for reasons other than having a lower payment.

EXERCISES 10.4

A. Calculating Lease Payments

1. Suppose you want to buy a new car, costing $21,575.
   a. Suppose you borrow the full price of the car with a 5-year loan at 7¾%. Calculate your monthly car loan payment.
   b. Suppose instead you lease the car for 2 years. The residual value is $17,453 and the leasing company uses a 7¾% interest rate to determine the payment. Find the monthly lease payment.

2. Hugo is looking at a new truck costing $18,345. Calculate his monthly payment:
   a. Assuming he buys the car, financing the full price with a 5-year loan at 9%.
   b. Assuming he leases the car for 3 years. Assume a $12,910 residual value and a 9% interest rate.

3. Vanessa is looking at leasing a new car for 2 years. The total cost is $24,075, the residual value would be $18,950, and the interest rate is 7½%. Calculate her monthly lease payment:
   a. Assuming she makes no down payment.
   b. Assuming she makes a $3,500 down payment.

4. A car dealer runs an ad for 3-year leases on two different car models. The cost for a new Cascadia sedan is $31,953 and the 3-year residual value is $25,750. The cost for a new Kiriana sedan is $27,935 and the 3-year residual value is $15,910. Calculate the lease payment for each of these vehicles, assuming an 8% interest rate. Is there anything surprising about your results?

B. Mileage Limits

5. The lease on Kim's SUV allows her 12,500 miles per year, with a 38 cents per mile penalty for overages. At the end of her 3-year lease, she returns the vehicle with 39,552 miles on it. How much of a mileage penalty, if any, does she owe?

6. The 2-year lease on my car allows me 15,000 miles per year plus 42 cents per mile over this limit. If I return the car with 43,552 miles on it, how much of a mileage penalty, if any, do I owe?

7. Suppose you have a 2-year lease on a car that allows 14,000 miles per year. The overmileage penalty is 37.5 cents per mile. In the first year you put 11,864 miles on the car, and in the second year you put 15,661 miles on it. How much of a mileage penalty, if any, will you owe?

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8. Suppose you have a 3-year lease on a truck that allows 12,000 miles per year, plus 44 cents per mile over this limit. In the first year you drive 10,563 miles, in the second year 11,997 miles, and in the third year 16,876 miles. How much of a mileage penalty will you owe?

C. Grab Bag

9. A leasing company estimates that the 3-year residual value on a new car costing $24,535 will be $18,025. It based its estimate on 12,500 miles per year. The company also offers a lease with a higher, 15,000-mile-per-year limit. At this higher mileage limit, the residual value of the car is estimated to be $16,295. On both leases the mileage penalty would be 28 cents per mile.

   Assuming an 8% interest rate:

   a. Calculate the lease payment for the 12,500-mile-limit plan.

   b. Calculate the lease payment for the 15,000-mile-limit plan.

   c. Suppose that you expect to drive the car 40,000 miles over the next 3 years. Which lease is the better deal for you in this case?

10. A college needs a new photocopier, which costs $4,575. It is considering a lease for the copier. An office equipment supply company offers a 3-year lease, with a limit of 25,000 copies per year. Each copy above this limit would be charged 3.5 cents. If the college takes this lease and makes 93,585 copies over the 3 years of the lease, how much extra would it have to pay?

11. Calculate the 3-year monthly lease payment on an office computer system costing $28,500, assuming a residual value of $3,500, an interest rate of 6.4%, and a $5,000 down payment.

12. Brandon wants to lease a new car costing $19,992. The 2-year residual value is $15,052 and the interest rate is 8½%. Calculate his lease payment.

D. Additional Exercises

13. Brandon wants to lease a new car costing $19,992. The 2-year residual value is $15,052 and the interest rate is 8½%. Brandon will trade in his current car, worth $8,500. Calculate his lease payment.

14. In all of the examples of this section where we compared buying to leasing, the term of the loan was quite a bit longer than the term of the lease. In fact, leases on cars seldom run for longer than 2 or 3 years, while car loans generally have much longer terms. Why do you suppose it is rare to see, say, a 5-year car lease?

15. In Exercise 4 we calculated the lease payments on two different model cars. Suppose that the dealer complains to the manufacturer that the lower-priced car has the higher payment, and asks for a reduction in the price of the new Kiriana so that its lease payment would be the same as for the Cascadia. What would the cost of the Kiriana need to be to make this happen?

Topic	Key Ideas, Formulas, and Techniques	Examples
Average Daily Balance, p. 420	• ADB is the weighted average of the daily balances. • Set up a table as shown in the text to aid in the calculation.	Joanna has a credit card whose billing period begins on the 17th day of each month. On July 17 her balance was $815.49. She made a $250 payment on July 28. She also made new charges of $27.55 on July 21, $129.99 on August 5, and $74.45 on August 8. Find her average daily balance. (Example 10.1.1)
Credit Card Interest, p. 423	• Interest is calculated as simple interest on the average daily balance. • ADB is used only to calculate interest; it is not the amount owed. • If payments are made in full within the grace period, interest may be waived altogether.	Suppose that Joanna's credit card (from Example 10.1.1) carries an interest rate of 15.99%. How much interest would she owe for the billing month from Example 10.1.1? What would her balance be on her August 17 monthly statement? (Examples 10.1.2 and 10.1.3)
Commissions, p. 425	• Card companies charge merchants fees called commissions for accepting credit card payment. • Commissions may be a flat amount or a percent.	Travis bought a pair of shoes for $107.79 and charged them to his credit card. The credit card company charges the shoe store 45 cents for each transaction, plus 1.25% of the amount charged. How much will the credit card company pay to the shoe store? (Example 10.1.5)
Choosing the Best Deal, p. 425	• To choose the best deal, you must consider whether the cost of a higher annual fee will make up for the expense of a higher interest rate. • The "best" choice depends on the average balance carried on the card.	How large a balance would I need to carry to prefer an $80 annual fee with a 9% interest rate over a $25 annual fee with a 15% interest rate? (Example 10.1.7)

(Continued)

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Topic	Key Ideas, Formulas, and Techniques	Examples
Home Equity and Loan to Value, p. 434	• Equity = value of property − amount owed on it. • Maximum loan to value is the maximum percent that can be owed against a given piece of real estate.	Les and Rhonda own a house worth $194,825. The balance they owe on their first mortgage is $118,548. They want to take out a second mortgage, and all of the lenders they have spoken with require a minimum equity of 5%. Find (a) their equity now, (b) the maximum amount they can borrow with the second mortgage, and (c) their equity if they borrow the maximum. (Example 10.2.1)
Escrow and PITI, p. 437	• Monthly escrow payment is 1/12 of the annual taxes and insurance. • PITI = mortgage payment + escrow payment + PMI.	Chantal's mortgage payment is $963.88. Suppose that her annual property taxes are expected to be $3,300 and her homeowners' insurance premium is $750. PMI is $45 a month. Find her total PITI. (Example 10.2.6)
Qualification Tests, p. 440	• 28% test requires that PITI must be less than 28% of monthly gross income. • 36% test requires that PITI plus long-term debt payments must total less than 36% of monthly gross income.	Antoine and Maria earn a combined annual income of $76,500. They are trying to qualify for a mortgage on a house for which the total monthly payment (PITI) would be $1,494.57. Do they pass the 28% test? If their PITI and long-term debt payments total $1,987.87, do they pass the 36% test? (Examples 10.2.7 and 10.2.8)
Up-Front Expenses, p. 441	• Up-front expenses include down payment, closing costs, points, and prepaids.	Drew and Joanne are buying a house for $128,550. They will make a minimum 3% down payment, and closing costs will total $2,100. Annual property taxes are $2,894 and homeowners' insurance is $757 annually. How much money will they need up front? (Example 10.2.9)
Points and Payback Period, p. 443	• Points may be paid to "buy" a lower interest rate. • Each point is 1% of the loan amount. Payback period is the time it takes to recover the cost of the points. • Payback period is cost of points divided by monthly savings from the lower-rate loan.	Determine the payback period for paying points given a choice between a no-point 30-year loan at 7.5% and a 2.5-point 30-year loan at 6.25% (Examples 10.2.10 and 10.2.11)

(Continued)

Topic	Key Ideas, Formulas, and Techniques	Examples
Installment Plan Payments, p. 449	• To find payment, add the principal and finance charge, then divide by the number of payments. • Finance charge may be expressed as a percent of the principal, called the carrying charge. • Finance charge may be expressed as a simple interest rate, calculated on the full principal for the full term.	Tonya bought an electronic piano for $1,850. She financed this purchase with a $50 down payment and a 2-year "simple interest loan" at an 8% rate and monthly payments. Find her monthly payment amount. (Example 10.3.2)
Rule of 78, p. 450	• On a loan with n payments, the percent of the total interest paid is (n − k) divided by the sum of the whole numbers from 1 to k. • The sum of the numbers from 1 to k is k(k + 1)/2. • To find payoff, calculate the remaining interest and subtract from the total remaining payments.	Suppose that after 10 months, Tonya decides to pay off her piano in full. How much would she need to pay off the loan at that point? (Example 10.3.4)
APR for Installment Loans, p. 452	• The "simple interest rate" for an installment plan severely understates the actual interest rate paid. • To find the actual interest rate, set up an amortization spreadsheet and guess-and-check to find the interest rate. Alternatively, use the approximation formula: $APR \approx 2nr/(n + 1)$.	Suppose that Bob's Super Rental World offers an LCD television that would normally sell for $375 on a 12-month rent-to-own contract for $35.00 a month. If this were an installment loan, what would the interest rate be? (Example 10.3.7)
Lease Payments, p. 459	• The payment on a lease consists of two pieces. • The payment on loss in value is the payment of an annuity whose PV is the loss in value over the lease's term. • The interest on residual is simple interest on the residual value.	Ajay is considering taking out a 2-year lease on a new car. The total price of the car would be $23,850, and the residual value is $18,350. He owns a car now, worth $4,800, which he will trade in when he gets his new vehicle. Calculate his lease payment, assuming an 8.7% interest rate. (Example 10.4.3)
Mileage Limits and Penalties, p. 461	• Lease's set limits on the maximum miles that can be driven over the term of the lease. • Miles over the limit are subject to an overage payment.	The terms of Ajay's 2-year lease allow for 12,500 miles per year, with a 32 cent per mile charge for any overages. How much will Ajay have to pay as a penalty if at the end of the lease his mileage is (a) 16,500 miles, (b) 28,743 miles, or (c) 72,400 miles? (Example 10.4.4)

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CHAPTER 11

International Business

"There is nothing quite wonderful as money,
There is nothing like a newly minted pound,
Everyone must hanker for the butchness of a banker,
It's accountancy that makes the world go round.
You can keep your Marxist ways
For it's only just a phase.
For it's money money money makes the world go round!"

—Monty Python, "The Money Song"

Learning Objectives

LO 1 Convert money amounts between U.S. and foreign currencies.

LO 2 Convert money amounts between two different foreign currencies.

LO 3 Calculate foreign exchange amounts using both published and retail exchange rates.

LO 4 Distinguish between spot and forward exchange rates.

Chapter Outline

11.1 Currency Conversion

It's a well-worn cliché to say that the world is getting smaller and business is becoming more global all the time, but this became a cliché honorably: it's true. Businesses manufacture, buy, and sell their products and services around the globe. There are plenty of products at your local shopping mall made in all sorts of exotic locations, and in fact you might be hard pressed to find any products that *aren't* made in all sorts of exotic locations. Someone residing in Chicago or Philadelphia can vacation in Italy almost as easily as in California. And the growth in the Internet and electronic commerce can bring international trade down to the individual consumer's level. Someone who has never personally been outside the United States can nonetheless easily buy a book from a shop in London, Toronto, or Singapore and have it delivered to her door just as easily as visiting a local bookstore.

Just as much of the world's business is conducted in English, much of the world's business in conducted in U.S. dollars. Business deals made outside the United States that do not involve any American individuals or companies can still be made in American dollars. There is little to prevent a Brazilian farmer from selling soybeans to a Korean food processor for a price in U.S. dollars, if that is the currency the two agree to use. Some countries (Ecuador, for example) have even gone so far as to give up on having a currency of their own, and have made the U.S. dollar official for the country.

Most countries, though, either have currencies of their own or participate with others in a shared currency (such as the euro used by many European countries), and business conducted in or with those countries ordinarily will involve that nation's currency. Furthermore, the U.S. dollar is hardly the only major currency in use in the world. That Brazilian farmer and Korean food company might make their deal in Japanese yen or European euros just as well as in U.S. dollars.

There are many interesting aspects to international trade and the world's many currencies, but our focus in this text is the mathematical challenges that they pose. The most important mathematical challenge for us to look at is converting money from one currency to another.

Suppose that you are an American vacationing in New Zealand. Even though New Zealand's currency still goes by the same name (dollar) and is still represented with the same "$" sign, New Zealand dollars are not the same as U.S. dollars. Though it may be possible to pay with U.S. dollars in some places (especially those who see a lot of American tourists or business travelers), New Zealand businesses will generally expect you to pay with New Zealand dollars. But are New Zealand dollars worth more, or less, than U.S. dollars? How much more or less? To be able to intelligently conduct any sort of business there, you will need to be able to convert between the two currencies.

To convert between currencies, you first must know the ***exchange rate***. An exchange rate is a statement of the value of one currency in terms of another. Exchange rates can vary over time, but tables of current exchange rates are published daily in the *Wall Street Journal, Financial Times, USA Today,* and in the business sections of many newspapers. Current rates can also be obtained from many banks, currency brokers, and online sources.

The table shown in Table 11.1 is typical:

TABLE 11.1
Exchange Rates as of 4 P.M. August 24, 2006

		US$ EQUIVALENT		CURRENCY PER US$	
Country	***Currency***	***Thursday***	***Wednesday***	***Thursday***	***Wednesday***
Argentina	Peso	0.324	0.324	3.0864	3.0864
Australia	Dollar	0.761	0.7628	1.3141	1.311
Bahrain	Dinar	2.6526	2.6526	0.377	0.377
Brazil	Real	0.4652	0.4649	2.1496	2.151
Canada	Dollar	0.9003	0.8986	1.1107	1.1128
1 month forward		0.9012	0.8994	1.1096	1.1119
3 months forward		0.9029	0.901	1.1075	1.1099
6 months forward		0.9055	0.9037	1.1044	1.1066
Chile	Peso	0.001869	0.00187	535.05	534.76
China	Renminbi*	0.1254	0.1254	7.9719	7.9719
Colombia	Peso	0.0004152	0.00042	2408.48	2380.95
Czech Rep.	Koruna	0.04524	0.04549	22.104	21.983
Denmark	Krone	0.171	0.1715	5.848	5.8309
Ecuador	U.S. dollar	1.0000	1.0000	1.0000	1.0000
Egypt	Pound	0.1741	0.1741	5.7432	5.7438
Hong Kong	Dollar	0.1286	0.1286	7.7766	7.7774
Hungary	Forint	0.00457	0.004607	218.82	217.06
India	Rupee	0.02154	0.02152	46.425	46.468
Indonesia	Rupiah	0.0001094	0.0001099	9141	9099
Israel	Shekel	0.2282	0.2288	4.3821	4.3706
Japan	Yen	0.008582	0.008594	116.52	116.36
1 month forward		0.008619	0.008631	116.02	115.86
3 months forward		0.008691	0.008706	115.06	114.86
6 months forward		0.008799	0.008813	113.65	113.47

(Continued)

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TABLE 11.1
(Continued)

		US$ EQUIVALENT		CURRENCY PER US$	
Country	***Currency***	***Thursday***	***Wednesday***	***Thursday***	***Wednesday***
Jordan	Dinar	1.4114	1.4114	0.7085	0.7085
Kuwait	Dinar	3.4576	3.4578	0.2892	0.2892
Lebanon	Pound	0.0006631	0.0006634	1508.07	1507.39
Malaysia	Ringgit	0.2718	0.2721	3.6792	3.6751
Malta	Lira	2.9724	2.9799	0.3364	0.3356
Mexico	Peso	0.0915	0.0917	10.929	10.9039
New Zealand	Dollar	0.6349	0.6398	1.5751	1.563
Norway	Krone	0.1575	0.1589	6.3492	6.2933
Pakistan	Rupee	0.01658	0.01656	60.314	60.387
Peru	New Sol	0.309	0.3091	3.2362	3.2352
Philippines	Peso	0.01944	0.01949	51.44	51.308
Poland	Zloty	0.3233	0.3249	3.0931	3.0779
Russia	Ruble	0.03731	0.03735	26.803	26.774
Saudi Arabia	Riyal	0.2666	0.2666	3.7509	3.7509
Singapore	Dollar	0.6338	0.6357	1.5778	1.5731
Slovak Rep.	Koruna	0.03377	0.03399	29.612	29.42
South Africa	Rand	0.1396	0.1396	7.1633	7.1633
South Korea	Won	0.0010425	0.0010464	959.23	955.66
Sweden	Krona	0.1386	0.1392	7.215	7.1839
Switzerland	Franc	0.8067	0.81	1.2396	1.2346
Taiwan	Dollar	0.03045	0.03051	32.841	32.776
Thailand	Baht	0.02653	0.02659	37.693	37.608
Turkey	New Lira	0.6692	0.6768	1.4943	1.4775
UK	Pound	1.8869	1.8936	0.53	0.5281
UAE	Dirham	0.2723	0.2723	3.6724	3.6726
Uruguay	Peso	0.0418	0.0419	23.923	23.866
Venezuela	Bolivar	0.000466	0.000466	2145.92	2145.92
EU	Euro	1.2758	1.2792	0.7838	0.7817

*The main *unit* of Chinese currency is the yuan. The term renminbi listed is the term for Chinese currency *in general*; it is not a currency unit.

Before this table can be used, some things demand explanation. There are two groups of columns, one labeled "US$ Equivalent" and one labeled "Currency per US$." Each gives the exchange rate, but in different forms. For example, for New Zealand, the "US $ Equivalent" columns tell us that on Thursday NZ$1 was worth US$0.6349. The "Currency per US$" columns tell us that on Thursday US$1 was worth NZ$1.5751. These are actually just two different ways of giving the same information, but we will see shortly that it is useful to have the information available in both forms.

Secondly, notice that in each column group there are two columns, one for Thursday and one for Wednesday. Most currencies have what are known as ***floating rates***, which means that their exchange rates are determined primarily or entirely by market forces, and so can change from day to day (or even moment to moment). The value of the currency is whatever a willing buyer will give a willing seller. When the demand for a country's currency is strong, the value of that currency will tend to rise, and when demand is weak, the currency's value will tend to fall. Under normal circumstances, day-to-day and even minute-to-minute exchange rates fluctuate a bit, but big changes in exchange rates don't usually occur quickly. You can see that most of the exchange rates in the table above differ between Wednesday and Thursday, but that the changes are slight.

A few countries have ***fixed*** currencies, where the exchange rate for the currency is set by the government and exchanging currency at anything other than the official rate is illegal.

A sign advertises exchange rates in downtown Montreal. Timothy J. Biehler

Few countries have fixed currencies today, though there are some notable exceptions. The most significant example is China, which fixes the value of its currency against the U.S. dollar, though China appears to be gradually moving toward a more flexible exchange rate. China is likely to eventually switch to a floating currency, and in fact by the time you are reading this it may already have done so. Fixed rates can still vary over time, as the government adjusts the rates, but will not normally vary much, if at all, from day to day. In the table, you can see that there was actually a slight change in the Chinese exchange rate between Wednesday and Thursday.

Most of the foreign currencies are listed by the name of their country. One important exception appears at the bottom of the table. The ***euro*** is a currency that has been adopted for common use by many European countries, including France, Germany, Italy, Spain and many others. These countries are not listed in the table. Not every European country uses the euro, however. The United Kingdom, Switzerland, and many Scandinavian and Eastern European nations do not at this time. It is likely, though by no means certain, that these other European countries will eventually adopt the euro. The euro is a particularly important currency to be familiar with, not only because of its use in major European nations, but also because it is seen by many as a significant rival to the U.S. dollar as an "international currency."

Converting from US$ to a Foreign Currency

When you want to convert an amount of money from US$ into a foreign currency, the most helpful form of the exchange rate is that given in the "Currency per US$" column of the table. Suppose that we want to know how much US$500 would be worth in New Zealand dollars. From the table, using the more recent Thursday rate, we found that US$1 was worth NZ$1.5751, which we can express by the equation

US$1 = NZ$1.5751

Common sense tells us that US$500 would be worth $500 times as much. Using this common sense together with the rules of mathematics, we can multiply both sides of this equation by 500 to get:

US$500 = NZ$787.55

which gives us the answer we were looking for.

Example 11.1.1 ***Convert US $3,724.59 into euros, using the Thursday rate.***

From the table, we find that on Thursday the exchange rate was

US$1 = €0.7838

Multiplying both sides of the equation by 3,724.59 we get

US$3,724.59 = €2,919.33

In this example, we made use of the symbol "€" for the euro, analogous to the symbol "$" used for the U.S. dollar. Knowing the symbols for major foreign currencies is not a bad thing. We will use them in this text at least for particularly widely used currencies, specifically:

Country	***Currency***	***Symbol***
United States	Dollar	$
European Union	Euro	€
United Kingdom	Pound	£
Japan	Yen	¥
China	Yuan	¥

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For other currencies, it is reasonable to simply write the name of the currency after the amount (e.g., 3,000 Swiss francs), or to look up and use the commonly accepted abbreviation for the currency (e.g., SFr 3,000). For countries that call their currency the dollar, it is appropriate to use the "$" sign, using the abbreviation for the country's name in front of the symbol if there is any potential for confusion, as we did when we wrote "NZ$" and "US$" to distinguish between United States and New Zealand dollars.

The "$" sign is also used in many places where the currency is not called a dollar. The Mexican peso is a common example of this; the dollar sign is commonly used for amounts in pesos. Just as with other country's dollars, we will distinguish amounts in pesos by writing "Mex$" or "M$" when needed for clarity's sake.

It is a common, and unacceptable, mistake, though, to just use the "$" in front of any money amount, regardless of the currency, even though many countries do use this symbol, including many whose currency is not called a "dollar". Still, in Example 11.1.1 it would be acceptable to write our final answer as "€2,919.33" or "2,919.33 euros"; it is not acceptable to write this answer as "$2,919.33" unless you know that "$" is the proper symbol for the currency. When you don't know the proper symbol, it is also reasonable to use the first letter of the currency name in front of the currency amount (e.g., "R5,000" for 5,000 Russian rubles), even though this may not be the official abbreviation.

Before moving on, we should mention something about the calculation done in Example 11.1.1. We could have gotten the answer more quickly by simply multiplying (3,724.59)(0.7838), without bothering to set up equations and multiply both sides. This is a perfectly reasonable way of doing the conversion. If you need to make conversions often, you will undoubtedly want to take this shortcut instead of writing out the equations. However, writing out the equations helps avoid the risk of accidentally using the rate from the wrong column (using 1.2758 instead of 0.7838), and is the safer path to take unless you are comfortable enough with currency conversion to be confident that you are always selecting the correct rate. If you prefer to take this shortcut, you will want to try it out with the homework exercises. If you find that you can consistently obtain correct answers by using the shortcut, it is reasonable to use it. If, however, you find that you sometimes make errors trying to take this shortcut, it is best to work things out with the equations.

Converting from a Foreign Currency to US$

Converting from a foreign currency into US$ works the same way, except that the more useful column is the "US$ Equivalent" in Table 11.1.

Example 11.1.2 ***Convert 348 Turkish new lira into US$, using the Thursday rate.***

From the table we have

$$L1 = US\$0.6692$$

Multiplying both sides of the equation by 348,000 we get

$$L348 = US\$232.88$$

As noted in the previous section, we could also have gotten this result simply by taking (348)(0.6692) to get 232.88. The same cautions apply. In this case, using the wrong rate would lead to the incorrect conclusion that the value is $520.02, quite far from the correct value.

Rounding Foreign Currencies

Like the U.S. currency, most foreign currencies have a main unit (comparable to the dollar) that can be divided into hundredths (comparable to cents). A British pound is equal to 100 pence, a Chinese yuan is equal to 100 fen,[1] and a Russian ruble equals 100 kopeks. So, like money in U.S. dollars, amounts of money in foreign currencies can usually be rounded to two decimal places.

[1]In practice, the value of 1/100 of a yuan is small enough that it is seldom actually used in practice. 1/10 of a yuan is called a jiao, and this is the more commonly used subunit.

Depending on the value of the currency, though, carrying values out to two decimal places may not make much sense. The main unit of Japanese currency, the yen, is worth a bit less than one U.S. penny. Given that, it doesn't make much sense to worry about decimal places for amounts in yen. Normally an amount expressed in Japanese currency would be rounded to whole numbers. It takes about 100 Indonesian rupiah to equal one U.S. cent, and so not only does it make no sense to express amounts in Indonesian currency to two decimal places, in most cases it would be logical to round to the nearest hundred rupiah (for example, rounding 187,584 rupiah to 187,600). There are no hard and fast rules that we can set forth here about when to use how many decimal places, but as with anything else you should use judgment and common sense about how far to round answers when converting currencies.

Example 11.1.3 ***Convert $87.59 into South Korean won.***

From Table 11.1:

$$US\$1 = W959.23$$

Multiplying both sides by 87.59 gives:

$$US\$87.59 = W84{,}018.96$$

Here we followed the usual two decimal places rule for currency. However, since it takes nearly 1,000 won to equal US$1, it is highly unlikely that these decimal places would actually be used. A more reasonable answer would be 84,018 won—or possibly even 84,000 won.

Forward Rates

You may notice that in the exchange rate table, beneath the listing for Japan there follow listings for "1 month forward," "3 months forward," and "6 months forward." The fact that exchange rates change over time can pose a real risk and challenge to businesses and financial institutions. ***Forward rates*** present an opportunity to lock in an exchange rate today for a currency exchange that will take place in the future.

Suppose that you work for an American company that does business in Japan. To some extent at least, you will have to deal with the exchange rate between dollars (the currency your business works in) and yen (the currency used in Japan). You of course can find out what the exchange rate is today, but you don't have a crystal ball to see what the exchange rate will be in the future. Yet you might be signing contracts to buy or sell products over the next several months. The deals you are making might look profitable at today's rates, but you could find that your profits disappear because of changes in the exchange rate between now and when payment is made.

Businesses and financial institutions sometimes try to protect themselves from this risk by making an agreement with another business or financial institution to exchange currencies at a future date at a rate agreed on today. These are the "forward rates" shown in the currency table. (The rates that are in effect at present are sometimes called "spot" rates.) If, for example, you made an agreement to exchange Japanese yen for U.S. dollars 3 months from now, right now the rate being offered in the market to do this is the 3-month forward rate shown in the table. This sort of deal allows you to protect yourself against an unforeseen change in the exchange rate, because 3 months from now you will be changing currency at the rate you've locked in today, regardless of what the actual exchange rate is then. Of course, if the exchange rate changes in your favor, you will miss out because you will be obligated to make the exchange at the agreed rate.

Example 11.1.4 ***In August 2006 Rylad Drugs, an American company, sold a large order of pharmaceuticals to a Japanese wholesaler. The price was set at ¥13,500,000, and it was agreed that payment would be made when the order was delivered in mid-November. At the current exchange rate Rylad is comfortable with the price, but it is worried that changes in the yen-dollar exchange rate over the next 3 months might make this deal less attractive. Rather than run the risk of being hurt by changes in the***

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exchange rate, suppose that on Thursday August 24, 2006, the company made a deal to convert this sum into U.S. dollars 3 months later, on November 24, 2006. When the exchange was made, how much did they receive in dollars?

Using the 3-month forward rate gives:

$$¥1 = \$0.008691$$
$$¥13{,}500{,}000 = US\$117{,}328.50$$

Whatever the exchange rate may actually have been on November 24, 2006, Rylad received $117,328.50. In reality, it may be that had the company not made this forward rate deal, it would have received more, or fewer, dollars. There was no way to know in advance whether the company would do better or worse by making the forward deal. The point, though, of this deal was to avoid the uncertainty, giving up the possibility of gain in exchange for avoiding the risk of loss.

Forward rates are an example of a ***futures contract***. For more discussion and other examples of these sorts of deals, see Section 6.3.

Converting between Currencies—Cross Rates

Of course, plenty of international trade does not involve U.S. dollars at all. What if we want to convert, say, £180.00 (British pounds) into Canadian dollars?

A **cross rate** is an exchange rate between two currencies, neither of which is the U.S. dollar. Tables of cross rates for many major currencies are published in major financial newspapers (though less commonly in other newspapers). Table 11.2 shows an example.

The table can be tricky to read, since it does not explain whether the "1.4171" in the Canada row and Euro column means 1 Canadian dollar is worth 1.4171 euros, or vice versa. As it happens, this particular table gives the value of the column currency in terms of the row currency, so that entry actually means that 1 euro is worth 1.4171 Canadian dollars. The easiest way to figure out which is which is to look at the rate between two currencies where the rate is obvious. The Japanese yen is a good choice for this. Since the value of 1 yen is so small relative to most other currencies, if you look in the yen column the small numbers in it tell you that the table must be giving the value of the column currency in terms of the rows.

Example 11.1.5 ***Convert C$8,000 Canadian dollars into Mexican pesos.***

From the table:

$$C\$1 = Mex\$9.83930$$

And so:

$$C\$8{,}000 = Mex\$78{,}714$$

TABLE 11.2

Currency Cross Rates as of 4 P.M. August 24, 2006

	US Dollar	*Euro*	*Pound*	*SFranc*	*Peso*	*Yen*	*Cdn Dollar*
Canada	1.11070	1.41710	2.09590	0.89600	0.10163	0.00953	1.00000
Japan	116.52000	148.66000	219.87000	93.99900	10.66200	1.00000	104.90600
Mexico	10.92900	13.94320	20.62200	8.81640	1.00000	0.09379	9.83930
Switzerland	1.23960	1.58150	2.33900	1.00000	0.11343	0.01064	1.11600
U.K.	0.53000	0.67610	1.00000	0.42750	0.04849	0.00455	0.47713
Euro	0.78380	1.00000	1.47900	0.63231	0.07172	0.00673	0.70567
U.S.	1.00000	1.27580	1.88690	0.80670	0.09150	0.00858	0.90030

Of course, a cross rate table is only practical for a limited number of currencies; a table that included many more would quickly become ridiculously large. We can closely approximate the conversion with our original table by using the U.S. dollar as an imaginary intermediate step. For example, to convert Russian rubles to Thai baht, we could first find the value of the rubles in dollars, and then convert those dollars to baht. (Of course, the intermediate step of converting to dollars need not *actually* take place. We simply pretend that it does to be able to calculate the values.)

Example 11.1.6 ***Convert 40,000 Russian rubles into Thai baht. Use the Thursday rates.***

First convert from rubles to U.S. dollars:

$$R1 = US\$0.03731$$
$$R40{,}000 = US\$1{,}492.40$$

Then convert from dollars to baht:

$$\$1 = B37.693$$
$$\$1{,}492.40 = B56{,}253$$

So we conclude that 40,000 Russian rubles equals 56,253 Thai baht.

Retail Foreign Currency Exchanges

Exchange rate tables such as the one we have been using above give a good indication of the values of different currencies, but there is no guarantee that you will be able to get those rates when making a foreign currency exchange. The rates published in the financial pages are used for very large exchanges between large financial institutions. Tables published in *The Wall Street Journal* and other financial publications note that the rates quoted "apply to trading among banks in amounts of $1 million and more." You may be able to get these rates, or rates close to them, when you pay for something by using a credit card or withdraw cash from an ATM in a foreign country.[2] They are not, though, the rates that you will be offered to exchange cash at a bank branch, a hotel front desk, or at an airport foreign exchange counter.

Suppose that you return home to Cleveland after a business trip to Vancouver and discover that you still have C$120 in Canadian currency in your wallet. You want to convert that money back into U.S. dollars. According to the tables that we used above, this would be worth $108.04. However, when you take this money to your local bank, you are disappointed to find that the bank offers you only $98.50 for it. What gives?

The fact of the matter is that, as we discussed at the start of this chapter, for most currencies the exchange rate is determined by the market: what a willing buyer will give a willing seller. The published exchange rates are not written in stone. The rate your bank offers will reflect its costs of processing the exchange and need to make a profit. A bank may also not be particularly excited about making a small, retail foreign currency exchange and the rates it offers may reflect this fact. In fact, your local bank branch may not even offer this service at all. If you don't like the rate offered, you don't have to accept it. You can keep your Canadian currency, or shop around for a better deal. You aren't forced to take the less attractive rate, but by the same token it really is not realistic to expect the bank to offer you the published rate.

Bitterly sulking home, grumbling about the rotten exchange rate, you might notice that there is a flip side to this rate. The bank's rate treated the Canadian dollar as "cheaper" versus U.S. dollars than it should have been. This worked against you as a seller of Canadian currency—but what if you instead went there to *buy* Canadian dollars? Could you then take advantage of the exchange rate to buy "cheap" Canadian dollars?

[2]You can't assume this, though, and may want to check with your credit card company or bank about what sort of exchange rates you can expect if you do this. There may be a fee charged for foreign currency transactions as well.

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Of course, the answer to this is no. As a buyer of Canadian dollars, the bank will most likely once again offer you an exchange rate that is less attractive than the published rates. For retail foreign currency exchanges, there are usually two separate rates. The ***bid rate*** (also called the ***offered*** or ***buying rate***) is the rate that the exchanger will give you to exchange your foreign currency for U.S. dollars (i.e., when it buys your foreign currency from you). The ***asked rate*** (also called the ***selling rate***) is the rate it will give you when you exchange your U.S. dollars for foreign currency (i.e., when it sells you foreign currency). This may seem unfair, but in fact it is no different from a retail store buying merchandise for less than it sells it for. The difference between the two rates is sometimes referred to as the ***spread***.

The table below is an example of a retail currency rate table:

TRANSNATIONAL NATIONAL CURRENCY EXCHANGE RETAIL RATES AS OF 8/25/06

Country/Currency	*Sell*	*Buy*
Euro	1.2954	1.2575
Japan/yen	0.008645	0.008414
Canada/dollar	0.9250	0.8650
Mexico/peso	0.0917	0.0912
South Korea/won	0.001105	0.001025
Switzerland/franc	0.8255	0.7945
UK/pound	1.9221	1.8567

Of course, we can still use the same *mathematics* to convert currencies. The difference between the published rates and retail rates means we use a different rate, but we don't need different mathematical tools to do the conversion.

There is, however, one complication that sometimes arises. For both its buy and sell rates, Transnational National Currency Exchange is giving the price for one unit of each currency in US$, or in other words the "US$ Equivalent" form of the rate. When the company is selling you euros, each euro will cost US$1.2954 . But when it is buying euros, it will only pay US$1.2575 for each. The difference between these two rates is how the currency exchange business makes its profit.

Since this table gives the US$ equivalent rate, converting from the foreign currency into dollars is no problem. For example, if you have €250 to exchange for U.S. dollars, we would use the buy rate from the table to find how much Transnational National would pay:

$$€1 = \$1.2575$$
$$€250 = \$314.38$$

On the other hand, if you want to exchange U.S. dollars for euros, the rate is not presented in the form we would want it to be. The following exercise will demonstrate how we can calculate this conversion.

Example 11.1.7 ***Suppose you want to exchange $200 for euros. Calculate the amount you would receive in euros from the retail table given above.***

Transnational National would be selling you euros, so we use the selling rate:

$$€1 = \$1.2954$$

We want this rate in a form that gives us the value of $1 in euros. We can get this by dividing both sides of the equation by 1.2954, to make the right side $1. When we do this, we get:

$$€0.771962328 = \$1$$

Once we have this, we can multiply both sides by 200 to get:

$$€154.39 = \$200$$

You would receive €154.39.

The method shown in this example is probably not what you would use if you needed to make these conversions often. It is also correct, and quicker, to simply divide 200/1.2954 and get the same result. The same caution about shortcuts given before applies here, but even more so. This gives the correct answer, but the mathematical reasoning behind it is not obvious, and for that reason it is quite easy to get confused about when to multiply and when to divide. If you worked somewhere where you needed to do these calculations often, they would become second nature, so you would be unlikely to make a mistake by using a shortcut. For those who need to make these conversions only occasionally, the method is a bit more cumbersome, but is less likely to lead to an error.

Sometimes, rate quotes will instead give only the "currency per US$" rate. Rate tables are often not particularly clear about which type or rate is being quoted. Knowing in advance whether the currency you are looking to exchange is worth more, or less, than a U.S. dollar can help in figuring this out.

Exchange Rates as a Percent (Optional)

There is yet another commonly used way of expressing exchange rates: as a percent. This is especially common at businesses located in places such as restaurants and shops near the U.S./Canadian border, where the two countries' currencies share a common name, are fairly close in value and are used on both sides of the border.

For example, a fast food restaurant in Canada might have a sign posted stating that "15% exchange on U.S. funds." What this means is that if you pay in U.S. currency, each U.S. dollar will be worth 15% "more" in Canadian dollars. This is the same as an exchange rate of US$1 = C$1.15. On the U.S. side of the border, a sign saying "15% exchange on Canadian funds" would be interpreted slightly differently. There, the meaning would be that each Canadian dollar is worth 15% "less" in U.S. dollars, equivalent to the exchange rate C$1 = US$0.85.

How are we supposed to know that in one case the 15% means more, and in the other case the 15% means less? Sometimes, signs will make this clear by using the word "premium" to indicate more ("15% premium for U.S. funds") or "discounted" to indicate "less" (as in "Canadian funds discounted 15%"). Often, though, this will not be stated explicitly. However, exchange rates are expressed in this way only in places where it is common knowledge which currency has the higher value. Anyone in a restaurant near the border would know that U.S. dollars have a higher value than Canadian dollars,[3] and so there is no need to give that extra information.

Example 11.1.8 ***A sign posted in a souvenir shop in Ganonoque, Ontario, states "U.S. exchange 25%." If you buy a sweatshirt there costing C$52.97, how much will it cost you if you pay in U.S. funds?***

The posted exchange rate means:

$$US\$1 = C\$1.25$$

Dividing both sides by 1.25 to get the value of C$1 gives us:

$$US\$0.80 = C\$1$$

Multiplying both sides by 52.97 gives

$$C\$52.97 = US\$42.38$$

Once again, we could also have got this by dividing 52.97/1.20 = 42.38.

Example 11.1.9 ***A souvenir shop in Niagara Falls, New York has posted "Canadian exchange: 15%". If you buy a framed poster there costing US$75.94 but pay in Canadian funds, how much will you pay?***

[3]At least this is true at the time I am writing this. If that changes in the future, and Canadian dollars become more valuable than U.S. dollars, the interpretation of the percentage would switch. But the relative values of the two currencies would still be common knowledge, and so the signs would still be just as well understood.

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Here, the exchange rate means:

$$C\$1 = US\$0.85$$

Using the same sort of mathematics as in the previous example, we can determine that the cost in Canadian dollars would be C\$89.34.

It is not possible to draw any general conclusions about the exchange rates offered by merchants on purchases. It is reasonable to expect that retailers would offer rates that are less attractive than the published rates for the same reasons that retail currency exchanges do, and this is generally true. On the other hand, though, businesses trying to attract tourists sometimes offer exchange rates for purchases that are considerably better than the published rates. Some American hotels, for example, offer to accept Canadian funds ***at par***, which means C\$1 = US\$1, to attract the business of Canadian tourists.[4] Exchange rates offered by businesses can offer a big cost advantage (or disadvantage) to customers paying in one currency or the other.

[4]Someone paying in U.S. dollars might be able to negotiate a better room rate, though.

EXERCISES 11.1

Except where another rate is provided in the problem, use the tables on pages 469–470 in answering the following questions. Use the more recent rate except where told to do otherwise.

A. Converting from U.S. to Foreign Currency

1. What is the value of 1 U.S. dollar in each of the following currencies?
   a. British (U.K.) pound
   b. Saudi riyal
   c. Mexican peso
   d. Thai baht
   e. Brazilian real

2. Convert \$590 into the currency of each of the following countries:
   a. Kuwait
   b. Turkey
   c. Argentina
   d. Malaysia
   e. Ecuador

3. Convert each of the following amounts from U.S. dollars to the specified foreign currency:
   a. \$20,000 into euros
   b. \$550 into Australian dollars
   c. \$8,735 into South African rand
   d. \$37.62 into Indian rupees
   e. \$54,769.25 into Russian rubles

4. On a trip to Philadelphia, Karol bought a sweatshirt for $37.98, which he charged on his credit card. Convert this price into Polish zloty.

B. Converting from Foreign Currency to US$

5. What is the value in U.S. dollars of each of the following:
   a. 1 Jordanian dinar
   b. 1 Israeli shekel
   c. 1 Peruvian new sol
   d. 1 Polish zloty
   e. 1 Chinese yuan

6. Convert 1,000 units of each of the following country's currencies into U.S. dollars:
   a. Brazil
   b. Russia
   c. India
   d. China
   e. Lebanon

7. Convert each of the following amounts into U.S. dollars:
   a. £35,000 (British pounds)
   b. €189.25 (European euros)
   c. ¥2,825 (Japanese yen)
   d. ¥350 (Chinese yuan)
   e. Rp75.50 (Indian rupees)

8. Laura bought stock in an Israeli company. Her total investment was 8,500 shekels. How much did her investment amount to in U.S. dollars?

C. Converting between Two Foreign Currencies

9. Use the cross rate table to make each conversion:
   a. €414.92 (European euros) into Canadian dollars
   b. ¥570,000 (Japanese yen) into British pounds
   c. SFr250 (Swiss francs) into Mexican pesos

10. At least one of the currencies in each of the following conversions is not listed in the cross rate table. Instead, use the U.S. dollar exchange rate table to make each conversion:
   a. NZ$708.59 (New Zealand dollars) into Chinese yuan
   b. 525 Mexican pesos into Pakistani rupees

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c. C$7,525 (Canadian dollars) into Venezuelan bolivars

d. €21.95 (European euros) into South Korean won

e. 735,960 Thai baht into Malaysian ringgit

D. Retail Foreign Currency Transactions

11. Globetrottin' Dave's Eclectic Currency Universe offers to buy and sell a number of currencies for U.S. dollars. His current rates are posted:

Country	Currency	Buy	Sell
Japan	Yen	0.00875	0.00919
European Union	Euro	1.186	1.253
U.K.	Pound	1.645	1.793
Canada	Dollar	0.7915	0.8508
Mexico	Peso	0.3416	0.3885
Australia	Dollar	0.6525	0.7303

Are the rates Dave is posting "US$ Equivalent" rates, or "Currency per US$"?

12. Using the table given in Exercise 11, what would you get at Globetrottin' Dave's for each of the following exchanges?

a. Exchange £300 (British pounds) for U.S. dollars

b. Exchange ¥20,000 (Japanese yen) for U.S. dollars

c. Exchange US$200 for Australian dollars

d. Exchange C$550 (Canadian dollars) for U.S. dollars

e. Exchange US$750 for Mexican pesos

13. Using the table given in Exercise 11, what would you get at Dave's for each of the following exchanges?

a. Exchange A$275 (Australian dollars) for U.S. dollars

b. Exchange US$475 for euros

c. Exchange US$150 for Japanese yen

d. Exchange €255.75 (euros) for U.S. dollars

14. Sam's local bank will exchange U.S. dollars and Mexican pesos. According to a sign in the lobby, the bank's current exchange rate is "Buy 11.00, Sell 10.50."

a. Is this the "Currency per US$" or the "US$ Equivalent" rate?

b. How much would Sam get at this bank if he converted US$100 into Mexican pesos?

c. How much would Sam get at this bank if he converted 500 Mexican pesos into U.S. dollars?

E. Exchange Rates as Percents

15. A sign in a shop in Toronto advertises "U.S. Exchange rate 20%." I went in and bought an Argos T-shirt for a total price of C$27.17, but I paid in U.S. funds. How much did I pay for the shirt?

16. A sign in a Singapore hotel states "Exchange for Australian currency 25%." Does this mean that A$1 = S$1.25, or that S$1 = A$1.25?

F. Grab Bag

17. A collector in the U.K. buys a collectible stuffed animal online for $75.38, charging the purchase to her credit card. Convert the price into British pounds.

18. While on vacation in Zurich, Devon stayed at a hotel for which his total bill was SFr 1,093.75 (Swiss francs.) How much is this in U.S. dollars?

19. As a sales promotion, a car dealer offers to give away 1 million South Korean won to a randomly selected customer. What would the value of this prize be in U.S. dollars?

20. Glenys lives in Christchurch, New Zealand. She bought several books over the Internet from a book shop in Barcelona, Spain, for €175.93 (euros). Convert this cost into New Zealand dollars.

21. A family from Djakarta is thinking about vacationing in Cabo San Lucas. The price of a vacation house rental is quoted as 25,000 Mexican pesos. Convert this cost into Indonesian rupiah.

22. A wine importer purchased a large shipment of South African wines for 837,250 rand. What was the cost in U.S. dollars?

23. The hit song "If I Had a Million Dollars" reflects the fact that "a million dollars" is thought of as the "magic number" for being rich. However, since the band responsible for this hit, the Barenaked Ladies, is Canadian, we can assume they meant 1 million Canadian dollars. For American audiences the song could be retitled "If I Had $900,300" (using the rates from the table in this section).

    Many different countries call their currency a "dollar". What is the value, in U.S. dollars, of "a million dollars" if the dollars in question are from:

    a. New Zealand
    b. Australia
    c. Taiwan
    d. Hong Kong
    e. Singapore

24. Some countries have unique names for their currencies (such as the Hungarian "forint", Malaysian "ringgit", or Israeli "shekel"), but other currency names, such as "dollar", "lira", "rupee", and "pound" are shared by different countries. Even if the names are the same, though, the currencies are not necessarily connected in any way.

    One widely used name for a currency is the "peso." Convert 12,500 Argentinean pesos into each of the following currencies:

    a. Chilean pesos
    b. Colombian pesos

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c. Mexican pesos
d. Philippine pesos
e. Uruguayan pesos

25. A sign in a Buffalo tavern states "C$1 = U.S.$0.85" If an order of volcanic chicken wings costs $6.75, what would the cost be if you paid in Canadian dollars?

26. The photo below shows the rates offered by an exchange bureau located in Canada. The rates shown are offers to buy and sell U.S. dollars and European euros. (The French word *achat* means "buy" and *vente* means "sell.")

a. How much, in Canadian dollars, would you get if you exchanged US$750 at this bureau?
b. How much, in U.S. dollars, would you get if you exchanged C$400 at this bureau.
c. Suppose that this exchange bureau converts 1,000 euros into Canadian dollars. Two hours later, another customer comes in and buys 1,000 euros with Canadian currency. Calculate the bureau's profit (in Canadian dollars) from this pair of transactions.

27. An engineering firm in Bangalore, India, bills an American company 325,700 rupees for consulting services. What is the cost in U.S. dollars?

28. In a Vancouver newspaper ad, a tour operator in Hobart, Tasmania (Australia) offers to accept "Canadian currency at par." Is this a good deal or not? Explain.

G. Additional Exercises

29. Prices in U.S. dollars are commonly carried out to two decimal places, because each dollar is divisible into 100 cents. Can you think of any place where prices in U.S. dollars are normally carried out to *more* than two decimal places?

30. In this chapter, we noted that like U.S. dollars, most currencies have a main unit that can be divided into 100 smaller units, such as our "dollars" and "cents". We also noted that the value of the main unit of some currencies is small enough that you would be unlikely to ever worry about smaller units.

    Are there any currencies in use in the world today that have a main unit that can be divided into more than 100 smaller units? For example, are there any currencies that are divided into 1,000 units (requiring three decimal places)? Research this question, using library and/or Internet resources.

31. Cattarauqua Ginseng Enterprises, an American company, made a large sale to a Japanese importer. The total amount of the order was ¥25,000,000 (in Japanese yen), payable on September 24, 2006. Not wanting to gamble on movements in foreign exchange rates, on August 24 the company entered into a futures contract to exchange this amount into U.S. dollars on September 24.

    a. Using the rates in the table from this section, what would the company receive in U.S. dollars?

    b. Suppose that on September 24 the spot rate for Japanese yen was as listed below:

U.S. Dollar Equivalent	***Currency per U.S. Dollar***
0.008588	116.4463

    How much would the company have received if they had instead made their exchange on the spot market?

    c. How much money did the company gain by making the futures exchange?

32. Some individuals and investment firms try to take advantage of the fluctuations in foreign currency rates to make profits. This practice is sometimes called "forex trading" ("forex" being short for foreign exchange.) Suppose that such a trader is working with large enough amounts of money to be able to obtain the rates published in the *Wall Street Journal*. Determine the profit (or loss) in U.S. dollars from each of these exchanges:

    a. Bought $1,000,000 worth of euros at Wednesday's closing price and converted back to U.S. dollars at Thursday's.

    b. Bought $2,000,000 worth of Colombian pesos at Wednesday's price and converted back to U.S. dollars at Thursday's

    c. Converted $1,600,000 into Singapore dollars at Wednesday's rate and back to U.S. dollars at Thursday's

33. A sign posted in the snack bar of a tennis club in Hamilton, Ontario, says "15% exchange on U.S. funds. Change given in Canadian funds only." Kevin's lunch cost C$8.43 and he paid with a US$10 bill. How much change did he get?

34. Andrew and his sister Alison both opened investment accounts on May 13, 1992. Andrew deposited $3,500 when he opened the account, and made no other deposits or withdrawals. His account has earned 4.8% interest compounded monthly.

    Allison made a deposit of $3,500, also on May 13, 1992, and also made no additional contributions to or withdrawals from her account. However, Allison was living in Canada at the time, and opened her account at a Canadian bank. Therefore, Allison first converted her US$3,500 into Canadian dollars, and then deposited the resulting Canadian money into the account. Her account has also earned 4.8% interest compounded monthly.

    The U.S.-Canadian exchange rates are given below:

Date	***U.S. Dollar Equivalent***	***Currency per U.S. Dollar***
On May 13, 1992	0.8314	1.2028
On May 13, 2002	0.6429	1.5555

    a. What was the value of Andrew's account on May 13, 2002?

    b. What was the value, in U.S. dollars, of Allison's account on May 13, 2002?

    c. Both Allison and Andrew contributed the same amount of money to their accounts, both earned the same compound interest rate, and both had their money on deposit for the same amount of time. Who came out ahead, as measured in U.S. dollars? Why didn't they both have the same amount?

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CHAPTER 11
SUMMARY

Topic	Key Ideas, Formulas, and Techniques	Examples
Converting from US$ into a Foreign Currency, p. 471	• Write an equation for the value of US$1 in the foreign currency, using the "Currency per US$" rate. • Multiply both sides by the number of dollars.	Convert US $3,724.59 into euros. (Example 11.1.1)
Converting from a Foreign Currency into US$, p. 472	• Write an equation for the value of one unit of foreign currency in US$, using the "US$ Equivalent" rate. • Multiply both sides by the number of units of foreign currency.	Convert 348 Turkish new liras into US$. (Example 11.1.2)
Converting between Two Foreign Currencies. p. 474	• Write an equation for the value of one unit of the currency you are converting from in the currency you are converting to, using a cross rate table. • Or convert the amount into US$ and then from US$ into the new foreign currency.	Convert 40,000 Russian rubles into Thai baht. (Example 11.1.6)
Conversions Using Retail Rates, p. 475	• If you are exchanging US$ for a foreign currency, use the sell rate. • If you are exchanging foreign currency for US$, use the buy rate. • If the rate is not given in the form you need, divide both sides by the appropriate number to change it into that form.	Suppose you want to exchange $200 for euros. The buy rate is €1 = $1.2575; the sell rate is €1 = $1.2954. Calculate the amount you would receive in euros. (Example 11.1.7)

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CHAPTER 12

Financial Statements

"The chief value of money lies in the fact that one lives in a world in which it is overestimated."

—H. L. Mencken

Learning Objectives

LO 1 Be familiar with the structure of income statements and balance sheets.

LO 2 Calculate values used on these financial statements.

LO 3 Complete vertical and horizontal analyses of financial statements and use them as a tool for interpretation.

LO 4 Calculate and interpret basic financial ratios.

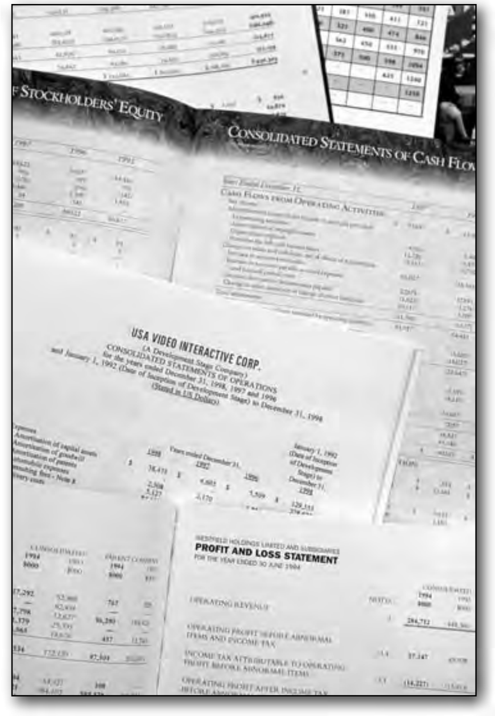
Chapter Outline

12.1 Income Statements

Former New York City mayor Ed Koch was famous for asking the public, "How'm I doing?" For a business, "how'm I doing?" financially is obviously an important question to be able to answer. ***Financial statements*** are reports that summarize the financial condition and performance of a business. There are many different types of financial statements, each designed to present certain aspects of the business's finances. The two most widely used types are the ***income statement***, intended to report the business's sales and profits, and the ***balance sheet***, intended to report the business's assets and debts.

Public companies (corporations whose stock is traded on the open market) are required to file financial statements with the U.S. Securities and Exchange Commission and send these statements to their stockholders. Other companies typically prepare financial statements as well, to report the company's financial results to its owners, for internal assessment and planning purposes, or to present information about the company's finances to current or prospective lenders or business partners. While different types of businesses may require different details to be reported in different ways, the layout and contents of financial statements are expected to follow certain rules. For statements filed with regulators or with the Internal Revenue Service, certain standards must be followed; for statements used internally by a business there may be more freedom but still certain standards are normally expected to be followed. In the United States, the set of regulations known as ***generally accepted accounting practices*** (***GAAP***; pronounced like the word "gap") are just what their name implies. It is worth noting, however, that for tax purposes businesses sometimes may, or even must, report their financial results in ways that differ from GAAP standards.

For all but the simplest businesses, financial statements are prepared by accountants (small businesses sometimes use computer software instead of hiring a human accountant). Obviously, this chapter is not designed to provide a complete understanding of financial statements, nor is it intended to equip you with everything you need to know to prepare a financial statement for a business. We will, however, explore some key points and concepts necessary to read and interpret a basic financial statement.



Financial statements are a valuable tool to analyze a business's performance. © David Young-Wolff/ PhotoEdit, Inc.

Basic Income Statements

The purpose of an income statement, as the name suggests, is to report on a business's financial performance over some period of time. An income statement will show the money taken in by the business (its ***gross revenues*** or ***gross sales***), the expenses incurred by the business, and the net profit or loss. The expenses are often separated into the ***cost of goods sold*** and ***expenses*** (costs such as wages, rent, utilities, advertising expenses, and so on). Subtracting the cost of goods sold from the gross income gives the ***gross profit***. Subtracting out the expenses leaves us with the ***net profit*** (or ***net income***). Here is a very simple example of an income statement:

Sammy's Lemonade Stand
Second Quarter 2007 Income Statement

Sales	$185.25
−Cost of goods sold	$75.35
Gross profit	$109.90
−Expenses	$45.00
Net income	**$64.90**

From this statement, we see that Sammy sold $185.25 worth of lemonade in the quarter. The lemonade he sold cost him $75.35 (for lemons, sugar, cups, etc.). Thus, his gross profit, what he would have made if he had no expenses (overhead), was $109.90. Since he did have expenses, though, we need to subtract that $45.00 to find that his net profit was $64.90.

Example 12.1.1 ***Cattarauqua Ginseng Enterprises had sales of $176,530 last year. The cost of the goods the company sold was $62,500 and its expenses totaled $78,595. What was the company's net income for the year?***

We can set this information up as a simple income statement to calculate the net income:

Sales	$176,530
−Cost of goods sold	$62,500
Gross profit	$114,030
−Expenses	$78,595
Net income	**$35,435**

Depending on the intended audience and use for the financial statement, a greater level of detail may be provided. The individual items that go into a category may be listed separately;

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their amounts are often put in a separate column to clearly separate the totals for each item from the total for the categories. Sammy might provide a bit more detail in his statement so that it looks like this:

Sammy's Lemonade Stand
Second Quarter 2007 Income Statement

Sales		$185.25
Cost of goods sold		
Lemons	$34.60	
Sugar	$12.25	
Cups	$20.15	
Other	$8.35	
Total cost of goods sold		$75.35
Gross profit		$109.90
Expenses		
Advertising	$12.00	
Wages	$31.00	
Other	$2.00	
Total expenses		$45.00
Net income		**$64.90**

While this income statement is lengthier and contains more detail than the first one, it is based on the same principles as the first. The amounts for each of the major categories (sales, cost of goods sold, and expenses) are the totals of their subcategories.

The expenses, for example, are equal to the sum of advertising, wages, and other expenses.

Example 12.1.2 ***Fill in the missing spaces in the income statement below:***

Latchman's Pharmacy
Third Quarter 2008 Income Statement

Sales		$248,535
Cost of goods sold		
Prescription medications	$94,350	
OTC medicines	$11,563	
Health and beauty	$23,505	
Other/sundries	$26,505	
Total cost of goods sold		(a)
Gross profit		(b)
Expenses		
Wages	$53,650	
Rent	$22,985	
Other	$18,754	
Total expenses		(c)
Net income		**(d)**

The total cost of goods sold can be found by adding the amounts in each of the four subcategories.

$$\text{Cost of goods sold} = \$94{,}350 + \$11{,}563 + \$23{,}505 + \$26{,}505 = \$155{,}923$$

The gross profit can be found as:

$$\text{Gross profit} = \$248{,}535 - \$155{,}923 = \$92{,}612$$

The expenses total:

$$\text{Total expenses} = \$53{,}650 + \$22{,}985 + \$18{,}754 = \$95{,}389$$

The net income is then:

$$\text{Net income} = \$92{,}612 - \$95{,}389 = -\$2{,}777$$

Since the expenses were larger than the gross profit, the pharmacy lost money in this quarter. It is conventional in accounting to indicate that an amount is negative by writing it in parentheses. This may be confusing at first since in mathematics ( )'s indicate multiplication, but it is always clear from context what is meant.

The completed income statement then should look like this:

Latchman's Pharmacy
Third Quarter 2008 Income Statement

Sales		$248,535
Cost of goods sold		
Prescription medications	$94,350	
OTC medicines	$11,563	
Health and beauty	$23,505	
Other/sundries	$26,505	
Total cost of goods sold		$155,923
Gross profit		$92,612
Expenses		
Wages	$53,650	
Rent	$22,985	
Other	$18,754	
Total expenses		$95,389
Net income		**($2,777)**

More Detailed Income Statements

The income statements we have been looking at above have been fairly simple. Complex businesses often require a greater breakdown of items into subcategories.

Sales: Reporting the sales of the business may require more detail than just the gross amount of money taken in. Gross sales may need to be adjusted for items that were returned, and for cash discounts that may have been given for early payments by customers who bought on credit (these discounts are discussed in detail in Chapter 8). It is unlikely that Sammy had to deal with customer returns or early payment discounts from lemonade sales (since payment would be made on the spot in any case). However, other businesses certainly do, and so these adjustments to sales need to be reflected on the financial statements. Subtracting these and any other appropriate reductions from the gross sales gives the ***net sales***.

Example 12.1.3 ***Cattarauqua Ginseng Enterprises had gross sales of $185,926 last year. It had one order of $5,300 of merchandise returned, and gave $4,096 in cash discounts for prompt payment. Calculate the company's net sales.***

$$\text{Gross sales} - \text{Returns} - \text{Cash discounts} = \text{Net sales}$$
$$\$185{,}926 - \$5{,}300 - \$4{,}096 = \$176{,}530$$

Depending on how specific the income statement needs to be, these items may or may not be listed separately on the income statement.

Cost of Goods Sold: On the face of it, this seems quite simple: this category should simply be the business's cost for the items it sold, pure and simple. However, this is complicated by the fact that the items a business buys in any given period are not exactly the items it sells in that period. Items are kept in inventory from one period to the next.

The cost of goods sold, then, is found by looking at the cost of the items in inventory at the start of the period. We then add the cost of new merchandise purchased within the period, to get the total cost of all goods that the business had in inventory at any time during the period. To find out how much was sold, we subtract the cost of the inventory in place at the end of the period.

The total cost of merchandise acquired may also give details of any reductions due to returned merchandise or savings due to cash discounts, just as we did in calculating the net sales.

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Example 12.1.4 At the start of last year, Cattarauqua Ginseng Enterprises had inventory that cost $37,923. Over the course of the year it made purchases of $48,923 before taking into account $1,850, which it received as a refund for a returned shipment and $1,175 of savings from cash discounts for early payment. At the end of the year, they had inventory whose total cost was $21,321. Calculate the company's cost of goods sold for the year.

We first determine the net cost of the company's purchases for the year:

$$\text{Gross purchases} - \text{Returns} - \text{Cash discounts} = \text{Net purchases}$$
$$\$48{,}923 - \$1{,}850 - \$1{,}175 = \$45{,}898$$

To calculate the cost of goods:

$$\text{Starting inventory} + \text{Net purchases} - \text{Ending inventory} = \text{Cost of goods sold}$$
$$\$37{,}923 + \$45{,}898 - \$21{,}321 = \$62{,}500$$

Expenses: We have already seen that a business may break down its expenses into subcategories; these subcategories are often grouped together into certain groupings as well.

Operating expenses (also known as ***overhead***) include most of the expenses of operating the business. Operating expenses include costs like rent, utilities, wages and salaries, insurance, and other similar costs. Operating expenses could also include ***depreciation***, an amount used to recognize the expense of equipment and other property used, not purchased, during the period in question. For example, a company that bought $500,000 worth of equipment that is expected to last 10 years might recognize this expense by claiming $50,000 of depreciation in each of those 10 years. (Depreciation is discussed in more detail in Section 8.4) Depreciation is intended to match up the cost of property and equipment with the time period in which it is actually used.

The two major expenses not generally included as operating expenses are ***taxes*** and ***interest*** paid by the company. Since the total income tax for the year cannot be known until the company completes its tax filings, the amount of taxes is often a projection listed as a "provision for taxes."

By subtracting the operating expenses from the gross profit we arrive at the business's ***earnings before interest and taxes***, not surprisingly often abbreviated as ***EBIT***. Subtracting all expenses from the gross profit gives the company's ***net income***.

Example 12.1.5 Cattarauqua Ginseng Enterprises had net sales of $176,530 last year, and the cost of goods sold was $62,500. Expenses for the year included $35,000 for salaries, $5,000 for depreciation, $23,000 for rent and utilities, an $8,500 provision for taxes, and $7,095 of interest expenses. Calculate the company's gross profit, total operating expenses, EBIT, and net income for the year.

First, we find the gross income:

$$\text{Net sales} - \text{Cost of goods sold} = \text{Gross income}$$
$$\$176{,}530 - \$62{,}500 = \$114{,}030$$

Next, we total up the operating expenses, which in this case include salaries, depreciation, rent, and utilities:

$$\text{Sum of operating expense subcategories} = \text{Total operating expenses}$$
$$\$35{,}000 + \$5{,}000 + \$23{,}000 = \$63{,}000$$

The EBIT can be found by subtracting these operating expenses from the gross income:

$$\text{Gross income} - \text{Operating expenses} = \text{EBIT}$$
$$\$114{,}030 - \$63{,}000 = \$51{,}030$$

Finally, to arrive at the net income we subtract the taxes and interest:

$$\text{EBIT} - \text{Taxes} - \text{Interest} = \text{Net income}$$
$$\$51{,}030 - \$8{,}500 - \$7{,}095 = \$35{,}435$$

Before moving on, let's put all of this together to complete a fairly detailed income statement example.

Example 12.1.6 ***Complete the financial statement shown below by filling in the missing items.***

Hawliss-Troomeny Motorsports Corporation
2007 Full Year Income Statement

I.	**Revenues**		
	Gross sales	$3,548,903	
	Less returns	$116,039	
	Less sales discounts	$24,350	
	Net sales		(a)
II.	**Cost of goods sold**		
	Inventory as of December 31, 2006	$1,275,908	
	Net purchases	$2,671,043	
	Less inventory as of December 31, 2007	$1,363,024	
	Cost of goods sold		(b)
III.	**Gross profit**		(c)
IV.	**Operating expenses**		
	Wages, salaries, and benefits	$427,903	
	Rent	$207,025	
	Utilities	$42,045	
	Insurance	$58,362	
	Depreciation	$35,000	
	Other	$14,445	
	Total Operating Expenses		(d)
V.	**Net income before interest and taxes**		(e)
VI.	**Other expenses**		
	Provision for taxes	$25,000	
	Interest	$32,500	
	Total interest and taxes		(f)
VII.	**Net income**		**(g)**

(a) Net sales = Gross sales − Returns − Cash discounts = $3,548,903 − $116,039 − $24,350 = $3,408,514.

(b) Cost of goods sold = Starting inventory + Net purchases − Ending inventory = $1,275,908 + $2,671,043 − $1,363,024 = $2,583,927

(c) Gross profit = Net sales − Cost of goods sold = $3,408,514 − $2,583,927 = $824,587

(d) Total operating expenses = Sum of operating expense subcategories = $427,903 + $207,025 + $42,045 + $58,362 + $35,000 + $14,445 = $784,780

(e) EBIT = Gross profit − Operating expenses = $824,587 − $784,780 = $39,807

(f) Total interest and taxes = Taxes + Interest = $25,000 + $32,500 = $57,500

(g) Net income = EBIT − Total interest and taxes = $39,807 − $57,500 = −$17,693

On the income statement, the net income might be written as ($17,693).

Note that in this example we kept items at the same level of detail lined up in columns.

This is common practice, making it easier for a reader to separate numbers that are totals of a category from numbers that represent subcategory amounts. Someone who is interested only in the "big picture" can get it quickly just by looking at the numbers in the rightmost column.

Vertical Analysis of Income Statements

Let's revisit Sammy's lemonade stand again. We can see from his income statement that Sammy made a nice profit from his business venture in the second quarter, and we can get a sense of how sales and expenses contributed to that profit. Businesses often are interested, though, in looking more precisely at the relative sizes of profit, various expenses, etc., in relation to the overall business activity. Sammy could enhance his financial statement by providing the amounts in each category as a percent of the business's sales.

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Sammy's Lemonade Stand
Second Quarter 2007 Income Statement

	Dollars		Percent of Sales	
Net sales		$185.25		100.00%
Cost of goods sold				
Lemons	$34.60		18.68%	
Sugar	$12.25		6.61%	
Cups	$20.15		10.88%	
Other	$8.35		4.51%	
Total cost of goods sold		$75.35		40.67%
Gross profit		$109.90		59.33%
Overhead				
Advertising	$12.00		6.48%	
Wages	$31.00		16.73%	
Other	$2.00		1.08%	
Total overhead		$45.00		24.29%
Net income		**$64.90**		**35.03%**

Having these percentages readily available can be very helpful in analyzing a business's performance. We can, for example, see that Sammy's gross profit is 59.33% of sales (this percent is called his ***gross profit margin***). If he knows that other neighborhood lemonade stands run at a 60% gross profit margin, he can see that he is doing about as well in that regard as others are. Similarly, if he sees that his overhead of 24.29% is much lower than an average of 40% for other similar businesses, he can conclude that he is managing his overhead costs well for his type of business. This analysis is much harder to do with the absolute dollar figures.

Looking at each category's percentage of the total net sales allows us draw conclusions about each area in proportion to the total. Since the items being compared are above/below each other, this is known as a ***vertical analysis***. This sort of analysis can help us to see how much each piece contributes to the overall picture. It does not, however, tell us much about the trends in the business, whether things are getting better or worse overall and in each category.

Horizontal Analysis of Income Statements

To be able to see trends in a business's sales, profits and/or expenses, we might instead want to look at a financial statement that compares the current period's results to a prior period. In that case, we would be more interested in percentages not of the total net sales, but rather of change from one period to the next. If we set up an income statement showing the results from the current period next to the results from a prior period to which we want to compare it, and then calculate the change from one period to the next, we would create a statement looking something like the one shown on the next page.

Horizontal analysis, so called because the items we are comparing are written next to each other in horizontal rows, enables us to see *trends* in the business. We can see that Sammy dramatically increased his spending on advertising, and this seems to have paid off in an 11.76% increase in sales. This increase in sales did not entirely translate into an increase in profits, though. Profits grew by a much more modest 3.92%. Why didn't profits grow as quickly as sales? There must have been an increase in some of the business's costs. Horizontal analysis helps spot where those cost increases came from. In addition to the increase in advertising spending, Sammy's cost for lemons shot up by $12.55, a 56.92% increase. Other costs stayed fairly stable, or decreased, so Sammy can readily identify which areas may require more attention as he manages his business.

An income statement may show both the horizontal period-to-period comparisons and vertical percent of sales comparisons. An example of a statement that includes this appears below.

Sammy's Lemonade Stand
Second Quarter 2007 Income Statement

	2007		2006		Increase/(Decrease)	
Sales		$185.25		$165.75	$19.50	11.76%
Cost of goods sold						
Lemons	$34.60		$22.05		$12.55	56.92%
Sugar	$12.25		$12.75		($0.50)	−3.92%
Cups	$20.15		$22.00		($1.85)	−8.41%
Other	$8.35		$8.50		($0.15)	−1.76%
Total cost of goods sold		$75.35		$65.35	$10.05	15.39%
Gross profit		$109.90		$100.45	$9.45	9.41%
Operating expenses						
Advertising	$12.00		$4.00		$8.00	200.00%
Wages	$31.00		$32.00		$1.00	−3.13%
Other	$2.00		$2.00		$0.00	0.00%
Total operating expenses		$45.00		$38.00	$7.00	18.42%
Net income		**$64.90**		**$62.45**	**$2.45**	**3.92%**

Example 12.1.7 ***Complete a vertical analysis and a horizontal analysis for 2007 for the company whose income statements for 2006 and 2007 are given below.***

Cattarauqua Ginseng Enterprises
2006 Income Statement

Sales	$176,530
−Cost of goods sold	$62,500
Gross profit	$114,030
−Expenses	$78,595
Net income	**$35,435**

Cattarauqua Ginseng Enterprises
2007 Income Statement

Sales	$196,235
−Cost of goods sold	$78,090
Gross profit	$118,145
−Expenses	$59,677
Net income	**$58,468**

Vertical Analysis: *A vertical analysis calculates each component of the income statement as a percent of net sales. We take the 2007 income statement, then, and calculate each category as a percent of net sales, placing the results in a new column to the right. For example, the percent for cost of goods sold would be $78,090/$196,235 = 39.79%. We do the same for the other categories. The result is:*

Cattarauqua Ginseng Enterprises
2007 Income Statement

	Amount	Percent of Net Sales
Sales	$196,235	100.00%
−Cost of goods sold	$78,090	39.79%
Gross profit	$118,145	60.21%
−Expenses	$59,677	30.41%
Net income	**$58,468**	**29.79%**

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Horizontal Analysis: *A horizontal analysis sets up the two periods side by side and calculates the change both as an amount and as a percent. For example, the change in sales was \$196,235 − \$176,350 = \$19,885 as an amount. As a percent change, we take \$19,885 and divide it by the 2006 amount of \$176,350 to get 11.28%. We do the same for the other categories. The result is:*

Cattarauqua Ginseng Enterprises
2007 Income Statement

			Increase (Decrease)	
	2007	**2006**	**Amount**	**Percent**
Sales	\$196,235	\$176,530	\$19,705	11.16%
−Cost of goods sold	\$78,090	\$62,500	\$15,590	24.94%
Gross profit	\$118,145	\$114,030	\$4,015	3.52%
−Expenses	\$59,677	\$78,595	(\$18,918)	(24.07%)
Net income	**\$58,468**	**\$35,435**	**\$23,033**	**65.00%**

Of course, there is more to an income statement than just correctly calculating the values that go into it. From the vertical analysis, we can see that this company enjoys quite good profitability as a percent of sales, and that in 2007 profits came in at nearly 30% of sales. From the horizontal analysis, we can see that sales increased from 2006 to 2007, but that the cost of goods sold increased at an even faster pace. While this might be cause for concern, we can also see that the company enjoyed a large drop in expenses, enabling them to experience a huge percent increase in overall profits.

Example 12.1.8 ***From the trends seen in the horizontal analysis above, can we make any predictions about Cattarauqua Ginseng Enterprises' profitability next year?***

The company's large increase in profits was due to a huge decrease in expenses, even in the face of an increase in the cost of goods sold. While we cannot tell from the income statement what caused such a large drop in expenses, it is probably unlikely that the company will see a similar decrease next year. It is unlikely then that the company's profits will be able to grow by the same percent next year. In fact, if the company continues to experience wholesale cost increases that outpace increases in their sales, their profits may well decline.

EXERCISES 12.1

A. Basic Income Statements

1. A company has net sales totaling \$91,585. The cost of goods sold is \$58,535 and the expenses are \$27,650. What was the company's net income?

2. Thomas Hydrometer Sales had 2007 gross profit totaling \$74,438,275 and expenses of \$69,657,424. What was the company's net income?

3. Calculate the missing values in the income statement below:

Sales	\$256,529
−Cost of goods sold	\$208,625
Gross profit	(a)
−Expenses	\$84,623
Net income	**(b)**

4. Calculate the gross profit and net income of the company whose incomplete income statement is given below:

Latchman's Pharmacy
Second Quarter 2007 Income Statement

Sales		$255,606
Cost of goods sold		
Prescription medications	$74,000	
OTC medicines	$8,509	
Health and beauty	$18,439	
Other/sundries	$22,001	
Total cost of goods sold		$122,949
Gross profit		
Total expenses		$115,063
Net income		

5. Suppose your company had 2006 net sales totaling $657,000. The cost of goods sold was $320,000. Expenses included $119,750 for wages and benefits, $58,000 for rent, $9,200 for utilities, $8,500 in depreciation, and $3,750 of other expenses. Calculate your company's gross profit and net income for 2006.

B. More Detailed Income Statements

6. A bike shop had gross sales of $513,400 in 2007, $14,900 in returns, and gave $5,380 in discounts for early payment. Calculate the shop's net sales for 2007.

7. A candy factory had gross sales of $2,956,720 in 2005. There were $396,000 in returns, and the company gave $18,430 in early payment discounts. Calculate the factory's 2005 net sales.

8. A bike shop had $186,500 worth of inventory at the start of 2007 and $164,850 worth at the end of the year. They made purchases of $173,145 during the year. Calculate their cost of goods sold for 2007.

9. A candy factory had $75,025 in inventory at the start of 2005 and $89,710 at the end of the year. They made purchases of $784,325 during the year. Determine their cost of goods sold for the year.

10. Calculate the 2007 gross profit for the bike shop from Exercises 6 and 8.

11. Find the 2005 gross profit for the candy factory from Exercises 7 and 9.

12. Ludd and Simms Electronics had net sales of $935,275 last year, and their cost of goods sold was $542,800. Expenses for the year consisted of $225,000 for salaries and benefits, $65,000 for depreciation, $98,355 for rent and utilities, a $20,000 provision for taxes, and $37,200 of interest expenses. Calculate the company's (a) gross profit, (b) total operating expenses, (c) EBIT, and (d) net income for the year.

13. Tokamak Home Generators had net sales of $6,450,036 last year. The company's cost of goods sold was $2,890,403. Expenses for the year included $803,419 for salaries and benefits, $816,400 for depreciation, $197,500 for rent and

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utilities and $45,500 in other operating expenses. The company's financial statements included an $800,000 provision for taxes, and $157,600 of interest expenses. Calculate the company's (a) gross profit, (b) EBIT, and (c) net income for the year.

C. Vertical and Horizontal Analysis of Income Statements

14. Complete a vertical analysis of the financial statement given below:

Net Sales	$19,508	(a)
Cost of goods sold	$6,409	(b)
Gross profit	$13,099	(c)
Operating expenses	$7,500	(d)
EBIT	$5,599	(e)
Interest and taxes	$2,000	(f)
Net income	**$3,599**	**(g)**

15. Fill in the missing items and complete a vertical analysis of the financial statement given below:

Net sales	$85,000	(d)
Cost of goods sold	$27,500	(e)
Gross profit	(a)	(f)
Operating expenses	$62,500	(g)
EBIT	(b)	(h)
Interest and taxes	$6,000	(i)
Net income	**(c)**	**(j)**

16. Complete a horizontal analysis for the income statement given below:

	2008	2007	Amount	Percent
Net sales	$27,500	$19,508	(a)	(h)
Cost of goods sold	$9,955	$6,409	(b)	(i)
Gross profit	$17,545	$13,099	(c)	(j)
Operating expenses	$6,445	$7,500	(d)	(k)
EBIT	$11,100	$5,599	(e)	(l)
Interest and taxes	$3,500	$2,000	(f)	(m)
Net income	**$7,600**	**$3,599**	**(g)**	**(n)**

17. Complete a horizontal analysis for 2007 for the company whose income statements for 2006 and 2007 are given below.

2006 Income Statement

Sales	$275,043
−Cost of goods sold	$192,343
Gross profit	$82,700
−Expenses	$51,000
Net income	**$31,700**

2007 Income Statement

Sales	$265,951
−Cost of goods sold	$144,043
Gross profit	$121,908
−Expenses	$72,500
Net income	**$49,408**

D. Grab Bag

18. If you are looking at trends from one year to the next, would you want to use vertical or horizontal analysis?

19. If you want to look at each item on the income statement as a percent of the company's total sales, would you want to use vertical or horizontal analysis?

Exercises 20 to 35 are based on the financial statement shown below:

	2007	2006
I. Revenues		
Gross sales	$425,000	$405,000
Less returns	$18,200	$18,300
Less sales discounts	$0	$2,500
Net sales		
II. Cost of goods sold		
Beginning inventory	$98,500	$118,000
Net purchases	$176,800	$137,500
Ending inventory	$127,300	$98,500
Cost of goods sold		
III. Gross Profit		
IV. Operating Expenses		
Wages, salaries, and benefits	$103,000	$98,500
Rent and utilities	$68,425	$66,500
Depreciation	$8,400	$8,700
Other	$15,000	$12,000
Total operating expenses		
V. EBIT		
VI. Interest and taxes		
Provision for taxes	$10,000	$7,500
Interest	$0	$0
Total interest and taxes	$10,000	$7,500
VII. Net income		

20. Calculate the 2006 net sales.

21. Calculate the 2007 net sales.

22. Calculate the 2006 cost of goods sold.

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23. Calculate the 2007 cost of goods sold.

24. Calculate the 2006 gross profit.

25. Calculate the 2007 gross profit.

26. Calculate the 2006 operating expenses.

27. Calculate the 2007 operating expenses.

28. Calculate the 2006 net income.

29. Calculate the 2007 net income.

30. Calculate the percent that would be used in a horizontal analysis of the company's wages, salaries, and benefits.

31. Calculate the percent that would be used in a horizontal analysis of the company's EBIT.

32. Calculate the percent that would be used in a vertical analysis of the company's 2006 cost of goods sold.

33. Calculate the percent that would be used in a vertical analysis of the company's 2007 net income.

34. By what percent did the company's returns change from 2006 to 2007?

35. By what percent did the company's sales discounts change from 2006 to 2007?

E. Additional Exercise

36. Joe did a vertical analysis of his company's financial statements, and calculated that the cost of goods sold was 59.4% of sales. Later, though, he realized that he accidentally used gross sales instead of net sales in this calculation. His company had no cash discounts, but did have returns amounting to 5.85% of gross sales. What should the cost of goods sold percent actually have been?

12.2 Balance Sheets

Income statements show the financial performance of a business over some period of time. They show profits, losses, expenses, and so on, but give no insight into the financial position of the business in terms of the value of what it owns or the total of its debts. ***Balance sheets*** are a type of financial statement used to consider the overall financial condition of the business from that point of view.

Basic Balance Sheets

A balance sheet is a financial statement that is intended to show a snapshot of a business's ***assets*** (what the business owns), ***liabilities*** (what the business owes), and ***(owner's) equity*** or ***net worth*** (the difference between the two). The balance sheet can be summed up with the equation:

$$\text{Equity} = \text{Assets} - \text{Liabilities}$$

This form of the equation is probably the most intuitive way of expressing the relationship. A mathematically equivalent expression of this relationship can be found by adding liabilities to both sides of this equation. This alternative form is more commonly used in accounting:

$$\text{Assets} = \text{Liabilities} + \text{Equity}$$

Example 12.2.1 ***A business has total assets of $1,503,055 and total liabilities of $932,055. What is the owners' equity?***

Working from the first form of the balance sheet equation gives:

$$\text{Equity} = \$1{,}503{,}055 - \$932{,}055 = \$571{,}000.$$

We could also have got this from the second form with just a bit of algebra:

$$\begin{aligned} \text{Assets} &= \text{Liabilities} + \text{Equity} \\ \$1{,}503{,}055 &= \$932{,}055 + \text{Equity} \\ \text{Equity} &= \$571{,}000 \end{aligned}$$

Example 12.2.2 ***A corporation reports that its assets total $20,500,000 and its stockholders' equity is $4,000,000. What are the corporation's total liabilities?***

Since the stockholders are the owners of the corporation, the term stockholders' equity is understood to mean the same thing as owners' equity. Working from the second form of the balance sheet equation gives:

$$\begin{aligned} \text{Assets} &= \text{Liabilities} + \text{Equity} \\ \$20{,}500{,}000 &= \text{Liabilities} + \$4{,}000{,}000 \\ \text{Liabilities} &= \$16{,}500{,}000 \end{aligned}$$

It would also be possible to get this from the first form of the equation, but the algebra is a bit more involved.

Just as with income statements, these broad categories may be listed out in some detail. Some of the more common terms used for this breakdown are defined below.

Balance sheets are commonly set up in two columns. One column lists the assets of the business, the other lists the liabilities and equity. Since assets = liabilities + equity, the totals of these two columns should be the same. They can be thought of as two sides of a balanced scale (hence the name balance sheet).

Here is a very simple example of a balance sheet for the company from Example 12.1:

Basic Example Corporation
Balance Sheet as of December 31, 2007

Assets		**Liabilities**	
Assets	$500,000	Total liabilities	$350,000
		Equity	
		Total equity	$150,000
Total assets	**$500,000**	**Total liabilities and equity**	**$500,000**

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Since the totals of each side should match, it is customary to call attention to these totals by doubly underlining them.

While it illustrates the basic set-up of a balance sheet, this example is far too simple to be of much use to anyone. To give a better picture of a company's situation, it is necessary to give more detail of just what sorts of assets and liabilities the company has. Just as with income statements, the degree of detail will vary, depending on the type of business and the audience for whom the financial statement is intended, but there are certain general categories that are commonly used.

Current assets are assets that the business expects to use or convert to cash within one year. Examples of current assets might include cash on hand, bank accounts, short-term notes (these three are all usually listed simple as "cash"), supplies used by in running the business (such as a bakery's supply of flour or a roofing contractor's inventory of shingles), or merchandise. Current assets also include ***accounts receivable***, monies owed to the business for sales made on credit.

Long-term assets are things owned by the business that are not intended to be used up or converted to cash in the near term. This category includes real estate, equipment, and other durable assets. The term ***property, plant, and equipment*** is sometimes used for long-term assets; "plant and equipment" refers to assets like machinery, fixtures, and so on that are used in the operation of the business.

A similar distinction between current and long-term is made for liabilities. ***Current liabilities*** are debts that the business must pay within one year. This category includes ***accounts payable***, monies that the business owes to suppliers or others for goods or services obtained on credit. ***Long-term liabilities*** are liabilities that do not meet the 1-year criterion to be considered current. This would include debts such as mortgages or other notes that do not come due within the next year.

Equity is usually broken into the subcategories of ***contributed capital*** (funds provided to the business by its owners' investment) and ***retained earnings*** (profits earned by the business that have not yet been paid out to its owners as dividends).

The following is an example of a balance sheet with a bit more detail than the first:

Hilbert Hotel Company
Balance Sheet as of December 31, 2007

Assets			**Liabilities**		
Current Assets:			Current liabilities:		
Cash	$527,500		Accounts payable	$404,029	
Accounts receivable	$245,904		Note due May 1, 2008	$300,000	
Prepaid expenses	$425,000		Salaries payable	$125,055	
Total current assets		$1,198,404	Total current liabilities		$829,084
Property, plant, and equipment:			Long-term liabilities:		
Buildings	$4,505,075		Mortgage notes:	$3,725,000	
Land	$725,000		Other long-term debt	$525,000	
Other plant and equipment	$2,750,000		Total long-term liabilities		$4,250,000
Total P, P, and E		$7,980,075	Total liabilities		$5,079,084
			Equity		
			Contributed capital	$750,000	
			Retained earnings	$3,349,395	
			Total equity		$4,099,395
Total assets		**$9,178,479**	**Total liabilities and equity**		**$9,178,479**

Balance Sheets and Valuation

It is a common and entirely understandable mistake to assume that the value of the assets shown on the balance sheet correspond to the actual value of those assets. Hilbert Hotels lists $725,000 worth of land on its balance sheet—doesn't this mean that the land owned by the company is worth $725,000? Surprisingly, the answer is no. Most assets are listed on the balance sheet at cost, not at their current market value. The land owned by Hilbert Hotels could be worth far more, or far less, than $725,000. In most circumstances, this amount would indicate that the land in question was originally purchased for $725,000.

With many items, though, even this "cost" is not exactly what we would normally assume it to be. The cost on the balance sheet is reduced as the funds spent to acquire the asset are counted against income as they are depreciated. Depreciation is covered in some detail in Chapter 8.4; we will give a brief explanation of how this works below. Land is not depreciated, but many other assets are.

Suppose that 7 years ago, Hilbert Hotels bought laundry equipment for $500,000, which was expected to be used for 10 years. Rather than count this expense entirely in the year in which the equipment was bought, the company instead might divide the total expense up over the equipment's useful life. It would then count $50,000 of this expense in each of these 10 years, to match the expense of acquiring this equipment up against the time when the equipment is being used. After 7 years, the company has then counted 7($50,000) = $350,000 of the expense of the equipment as expenses against income. This means that the remaining "cost" of the equipment is $500,000 − $350,000 = $150,000. This equipment then would be listed on the balance sheet as an asset worth $150,000. Note that this is not in any way intended to be an estimate of how much the equipment would sell for today; it simply indicates that the company's accounting attributes $150,000 of unused value to that laundry equipment.

Example 12.2.3 ***Suppose that Zarofire Systems purchased computer workstations for $450,000 two years ago. The expected life of the workstations is 5 years and no salvage value is assumed, and so they are taking depreciation for them at a steady rate of $90,000 per year. What will the value of this asset be on the company's balance sheet? How much are the workstations actually worth?***

Zarofire has taken 2($90,000) = $180,000 in depreciation on these workstations. So on the balance sheet they are listed with a value of $450,000 − $180,000 = $270,000. From this information, it is not possible to determine the actual fair market value of these workstations.

Example 12.2.4 ***After 5 years, what will be the value of the workstations on the company's balance sheet? What will the workstations actually be worth?***

After 5 years, Zarofire will have taken 5($90,000) = $450,000 in depreciation. So the entire cost will have been taken as an expense against income. This leaves their balance sheet value at $0. Even though the workstations are listed as having no value on the balance sheet, they may or may not be actually worthless at that time.

Vertical and Horizontal Analysis of Balance Sheets

Vertical and horizontal analyses of balance sheets can be done in much the same way as we did them with income statements. For the vertical analysis, each item on the asset side of the balance sheet is listed as a percent of total assets; each item on the liabilities and equity side is listed as a percent of the total liabilities and equity. Note, though, that since total assets equal total liabilities and equity, we will end up dividing everything by the same number.

For horizontal analysis, the idea is the same as for income statements. The current period's values are listed next to the prior period's. For each category we then compute the change both as an amount and as a percent.

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Formatting becomes an issue, though, since we are already pressed for space listing the assets next to the liabilities and equity. Adding additional columns would be impossible to reasonably do on a normal 8.5 × 11-inch sheet of paper. It is common practice, therefore, to list the assets above the liabilities and equity. This loses the visual "balance" of listing them next to each other, which is unfortunate, but is necessary.

It is also common, again for the sake of space, to forego putting the subcategory and category total amounts in separate columns. An underline is placed below the last amount before a total is taken, though, so category totals can be quickly found by looking for the underlines.

The following examples will illustrate.

Example 12.2.5 ***Complete a vertical analysis balance sheet for Hilbert Hotels as of December 31, 2007.***

To do this, we first realign the spreadsheet as discussed above. Then, we add a column for percent. Since each item will be expressed as a percent of the total assets (or total liabilities and equity), we find these percents by dividing. For example, we find the percent for cash by dividing $527,500/$9,178,479 = 5.75%.

We continue through the balance sheet, dividing each amount by $9,178,479 to find the percent. The result is:

Hilbert Hotel Company
Balance Sheet as of December 31, 2007

Assets	**Amount**	**Percent**
Current assets:		
Cash	$527,500	5.75%
Accounts receivable	$245,904	2.68%
Prepaid expenses	$425,000	4.63%
Total current assets	$1,198,404	13.06%
Property, plant, and equipment:		
Buildings	$4,505,075	49.08%
Land	$725,000	7.90%
Other plant and equipment	$2,750,000	29.96%
Total P, P, and E	$7,980,075	86.94%
Total assets	**$9,178,479**	**100.00%**
Liabilities	**Amount**	**Percent**
Current liabilities:		
Accounts payable	$404,029	4.40%
Note due May 1, 2008	$300,000	3.27%
Salaries payable	$125,055	1.36%
Total current liabilities	$829,084	9.03%
Long-term liabilities:		
Mortgage notes	$3,725,000	40.58%
Other long-term debt	$525,000	5.72%
Total long-term liabilities	$4,250,000	46.30%
Total liabilities	$5,079,084	55.34%
Equity	**Amount**	**Percent**
Contributed capital	$750,000	8.17%
Retained earnings	$3,349,395	36.49%
Total equity	$4,099,395	44.66%
Total liabilities and equity	**$9,178,479**	**100.00%**

Example 12.2.6 ***The balance sheet for Hilbert Hotels as of December 31, 2006, is given below. Use this, together with the December 31, 2007, balance sheet given previously, to complete a horizontal analysis comparing 2007 to 2006.***

Hilbert Hotel Company
Balance Sheet as of December 31, 2006

Assets			Liabilities		
Current assets:			Current liabilities:		
Cash	$835,043		Accounts payable	$713,515	
Accounts receivable	$185,001		Salaries payable	$87,530	
Prepaid expenses	$108,302				
Total current assets		$1,128,346	Total current liabilities		$801,045
Property, plant, and equipment:			Long-term liabilities:		
Buildings	$4,832,075		Mortgage notes	$3,785,000	
Land	$725,000		Other Long-term debt	$895,000	
Other plant and equipment	$3,286,323		Total long-term liabilities		$4,680,000
Total P, P, and E		$8,843,398	Total liabilities		$5,481,045
			Equity		
			Contributed capital	$750,000	
			Retained earnings	$3,740,699	
			Total equity		$4,490,699
Total assets		**$9,971,744**	**Total liabilities and equity**		**$9,971,744**

To complete the horizontal analysis, we take the reformatted 2007 sheet and add a column for the 2006 values. Then, we subtract the 2006 values in each category from the 2007 values to get the amount of change. We divide the amount of change by the 2006 value to get the percent change.

For example, Cash was $527,500 in 2007 and $835,043 in 2006. The change is $527,500 − $835,043 = −$307,543, which we list as ($307,543) in the table. To find the percent, we divide −$307,543/$835,043 to get −36.83%. We do the same for each line.

Note, however, that since the note due May 1 2008, was listed on the 2007 but not 2006 balance sheet it is impossible to calculate a change. If we consider the 2006 amount to be zero, we could say that the amount change was $300,000 − 0 = $300,000. However, to calculate a percent we would have to divide $300,000/$0 which is mathematically impossible. Therefore, we simple place "n/a" for "not applicable" in the change column for this line.

The result looks like this:

Hilbert Hotel Company
Balance Sheet as of December 31, 2007

Assets			Change	
	2007	**2006**	**Amount**	**Percent**
Current assets:				
Cash	$527,500	$835,043	$(307,543)	−36.83%
Accounts receivable	$245,904	$185,001	$60,903	32.92%
Prepaid expenses	$425,000	$108,302	$316,698	292.42%
Total current assets	$1,198,404	$1,128,346	$70,058	288.51%
Property, plant, and equipment:				
Buildings	$4,505,075	$4,832,075	$(327,000)	−6.77%
Land	$725,000	$725,000	$ -	0.00%
Other plant and equipment	$2,750,000	$3,286,323	$(536,323)	−16.32%
Total P, P, and E	$7,980,075	$8,843,398	$(863,323)	−9.76%
Total assets	**$9,178,479**	**$9,971,744**	**$(793,265)**	**−7.96%**

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Liabilities

	2007	2006	Change Amount	Change Percent
Current liabilities:				
Accounts payable	$404,029	$713,515	$(309,486)	−43.37%
Note due May 1, 2008	$300,000	n/a	n/a	n/a
Salaries payable	$125,055	$87,530	$37,525	42.87%
Total current liabilities	$829,084	$801,045	$28,039	3.50%
Long-term liabilities:				
Mortgage notes:	$3,725,000	$3,785,000	$(60,000)	−1.59%
Other long-term debt	$525,000	$895,000	$(370,000)	−41.34%
Total long-term liabilities	$4,250,000	$4,680,000	$(430,000)	−9.19%
Total liabilities	$5,079,084	$5,481,045	$(401,961)	−7.33%

Equity

	2007	2006	Change Amount	Change Percent
Contributed capital	$750,000	$750,000	0	0.00%
Retained earnings	$3,349,395	$3,740,699	$(391,304)	−10.46%
Total equity	$4,099,395	$4,490,699	$(391,304)	−8.71%
Total liabilities and equity	**$9,178,479**	**$9,971,744**	**$(793,265)**	**−7.96%**

Other Financial Statements

You might have wondered when we were discussing depreciation how Hilbert Hotels would reconcile the listing the $50,000 expense over 10 years with the fact that they actually paid $500,000 when they bought the equipment. Income statements and balance sheets are by no means the only financial statements in common use. ***Cash flow statements***, for example, detail the actual flow of dollars into and out of the business, ignoring income and expenses that do not involve the direct actual flow of cash (such as depreciation).

While we have provided a basic introduction to financial statements and the calculations that go into them, there is much more to the story. Students who have a professional or personal interest in a deeper understanding of these subjects are encouraged to pursue more in-depth coursework in accounting.

EXERCISES 12.2

A. Basic Balance Sheets

1. A company's assets are $515,000 and its liabilities are $375,000. What is the owners' equity?

2. A corporation has assets of $15,763,909 and liabilities are $12,444,929. What is the shareholders' equity?

3. A company has liabilities of $62,500 and equity of $44,900. What are the company's total assets?

4. A company has liabilities of $873,944 and shareholder's equity is $540,875. What is the total of the assets listed on the balance sheet?

5. A company has assets of $994,202 and equity of $374,009. Find the total liabilities.

6. Calculate the total liabilities of a corporation that has assets of $4,802,559 and shareholder's equity of $2,022,222.

7. Fill in the missing items on the balance sheet shown below:

Balance Sheet as of December 31, 2007

Assets		**Liabilities**	
Assets	$729,943	Total liabilities	$652,043
		Equity	
		Total equity	(a)
Total assets	**$729,943**	**Total liabilities and equity**	**(b)**

8. Fill in the missing items on the balance sheet shown below:

Balance Sheet as of December 31, 2007

Assets		**Liabilities**	
Assets	(a)	Total liabilities	(b)
		Equity	
		Total equity	$30,000
Total assets	**$91,400**	**Total liabilities and equity**	**(c)**

B. Balance Sheets and Valuation

9. A company paid $600,000 for fixtures, which it is depreciating at a $50,000 annual rate. After the fixtures have been in use for 5 years, what will their value be on the company's balance sheet? What will their actual market value be?

10. A company paid $750,000 for equipment. Since buying this equipment, they have taken $147,935 in depreciation for it. At what value is this equipment listed on the company's balance sheet?

C. Vertical and Horizontal Analysis of Balance Sheets

Exercises 11 to 12 are based on the balance sheet given below.

Assets	**2007**	**2006**
Current Assets:		
Cash	$85,000	$95,000
Accounts receivable	$255,000	$210,000
Total current assets	$340,000	$305,000
Property, plant, and equipment:		
Buildings	$362,000	$375,000
Land	$250,000	$250,000
Other plant and equipment	$1,875,000	$1,569,000
Total P, P, and E	$2,487,000	$2,194,000
Total Assets	**$2,827,000**	**$2,499,000**

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Liabilities	**2007**	**2006**
Current liabilities:		
Accounts payable	$76,500	$124,200
Other	$62,500	$55,000
Total current liabilities	$139,000	$179,200
Total long-term liabilities	$1,675,000	$1,687,500
Total liabilities	$1,814,000	$1,866,700
Equity	**2006**	**2007**
Contributed capital	$250,000	$250,000
Retained earnings	$763,000	$382,300
Total equity	$1,013,000	$632,300
Total liabilities and equity	**$2,827,000**	**$2,499,000**

11. Complete a vertical analysis of this company's 2007 balance sheet.

12. Complete a horizontal analysis of this company's 2007 versus 2006 balance sheets.

D. Grab Bag

13. In 2006 Tokamak Home Generators had assets of $3,950,000 and total liabilities of $3,454,000. In 2007 the company had assets of $4,508,000 and liabilities totaling $3,299,995. Complete a horizontal analysis of the company's balance sheet.

14. In 2007 Tokamak Home Generators had assets of $4,508,000 and liabilities totaling $3,299,995. Complete a vertical analysis of the company's balance sheet.

E. Additional Exercise

15. a. Suppose that MinusCo's balance sheet lists assets of $467,925 and liabilities of $678,900. What is the owners' equity?

    b. If the company sold off all of its assets, would it have enough money to cover all of its liabilities?

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12.3 Financial Ratios

In the previous sections, we introduced the basics of accounting statements, and considered some ways of analyzing their contents, such as vertical and horizontal analysis. In this section we will define and discuss the interpretation of certain ***financial ratios***, measurements derived from items on financial statements that can be used as a tool for summarizing and analyzing certain aspects of a business's financial strength.

In the examples of this section, we will refer to the financial statements given below.

Thomas Hydrometer Sales Corp.
2007 Full Year Income Statement

I. Revenues		
Gross sales	$75,938,275	
Less returns	$1,500,000	
Net sales		$74,438,275
II. Cost of goods sold		
Inventory as of December 31, 2006	$32,535,000	
Net purchases	$38,300,000	
Less inventory as of December 31, 2007	$22,982,651	
Cost of goods sold		$47,852,349
III. Gross profit		$26,585,926
IV. Operating expenses		
Total operating expenses		$20,752,149
V. Net income before insurance and taxes		$5,833,777
VI. Other expenses		
Provision for taxes	$735,926	
Interest	$317,000	
Total interest and taxes		$1,052,926
VII. Net income		**$4,780,851**

Thomas Hydrometer Sales Corp.
Balance Sheet as of December 31, 2007

Assets			**Liabilities**		
Current assets:			Current liabilities:		
Cash	$1,082,925		Accounts payable	$2,175,923	
Accounts receivable	$1,242,075		Other	$143,997	
Inventory	$22,982,651				
Total current assets		$25,307,651	Total current liabilities		$2,391,920
Property, plant, and equipment:			Long-term liabilities:		
Buildings	$2,167,349		Mortgage notes:	$1,250,000	
Land	$100,000		Other Long-term debt	$9,073,055	
Other plant and equipment	$5,675,000		Total long-term liabilities		$10,323,055
Total P, P, and E		$17,925,000	Total liabilities		$12,714,975
			Equity		
			Contributed capital	$800,000	
			Retained earnings	$19,735,025	
			Total equity		$20,535,025
Total assets		**$33,250,000**	**Total liabilities and equity**		**$33,250,000**

Income Ratios

Income ratios are, generally speaking, measurements of the financial results from a business's ongoing activities. They are primarily based on the items listed on a business's income statement. We can further categorize income ratios into several subcategories.

Profitability ratios are, as their name suggests, measurements of the how well the business's sales translate into profits. The two most commonly used profitability ratios are the ***gross profit margin*** and the ***net profit margin***. Both of these ratios are discussed in Chapter 8 in terms of their use in setting prices. To calculate these ratios from the balance sheet:

Definitions 12.3.1 and 12.3.2

Ratio	***Formula***	***What It Tells You***
Gross profit margin	$\frac{\text{Gross profit}}{\text{Net sales}}$ expressed as a percent	The percent of net sales that goes toward profit or costs other than cost of merchandise. A high gross profit margin indicates that the business is able to charge prices for merchandise well above cost.
Net profit margin	$\frac{\text{Net income}}{\text{Net sales}}$ expressed as a percent	The percent of net sales that ends up as actual profit. A high net profit margin indicates that the business earns high profits in relation to sales.

Both gross and net profit margins can vary quite a bit, depending on the type of business. A profit margin that might be considered very high for a grocery store might be considered quite low for a specialty clothing boutique, for example.

Example 12.3.1 ***From the financial statements given at the start of this chapter, determine Thomas Hydrometer Sales' gross profit margin and net profit margin for 2007. Explain what these ratios tell you about the company's financial performance.***

$$\text{Gross profit margin} = \frac{\$26,585,926}{\$74,438,275} = 35.72\%$$

$$\text{Net profit margin} = \frac{\$5,833,777}{\$74,438,275} = 7.84\%$$

The gross profit margin of 35.72% of the company's sales represents income over their merchandise cost. After expenses are taken out, 7.84% of the company's sales translated into net profit.

Effectiveness ratios are intended to measure how effectively the business puts its resources to work to generate profits. The commonly used effectiveness ratios are ***return on assets*** and ***return on equity***. The values of these ratios are calculated by using an annual financial statement.

Definitions 12.3.3 and 12.3.4

Ratio	***Formula***	***What It Tells You***
Return on assets (ROA)	$\frac{\text{Net income}}{\text{Total assets}}$ (usually expressed as a percent)	ROA compares the business's profits against the assets used to generate those profits. A high ROA means that the company is putting its assets to work to generate a comparatively large profit.
Return on equity (ROE)	$\frac{\text{Net income}}{\text{Owners' equity}}$ (usually expressed as a percent)	ROE compares profits against the net assets used to generate those profits. ROE is intended to measure how effectively the business's owners "own" assets are being put to use. A high ROE means that profits are large in comparison to net assets.

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Example 12.3.2 ***Calculate Thomas Hydrometer Sales' return on assets and return on equity for 2007. Explain what these ratios tell you about the company's performance.***

$$ROA = \frac{\$4,780,851}{\$33,250,000} = 14.38\%$$

$$ROE = \frac{\$4,780,851}{\$20,535,025} = 23.28\%$$

The company's profits for 2007 represented a 14.38% return on the balance sheet value of its total assets. As a return on equity, that is, the original capital and retained earnings of the investors in the company, the rate of return was 23.28%.

Return on assets is based on the company's assets, without consideration of how much debt the company may have taken on to acquire those assets. By using the owners' equity instead of assets, return on equity takes into account the company's debt. Both numbers can be misleading, though, if the company's assets have a market value that differs substantially from the balance sheet value—for example, if the company has a large amount of land bought many years ago that has appreciated in value over the years but is still listed at cost on the balance sheet.

Like profit margins, a reasonable value for these ratios can vary quite a bit, depending on the type of business. A software development company might be expected to have higher returns on assets and equity than a manufacturer, since manufacturing requires larger investments in factories and equipment.

Conversion ratios are measurements of how quickly inventory is converted in to sales, or receivable are converted into receipts.

Definitions 12.3.5 and 12.3.6

Ratio	***Formula***	***What It Tells You***
Inventory turnover	$\frac{\text{Cost of goods sold}}{\text{Average inventory}}$	How many times the company turned over its inventory in the period reported on the financial statements. A high inventory turnover ratio indicates that the business is moving its merchandise quickly, on average.
Average collection days	$\frac{\text{Accounts receivable}}{\text{Net sales per day}}$	How many days' worth of sales the receivables represent. A high value of this ratio indicates that the business is taking a long time to collect payment for purchases.

Each of these ratios requires some preliminary work. Both require a quantity that is not usually listed on the financial statements.

To calculate the ***average inventory***, required for the inventory turnover ratio, we take the average of the starting and ending inventories.

$$\text{Average inventory} = \frac{\text{Starting inventory} + \text{Ending inventory}}{2}$$

To find the ***net sales per day***, we take the net sales from the income statement and divide by the number of days in the period covered by the financial statement. If the income statement is annual, we might divide by 365 days per year (or 366 days if it is a leap year); however, it is common practice to use bankers' rule and divide by 360. If the financial statement is quarterly, we divide by an assumed 90 days per quarter.

$$\text{Net sales per day} = \frac{\text{Annual net sales}}{360} \text{ or } \frac{\text{Quarterly net sales}}{90}$$

Example 12.3.3 ***Calculate the inventory turnover at cost and the average collection days for Thomas Hydrometer Sales. Explain what these ratios tell you.***

To calculate the inventory turnover, we first calculate the average inventory:

$$\text{Average inventory} = \frac{\$32{,}535{,}000 + \$22{,}982{,}651}{2} = \$27{,}758{,}826$$

The inventory turnover is then:

$$\text{Inventory turnover} = \frac{\$74{,}438{,}275}{\$27{,}758{,}826} = 2.68 \text{ times}$$

The company sold 2.68 times as much merchandise in the past year as its average amount of merchandise in inventory.

To find the average collection days, we first calculate the net sales per day:

$$\text{Net sales per day} = \frac{\$74{,}438{,}275}{360} = \$206{,}773$$

Then, to calculate the average collection days:

$$\text{Average collection days} = \frac{\$1{,}242{,}075}{\$206{,}773} = 6 \text{ days}$$

It takes on average 6 days from the date of sale to collect payment for the sale.

As with our other ratios, what we would consider good values for these ratios can depend quite a bit on the type of business. A retail fruit market selling perishable fruits and vegetables would hopefully be turning its inventory over very quickly, and hence have a much higher turnover ratio, than a jeweler who might take more time to sell his stock. Likewise, the average collection days can vary, depending on how much a given type of business would be expected to be selling on credit.

Balance Sheet Ratios

Balance sheet ratios are measurements of a business's financial strength, based on values listed on the company's balance sheet. Three commonly used ratios are the ***current ratio*** and ***quick ratio*** (or ***acid test***), both measurements of the company's ability to make good on its near term financial obligations, and the ***liabilities-to-equity ratio***, a measurement of how much debt the company has in relation to its overall financial position.

These three ratios are defined as follows:

Definitions 12.3.7, 12.3.8, and 12.3.9

Ratio	*Formula*	*What It Tells You*
Current ratio	$\dfrac{\text{Current assets}}{\text{Current liabilities}}$	A comparison of current assets to current liabilities. A ratio greater than 1 means the business has more current assets than liabilities. A ratio less than 1 raises questions about the company's ability to cover near-term obligations.
Quick ratio	$\dfrac{\text{Current assets} - \text{Inventory} - \text{Prepaid expenses}}{\text{Current liabilities}}$	A comparison of assets that can be put to immediate use against near-term liabilities. The interpretation is similar to current ratio, except that current assets less easily converted to cash don't count.
Liabilities-to-equity ratio	$\dfrac{\text{Total liabilities}}{\text{Owners' equity}}$	A measurement of how deeply the business is in debt. As a rule of thumb, a ratio greater than 1 indicates a high debt load.

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Recall that current assets are assets that can be readily converted to cash (within 1 year) and current liabilities are obligations which must be met near term (again, within one year). The current ratio seeks to evaluate how well equipped the business is to meet its near-term obligations. If the current ratio is less than 1, this means that current liabilities exceed current assets, which would raise doubts about the company's ability to meet its near-term financial obligations. If the current ratio is more than 1, current assets exceed current liabilities; the higher the current ratio, the greater the confidence that the company will be able to meet its near-term financial obligations.

The quick ratio seeks to accomplish the same thing as the current ratio. The quick ratio, however, recognizes that some current assets may not be usable to pay off the company's obligations. The quick ratio is a stricter measure of the company's ability to meet those obligations. A company with a quick ratio below 1 may still not run into any difficulty, since it would be expected that in the normal course of business inventory is being turned into sales, but a quick ratio over 1 inspires far more confidence.

The debt-to-equity ratio measures just how far in debt the business is relative to its overall resources. The larger this ratio is, though, the larger equity is in proportion to liabilities, and so the more "breathing room" the company has in managing its debt. Since Equity = Assets − Liabilities, as long as the liabilities are less than the assets this ratio will be a positive number.

Example 12.3.4 ***Calculate the current, quick, and debt-to-equity ratios for Thomas Hydrometer Sales and explain what they tell you about the company's financial health.***

$$\text{Current ratio} = \frac{\$25{,}307{,}651}{\$2{,}391{,}920} = 10.58$$

The current ratio indicates that the company has 10.48 times as much in current assets as current liabilities. This suggests that it is in a good position to meet its near-term financial obligations.

$$\text{Quick ratio} = \frac{\$25{,}307{,}651 - \$22{,}982{,}651}{\$2{,}391{,}920} = \frac{\$2{,}325{,}000}{\$2{,}391{,}920} = 0.97$$

The quick ratio is less than 1, indicating that the company does not have enough assets immediately available to cover its near-term liabilities. We can see that most of the company's current assets are in inventory, explaining the difference between current and quick ratios. While a quick ratio below 1 is potential for concern, the company probably will not run into problems from this since it is not far below 1, the company can be expected to continue selling its inventory, and the current liabilities are unlikely to all come due at once anyway.

$$\text{Liabilities-to-equity ratio} = \frac{\$12{,}714{,}975}{\$20{,}535{,}025} = 0.62$$

The company's total liabilities amount to 62 cents for each \$1 of equity. This suggests that the company's debt load is not excessive.

Valuation Ratios

An investor evaluating a business as a potential investment needs to be able to determine whether the price for the company's stock is reasonable on the basis of its financial position. There is no absolute way of determining the fair price for a business, but investors can employ a number of different measurements to help in determining a reasonable price.

In looking at the price per share of a company's stock, it is helpful to know the company's earnings, assets, liabilities, sales, and so forth on a ***per share*** basis. Knowing how much the company earned as a whole does not tell you much about the value of a single share of stock, unless you also take into account how many shares those profits are divided among.

Any financial statement value can be converted to a per share value by dividing by the overall value for the entire company by the number of shares. Some of the more commonly used per share values are:

Definitions 12.3.10 and 12.3.11

Ratio	***Formula***	***What It Tells You***
Earnings per share (EPS)	$\frac{\text{Total net income}}{\text{Number of shares}}$	How much did the business earn per share of stock?
Book value per share	$\frac{\text{Total owners' equity}}{\text{Number of shares}}$	How much equity is attributable to each share of stock?

Example 12.3.5 ***Thomas Hydrometer Sales Corp is a public company with 3,000,000 total shares outstanding. Calculate the company's 2007 earnings per share and book value per share as of December 3, 2007.***

$$\text{Earnings per share} = \frac{\$4,780,851}{3,000,000} = \$1.59 \text{ per share}$$

$$\text{Book value per share} = \frac{\$20,535,025}{3,000,000} = \$6.85 \text{ per share}$$

To compare the price of a company's stock to its financial position, several ***per-share ratios*** are in common use. Two of the most commonly used per share ratios are:

Definitions 12.3.12 and 12.3.13

Ratio	***Formula***	***What It Tells You***
Price-to-earnings (PE) ratio	$\frac{\text{Price per share}}{\text{Annual earnings per share}}$ [If earnings are positive; the ratio is undefined if earnings are negative (i.e., the business lost money)]	How much someone buying the company stock is paying for each dollar of annual earnings. A high PE ratio indicates that the stock price is expensive relative to current earnings.
Price-to-book ratio	$\frac{\text{Price per share}}{\text{Annual earnings per share}}$	How much someone buying the company's stock is paying for each dollar's worth of equity. A high price-to-book ratio indicates that the stock is expensive relative to the company's owners' equity.

Example 12.3.6 ***As of the close of trading last Friday, Thomas Hydrometer Sales' stock was selling for $37.45 per share. Calculate the PE and price-to-book ratios. Explain what these ratios tell you about the company.***

$$PE = \frac{\$37.45}{\$1.59} = 23.55$$

$$\text{Price to book} = \frac{\$37.45}{\$6.85} = 5.47$$

An investor buying stock in this company would be paying $23.55 for each $1 of earnings; or, in other words, the market value of the company is 23.55 times its 2007 earnings. Similarly, an investor would pay $5.47 for each dollar of equity in the company, or the total market value of the company is 5.47 times its owners' equity.

The price-to-earnings and price-to-book ratios do not have to be calculated on a per share basis. They can also be calculated by using the total market price for the business divided by the total earnings.

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Example 12.3.7 ***Magnum Opus Corporation is considering buying a regional chain of muffler shops for $300,000,000. Last year, this chain earned $20,000,000, and the owners' equity is presently $85,000,000. Calculate the price-to-earnings and price-to-book ratios at this proposed purchase price.***

$$\text{Price to earnings} = \frac{\$300{,}000{,}000}{\$20{,}000{,}000} = 15$$

$$\text{Price to book} = \frac{\$300{,}000{,}000}{\$85{,}000{,}000} = 3.53$$

A company with a high price-to-earnings ratio is commanding a higher selling price for each dollar of its earnings than a company with a lower PE ratio. Likewise, a company with a high price-to-book ratio is more expensive relative to its owners' equity than a company with a lower price-to-book ratio.

When deciding a fair price for a company, investors must take into account not just the present financial position of a company but also the company's future growth prospects. Suppose that you are considering investing in two companies, both of which earned $1.50 per share last year. Both are solid and well-managed companies. One company is an established grocery store chain, while the other is a rapidly growing biotechnology company. In all likelihood, you would expect to pay more for the biotech company because, even though both companies have the same earnings today, the biotech company's future earnings seem likely to grow more quickly than the grocery chain's would. In determining the appropriate PE ratio for a business, investors take into account factors such as future earnings growth potential and risks to the business. While there is no absolute way of determining what the PE ratio should be, investors often look to the PE ratios of similarly situated businesses as a guide.

Example 12.3.8 ***Magnum Opus Corp. is considering buying out Southeastern Neighborhood Apothecaries, a chain of retail pharmacies. Magnum Opus' financial analysts have looked at the market prices of other similar pharmacies and found that similar businesses sell at a PE of 18 on average. If Southwestern Neighborhood Apothecaries is earning $6.5 million this year, what would be a reasonable price for Magnum Opus to expect to pay?***

$$PE = \frac{\text{Price}}{\text{Earnings}}$$

$$18 = \frac{\text{Price}}{\$6.5 \text{ million}}$$

Multiplying both sides by $6.5 million gives:

$$\text{Price} = \$117 \text{ million}$$

Of course this number is not absolute. Magnum Opus may be willing to pay more, or less, depending on its business goals and objectives, and on the specific details of this particular pharmacy chain that might make it command a higher or lower price. Magnum Opus may also determine the price it is willing to pay on considerations other than earnings, such as the value of the assets the company owns or the benefits that acquiring this business may offer to the company's overall business strategy.

Notice that in this example we ended up multiplying the company's profits by the presumed PE ratio. For this reason, the PE ratio is sometimes referred to as the ***earnings multiple***.

The price-to-book ratio can be helpful in some situations, but can also be a misleading measurement of the company's value. Recall that the owners' equity is meant to be a sort of net worth of the company; the value of the business's assets less its debts. A company that has a book value per share of $17.53 might then be thought to have net assets worth $17.53 per share. The difficulty with using this value, though, is the fact that, as we discussed in Section 12.2, assets are listed on the balance sheet at cost or at depreciated cost, values that do not necessarily match actual market value. Using the price-to-book ratio effectively as a valuation tool often requires a bit of financial detective work, requiring an analysis of how closely the balance sheet values of a business's assets match up to the real market value of those assets.

EXERCISES 12.3

Many of the exercises in this section refer to the 2007 financial statements of the Tastee Lard Donut Shoppe, shown here:

Tastee Lard Donut Shoppe 2007
Full Year Income Statement

I. Revenues		
Gross sales	$425,913	
Less returns	$5,000	
Net sales		$420,913
II. Cost of goods sold		
Inventory as of December 31, 2008	$24,700	
Net purchases	$89,650	
Less inventory as of December 31, 2007	$28,315	
Cost of goods sold		$86,035
III. Gross profit		$334,878
IV. Operating expenses		
Total operating expenses		$282,308
V. Net income before insurance and taxes		$52,570
VI. Other expenses		
Provision for taxes	$8,000	
Interest	$2,500	
Total interest and taxes		$10,500
VII. Net income		**$42,070**

Tastee Lard Donut Shoppe
Balance Sheet as of December 31, 2007

Assets			**Liabilities**		
Current assets:			Current liabilities:		
Cash	$72,700		Accounts payable	$22,650	
Accounts receivable	$8,500		Other	$6,700	
Inventory	$28,315				
Total current assets		$109,515	Total current liabilities		$29,350
Property, plant, and equipment:			Long-term liabilities:		
Buildings	$125,000		Mortgage notes:	$129,000	
Land	$25,000		Other long-term debt	$12,500	
Other plant and equipment	$80,000		Total long term-liabilities		$141,500
Total P, P, and E		$230,000	Total liabilities		$170,850
			Equity		
			Contributed capital	$120,000	
			Retained earnings	$48,665	
			Total equity		$168,665
Total assets		**$339,515**	**Total liabilities and equity**		**$339,515**

A. Income Ratios

1. Last year, Stan's Taqueria Supply Company had net income of $48,925 on net sales of $328,535. The cost of goods sold was $142,675.
   a. Calculate the company's net profit margin.
   b. Calculate the company's gross profit margin. (*Hint:* First calculate the gross profit.)
   c. If Stan's main competitor had roughly the same gross profit margin but a higher net profit margin, what can you say about Stan's overhead expenses?

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2. Calculate the gross and net profit margins for the Tastee Lard Donut Shoppe.

3. If a company has a net profit margin of 5.25% on sales totaling $425,575, what is the company's net income?

4. Dave's Magic Shop has net assets of $316,000. The company earned $26,500 last year. The company's liabilities total $207,000. Calculate (a) the return on assets and (b) the return on equity.

5. Calculate the return on assets and the return on equity for the Tastee Lard Donut Shoppe.

6. a. HighDebtCorp Inc. has assets totaling $25,000,000. The company earned $5,400,000 last year. The company's liabilities total $23,000,000. Calculate the company's ROA and ROE.
   b. LowDebtCorp Inc. also has total assets of $25,000,000. The company also earned $5,400,000 last year. The company's liabilities total $5,000,000. Calculate this company's ROA and ROE.
   c. Compare your results in (a) and (b). What does this suggest about companies whose ROA and ROE are close together? Far apart?

7. Suppose your company's return on assets was calculated to be 14%. The average ROA in your industry 19%. Does this suggest that your company is doing a good job of putting assets to use, or that your company could be doing better? Explain your reasoning.

8. Calculate the inventory turnover ratio for the Tastee Lard Donut Shoppe.

9. Constant Philatelic Sales had an inventory valued (at cost) at $418,500 at the start of last year and $396,000 at the end of the year. During the year, the company sold merchandise that had a total cost of $296,500.
   a. Calculate the inventory turnover ratio.
   b. Cohen Philatelic Sales, a close competitor, had an inventory turnover ratio of 1.15. Which company is selling its merchandise more quickly?

10. a. Suppose your company's accounts receivable total $47,530 and your annual sales are $235,000. Calculate your company's average collection days.
    b. Suppose that, for the type of business you are in, the typical average collection days is 115 days. Are you doing a better or worse than average job of collecting on your receivables.

11. Calculate the average collection days for the Tastee Lard Donut Shoppe. Why do you suppose the value is so low?

B. Balance Sheet Ratios

12. Calculate the current ratio and the quick ratio for the Tastee Lard Donut Shoppe.

13. The Highly Liquid Bottled Water Company has current assets of $485,500, of which $127,500 is inventory, $3,950 is prepaid expenses, and the rest is cash and receivables. Current liabilities are $172,300. Calculate the company's current and quick ratios.

14. The Highly Illiquid Ice Company has current assets of $755,929. Of this, $433,043 is in cash and receivables, the remainder is in inventory. Current liabilities total $748,889. Calculate the company's current and quick ratios.

15. Suppose that your business makes a large amount of sales on credit to two large customers. Company A has a quick ratio of 4.35, while Company B has a quick ratio of 0.72. Which customer are you more likely to have trouble collecting from?

16. If a company's current ratio is much larger than its quick ratio, and the company has no prepaid expenses on its books, what can you say about its inventory?

17. Calculate the liabilities-to-equity ratio for the Tastee Lard Donut Shoppe.

18. a. HighDebtCorp Inc. has net assets of $25,000,000. The company's liabilities total $23,000,000. Calculate the company's equity and the company's liabilities-to-equity ratio.

    b. LowDebtCorp Inc. also has net assets of $25,000,000. The company's liabilities total $5,000,000. Calculate the company's liabilities-to-equity ratio.

C. Valuation Ratios

19. Suppose that ExampleCo Inc. earned $575,923 last year and has shareholders' equity of $3,545,926. The company has 425,000 shares outstanding. Calculate the company's EPS and book value per share.

20. Suppose that the Tastee Lard Donut Shoppe is a corporation with 750 shares outstanding. Calculate its 2007 earnings per share and book value per share.

21. Texas Hypothetical Supply Corp. lost $1,953,026 last year. The company's equity is $12,753,000. The company has 810,000 shares outstanding. Calculate the EPS and book value per share.

22. If a company earns $415,000 and is sold for $7,535,025, what is the price-to-earnings ratio?

23. Suppose that the current market price of each share of ExampleCo Inc. (from Exercise 19) is $17.55. Calculate this company's PE and price-to-book ratios.

24. A company's earnings per share is $3.79 and the stock price is $82.50. What is the PE ratio?

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25. Suppose that the stock price for Texas Hypothetical Supply Corp. (from Exercise 21) is $58.55. Calculate the PE and price-to-book ratios.

26. A company's book value is $48.95 per share. The stock price is $137.22. What is the price-to-book ratio?

27. The owner of the Tastee Lard Donut Shoppe believes that the business could be sold for $870,000. If this is so, what is the company's PE ratio?

28. If a company earns $4.74 per share and its PE ratio is 22.5, what is its stock price?

29. Magnum Opus Corp. is considering buying out a chain of video rental stores, which earned $3,725,543 last year. The company's management believes that a PE multiple of 16.5 is appropriate for this business. On this basis, what price would management offer for the chain?

D. Grab Bag

30. For each of the ratios listed below, state whether a business is generally better off if its value of the ratio is high or low.
    a. Return on assets
    b. Inventory turnover ratio
    c. Current ratio
    d. Liabilities-to-equity ratio

31. For each of the ratios listed below, state whether a business is generally better off if its value of the ratio is high or low.
    a. Net profit margin
    b. Return on equity
    c. Quick ratio
    d. Average collection days

32. a. Suppose that a company has annual earnings per share of $2.56 and the stock sells with an 18.5 PE ratio. What is the stock price per share?
    b. Suppose that a company has annual earnings per share of $2.56 and the stock sells with a 22.5 PE ratio. What is the stock price per share?

33. Five years ago you bought stock in a company. The PE ratio was 15. Today the PE ratio is 25. Have you made a profit?

34. If you are trying to determine whether a company is carrying too heavy a debt load, which of the ratios defined in this section would be the most useful to know?

35. If you are trying to determine whether a company is likely to have trouble paying its bills in the near term, which of the ratios defined in this section would be the most useful to know?

36. If you are trying to determine how effectively a company's management is putting its resources to use, which of the ratios defined in this section would be the most useful to know?

37. If you are trying to determine whether or not a company's stock price is high relative to the value of its assets, which of the ratios defined in this section would be the most useful to know?

38. If you are trying to determine how effective a company is at collecting on its receivables, which of the ratios defined in this section would be the most useful to know?

39. If you are trying to determine how quickly a company is moving its merchandise, which of the ratios defined in this section would be the most useful to know?

40. Shender Chemical Corp has assets of $3,875,000 and liabilities of $2,935,025. The company's current assets are $2,245,000, of which $1,000,000 is in receivables and $500,000 is in inventory. Current liabilities are $899,975. Last year, the company's inventory stood at $650,000.

    The company earned a gross profit of $1,525,075 last year and had operating expenses of $1,115,015, on sales of $6,350,000. Calculate the company's

    a. gross profit margin
    b. net profit margin
    c. current ratio
    d. quick ratio
    e. liabilities-to-equity ratio
    f. inventory turnover ratio
    g. average collection days
    h. ROA
    i. ROE

E. Additional Exercises

41. Suppose a company has current assets of $750,000 and a current ratio of 4.25. Inventory and prepaid expenses total $350,000. What is the quick ratio?

42. Suppose a company has an ROA of 13.75% and a ROE of 18.53%. The company's equity is $583,901. What are the company's total liabilities?

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CHAPTER 12
SUMMARY

Topic	Key Ideas, Formulas, and Techniques	Examples
Basic Income Statements, p. 487	• Income statements provide information on a business's financial performance and profits. • A basic statement lists revenues, expenses, and the resulting profit (or loss).	Cattarauqua Ginseng Enterprises had sales of $176,530 last year. The cost of the goods it sold was $62,500 and its expenses totaled $78,595. What was the company's net income for the year? (Example 12.1.1)
More Detailed Income Statements, p. 489	• Depending on the nature of the business and intended audience of the income statement, sales, expenses, and other items may be broken down into subcategories.	At the start of last year, Cattarauqua Ginseng Enterprises had inventory which cost $37,923. Over the course of the year it made purchases of $48,923 before taking into account $1,850 that it received as a refund for a returned shipment and $1,175 of savings from cash discounts for early payment. At the end of the year, they had inventory whose total cost was $21,321. Calculate the company's cost of goods sold for the year. (Example 12.1.4)
Vertical Analysis of Income Statements, p. 491	• Vertical analysis of an income statement requires expressing each of the items as a percent of the overall net sales. • Vertical analysis is most useful for determining what proportion each item takes as a percent of the total.	Complete a vertical analysis for a given income statement. (Example 12.1.7)
Horizontal Analysis of Income Statements, p. 492	• Horizontal analysis requires listing the current period amounts next to the amounts for a prior comparison period. • The difference between the two periods is calculated both as an amount of increase (decrease) and percent increase (decrease).	Complete a horizontal analysis for a given income statement. (Example 12.1.7)
Basic Balance Sheets, p. 498	• The balance sheet lists the business's assets and liabilities as of a particular point in time. • Assets = Liabilities + Equity.	A corporation reports that its assets total $20,500,000 and its stockholders' equity is $4,000,000. What are the corporation's total liabilities? (Example 12.2.2)
Depreciation and Balance Sheet Valuation, p. 501	• Assets are not listed on the balance sheet at their actual fair market value. • Assets are listed at depreciated cost.	A company purchased computer workstations for $450,000 two years ago. It is taking $90,000 depreciation for them each year. What will the value of this asset be on the balance sheet? (Example 12.2.3)
Vertical and Horizontal Analysis of Balance Sheets, p. 501	• Vertical analysis of balance sheets is similar to income statements, except that items are expressed as a percent of total assets. • Horizontal analysis is done in the same way as for income statements.	Complete vertical and horizontal analyses of a given balance sheet. (Examples 12.2.5 and 12.2.6)

(Continued)

Topic	Key Ideas, Formulas, and Techniques	Examples
Income Ratios, p. 508	• Gross profit margin $= \frac{\text{Gross profit}}{\text{Net sales}}$ • Net profit margin $= \frac{\text{Net income}}{\text{Net sales}}$	Calculate the gross and net profit margins based on given financial statements and interpret the results. (Example 12.3.1)
Effectiveness Ratios, p. 509	• ROA $= \frac{\text{Net income}}{\text{Total assets}}$ • ROE $= \frac{\text{Net income}}{\text{Owners' equity}}$	Calculate the ROA and ROE based on given financial statement and interpret the results. (Example 12.3.2)
Conversion Ratios, p. 510	• Inventory turnover $= \frac{\text{Cost of goods sold}}{\text{Average inventory}}$ • Average collection days $= \frac{\text{Accounts receivable}}{\text{Net sales per day}}$	Calculate the inventory turnover and average collection days ratios based on given financial statements and interpret the results. (Example 12.3.3)
Balance Sheet Ratios, p. 511	• Current ratio $= \frac{\text{Current assets}}{\text{Current liabilities}}$ • Quick ratio $= \frac{\text{Current assets} - \text{inventory} - \text{prepaid expenses}}{\text{Current liabilities}}$ • Liabilities-to-equity ratio $= \frac{\text{Total liabilities}}{\text{Owners' equity}}$	Calculate the balance sheet ratios based on given financial statements and interpret the results. (Example 12.3.4)
Valuation Ratios, p. 512	• PE $= \frac{\text{Price per share}}{\text{Annual earnings per share}}$ • Price-to-book $= \frac{\text{Price per share}}{\text{Annual earnings per share}}$	Calculate the PE and price-to-book ratios for a company given its financial statements, stock price, and number of shares. (Example 12.3.6)

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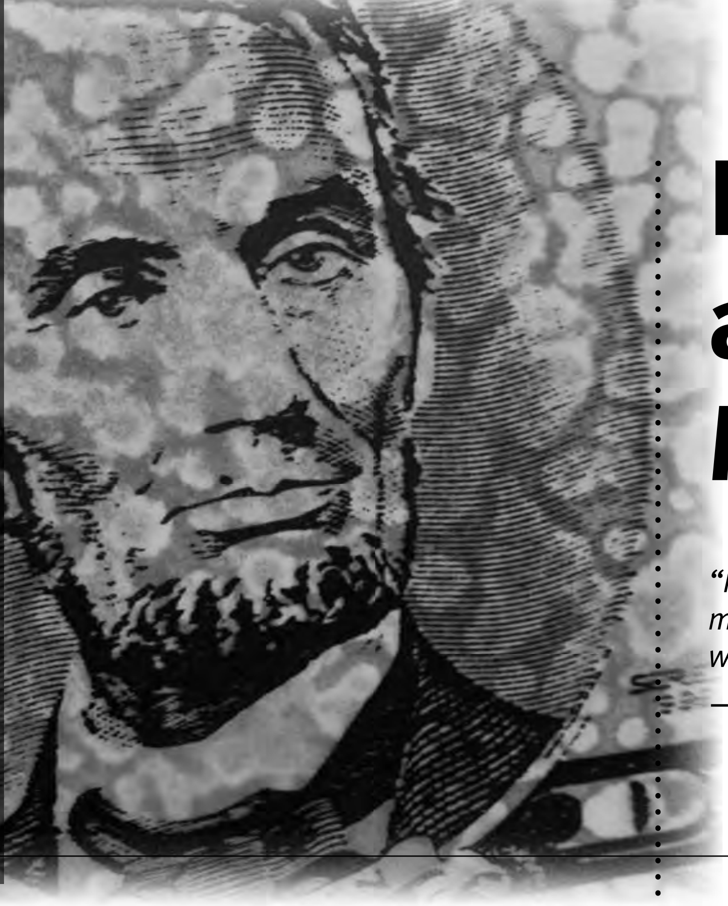
CHAPTER 13



Insurance and Risk Management

"It's a kind of spiritual snobbery that makes people think they can be happy without money."

—Albert Camus

Learning Objectives

LO 1 Relate the law of large numbers to insurance rating and underwriting.

LO 2 Be familiar with the various types of insurance coverage.

LO 3 Calculate insurance premiums from a rate table.

LO 4 Apply deductibles, coinsurance, coverage limits, and out of pocket maximums to determine the amount to be paid on a claim.

LO 5 Understand the basic principles behind rate calculations, and apply these to calculate premiums for simple situations.

LO 6 Calculate the employees' costs for insurance benefits based on the employer's contribution formula.

Chapter Outline

13.1 Property, Casualty, and Liability Insurance

Risk is a fact of life. Every decision made and every action taken carries with it the risk that things will not work out as we want or expect. A big part of success in business (or in life for that matter) is the ability to wisely assess the risks in different courses of action, weigh them against their potential rewards, and manage these risks effectively.

Sometimes managing risks is mainly a matter of making good choices. A restaurant manager might be able to save expenses by cutting corners on food safety practices, but this would come at the risk making customers sick or facing a very bad health inspection, either of which could be disastrous for the business. Risking financial disaster to save a few dollars of expense would be a bad business decision[1] (not to mention the bad ethics of putting your customers' health at risk). The restaurant manager has the power to control this risk, though; by choosing to follow appropriate health and safety practices, the risk of a disastrous outcome can be minimized.

It is of course just good business management to take care to avoid unnecessary risks. It is impossible, though, to avoid risk altogether. It is always possible that despite every prudent precaution, something unfortunate will happen due to oversight, someone else's actions, or just plain bad luck. It is necessary to be able to find ways to protect our businesses (and ourselves!) from potential bad consequences of those risks which we cannot avoid. ***Insurance*** is a financial tool for accomplishing this. Of course insurance cannot prevent bad things from happening, but it can provide money to help pay the financial consequences when they do.

Insurance can't stop bad things from happening, but it does provide the financial means to recover from them. © U.S. Air Force photo by Master Sgt. Michael A. Kaplan/DIL

As you are no doubt well aware, there are many different types of insurance, each designed to deal with a specific type of risk. In this chapter, we will take a look at several of the most common types, and their uses for both business and personal finance. We will begin by looking at various types of ***property, casualty, and liability insurance***. Insurance that falls into this broad category includes business and professional liability, disability, automobile, homeowner's, flood, earthquake and malpractice insurance, as well as many others. In fact, it is probably easier to define this category by what it does not include rather than what it does. Life and health insurance are not included in this category, though life and health insurance products do have much in common with these other types of insurance.

Insurance contracts can be purchased for all sorts of risks, ranging from the everyday risks we all face (such as the risk of a fire or storm damage or car accident) to incredibly specialized policies (such as the risk that a college athlete with a promising future as a professional might suffer a career-ending injury while still playing at the collegiate level). A treatment of the full spectrum of insurance products, and the issues surrounding those products, lies far beyond the scope of this book. In this chapter, though, we will consider the basic principles, opportunities, and challenges that businesses and individuals face in the insurance market.

Basic Terminology

An ***insurance policy*** is a financial contract in which an ***insurer*** promises to pay specific financial ***benefits*** to an individual or business, called the ***policyholder***, if and when certain events (called ***covered risks*** or ***covered events***) occur. The policyholder pays the insurer a ***premium*** in exchange for this coverage. If a covered event occurs, the policyholder files a ***claim*** with the insurer, requesting the financial benefits due under the terms of the policy. The insurance company may send a ***claims adjuster*** to verify the circumstances surrounding the claim, and to determine the amount of the benefit that is to be paid.

Insurers may be organized as ***mutual companies*** or as ***stock companies***. A mutual insurance company is owned by its policyholders, whereas a stock company is a corporation owned by investors. A mutual company is operated to provide insurance protection to its policyholder/owners, but not to earn a profit for any investors. If a mutual company

[1]Just because something is a bad decision, though, doesn't mean that no one will ever make it. It is not hard to find examples of businesses that have failed and/or caused serious problems for their customers, employees, and community by taking shortsighted risks.

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earns a profit, it may pay some of that profit to its policyholders as a ***dividend***. Many mutual companies seldom do this, though, instead retaining their profits to maintain financial strength or to allow for lower premiums. A stock company, on the other hand, is a for-profit business, operated to earn a profit for its stockholders by providing coverage to its policyholders. State Farm and Nationwide Mutual are examples of large, well-known mutual companies; Allstate and GEICO are examples of well-known stock companies.

In theory, a mutual company might be expected to be able to offer lower rates since a mutual insurer does not need to try to make a profit for its stockholders. However, a well-managed stock company may be able to offer equal or better coverage at an equal or lower price. In most cases it makes little if any difference to the policyholder whether the company is organized as a mutual or stock company, so long as the premium is competitive and the company is financially strong. In recent years, many mutual companies, such as MetLife and John Hancock, have decided to convert themselves into stock companies through a process known as ***demutualization***.

There are also some other types of insurers. A few sizable insurers are organized as ***fraternal companies***, in most respects the same as mutual companies except that they offer coverage only to members of a specific organization, such as a trade union or religious group. Thrivent Financial for Lutherans is an example of a very large fraternal insurer. ***Cooperative insurers*** are similar to fraternals and may be set up by a group of people sharing a common need or interest, though they are usually quite small in size.

Insurance policies are normally sold by an ***insurance agent*** or ***broker***. Agents/brokers generally must be licensed by the state in which they work. There are some distinctions between an agent and a broker, but in practice the two terms are often loosely used interchangeably. Some agents represent only one company, while others offer products from many different companies. An insurance agent may be an employee of an insurer, or may operate as an independent business person. Some agents sell only certain types of insurance, specializing in life or business insurance, for example, while others offer a full range of insurance products. Insurance may also be offered for sale through banks or other financial institutions. However, in the United States, banks are generally not allowed to be in the insurance business themselves; many banks market insurance products to their customers, but these products are actually then sold by licensed agents.

To ensure that insurance companies and agents fairly represent the products they are offering for sale, and that insurance companies have the financial resources necessary to meet the obligations of the policies they sell, the insurance industry is subject to a considerable amount of governmental regulation. It may be surprising, though, that virtually none of this regulation is done by the federal government. Insurance is regulated by the states, with each state having its own ***insurance department*** that regulates the insurance business within that state. Insurance regulations are reasonably consistent from state to state, but still there are differences. Most large insurers are licensed to do business in every state, but some smaller companies may opt to do business in only a few rather than face the burden of complying with the requirements of 50 different insurance departments. Also, some of the specific details of the policies offered may vary from state to state to address specific regulatory requirements.

There are an enormous number of different types of insurance policies. Some of the most commonly seen types are listed in the table below:

Policy Type	***What It Covers***
Automobile collision	Damage that you cause to your own vehicle. Not required to operate a vehicle, but may be required by bank that finances the auto loan.
Automobile comprehensive	Covers theft of a vehicle or its contents. Not required to operate a vehicle, but may be required by the bank that finances an auto loan.
Automobile liability	Injuries to other people or damages to their property that you cause while operating an automobile. A minimum level of liability insurance is required to operate a vehicle in most states and situations.
Business disruption	Provides benefits to a business in the event that it is unable to function as a result of natural disasters or other causes.

Policy Type	***What It Covers***
Disability	Provides financial benefits if you become disabled. Some provide benefits only if you are unable to work at all, while others provide benefits if you are unable to work at your usual profession. Benefits are often set as a percent of predisability income.
Earthquake	Covers damages caused by earthquakes, which is not covered by most homeowner's insurance policies. May be purchased as a separate policy, often connected to a homeowner's policy.
Flood	Covers damage due to floods, which is not covered by homeowner's insurance policies.
Health	Also called ***medical*** insurance, health insurance covers the cost of medical treatment. Health insurance policies may be comprehensive, covering most types of medical costs, or may only cover specific types of medical treatments. ***Dread disease*** policies provide coverage only for a specific type of diagnosis.
Homeowner's	Covers damage to your home and property due to fire, weather, theft, and many other causes. Also usually includes coverage for liability for injuries visitors suffer on your property. Homeowner's policies do not cover damage from flooding or earthquakes and may have other exclusions as well, such as limits on the amount of coverage for jewelry or computers. Policyholders may be able to purchase ***riders***, additional coverage for risks that would otherwise be excluded.
Liability	Covers financial losses due to claims and lawsuits against an individual or business for injuries or other damages for which it may be held financially responsible. Some policies may cover only certain types of liabilities, while ***general liability*** insurance covers a broad, but not necessarily all-inclusive, range of possible claims. Individuals often have liability coverage through their homeowner's and auto insurance policies; businesses usually purchase separate liability policies.
Life	Pays a financial benefit to your family (or others that you specify) in the event of your death.
Professional liability	***Professional liability*** covers liability for damages to others while working in your profession. ***Malpractice insurance***, most commonly associated with physicians, is a type of professional liability coverage.
Property	Similar to homeowner's insurance, but covers property such as business offices or rental property. May not include liability coverage as homeowner's does. Property insurance may also be issued for specific types of risks, such as ***fire insurance***.
Renters	Similar to homeowner's, but does not cover damage to the property.
Umbrella	Umbrella policies are purchased in connection with auto and homeowner's insurance. Umbrella policies offer coverage for liability claims that exceed the coverage limits on auto and homeowner's policies.
Unemployment	Provides income when you lose a job. Unemployment insurance is a government program; it is not provided by insurance companies.
Workers' compensation	Provides compensation for injuries to a business's workers sustained while on the job.

Insurance and the Law of Large Numbers

Before talking more about insurance itself, we need to understand the mathematical principles that make insurance work. The most significant is the ***law of large numbers***. The law of large numbers assures us that, while it may be impossible to predict whether or not a random event will happen in any individual situation, it may be possible to make a reasonable prediction about how may times it will happen *over a large number of situations.*

One familiar example may help to illustrate this. When you flip an ordinary coin, there are two possible results: heads or tails. There is a 50/50 chance of getting either heads or tails on any given flip, so we would expect the result should be heads half the time and tails the other half of the time. Now suppose you flip a coin twice. This suggests that one flip should come up heads and one should come up tails. This may happen, but you certainly wouldn't be all that surprised if both flips came up heads or both came up tails. With just two flips, everyday experience tells us not to expect the proportion of heads or tails to necessarily be exactly 50%; 0% or 100% are both completely reasonable outcomes.

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Now suppose you flip the coin 10 times. All heads or all tails is still within the realm of possibility, but far less likely to happen than with only two flips. Still, though, you would not be surprised to get four heads and six tails, or seven heads and three tails, and so on. With 10 flips we still don't necessarily expect the proportion to necessarily be 50%; but we do have a much greater expectation of it being closer to 50% with 10 flips than we do with two.

Suppose now that we flip the coin 1,000,000 times. All heads or all tails is again still *possible,* but the likelihood of that happening is so small that it's not worth considering. It is still possible to get 30% or 40% heads or tails, but the chance of that happening seems pretty remote. We still don't necessarily expect an exactly 50/50 split—499,563 heads/500,437 tails certainly would not be a surprising outcome, for example. But with so many flips, even though we can't expect exactly 50%, we most certainly can expect the proportion of heads or tails to be very close to 50%. This is the idea of the law of large numbers—the greater the number of coin flips, the more predictable the proportions of heads and tails.

Of course, we don't buy insurance for coin flips. But the law of large numbers works just as well with the things we do buy insurance for. For example, it is unlikely that any one specific person will become disabled in a given year. But people do become disabled all the time. Though the likelihood that you will be one of the people who does face disability in the next year may be small, the financial consequences of being unable to work for a living could be disastrous to you. If you buy a disability insurance policy, neither you nor the insurer can know in advance whether or not you will be one of the people who does become disabled. But if the company sells lots of disability policies, it can reasonably predict how many of its policyholders will face disability, even though it can't predict the specific ones. So the company can make a reasonable projection of how much it will need to pay out in claims *overall* and can use this projection to determine how much premium it will need to charge to be able to pay those claims. By buying a disability insurance policy, you can take advantage of the insurance company's ability to take advantage of the law of large numbers to protect yourself financially.

As a simple illustration of how this works, imagine that an insurance company offers a policy that will pay the policyholder a benefit of $200,000 if he becomes permanently disabled within the next year.[2] From past experience, the company expects that 1 out of every 5,000 policyholders will be entitled to make a claim against their policies. This works out to $\$200{,}000(1/5{,}000) = \$200{,}000/5{,}000 = \$40$ per policy. Thus, if the insurer collects $40 from each policyholder, and if the number of policies sold is large enough that we can expect the proportion of actual claims to match this 1 in 5,000 frequency, the premiums collected should be enough to provide for the predicted claims expense. Even though on any one individual policy the insurer would either pocket the entire $40 or pay out a $200,000 claim, *overall* the premiums collected and claims paid would be expected to balance out. This $40 per policy is sometimes referred to as the ***pure premium***[3] for the policy. The actual premium charged to the policyholders would of course be higher, though, since the company must also add on charges for its operating expenses, agents' commissions, taxes, profit, and other costs.

Even though the company expected the rate of claims to be 1 claim per 5,000 policies, it is possible that more or fewer than the predicted number of claims will occur. The pure premium does not guarantee that premiums will balance out with claims. The actual cost of claims divided by the number of policies sold could work out to be more, or less, than $40. Yet, the more of these policies they sell, the closer to the predicted $40 we would expect

[2]This is not entirely realistic. A disability policy would be much more likely to pay a certain income per week or month while the person is disabled, not a single lump sum payment. The insurer would then need to take into account not only the likelihood of someone filing a claim but also the likely duration of the disability. However, this simple example will serve to illustrate the concept just fine without getting bogged down.

[3]This is also referred to as the ***expected claims cost***. This can be confusing at first, though, because the word *expected* is being used in a special technical sense that does not mean quite the same thing as it does in ordinary speech. This use of the term *expected* is discussed in more detail in Chapter 16.

the claims per policy to work out to be (just as the greater the number of coin flips, the closer to 50% we expect the proportion of heads to be). Also, the company will no doubt be selling many other policies as well, and so higher-than-predicted costs from one policy type may be offset by lower-than-predicted costs from another. (Smaller insurers, who do not sell enough policies to be confident of the law of large numbers' protection, often make financial arrangements to share risks with larger companies or with each other; these sorts of arrangements are known as ***reinsurance***.)

Example 13.1.1 ***An insurance company offers a policy that pays a flat $5,000 if a policyholder is a victim of identity theft. The company believes that on average one out of every 287 policyholders will have a claim on their policies each year. Calculate the pure premium for this policy for one year.***

$$\text{Pure premium} = \frac{\$5{,}000}{287} = \$17.42$$

Note that the pure premium is not the actual premium that will be charged for the policy. A large percentage, usually the largest percentage, of the premium paid for an insurance policy may be the pure premium, but in addition the premium must cover administrative expenses, sales commissions, and room for profit. While the pure premium is $17.42, the actual premium for the policy will be higher.

Insurance Rates and Underwriting

The policy we considered in Example 13.1.1 was a very simple one: coverage was provided for a single type of event (identity theft) and for a specified dollar amount ($5,000). Most insurance policies are far more complex. An automobile liability policy, for example, might deal with claims ranging from a fatal accident resulting in a million dollar wrongful death lawsuit down to comparatively minor repair costs resulting from a fender-bender. To predict the average claims cost, the insurer must assess all the many different types of events that could lead to a claim against the policy, considering both how often each event will happen (the ***frequency*** of that type of claim) and the amount that could have to be paid to settle such a claim (the claim's ***severity***). Obviously, this is far more complicated than our initial example, where the single kind of claim covered had only one possible benefit payment. Determining the pure premium for a policy covering many different types of claims, each of which could result in a range of possible claim amounts, requires significant analytical effort. Insurance companies employ financial professionals known as ***actuaries*** to work this all out.

An insurer must also take into account the fact that, even though it is not possible to say in advance which individual policyholders will have claims, it may be possible to say in advance which policyholders are *more likely* than others to file a claim. In our disability policy example, we quietly assumed that each policy was equally likely to produce a claim. In reality, though, lion tamers are more likely than librarians to suffer a permanent disability in the course of plying their trade. Reflecting this reality, an insurer will most likely charge a higher premium to those who pose a greater risk of claims. In fact, in a competitive market the insurer cannot afford not to do this; if Insurer A charges both the same rate, while Insurer B charges different rates, lion tamers will buy their policies from Insurer A and librarians will buy theirs from Insurer B. This will most likely lead to higher than predicted claims at Insurer A, causing financial losses and possibly even jeopardizing the insurer's ability to survive and meet the benefit obligations it has promised.

Insurance companies group potential policyholders into different categories based on their potential for filing claims. These groupings are called ***rating classes*** or ***underwriting classes***. Most readers are probably familiar with rating classes for automobile insurance. The rates that you pay for auto insurance may be affected by your sex, your age, your marital status, how far you commute to work, the city where you live, your driving record, the type of vehicle you drive, and other factors. In recent years, a number of insurers have even taken to basing auto insurance rates on your credit rating (on the theory that people with

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good credit ratings are less likely to have accidents than people who don't). Even though insurers may consider a wide variety of factors in determining rating classes, they do not have complete freedom in what factors to consider for rating classes. State regulations and federal antidiscrimination laws may prohibit certain types of rating classification (such as basing rates on race or religion). Making matters more complex, one state may disallow a rate classification that might be used in another state. For example, even though charging different auto insurance rates based on gender and marital status is common practice in most states, in others doing so is illegal. (Montana is an often-cited example of a state which has prohibited charging different auto insurance rates based on gender.)

Once the insurer has established its underwriting guidelines and rating classes, the rates for a particular type of insurance will be set for each rating class. Sometimes this is expressed as separate rates for each class, sometimes the company will set a basic rate and then have adjustments made to this rate for certain classes.

However the rates are set, the insurer will set them in a ***rate book***. The term *rate book* is still in use even though today rate quotes are usually done with computer software. Different insurers may set up their rate formulas in various ways. The following is a hypothetical example intended to demonstrate mathematically how these may be set up.

AFFILIATED BENEVOLENT MUTUAL INSURANCE COMPANY OF OHIO
STANDARD PERSONAL AUTOMOTIVE LIABILITY COVERAGE
SEMIANNUAL RATES

Rating Region	*Workday Commute*	*Basic Rate*	*Long-Term Client Rate*
A	<5 miles	$182.50	$172.50
	5–20 miles	$215.55	$203.55
	>20 miles	$245.00	$230.00
B	<5 miles	$208.75	$198.75
	5–20 miles	$242.15	$230.15
	>20 miles	$271.10	$256.10
C	<5 miles	$163.50	$153.75
	5–10 miles	$181.01	$169.01
	10–20 miles	$192.05	$180.05
	>20 miles	$201.42	$186.42

In addition to a table setting basic rates, the rate book may include adjustments for different underwriting classes.

AFFILIATED BENEVOLENT MUTUAL INSURANCE COMPANY OF OHIO STANDARD PERSONAL AUTOMOTIVE LIABILITY COVERAGE—MULTIPLE OF BASIC PREMIUM

Category	*Male*	*Female*
Age 16–19 (no driver training)	2.55	2.05
Age 16–19 (driver training)	2.25	1.85
Age 20–24 single	2.20	1.80
Age 20–24 married	1.90	1.60
Age 25–29 single	1.90	1.65
Age 25–29 married	1.45	1.35
Age 30–64	0.95	0.85
Age 65+	1.10	1.00

The values in this table are given as multiples of the basic premium. To determine the premium that would be charged to a given individual, you would need to look up the rate in the first table based on region and commuting distance, and then multiply by the number in the multiples table based on age and sex.

Example 13.1.2 ***Using the tables above, determine the semiannual rate for a policy for:***

(a) A 17-year-old male with no driver training who lives in region A and commutes 4 miles per day

(b) A 27-year-old married female living in Region C with a 17-mile daily commute

(c) A 67-year-old single man who lives in Region B with a 12-mile commute who qualifies for the long-term rate

(a) From the basic rate table, we find a basic rate of $182.50. From the multiplier table, we see a multiple of 2.55. So the rate would be ($182.50)(2.55) = $465.38.

(b) The basic rate is $192.05, the multiplier is 1.35. Rate = ($192.05)(1.35) = $259.27.

(c) The basic rate is $230.15, the multiplier is 1.10. (The fact that he is single is irrelevant in this company's rate book for ages 65+.) Rate = $230.15(1.10) = $253.17.

The multipliers function in the same way as a percent increase or decrease applied to the basic premium. This provides an alternative way of stating the effect of rating classifications, and they are sometimes expressed in that way.

Example 13.1.3 ***What percent over the basic rate does a 26-year-old married man pay? What percent below the basic rate does a 54-year-old-woman pay?***

The man's multiplier is 1.45, so he would pay 145% of the basic rate. So he pays 45% over the basic rate.

The woman's multiplier is 0.85, so she would pay 85% of the basic rate. She pays 15% below the basic rate.

The rate book and examples used above are intended to represent a typical rate book and its use; it should be clearly understood though that the example is completely hypothetical, and any given insurance company may set its tables up differently. For example, the lower "long-term client" rate might be given as a percent discount instead of a separate rate in the book, the workday commute might be dealt with as a multiplier instead of a separate rate, or separate rates might be given by gender rather than have multipliers to deal with genders. The company may also charge a higher premium for customers who have been involved in traffic accidents. (See Exercises 15-18 for more on this.) While the specifics might differ, though, this is illustrative of the general way a rate book might be set up.

Deductibles, Coinsurance, and Coverage Limits

Insurance policies seldom provide complete and unlimited coverage of a claim, regardless of its size. A ***deductible*** is the amount of expense that the policyholder must pay herself before the insurer will pay any benefits.

Example 13.1.4 ***Adam has comprehensive coverage on his car. The deductible is $500. The car's value is $13,200, and it is stolen. How much will the insurer pay on this claim?***

Adam is responsible for the $500 deductible; the insurer pays the claim beyond that. So the insurer will pay $13,200 − $500 = $12,700.

Example 13.1.5 ***Kyle and Sara's homeowner's insurance policy has a $1,000 deductible. Some of the house's siding is blown off in a windstorm, costing $735.22 to repair. How much will the insurer pay on this claim?***

Kyle and Sara are responsible for the first $1,000 in claims. Since the siding damage was less than this, the insurance company will pay nothing.

Coverage limits specify a maximum amount that the insurer will pay for claims on a policy. For example, a business should carry liability insurance to financially protect itself against lawsuits,

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but no insurer will offer a policy that will provide unlimited coverage. Liability policies specify a maximum amount that the insurer will pay regardless of the size of the claim.

> ***Example 13.1.6*** ***Trenyce owns a clothing boutique. She carries a $1,000,000 liability insurance policy for the business. A customer slips and falls over a power cord while shopping in the store and sues for $125,000 in damages. Will the coverage limit be an issue for Trenyce?***
>
> *Her insurer will cover claims up to $1,000,000. The coverage limit will not be an issue.*

While unlimited coverage is not generally available, it is usually possible to purchase insurance up to a limit that is more than adequate to cover any reasonably imaginable liability. For obvious reasons, though, higher coverage limits mean higher premiums. Coverage limits are more often an issue when someone tries to save money by buying a lower limit policy. Having insurance does not guarantee that you are protected from having to deal with a large financial obligation if the coverage limits are too low.

> ***Example 13.1.7*** ***Rick runs a tree service business. He carries a $50,000 liability policy. When cutting down a tree from a residential property, the tree falls onto a neighbor's house, leading to $135,000 in damages. How much of this liability will Rick be responsible for?***
>
> *Rick's insurance will only cover $50,000 of the claim. Rick is left with $135,000 − $50,000 = $85,000 in responsibility.*

Limits of coverage can be a bit tricky with motor vehicle liability insurance. A motor vehicle liability policy will usually have a coverage limit expressed in a form such as "25/75." The first number indicates the limit of liability for harm done to any one individual (in thousands), the second number states the overall limit of liability. This is sometimes further extended to the form "25/75/50," where the last number indicates the limit of coverage for damage to property (as opposed to people).

> ***Example 13.1.8*** ***Kelsi has a 25/75/50 motor vehicle liability policy. Kelsi caused a car accident in which she injured two people and caused $18,500 in property damage. A court awarded one of the victims a $35,000 judgment, and $15,000 to the other. How much will her insurance company pay?***
>
> *The property damage falls under the limit, so it is covered in full. Also, the $15,000 judgment falls under the per person limit, so that is also covered. The $35,000 judgment, though, will be covered only up to $25,000. So the insurance company will pay only $25,000. In total, she is covered for $18,500 + $25,000 + $15,000 = $58,500 of the claim. Kelsi herself will be responsible for the $10,000 that was over the limit.*

Each state has its own minimum requirements of how much liability insurance a driver must carry, but these limits are usually quite low in comparison to the damages that can result from even a fairly modest accident. It really isn't all that hard to do a few hundred thousand dollars of damage with a car. In Kelsi's case, even though she had insurance, she is left with a major financial responsibility to bear. This is a problem for her, but it is also a concern for the accident victim. If Kelsi doesn't have the resources to come up with that $10,000, it is unlikely that the victim will actually be able to collect. If Kelsi had caused a million dollar accident, it is nearly certain that the damages will put Kelsi in bankruptcy court and the full damages will never actually get paid. For this reason, many insurers offer ***uninsured/underinsured motorist coverage***, which protects the policyholder (and her passengers) in case of injuries caused by another motorist who does not have adequate liability insurance coverage. Some states have attempted to deal with this issue by requiring ***no-fault insurance***. When an accident happens, each policyholder's insurance pays the policyholder's claim, regardless of who was at fault, making the amount of insurance carried by the other party in an accident irrelevant.

In some cases, an insurance policy will offer coverage only for a certain percent of the dollar amount of a claim. This is called ***coinsurance*** because the financial risk of the claim is in effect shared by the insurer and the insured. A policy in which the insurer pays 75% of the cost of any claims would be an example. This type of policy might commonly be described as a "75% coinsurance" policy, or as a "75/25" policy.

Coinsurance is very often a feature of health insurance policies, and will be discussed in that context in the next section. It is less common with casualty policies, though there are exceptions. Property insurance policies often carry a provision for coinsurance if the property is insured for less than its full value. For example, suppose that you own a warehouse that, together with its contents, is worth $2,500,000. It is very unlikely that you would ever suffer a *complete* loss—even a serious fire would probably not completely destroy the entire warehouse and all of its contents. You might decide, then, to save some money on the premium by buying insurance coverage only up to $1,500,000.

The problem that this presents, though, is that the insurance premium for a $1,500,000 policy was most likely based on the potential claims of a property worth $1,500,000. The likelihood of a claim running near that coverage limit on your facility is much greater than the probability of a claim running near the limit for a $1,500,000 facility. Therefore the $1,500,000 premium would not adequately cover the potential for claims on your warehouse.

Insurers may address this issue by including a ***coinsurance clause***, a policy provision stating that if the property is insured for less than its full value, the insurer will provide coverage only for claims at the percent of full value for which the property is insured.

Example 13.1.9 ***Your warehouse and its contents are worth $2,500,000. You insure the property for $1,500,000. Your policy contains a coinsurance clause. There is a fire, and you incur a $1,200,000 loss. How much will your insurance pay?***

You have insured the property for

$$\frac{\$1,500,000}{\$2,500,000} = 0.60 = 60\%$$

of its full value. Your insurance will cover 60% of the claim: (60%)($1,200,000) = $720,000.

Coinsurance clauses often include a provision that coinsurance will not apply if the property is insured for more than a given percent of its full value. Eighty percent is a common minimum; a policy that includes this provision would be referred to as having an "80% coinsurance clause." Note that there is potential for confusion with this wording; the phrase "80% coinsurance" could also be interpreted to mean that the policy carries 80/20 coinsurance on all claims. It will usually be clear from context which way it should be interpreted.

Example 13.1.10 ***Your warehouse and its contents are worth $2,500,000. You insure the property for $2,000,000. Your policy contains an 80% coinsurance clause. There is a fire, and you incur a $1,200,000 loss. How much will your insurance pay?***

You have insured the property for

$$\frac{\$2,000,000}{\$2,500,000} = 0.80 = 80\%$$

of its full value. This meets the minimum requirement, so coinsurance does not apply. Your claim is covered in full.

How Deductibles, Coverage Limits, and Coinsurance Affect Premiums

It stands to reason that the higher a policy's deductible, the lower its premium, and vice versa. Similarly, the higher the coverage limit, the higher the premium. Insurers may provide for this by presenting rates as a formula based on coverage limits and deductibles, by

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simply listing different rates for difference coverage levels, or by providing multipliers similar to the ones used in previous examples for underwriting classifications. The following examples will serve to illustrate:

Example 13.1.11 ***The rates for an umbrella liability policy based on the coverage limits are given in the table below:***

GLOBAL RISK MUTUAL LIABILITY INSURANCE COMPANY
UMBRELLA LIABILITY COVERAGE ANNUAL PREMIUMS

Basic \$1,000,000	\$195.00
Each additional \$1,000,000 (up to \$5 million)	\$125.00
Each additional \$1,000,000 (\$5 million to \$10 million)	\$105.00

(a) Determine the premium for a \$3,000,000 policy.

(b) Determine the premium for a \$10,000,000 policy.

(a) The first \$1,000,000 costs \$195.00. We need \$2,000,000 beyond that to get to the desired total. So the total cost will be \$195.00 + 2(\$125.00) = \$445.00.

(b) \$195.00 + 4(\$125.00) + 5(\$105.00) = \$1,220.00.

Note that the premium rates per million go down as the coverage limit goes up. This may seem surprising at first. The reason for this is that even though a higher limit policy does expose the insurer to a higher limit on each claim, most claims fall nowhere near the coverage limit.

Example 13.1.12 ***Right now, Dave has a 50/100 motor vehicle liability policy, and a \$200 deductible collision policy. His liability premium is \$273.50 and his collision premium is \$255.00, for a total of \$528.50. He is considering changing these coverages. His insurance agent provides him with a table of adjustment multipliers.***

New Liability Coverage	*Multiplier*	*New Collision Deductible*	*Multiplier*
25/50	0.925	\$50	1.355
75/200	1.055	\$300	0.925
100/300	1.125	\$500	0.845
250/500	1.235	\$1,000	0.725

Calculate Dave's total premium for these coverages if (a) he changes his liability limit to 100/300 and his deductible to \$50 and (b) if he increases his liability limit to 250/500 and his deductible to \$1,000.

(a) The multiplier for his liability coverage is 1.125. So his new liability premium would be (1.125)(\$273.50) = \$307.69. For his collision deductible, the multiplier would be 1.355, so his new collision premium would be (1.355)(\$255.00) = \$345.53. The total for these coverages would be \$307.69 + \$345.53 = \$653.22.

(b) His new liability premium would be (1.235)(\$273.50) = \$337.77. His new collision premium would be (0.725)(\$255.00) = \$184.88. The total is \$337.77 + \$184.88 = \$522.65.

When buying insurance, consumers are often more focused on deductibles than on coverage limits. It is often a good idea, though, to focus more on coverage limits than deductibles. In part (b) of this example, Dave could actually lower his overall premiums while raising his coverage limits dramatically if he is willing to accept a higher deductible. Dave may want to consider the adjustments. Any significant accident could far exceed his current coverage limits and wipe him out financially, so higher coverage limits would provide him with greater protection. An increase in the deductible would be unpleasant should he have a claim, but it would not be financially disastrous.

EXERCISES 13.1

A. Pure Premium and the Law of Large Numbers

1. An insurance policy will pay a lump sum of $500,000 in the event that a policyholder becomes permanently disabled. The insurer believes that on average one out of every 2,500 people buying this policy will have a claim. What is the pure premium for this policy?

2. An insurance policy will pay a business a lump sum of $2,800,000 in the event that its chief executive dies in the next year. The insurer believes that there is a 1 in 375 chance that this will happen. Calculate the pure premium for this policy.

3. An auto insurance company has determined from past experience that it can expect 1 out of every 189 policyholders to have a liability claim in the next year, and that the average claim size will amount to $87,532. Calculate the pure premium for this liability policy.

4. From past experience, an insurance company believes that the average claim on a $1 million umbrella liability policy will be $436,000, and that on average 1 out of every 7,350 policy holders will file a claim in the next year. What would be the pure premium for this policy?

5. A group of 37 business owners in Cascadilla Falls is considering forming a cooperative insurance company to provide themselves with liability insurance. The likelihood of any of these business owners having a liability claim in any given year is very small, but if a claim is made, it could be for a very large amount of money. Is their plan reasonable?

B. Underwriting and Premium Calculations

For Exercises 6 to 18, use the Affiliated Benevolent Mutual automobile liability insurance rate table and demographic factors table given on page 528 of this section. Note that the premiums listed in the table are semiannual rates.

6. Calculate the semiannual premium for a 35-year-old unmarried man who lives in Region A and commutes 10 miles to work.

7. Calculate the semiannual premium for a 43-year-old married woman who lives in Region C, and commutes 8 miles to work.

8. Calculate the semiannual premium for a 17-year-old male who has not taken driver training, lives in Region B, and commutes 3 miles.

9. Calculate the semiannual premium for a 21-year-old married woman who lives in Region A and has no daily commute.

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10. Calculate the semiannual premium for a 28-year-old unmarried woman who lives in Region A, commutes 35 miles, and qualifies for the long-term customer discount.

11. Calculate the semiannual premium for a 68-year-old married man who lives in Region B, commutes 20 miles, and qualifies for the long-term customer discount.

12. Calculate the semiannual premium for a 55-year-old divorced man who lives in Region B and commutes 2 miles.

13. What percent over the basic rate does a 17-year-old male who has taken driver training pay?

14. Suppose that Affiliated Benevolent Mutual offers a 5% discount to customers who also have a homeowner's insurance policy with the company. John is a 33-year-old married man living in Region B who commutes 9 miles each day. He qualifies for the long term-customer discount, and also has his homeowner's policy with the company. Calculate his semiannual rate.

Use the following information for Exercises 15 to 18. Affiliated Benevolent Mutual increases its rates for customers who have been involved in traffic accidents, on the basis of the number of "points" charged to the customer's license. These extra charges are given in the table below:

Number of points	***Semiannual premium increases by***
1–2	No increase
3	25%
4–5	50%
6–7	75%
8–9	100%
10 or more	Policy will be cancelled

15. Calculate the semiannual premium for an unmarried male aged 33 living in Region A, who commutes 16 miles, and who has 4 points on his license.

16. Calculate the semiannual premium for a married 27-year-old woman living in Region C with no daily commute who has 8 points on her license.

17. Calculate the semiannual premium for an unmarried 20-year-old woman living in Region B with a 12-mile commute and 10 points on her license.

18. Calculate the semiannual premium for a married 64-year-old man who lives in Region A, has no daily commute, qualifies for the long-term customer discount, and has 7 points on his license.

C. Deductibles, Coinsurance, and Coverage Limits

19. Teddy has a renter's insurance policy with a $350 deductible. Someone broke into his apartment and stole computer equipment worth $2,400. This theft is covered by his policy. How much will his insurance company pay on this claim?

20. Lana has comprehensive coverage on her car, which is stolen. The car's value was $18,735 and her deductible was $500. How much will her insurance company pay for this claim?

21. Ellen has a homeowner's insurance policy with a $750 deductible. Vandals break one of the windows, causing $450 in damage. How much will her insurance policy pay if she files a claim for this?

22. Landervan Home Improvements carries a $75,000 liability insurance policy. While doing electrical work on a home, one of the company's workers caused a fire that led to $40,000 in damages. How much of the company's liability will be covered by the policy?

23. A restaurant carries a $250,000 general liability policy. One morning, a waitress tripped while carrying a pot of very hot coffee and spilled it on a customer, causing serious burns. The customer sued the restaurant and was awarded $380,000 in damages. How much of these damages will the business have to pay?

24. Sylvia carries a 200/500 automobile liability policy. In an accident, she injures someone who is awarded $280,000 in damages for his injuries. How much of these damages will her policy cover?

25. Jake carries a 100/250 automobile liability policy. He injures three people in a car accident, each of whom is awarded $75,000 for injuries. How much of these damages will Jake's policy cover?

26. Efren carries a 75/250 automobile liability policy. In an accident, he is responsible for injuring three people, who are each awarded $80,000 in damages for their injuries. How much will his insurance policy cover? How much will he have to pay himself?

27. Sarah carries a 50/150 automobile liability policy. She injures two people in an accident, one of whom is awarded $75,000 in damages, and the other of whom is awarded $45,000. How much will her insurance cover? How much will she have to pay herself?

28. Dana carries a 25/50/35 automobile liability policy. In an accident for which she is found to be responsible, the other motorist was injured and was awarded $48,000 for his injuries and $10,000 for damage to his car. How much of these damages will Dana's insurance cover? How much will she be left to pay herself?

29. Glenn carries a 100/300/50 automobile liability policy. He is responsible for an accident in which three people are injured. One person is awarded $125,000 in damages for her injuries, while the other two are each awarded $50,000. He also did $25,700 worth of damage to the other motorist's cars. How much of these damages will Glenn's insurance cover? How much will he be left to pay himself?

30. Suppose that Sylvia (from Exercise 24) also has a $1 million umbrella policy. How does that change your answer to Exercise 24?

31. Suppose that Glenn (from Exercise 29) also has a $1 million umbrella policy. How does that change your answer to Exercise 29?

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32. Suppose that your company's offices and their contents are worth $1,400,000. You carry a $1,200,000 fire insurance policy with a 75% coinsurance clause. There is a fire, which does $1,000,000 of damage to your offices. How much of this damage will be covered by your policy?

33. Suppose that your company's offices and their contents are worth $1,400,000. You carry a $1,000,000 fire insurance policy with a 75% coinsurance clause. There is a fire, which does $1,000,000 of damage to your offices. How much of this damage will be covered by your policy?

D. Grab Bag

Exercises 34 to 36 are based on the rate table given here:

GLOBAL RISK MUTUAL LIABILITY INSURANCE COMPANY
GENERAL BUSINESS LIABILITY COVERAGE ANNUAL PREMIUMS

Basic $500,000	$975.00
Each additional $250,000 (up to $2 million)	$125.00
Each additional $500,000 ($2 million to $10 million)	$175.00

34. Calculate the annual premium for a $750,000 business liability policy.

35. Calculate the annual premium for a $2,000,000 business liability policy.

36. Calculate the premium for a $4,000,000 business liability policy.

37. Suppose that an insurance company sells flight insurance policies at an airport kiosk. The policy will pay $5,000,000 to your heirs in the event that your airplane crashes and you die as a result. The company sells the policies for $32.50. If the likelihood that your plane will crash resulting in your death is 1 in 12,500,000, what is the pure premium for this policy? (See Exercise 44 for more on this type of policy.)

38. Bruce has a 100/300/75 auto liability policy. He is involved in a car accident, where he is held responsible for injuries to four people. Each of the four is awarded $90,000 in damages as a result. How much of these damages will Bruce's policy pay? How much will he be left to pay himself?

39. Bruce has a 100/300/75 auto liability policy. He also has a $2 million umbrella policy. He is involved in a car accident, where he is held responsible for injuries to four people. Each of the four is awarded $90,000 in damages as a result. How much of these damages will Bruce's policy pay? How much will he be left to pay himself?

Exercises 40 to 43 refer to the table below. In addition, these exercises also use the Affiliated Benevolent Mutual auto insurance rate tables given on page 528. Assume that the rates in those tables are for a 50/150 liability policy.

New Liability Coverage	*Multiplier*	*New Collision Deductible*	*Multiplier*
25/50	0.91	$100	1.355
75/200	1.05	$500	0.925
100/300	1.12	$750	0.845
250/500	1.20	$1000	0.725

40. Tracey is a 28-year-old unmarried man living in Region A. He commutes 38 miles to work and qualifies for the long-term customer discount.
    a. Calculate Tracey's semiannual liability premium.
    b. What would Tracey's premium be if he switched to a 250/500 policy?

41. Marissa is a 38-year-old married woman living in Region C. She commutes 5 miles to school.
    a. Calculate Marissa's semiannual liability premium.
    b. What would her premium be if she switched to a 100/300 policy?

42. Suppose that Tracey (from Exercise 40) also carries collision coverage, and his semiannual premium is $115.43 with a $250 deductible. How would increasing his deductible to $750 change his premium?

43. Suppose that Marissa (from Exercise 41) also carries collision coverage with a $250 deductible. Her semiannual premium is $182.43. How much would she save by increasing her deductible to $1,000?

E. Additional Exercises

44. In Exercise 37 the difference between the pure premium and the actual premium charged for the policy was enormous. Why do you suppose people actually buy these policies when the gap between their cost and the pure premium is so large?

45. Six months ago Jennifer bought a new car for $23,500. Today, the car is worth $19,500, but she owes $21,925 on her car loan for it. She has collision insurance with a $500 deductible. Jennifer has a serious accident, and her car is totaled (meaning that the damage is bad enough that it cannot be repaired). Her insurance will reimburse her for the value of the car, and then she will need to pay off the balance of her loan. How much of her own money will she need to be able to pay off the loan?

13.2 Health Insurance and Employee Benefits

In many countries, health care is primarily funded through a governmental program. This sort of arrangement is often referred to as ***socialized medicine*** or ***universal health care***. Every citizen is provided with the same basic package of benefits, paid for through a governmental agency with funds collected in taxes (or borrowing). The government program may not cover all medical expenses (prescription drugs, for example) or may not provide access to all the health care providers or services that an individual may want. But since basic coverage is automatic, insurance for health care expenses not covered by the government plan may be desirable, but is not a critical issue for most people.

In the United States, on the other hand, there is no national universal program for health insurance coverage. ***Medicare*** provides coverage for Americans over the age of 65, and ***Medicaid*** provides coverage for the poor, but for the rest, paying for medical expenses is another matter. While paying cash for medical expenses is of course an option—theoretically—even a comparatively minor illness can easily cost thousands of dollars, and the cost of treating a major illness can be staggering. In this environment, having health insurance coverage becomes essential to be able to access medical care without facing the prospect of financial ruin.

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Health insurance can be purchased as an ***individual policy*** by an individual to cover himself and his immediate family. Traditionally, though, health insurance coverage has been provided as a benefit of employment. In that case, the insurance company offers a policy to a company's employees as a group, as a ***group policy***. The connection of this very important insurance coverage to employment makes health insurance a particularly challenging issue for businesses. The cost of providing health insurance to its employees can be a huge expense for a business; at the same time, employers who do not offer good health benefits run the risk of losing their best workers to competitors who offer better benefits.

Regardless of how it is purchased, health insurance is an expensive proposition. Costs for hospitalization, prescription drugs, and the services of doctors and other health care professionals can quickly add up to shockingly large totals, and so the insurance coverage to pay these costs is similarly expensive. For those fortunate enough to have group insurance coverage with premiums subsidized or entirely paid for as a benefit of employment, these costs are not felt so directly, though the business providing that coverage certainly feels the expense. For the self-employed or those whose employers do not provide group coverage, the cost can be exorbitant.

The prohibitive cost of this insurance actually makes the cost picture worse; since the cost is so high, many people choose to "do without." Yet the people most likely to feel that they can afford to do without are also the people least likely to need the coverage. When the healthiest people choose to opt out of insurance coverage, they do not remove much cost from the system, leaving nearly the same total cost to be spread over a smaller number of policies. This raises the premium costs ever higher, potentially creating a vicious circle where costs keep rising, increasing the incentive for more people to opt out of health coverage, leading to even higher costs per policy, and so on.

The traditional approach to health care funding in the United States is the source of much debate, much of it highly political and much of that bitter and angry. It is likely that much will change in the health care funding arena in the next decade, and it is certain that any changes that do occur will make some people very happy and others very unhappy. What those changes will actually turn out to be, though, is anyone's guess at this point. What is certain, though, is that health insurance coverage is an enormously important consideration both for employees and for employers. Whether you are looking at this area primarily as an individual looking to better understand your own options, or as a business owner looking to control your costs while at the same time attracting and holding on to the best employees, health insurance is an important consideration.

Types of Health Insurance—Indemnity Plans

The most traditional type of health insurance coverage is an ***indemnity plan***. While many variations on this type of plan exist, one common form was a ***hospitalization policy*** in combination with a ***major medical policy***. (The well-known Blue Cross and Blue Shield plans were originally of this type, with the Blue Cross providing the hospitalization coverage and the Blue Shield offering the major medical.) As the name suggests, a hospitalization policy is intended to cover the costs of hospitalization, sometimes also with a basic medical plan that provides a certain dollar benefit toward certain specified services. A major medical policy covers medically necessary services and items not covered by the basic medical and hospitalization policy, such as prescription drugs. Major medical plans often come with a ***deductible*** (an amount that you must pay yourself before coverage kicks in) and ***coinsurance*** (a percent of the claim that you must pay yourself).

When an insured person has a medical claim to be covered under a deductible and coinsurance plan, she submits that claim to the insurer. The insurer then verifies that the claim is a legitimate one for a covered service, and then reimburses the insured for the amount required under the policy.

Example 13.2.1 ***Mary has a "deductible and coinsurance" health insurance policy with a $250 deductible and 80/20 coinsurance. She submits claims for $650.79 of medical expenses covered under her contract. How much will she receive from her insurer in reimbursement for these claims?***

The $250 deductible must first be satisfied, so the insurer will cover $650.79 − $250 = $400.79. This is covered at 80%, so she will be reimbursed (80%)($400.79) = $320.64.

Example 13.2.2 ***Suppose that later in the same year, Mary submits an additional $708.25 in covered claims. How much will she be reimbursed for these?***

She has already satisfied her deductible, so the entire remaining claim is covered: (80%)($708.25) = $566.60.

Deductible and coinsurance plans often include an ***out-of-pocket maximum***. If the total of the insured's share of medical costs (from the deductible and coinsurance) exceeds the out-of-pocket maximum, the insurer covers all additional claims in full.

Example 13.2.3 ***Mary's policy has a $1,000 out-of-pocket maximum. Before the end of the year, she has some significant medical expenses, and runs up claims totaling $6,000, which she submits to the insurer. How much will the insurer pay?***

Since her deductible is satisfied, the insurer would normally cover 80%($6,000) = $4,800. However, the out-of-pocket maximum applies. So far this year, Mary has been left to pay $650.79 − $320.64 = $330.15 on the first claim and $708.25 − $566.70 = $141.65 on the second, for a total of $330.15 + $141.65 = $471.80.

Thus, the most she can be left to pay this year is $1,000 − $471.80 = $528.20. Her insurer will pay $6,000 − $528.80 = $5,471.80.

Indemnity policies also may have a maximum amount that the insurer will pay on the contract. Unlike the deductible, which resets every year, these coverage maximums are often a maximum over the entire lifetime of the contract. Usually this limit is quite high (a million dollars or more), in which case it is not much of a concern. Most policies offered through an employer have high lifetime maximums (if they have them at all). However, there are policies sold with lower lifetime maximums, and with those policies the maximum may be a significant issue. For example, policies sold to college students through a campus health office often carry limits of $10,000 or $25,000 or some similar amount. This maximum is more than adequate to cover a broken leg or minor illness, but the policy could be exhausted very quickly in the event of a more serious injury or illness.

A health insurance policy may be an ***individual policy*** covering just one single individual, or a ***family policy*** covering an individual and his immediate family. Spouses and dependent children (up to a certain age limit) may be covered under almost any family policy; some policies will also cover an unmarried domestic partner. When more than one individual is covered under a policy, a separate deductible may apply to each individual up to a certain overall policy deductible. For example, a "$250/$500 deductible" indicates that the deductible is $250 per person, up to a maximum $500 deductible for the contract as a whole.

Example 13.2.4 ***Ken is a single father with three children covered under a family health insurance policy. The policy provides 80/20 coinsurance with a $250/$500 deductible. Ken submits claims of $840 for himself, $100 for one child, and $670 for another child. He has not submitted any other claims this year. How much will his health insurance pay?***

Ken has a $250 deductible for himself, so the insurer will cover $840 − $250 = $590 of his claim.

His first child's claim of $100 falls under the $250 deductible, so none of it is covered.

His second child's $670 claim is over the deductible, so it will be covered. However, it is not the $250 per person deductible that applies. Ken and his first child have already satisfied

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\$350 of the overall deductible, leaving just \$500 − \$350 = \$150 to satisfy overall. So his second child's claim will be covered for \$670 − \$150 = \$520.

The total covered claims are \$590 + \$520 = \$1,110. Then (80%)(\$1,110) = \$888 is the amount his insurance will pay.

Example 13.2.5 Ken later submits a claim of \$200 for his third child, for whom he had not yet submitted any claims. How much will the insurance pay for this claim?

\$200 falls below the \$250 per person deductible, but since the overall family deductible has been satisfied, the insurer will pay (80%)(\$200) = \$160.00.

Types of Health Insurance—PPOs, HMOs, and Managed Care

The ever-increasing costs of medical care pose a challenge to everyone involved in paying for them. For employers providing health insurance, the rising costs of health insurance pose a major expense, while employees contributing a portion of the cost of their health coverage find this cost absorbing an ever-increasing proportion of their paychecks. For small business owners, the self-employed, and workers whose employers do not offer health insurance as a benefit of employment, the costs often place obtaining health insurance out of reach.

In an attempt to provide a desirable level of coverage while keeping costs under control, a number of alternative types of health insurance have been developed. A ***preferred-provider organization*** (PPO) is a type of health insurance coverage that encourages insured members to use certain "preferred providers" with whom the insurer has arranged favorable terms, by offering a greater level of coverage when preferred providers are used. A ***health maintenance organization*** (HMO) may provide coverage only for care received from specific health care providers, and also will require its members to arrange for all care through a ***primary care provider***. Members of HMOs often must obtain a ***referral*** from their primary care provider in order to have their medical expenses covered under the policy. An alternative to the HMO is a ***point-of-service plan*** (POS) that offers a combination of an HMO with a traditional indemnity plan; members who obtain services from nonapproved providers, or who do not follow the rules of the HMO and obtain referrals for services from covered providers, receive "out-of-network" benefits based on a traditional deductible/coinsurance plan offering less generous benefits than the "in-network" HMO.

Plans that attempt to control their costs by negotiating favorable financial arrangements with health care providers and by managing which services their members' access and how they access them are referred to as ***managed care plans***.

There is an enormous range of types of these plans, each with its own distinctive features, pros, and cons. In contrast to traditional indemnity plans, many managed care plans emphasize preventive care, often providing much greater coverage of things like annual checkups or nutritional counseling.

Most of these types of plans do not require a deductible or coinsurance for most benefits. Typically, managed care plans come with an array of ***copayments*** for each covered service. A copayment is a flat fee charged to the member, usually when the service is provided. For example, a policy might have a \$10 copayment for doctor's office visits. In that case, when you visit your doctor you would pay \$10 then and there, and the remaining cost of your visit would be paid by the plan directly to the doctor. In recent years, ***tiered copays*** have become common for pharmacy benefits, calling for a higher copay for certain more expensive drugs. For example, a 10/25 copay for prescriptions might require a \$10 copay for a generic drug, but a \$25 copay for a brand name drug.

As a general rule, it is normally true that the higher the copayments, the lower the plan premium is likely to be. This is true because a higher copay means that more of the cost of each service will be borne by the member, and hence less by the insurer. It is also true because if copayments are high, members are assumed to be more judicious about when they access services. Someone who pays a \$2 office visit copay will be more likely to see a doctor for a case of the sniffles than someone who must pay \$30. Yet, on the other hand, if copays are too high costs may actually escalate if members avoid seeking care until problems are severe, and thus conditions that might have been easily and inexpensively treated instead grow into more complicated and expensive ones.

Calculating Health Insurance Premiums

Premiums for group health insurance may be determined by ***community rating***, ***adjusted (or modified) community rating***, or ***experience rating***.

When rates are determined by ***community rating***, the insurer charges the same rates to all employers in a given area for a given level of benefits. Strict community rating is not often used with large employer groups, since groups who could qualify for a better rate with another rating method usually would not be interested in a community rate, which would leave the insurer with a "community" heavily slanted toward higher cost groups. However, community rating may be used with small employers or coverage offered for sale to individuals, since smaller groups are often not in a position to justify a better than community rate regardless of their situation.

With ***adjusted (modified) community rating***, the insurer may begin with the community rate and then modify it on the basis of the age/sex demographics of the group covered, the type of business, or other factors. For adjusted community rates, the insurer will have a set of factors that are to be multiplied by the community rates to determine the rates for a given group.

Example 13.2.6 ***Greatest Lakes Health Insurance Corp.'s community rate for its standard HMO plan in the Detroit metropolitan area is $280 single, $590 family. A Detroit area school district wants to offer this HMO to its employees. Greatest Lakes uses adjusted community rating. The rate factor for public schools is 1.15, while the demographic factor for this school district is 0.844. Determine the premium for this school district.***

$$\text{Single Rate} = (\$280.00)(1.15)(0.844) = \$271.77$$
$$\text{Family Rate} = (\$590.00)(1.15)(0.844) = \$572.65$$

The type-of-business factor is usually determined by industry statistics. It can be demonstrated that employees of certain types of businesses tend to generate greater expenses than those of other types of businesses (though it is sometimes surprising which businesses produce the higher claims). The demographic factor is usually calculated as a weighted average of factors based on the typical expense of covering individuals of different ages and sexes. The details of how these factors are determined will not be covered in this text, other than observing that a group that includes a higher percentage of young members would usually expect to see a lower factor than a group with more members who are older. While in Example 13.2.6 adjustment was made for both industry group and demographics, a given insurer may adjust for only one but not both of these, or may take into account other factors instead.

With ***experience rating***, the rates for a group insurance policy are based on the past medical claims experience of that specific group. To set the rates, the insurer begins with the medical claims from the group over some period of time (typically the prior 1 to 3 years), adjusts them upward for assumed medical cost and utilization trends, and then applies a ***load*** (essentially a markup) for administrative costs, a margin of error in case claims run unusually high, and profit. There can be many other variations and adjustments made as well, but this description provides a general summary of experience rating methods.

Example 13.2.7 ***Zarofire Systems offers a medical plan to its employees. Last year, the 800 employees who participated in the plan incurred medical costs totaling $3,457,089.***

The insurer uses a rating formula that bases each year's rates on the prior year's claims, and then applies load factors of 1.05 for cost and utilization trends and 1.15 for administrative costs, including margin of error and profit. Determine the rate per employee for the coming year.

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Applying the factors to the prior year's rates gives a total of ($3,457,089)(1.05)(1.15) = $4,174,435. Dividing this among the 800 employees gives $4,174,435/800 = $5,218.04 per year. Since rates are usually given per month, we divide this figure by 12 to get $5,218.04/12 = $434.84.

In this example, costs were given per employee. Typically, different rates are charged for single or family contracts, sometimes with further distinctions among the family contracts, depending on whether or not the "family" includes a spouse or children. The details of how these costs are split among single versus family contracts fall beyond the scope of this book, other than mentioning that this split is almost never done by separating the actual claims filed for single contracts from those of family contracts. However, unless the method of splitting the costs changes from year to year, we can readily estimate rates on the basis of the prior year's.

Example 13.2.8 ***Last year, the premiums for the Zarofire plan were $258.10 for a single contract and $787.50 for a family contract. Those premiums were based on projected claims of $3,108,103. The loads did not change from last year to this year. What will the rates be this year?***

The projected claims for last year were $3,108,103. This would include the assumed cost and utilization trend, but not the administration. Building in the administrative load, we calculate that the total assumed plan cost would have been $3,108,103(1.15) = $3,574,318.

Then $4,174,435/$3,574,318 = 1.1679, so next year's projected costs are a 16.79% increase over the projection on which this year's costs were based. So we would expect next year's premiums to increase by 16.79%. Therefore, we expect the single rate to be ($258.10)(1.1679) = $301.43, and the family rate to be ($787.50)(1.1679) = $919.72.

We do not take these rates to be the exact ones for the next year, since the method of splitting costs between single and family contracts may not produce rates in exactly the same proportion next year. But we can reasonably expect these figures to be good estimates of next year's rates.

As we have discussed previously, insurance is based on the law of large numbers, which requires large numbers! While the group from the previous two examples might be large enough for us to confidently use their past experience as a predictor of their future experience, this would not be true for a smaller company. If a company had only 100 employees, we would not be able to realistically base a rate on its experience, since the group is not large enough for us to be confident that the experience seen is large enough to smooth out any "blips." On the other hand, 100 employees is probably large enough to not ignore their claims experience entirely.

Insurers using experience rating make use of ***blended rates*** to account for this. The rate for smaller experience-rated groups is determined entirely on the basis of the group's experience, and then blended with a community rate. The percent of the group's own experience to be included in the final rate is based on its size. A ***credibility table*** gives the percent to be used.

An example of a hypothetical credibility table is given below:

GREATEST LAKES HEALTH INSURANCE COMPANY—CREDIBILITY TABLE

Number of covered employees	<100	100–200	201–300	301–400	401–500	501–750	>750
Credibility	0%	20%	35%	50%	70%	85%	100%

From this table, we can see that groups with fewer than 100 covered employees would pay the community rate; groups with more than 750 covered employees would not have their rate blended with the community rate at all.

Example 13.2.9 ***On the basis of experience, Greatest Lakes' health insurance rates for the health insurance plan offered to Burned Beans Coffee Roasters would be $108 single/$242 single plus spouse or partner/$455 family. The community rates are $175/$360/$700. The company has 125 covered employees. What rates will they be charged?***

Using the table, we see that their experience is given 20% credibility. So their actual rate will consist of 20% their own experience and 80% the community rate.

For a single contract, the rate would be (20%)($108) + (80%)($175) = $161.60. Similarly, the employee + spouse/partner rate would be $336.40 and the family rate would be $651.00.

The company might feel that this rate is unfair, since the rate it will pay is much higher than the rate that its experience would suggest. However, if even one or two employees experienced a serious illness, the company's experience for that year could run much higher than the community rate. Blending means that the company will pay a higher rate than their experience in good years, but it also means they will pay a lower rate than their experience in bad years.

Health Care Savings Accounts

Despite the attempts of managed care plans to control costs, costs have nonetheless continued to escalate. ***Health care savings accounts (HSAs)*** are a type of plan that is often proposed as an alternative to provide health care coverage and control costs. An HSA is similar to a traditional deductible and coinsurance plan, except that the deductible is very large. This high-deductible policy is paired with a tax-advantaged savings account, into which the employee and/or employer makes regular deposits, which earn interest.

The idea of an HSA is that most medical claims are to be paid directly from the savings account (or from other financial sources), with the insurance policy kicking in only for extraordinarily large medical expenses. While HSAs generate plenty of discussion and political advocacy, they have met with limited success in the health care market. How significant a role they will play in future health care funding remains to be seen.

Self-Insurance

Many large companies prefer to bypass insurance coverage altogether and instead choose to ***self-insure***. With a fully self-insured plan, there is no insurance coverage in place at all. The employer offers a medical benefit package that may look to employees exactly like any other health insurance plan, but in fact the responsibility for funding the claims falls entirely on the employer. Many employers choose to back up a self-insured plan with ***stop loss coverage***, an insurance contract that kicks in only for unexpectedly large claims levels. Most of the risk is still borne by the employer, but a stop loss policy protects against catastrophically large claims.

Companies choose to self-insure for a variety of reasons. If the company believes that its employees' medical claims are likely to fall well below average, it may want to self-insure in the belief that it will pay less by funding the claims than it would by paying insurance premiums, even under an experience-rated contract. The company may believe that it can save on administrative costs by hiring a company to administer the plan's claims for less than an insurance company would build in for administrative and profit loads. Another motivator is to escape state insurance regulations. Insurance is regulated at the state level, and some states place restrictions or mandates on insurance plans that the employer may want to avoid. While self-insured health plans are subject to federal governmental regulations as an employee benefit plan, since they are not insurance, they are not subject to state laws that may mandate coverage for controversial medical procedures or limit the employer's ability to provide benefits that it wants to offer.

Health Insurance as an Employee Benefit

Some employers still bear the entire cost of health insurance coverage, but most now expect their employees to chip in for at least some of the cost of health coverage. There are numerous methods used to decide how much of the cost will be borne by the employee. Sometimes an employer will contribute a set amount of money, usually dependent on the employee's

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tier level (such as "$150 for single coverage, $450 for family"). Sometimes an employer will contribute some set percent of the cost (such as "65% of the cost"). And sometimes a more complicated formula may be used (such as "the cost of a single contract paid in full, plus half of the difference between single and family cost for a family contract").

Example 13.2.10 ***Suppose that the cost of a single health insurance contract is $300 and a family contract is $750 per month. How much would you pay per month if your employer's contribution toward a family contract is (a) a flat $600 per month, (b) 75% of the cost, or (c) the cost of a single contract, plus half of the difference between single and family costs.***

(a) If the employer pays $600, you are left to pay $750 − $600 = $150.

(b) If the employer pays 75%, you pay 25%, or (25%)($750) = $187.50.

(c) If the employer pays the single cost plus half the difference, that leaves half the difference for you to pay. The difference is $750 − $300 = $450. So you pay (50%)($450) = $225.00.

If the premium rates rise, as they almost always do from one year to the next, the effect on you as the employee can be quite a bit different, depending on the formula used.

Example 13.2.11 ***Suppose that the health insurance cost increases to $350 for a single and $875 for a family contract. What would your cost then be for the formulas in Example 13.2.10?***

(a) The employer still pays $600. You are left to pay $875 − $600 = $275.

(b) (25%)($875) = $218.75.

(c) The difference is $875 − $350 = $425. So you pay (50%)($425.00) = $212.50.

Notice that with the flat dollar formula, the employer's cost does not increase when the premium does; all of the increase flows to the employee. (Unless, of course, the employer decides to increase its $600 contribution.) With the other two formulas, the employer and employee share in the cost increase, though not necessarily to the same degree. Which type of formula is preferable depends on whether you are the employer or the employee.

Many employers offer their employees a choice of different health care plans. An employer may offer a choice of a traditional deductible and coinsurance plan or an HMO, or may offer multiple different managed care plans for employees to choose from, or some other array of alternatives. Again, the employer may set its contributions according to any number of different formulas. As the following example will illustrate, though, the formula that a business uses to determine its contributions may have a very big effect on which plans its employees choose.

Example 13.2.12 ***Suppose that a company offers its employees a choice of two different health insurance plans. The company offers a point-of-service plan or an HMO. The monthly rates for the point-of-service plan are $298.75 single and $786.92 family. The rates for the HMO are $244.08 single and $703.92 family.***

Calculate the employee costs for each plan if the employer's contribution formula is (a) 70% of the premium for either plan or (b) 90% of the cost of the less expensive plan, applied to either choice.

(a) Under the 70% formula, the employee pays 30% of the cost of any of the options. The employee costs are then:

POS:	*Single:*	*(30%)($298.75) = $89.63*
	Family:	*(30%)($886.92) = $266.08*
HMO:	*Single:*	*(30%)($244.08) = $73.22*
	Family:	*(30%)($703.92) = $211.18*

(b) Under the 90% of the less expensive plan, the employer would contribute 90% of the HMO costs to either option. For singles, this is (90%)($244.08) = $219.67; for families

this is (90%)($703.92) = $633.53. The employee is left to pay the remainder, so the costs are:

POS:	*Single:*	*$298.75 − $219.67 = $79.08*
	Family:	*$886.92 − $633.53 = $253.39*
HMO:	*Single:*	*$244.08 − $219.67 = $24.41*
	Family:	*$703.92 − $633.53 = $70.39*

Notice that under the formula in (a), there is much less difference in the employee's cost between the two plans than under the formula in (b). Under option (b) employees have a much greater financial incentive to choose the HMO plan. If the company wants to steer its employees toward the lower cost HMO plan, formula (b) would be a much better contribution formula to use.

Flexible Spending Accounts

While they are not truly an insurance coverage, many employers offer the opportunity to participate in ***flexible spending accounts (FSAs)*** for health or child care expenses. With an FSA, the employee has money deducted from her paycheck and deposited to a special account. When she pays for qualifying health or child care expenses, she can then submit a claim to the account administrator and be reimbursed for the amount paid.

Allowable expenses for medical FSAs include copayments, deductibles, and coinsurance from health insurance, as well as many other medically related expenses such as eyeglasses.

The advantage of this type of plan is that the money withheld is not subject to income tax. The avoided income taxes can provide substantial savings.

Example 13.2.13 Susannah is in the 28% tax bracket for her federal taxes and pays 6% in state income taxes. Suppose that she expects to pay about $500 in reimbursable medical expenses, and $4,000 in reimbursable daycare expenses for her daughter.

If she takes advantages of the medical and child care FSAs her employer offers, how much will she save in taxes?

If she deposits $4,500 into the FSAs, she will avoid paying state and federal income taxes totaling 34% on this money.[4] So her savings would be (34%)($4,500) = $1,530.

On the other hand, deposits made to an FSA that are not used by the end of the plan year are forfeited, so careful consideration has to be given to the amount of claims that will be made.

Exercise 13.2.14 Tung is in the 28% tax bracket for his federal taxes and 8% for state taxes. He thinks his family's reimbursable medical costs will total $720 this year, but is not sure that the total will be quite this much. If he goes ahead and contributes $720, how much would his expenses have to total in order for him not to lose money?

By depositing $720 to the FSA, he avoids (36%)($720) = $259.20 in taxes. So the deposits actually cost him $720 − $259.20 = $460.80. If his reimbursable expenses total $460.80 or more, he comes out ahead, even if some of the money in his account is forfeited.

Other Group Insurance Plans

In addition to health insurance coverage, many employers offer other types of insurance as an employee benefit. Some examples include life and dental insurance. Dental insurance takes forms similar to the medical insurance plans discussed in this chapter, though they cover only dental expenses. Life insurance is discussed in the next section.

[4]We are assuming here that removing this $4,500 from her taxable income does not drop her into a lower tax bracket. See the Additional Exercises of this section for an example where FSA contributions change someone's tax bracket.

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While most people tend to focus on their rate of pay, benefits often represent a very significant part of a worker's overall compensation, and also often represent a very significant cost to the employer. Not every employee has the same benefits needs, however. A worker who is the sole breadwinner for a family will place great value on health insurance coverage provided at a reasonable cost. For a married person whose spouse gets fully paid family coverage at work, this benefit may be of no interest at all. In fact, if he can get health coverage at no cost or lower cost through his spouse, he probably would prefer to forego this coverage and obtain other benefits instead. To address the different needs of different employees, some employers offer ***cafeteria plans***.

Under a cafeteria plan, each worker is given a range of benefit choices, and a certain amount of "benefit allowance." The employee can select the benefits she wants, using her benefit allowance toward the cost of each benefit. (If the costs of the desired benefits exceed the allowance, she can pay for these costs as a deduction from salary.)

Cafeteria plans may offer a spectrum of different insurance benefits, as well as other options, such as purchasing additional vacation time.

Because of the costs of administering these types of plans (including ensuring compliance with tax laws, which can become complicated with these sorts of plans), cafeteria plans are much more common among large companies. Cafeteria plans are discussed in more detail in Chapter 15.

EXERCISES 13.2

A. Deductible and Coinsurance Plans

In each of Exercises 1 to 10, assume that the people described are covered by a deductible and coinsurance health insurance plan with a $350/$700 deductible and 80% coinsurance. The out-of-pocket maximum is $3,000. Assume that none of the individuals have submitted any insurance claims for the year prior to the ones described.

1. Dave submits claims for himself totaling $675. How much of these claims will his insurance cover?

2. Nehan submits claims for herself totaling $325. How much of these claims will her insurance cover?

3. Bob submits claims of $775 for himself and $1,560 for his daughter. How much will his insurance cover?

4. Sid submits claims of $425 for one of his sons, $535 for his other son, and $1,250 for his wife. How much will his insurance cover?

5. Brody submits a claim of $10,000 for his wife. How much will the insurance cover?

6. Gideon submits claims totaling $585 for his wife, $355 for his son, and $820 for himself. How much will insurance cover?

7. Tessa submits claims of $300 for her husband and $400 for herself. How much will her insurance cover?

8. Annamaria submits a claim of $650 for herself. How much will insurance cover?

9. Chet has claims totaling $28,500. How much will this insurance pay for these claims?

10. Rachael has claims totaling $18,600 for the year for herself. How much will this insurance cover?

B. Calculating Insurance Premiums

11. Trustworthy Healthcare Insureco uses adjusted community rating for its PPO plan. The community rate for this plan is $195 single, $390 for a double (single person with spouse or partner), and $635 for family. Deressigt Corp. wants to offer this plan to its employees. The company's industry factor is 1.05 and its demographic factor is 0.983. Calculate the premium Deressigt Corp. would pay for this plan.

12. Suppose that health insurance rates are to be determined by adjusted community rating for a company whose demographic factor is 1.035 and whose industry factor is 0.985. The single community rate is $303.92 and the family rate is $765.55. Calculate the company's rates.

13. Second Law Capital Management offers a medical plan to its employees. Last year, the 937 employees who participated in the plan incurred medical costs totaling $5,213,929. The insurer uses a rating formula that bases each year's rates on the prior year's claims, and then applies load factors of 1.08 for cost and utilization trends and 1.15 for administrative costs, including margin of error and profit. Determine the rate per employee for the coming year.

14. Suppose that health insurance rates are to be determined by an experience rating formula. The insurer calculates the new year's rates by starting with the last year's claims, and then applies load factors of 1.10 for cost and utilization and 1.18 for administrative expenses including margin and profit. Last year's claims came to $385.96 per employee per month. Determine the rate per employee per month for the coming year.

15. Last year, Contrapolar Power Corp's health insurance rates were $293.75 single and $755.92 family. These rates were based on projected claims of $2,935,910. This year's rates are based on projected claims of $3,333,145. The administrative loads did not change, and neither did the rating formula. What will the company's rates be this year?

16. Suppose that health insurance rates are to be determined by experience rating. Last year's rates were $372.00 single and $779.55 family. Last year's projected claims were $3,500,000, while this year's projection is $3,378,904. Assuming no changes in the administrative loads or rating formula, what will this year's rates be?

Exercises 17 to 18 use the credibility table given on page 542.

17. Smallco Industries has 145 employees who participate in the company's health insurance plan. The company's rates based on full experience rating would be $168.52 single and $390.75 family. The community rate is $242.75 single and $506.75 family. Calculate the company's health insurance rates.

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18. A company has 266 employees, all of whom are covered by the company's health insurance plan. The company's rates based on full experience rating would $285 single and $690 family; the community rates are $205 single and $625 family. Calculate the company's health insurance rates.

19. While you are having lunch with your old friend, the president of Smallco Industries (from Exercise 17), she complains to you that her company is paying insurance rates that are higher than her employees' claim experience really justifies. She is thinking about self-insuring. Is this a good idea?

C. Health Insurance As an Employment Benefit

Base your answers to Exercises 20 to 23 on a monthly single rate of $255.95 and a family rate of $755.92.

20. Calculate the (a) single and (b) family employee monthly health insurance costs if the employer contributes 60% of the cost.

21. Calculate the (a) single and (b) family employee monthly health insurance cost if the employer contributes $200 toward single and $500 toward family coverage.

22. Calculate the (a) single and (b) family employee monthly health insurance cost if the employer contributes the full cost of a single contract plus half the difference between single and family costs.

23. Calculate the (a) single and (b) family monthly employee monthly health insurance cost if the employer contributes 90% of the cost of a single contract plus 75% of the difference between single and family costs.

D. Flexible Spending Accounts

In Exercises 24 to 25, assume that participating in an FSA does not change either person's tax bracket.

24. Lorraine anticipates having $750 in reimbursable medical and $3,000 in reimbursable child care expenses. If she is in the 28% federal and 4.5% state income tax brackets, how much could she save by using a flexible spending account for these expenses?

25. Drew anticipates $475 in reimbursable medical and $2,500 in reimbursable child care expenses. If he is in the 25% federal and 7% state income tax brackets, how much could he save by using a flexible spending account for these costs?

E. Grab Bag

Keely Powertrain Corp. offers its employees a choice of two different health care plans. The company has just been given the rates for these two plans for next year. They are given in the table below.

	Traditional Indemnity Plan	***Point of Service Plan***
Single	$337.50	$219.45
Family	$955.43	$632.36

Use this table in answering Exercises 26 to 30.

26. Keely Powertrain currently pays 85% of the cost of a single contract and 65% of the cost of a family contract. According to this formula, how much would the employees' monthly costs be for these plans?

27. The company is considering changing its formula for how much it will pay toward health insurance. If company switches to a formula where it will pay the full cost of either plan for single coverage and it will contribute the full cost of a single contract plus half the difference between single and family for family coverage, how much would the employees' monthly costs be with this new formula?

28. Suppose that the company decides that it will contribute the full single or family cost of the lower cost plan toward either of the two plans. What would the employee's monthly costs be then?

29. Keely Powertrain's management is very concerned about managing its health care benefit costs, and would like to strongly encourage its employees to select the lower cost point-of-service plan. Which of the employee cost formulas proposed in Exercises 26 to 28 would be the best choice if management wants to do this?

F. Additional Exercises

30. Suppose that Pedro's income puts him $1,850 over the cutoff for the 25% federal income tax bracket. The next lower bracket is 15%. He lives in Washington, which has no state income tax. He anticipates paying $2,500 for child care next year. How much would he save by using a flexible spending account for these expenses?

31. Some health insurance companies will offer their insurance to an employer group only if the employer contributes at least half of the cost of the insurance. Why do you suppose this is?

32. Some health insurance companies will not offer a low copayment plan to an employer if the employer also offers a less-expensive, higher copayment plan. Why do you suppose this is?

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13.3 Life Insurance

Your own death is not a pleasant subject to think about, nor is the death of your loved ones. But that doesn't change the unavoidable fact that we will all, sooner or later, die. The purpose of ***life insurance*** is to provide financial protection to the people who financially depend on you when you pass away.

A life insurance policy is a contract between a policyholder (also called a ***policy owner***) and an insurer in which the insurer agrees to pay a set amount of money, called the ***death benefit***, to a specified person or group of people, called the ***beneficiaries***, on the death of a certain individual, called the ***insured***. The insured and policyholder are often the same person, but they don't have to be; a woman might own an insurance policy on her husband (or vice versa), for example.

The purpose of life insurance is to protect the survivors from the financial consequences of the insured's death. The size of policy needed therefore depends on what those financial consequences are likely to be. A person might have a relatively small $20,000 policy intended to provide for funeral and burial expenses, or a family's breadwinner might have a $500,000 policy intended to financially support the surviving family. A small business may have insurance on one of its partners to protect the business and surviving partners from the financial consequences of losing a key person.

There are several different types of life insurance. Regardless of the type, though, when an insurance policy is applied for, the insurance company will go through a process of ***underwriting***. Just as an auto insurer charges different rates for different people, depending on factors used to assess the likelihood of claims, life insurers charge different rates, depending on factors that may indicate the likelihood of a claim. In this case, the "likelihood of a claim" means the likelihood of dying in the near term. Age and sex are the most basic factors; in general, an older person will pay more for life insurance than will a younger one. Also, since women tend to live longer than men, life insurance premiums for women tend to be lower. A third major characteristic taken into account by insurance underwriters is smoking (or other tobacco use). Smokers pay significantly higher rates for life insurance than do nonsmokers.

In their rate books, life insurers set a rate (usually expressed as a rate per $1,000 of death benefit) for each age, sex, and smoking status. These are not the only underwriting considerations, however. Life insurers may also take into account other indicators of health, such as weight, cholesterol level, family health history, dangerous occupations or hobbies, or diagnosed medical conditions. Each insurer has its own criteria for what is necessary to qualify for a ***standard*** policy, issued on the basis of the book rates. The underwriting criteria may be much more stringent for large policies than for small ones, since the insurer's risk is so much greater on a large policy.

An individual who does not meet these standards may be sold insurance at a higher rate (sometimes called a ***substandard rating***), or may be denied coverage altogether. Some insurers also have a ***preferred*** underwriting category, with lower rates offered to potential insureds who, according to their current health and family health history, appear to present an especially low risk of a claim.

When insurance is sold as a group policy, the insurer may be less stringent in its underwriting than for individual policies. Often, the insurer will charge rates based only on the age/sex/smoking status of the individual insureds, excluding only those who are in particularly poor health. An insurer may even offer a ***guaranteed issue*** contract, agreeing to cover all members of the group without exception. In cases where the group members have a choice of how much coverage to purchase, the insurer may agree to offer a certain level of death benefit guaranteed issue (such as "1 year's salary"), but more carefully underwrite members of the group who choose to purchase larger policies.

Term Insurance

The simplest form of life insurance is ***term insurance***. Term insurance provides coverage for a set period of time. If the insured dies within that period of time, the death benefit is paid; if not, the policy expires. (Term policies often contain a provision that they may be renewed without having to undergo underwriting at the end of the term.) ***One-year term*** is by far the simplest form; as the name suggests, it provides coverage for a term of 1 year. Term policies are also commonly issued with terms of 5, 10, or 20 years. ***Level term*** policies have a premium that does not increase during the policy's term. As you might expect, level term policies tend to carry higher premiums than policies whose premiums increase with time, to compensate for the fact that the premiums do not increase with age even though the risk of near-term death does.

Premiums for term policies can be quite economical. While a working person's death might be financially disastrous for her family, the likelihood of a reasonably young person, in good enough health to pass underwriting, dying in the near term is quite low. Even after factoring in agent commissions, expenses, profit, and other costs, this low likelihood of a claim can produce surprisingly low premiums.

Rates are typically expressed per $1,000 of coverage, sometimes varying depending on the size of the policy. There may also be a flat amount added to all policies, regardless of the policy size. The following is a typical example of a portion of a term insurance rate book. (Only a few ages are shown.)

TRUSTWORTHY MUTUAL LIFE INSURANCE COMPANY
ANNUAL RATES PER $1,000 PLUS $25 ANNUAL POLICY CHARGE
RATES FOR MINIMUM $100,000 MAXIMUM $249,999
PER $1,000 RATES REDUCED BY 3% FOR ISSUE AMOUNTS $250,000 TO $500,000

ANNUALLY RENEWABLE TERM

	MALE			FEMALE		
Age	***Preferred***	***Standard***	***Smoker***	***Preferred***	***Standard***	***Smoker***
35	$0.89	$1.02	$1.85	$0.78	$0.89	$1.62
36	$0.89	$1.04	$1.88	$0.78	$0.91	$1.64
37	$0.92	$1.05	$1.90	$0.80	$0.92	$1.66
38	$0.93	$1.06	$1.94	$0.81	$0.93	$1.70
39	$0.95	$1.06	$1.96	$0.83	$0.93	$1.71
40	$0.95	$1.08	$2.01	$0.83	$0.94	$1.76

20-YEAR LEVEL TERM

	MALE			FEMALE		
Age	***Preferred***	***Standard***	***Smoker***	***Preferred***	***Standard***	***Smoker***
35	$1.26	$1.45	$2.63	$1.11	$1.26	$2.30
36	$1.26	$1.48	$2.67	$1.11	$1.29	$2.33
37	$1.31	$1.49	$2.70	$1.14	$1.31	$2.36
38	$1.32	$1.51	$2.75	$1.15	$1.32	$2.41
39	$1.35	$1.51	$2.78	$1.18	$1.32	$2.43
40	$1.35	$1.53	$2.85	$1.18	$1.33	$2.50

Example 13.3.1 ***Using the table above, determine the annual premium for (a) a $200,000 annually renewable term policy for a male smoker aged 38 and (b) a $350,000 20-year level term policy for a female preferred age 35.***

(a) The rate per $1,000 is $1.94, plus a flat policy charge of $25. So the premium would be (200)($1.94) + $25 = $388.00 + $25.00 = $413.00.

(b) The rate per $1,000 is $1.11. Since the table states that there is a 3% discount for death benefits above $250,000, the actual rate would only be 97% of this however. The premium would then be (350)(0.97)($1.11) + $25 = $376.85 + $25 = $401.85.

Insurance premiums may be paid annually, or may be paid at other common frequencies such as semiannually, quarterly, or monthly. Insurance rate books may contain factors that can be multiplied by the annual rate to determine premiums for other payment frequencies. For example:

TRUSTWORTHY MUTUAL LIFE INSURANCE COMPANY
PERIODIC PAYMENT FACTORS

If premium is paid:	***Annually***	***Semiannually***	***Quarterly***	***Monthly***
Multiply annual rate by:	1	0.52	0.265	0.135

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Example 13.3.2 ***If the annual premium for a term insurance policy is $413.00, what would the premium be from the table above if the premium is paid (a) semiannually, (b) quarterly, and (c) monthly.***

(a) (0.52)($413.00) = $214.76 semiannually

(b) (0.265)($413.00) = $109.45 quarterly

(c) (0.135)($413.00) = $55.76 monthly

The semiannual rate is not exactly half the annual rate, nor is the quarterly rate a quarter, nor the monthly rate one-twelfth. This difference reflects the insurance company's higher costs of processing multiple payments, the fact that the insurance company loses the opportunity to earn as much interest when the entire premium is not paid up front, and the possibility that, if a claim is filed during the year, the insurer will not collect any premium after the claim is filed.

As mentioned previously, if an applicant for life insurance does not meet the insurer's usual underwriting criteria, the insurer may still offer to issue a "substandard" policy at an increased rate. The amount of the increase is indicated by an underwriting classification called a ***substandard rating*** or ***table rating***. Insurers do not all use the same terminology to indicate table ratings. The rating is often expressed as a percent of the standard premium. Another common method is to use letters of the alphabet, with each successive letter representing a 25% increase over the standard premium.[5] For example, a "Table C" rating would mean a rate that is 75% over the standard premium. This rating could also be expressed as a "175% table rating."

Normally, the increase applies only to the portion of the rate which is based on policy size; it does not apply to the flat per policy fee.

Example 13.3.3 ***Amy is a 38-year-old smoker who has applied for a $180,000 annually renewable term policy with Trustworthy Mutual. Because of her high cholesterol, the company's underwriter has rated her at Table B. What would the quarterly premium be for this policy?***

Table B means 150% of the usual premium (not including the policy fee). So Amy's annual premium would be (1.50)(180)($1.70) + $25 = $484.00. Her quarterly premium would then be (0.265)($484.00) = $128.26.

The rates shown in the examples above are intended to illustrate the general method of calculating term insurance premiums; they should not be taken as an indication of the actual rates that anyone might be expected to pay. As with other types of insurance, rates can vary considerably from one company to another.

Whole Life Insurance

Term insurance is intended primarily to provide life insurance coverage for a limited period of time, particularly for times of life when death is unlikely. Though many term policies can be renewed annually, as the insured gets older the costs become prohibitively high. The risk of a 35-year-old who has passed underwriting standards dying in the near term is remote. The risk of a 75-year-old who passed underwriting 40 years ago dying in the near term is not nearly so remote. If the life insurance is bought as protection against premature death during one's working years, or until children are grown, or until the mortgage is paid off, then term may be the best fit. Though most such policies expire without ever paying their death benefits, they provide enormous peace of mind while they are in force. However, for someone who wants a policy that will remain in force until death, and pay a death benefit regardless of when death occurs, term insurance may be poor fit.

Whole life insurance policies are policies that are intended to remain in force for the insured's entire remaining life. With a whole life policy, the premium is set when the policy

[5] The 25% per letter increment is common, but not always used. "Table A" normally means 25% above the usual premium, but for another insurer it may mean 50% or some other percent. In this text we will assume the 25% per letter increment, but the reader should be aware that this cannot always be assumed in practice.

is issued, and does not increase as the insured gets older. The premium in the early years of a whole life policy is much higher than the rate for a comparable death benefit term policy, but in the later years that premium will be much lower than what a comparable term policy would cost. The basic idea is that the extra premium paid in the early years is used to "save up" to subsidize the cost of coverage in the policy's later years.

Determining the appropriate rate for a whole life policy is a challenge, requiring the insurer to take into account the probability of death at different ages, as well as the future rate of investment return on the "extra" premium paid in the early years. Insurance companies employ financial professionals know as ***actuaries*** to perform these calculations. Once an actuary has determined the appropriate rates, though, whole life insurance rates may be given in tables similar to those used for term insurance.

Whole life policies require the policyholder to pay the required premium throughout the insured's life. For typical life expectancies, a whole life policy bought on, say, a 25 year old insured may run for 50 years or more before paying its death benefit. A lot can happen over the course of 50 years. Regardless of the policyholder's intent when the policy was purchased, it is not unusual for a whole life policy to not remain in force until the insured's death. If the policy premium is not paid, the policy will not remain in force, in which case it is said to ***lapse***. If the policy owner decides that he no longer wants to continue the policy, we say that he ***surrenders*** the policy.

This poses a question, though. If a whole life policy lapses, what happens to the "savings" that were building up to subsidize the policy in later years? It does not seem quite fair that these should be entirely forfeited, and in fact they are not. A distinctive feature of a whole life policy is that it accumulates a ***cash value***. If the insurance policy lapses, the policy owner is entitled to receive the policy's cash value. In the early years of a whole life policy the cash value will be quite small, but over time the cash value grows as additional premiums are paid and invested by the insurer. When a policy is first issued, the policyholder is provided with a table of the policy's cash value over the course of its life.

Whole life policies offer their policyholders alternatives to taking the cash value in cash. Since these represent the alternatives to simple forfeiting the built-up policy values, these are referred to as ***nonforfeiture options***. The policyholder may select ***reduced paid up (RPU)*** insurance, a life insurance policy with a reduced death benefit that is considered paid in full by the cash value. An RPU policy cannot lapse as a result of nonpayment of premium, since there is no premium due. Another common option is ***extended term*** coverage. With extended term, the whole life policy is replaced with a term policy having the same death benefit. The cash value is used to pay for this term policy for a set period of time. Extended term is often the default option in the event that a policy lapses and the insurance company is unable to contact the policyholder. (It happens.) A table of reduced paid up and extended term rates for a whole life contract will be provided when the policy is issued. An example of a typical table is shown below:

End of Contract Year	***Cash Value (per $1,000)***	***RPU (per $1,000)***	**EXTENDED TERM**	
			Years	***Days***
1	$0	$0	0	0
2	$0	$0	0	0
3	$9	$25	3	104
4	$23	$61	8	82
5	$36	$94	11	263
6	$50	$127	14	148
7	$65	$162	16	189
8	$80	$195	18	21
9	$95	$226	19	66
10	$110	$256	20	7
Age 65	$610	$819	15	265

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Example 13.3.4 ***Suppose that the nonforfeiture table shown above is for a $150,000 death benefit policy. If the policyowner decides to surrender the policy at the end of the 8th contract year, what would he receive if he chose to surrender the policy for (a) cash, (b) RPU, or (c) extended term.***

(a) The cash value at the end of year 8 is $80 per thousand. So the cash value is (150)($80) = $12,000.

(b) The RPU at the end of year 8 is $195 per thousand. So the policy could be surrendered for a paid-up policy with a (150)($195) = $29,250 death benefit.

(c) The $150,000 death benefit could be continued with extended term for 18 years and 21 days.

The insurer does not have the opportunity to change whole life insurance premiums once a policy is issued. When setting rates, the actuary must make predictions about investment rate of return and life expectancies far into the future. If the assumed rate of return or of claims proves overly optimistic, the insurer has no opportunity to adjust the premiums to reflect this. Thus, insurers have to be fairly conservative about the assumptions they use in calculating whole life premiums.

But this conservatism makes it likely that the rates of return and of claims will actually turn out to be quite a bit better than the assumed ones. With a ***participating*** whole life policy, the policy owner may be paid dividends based on the insurer's actual experience with policies of a given type. The dividends may be paid in cash, may be used to reduce premiums, or may be used to purchase additional chunks of paid-up death benefit (known as ***paid-up additions***). Not all life insurance policies are participating, though. With a ***nonparticipating*** policy, the policy owner does not receive any such dividends. A participating policy may carry a higher up-front premium than a nonparticipating one, but in the long run the ability to earn dividends may more than make up for the higher premium.

Rates for whole life insurance are usually calculated from a rate table in much the same way as term insurance rates are. If a substandard rating is applied, however, it may not result in a simple percent increase as it did with a term policy.

A third type of life insurance is also quite popular; in addition to being important in its own right, it can provide some insight into how whole life insurance works.

Universal Life Insurance

Traditional whole life insurance does not offer much flexibility in how the premiums are paid, and the whole process of determining the cash values and dividends is not at all transparent. ***Universal life insurance*** provides an alternative to traditional whole life. Universal life insurance functions much like an investment account with a term insurance policy attached to it. When a premium is paid on a universal life policy, the funds are deposited to an investment account (often less a percent charge called a ***load***, and, in some states, premium taxes). Each month interest is credited to the account and a deduction called a ***mortality charge*** or ***term cost*** is made for the cost of the insurance coverage (comparable to the cost of a term insurance policy).

For example, suppose that you have a newly purchased universal life insurance policy. The death benefit is $100,000, and you pay a $2,000 premium up front. The policy charges a 3% load on all premiums, and pays 6% interest compounded monthly. The monthly insurance charge is given as $0.043 per thousand, plus a $2.50 monthly service charge. We can illustrate the first few months of this policy by means of a table:

Month	Premium (Less Load)	Mortality Charge	Fees	Interest	End Balance
1	$1,940.00	$4.22	$2.50	$0.00	$1,933.28
2	$0.00	$4.22	$2.50	$9.67	$1,936.23

In the first month, the $2,000 premium is paid, which leaves $1,940.00 after the 3% load is taken off. The mortality charge is calculated as follows. The death benefit is $100,000. However, since the insurance company is holding an account value of $1,940 on the policy, the insurance needs to provide only $100,000 − $1,940 = $98,060. (This amount is called the ***net amount at risk***.) So the term cost of insurance applies only to that amount of a benefit; therefore it will cost (98.060)($0.043) = $4.22. No interest is earned at the start of the first month since the money has just been deposited. So the balance of the account value is $1,940 − $4.22 − $2.50 = $1,933.28. In the second month, the same process is repeated, except that one month's worth of interest will be paid on the $1,933.28 that was on deposit for the month. At the end of the second month, then, the account value is $1,933.28 − $4.22 − $2.50 + $9.67 = $1,936.23.

Whole life policies can be thought of as working in much the same way; the difference is that with a whole life policy none of this is apparent to the policyholder. With a whole life policy, the insurer sets a fixed premium, which has been determined in advance to be adequate to carry the policy through the insured's lifetime, with interest credited at an assumed rate and the mortality costs charged at an assumed rate. With universal life, all of this is transparent to the policyholder; the policy's annual statement will show the interest paid and fees charged directly. Universal life also offers the advantage that, while a set premium may be scheduled for the policy, the policyholder may choose to pay more or less than the scheduled premium.

Example 13.3.5 ***Calculate the account value at the end of the third month for the universal life insurance policy shown above. (Assume that no additional premiums are paid.)***

In the third month, the net amount at risk is $100,000 − $1,936.19 = $98,063.81. So the term cost is (98.06381)(0.043) = $4.22. We also deduct the $2.50 monthly fee. Interest will be earned equal to ($1,936.23)(0.06)(1/12) = $9.68. Putting this all together, the account value will be $1,936.23 − $4.22 − $2.50 + $9.68 = $1,939.19.

Month	Premium (Less Load)	Mortality Charge	Fees	Interest	End Balance
1	$1,940.00	$4.22	$2.50	$0.00	$1,933.28
2	$0.00	$4.22	$2.50	$9.67	$1,936.23
3	$0.00	$4.22	$2.50	$9.68	$1,939.19

A spreadsheet could be set up similar to the ones used in Chapter 5 to carry this out to future months.

Notice that, since the monthly interest is higher than the cost of insurance, the policy's value is growing from month to month. This may, but does not need to, happen. If instead you had deposited $500 in premium, the interest earnings would be less than the insurance cost, and so the policy value would decline from month to month. However, a $500 premium would still be adequate to cover the insurance costs in the near term. One of the advantages of universal life is that it allows a great degree of flexibility in how premiums are paid. A policyholder who wants to pay a large premium up front, but then not pay much thereafter, can. A policyholder who wants to make smaller but more frequent payments can as well. Unlike whole life insurance, though, the policyholder can choose to vary the premium amounts and timing to suit his needs.

To reflect the fact that the insurer undertakes large expenses of commissions, underwriting, and policy issue expenses up front, universal life policies usually carry a ***surrender charge***. If the policy is surrendered, the policyholder receives the policy's account value less the surrender charge. The account value less the surrender charge represents the cash value of the policy at any given point in time. Surrender charges are usually huge in the early years of the policy, declining to zero over a period of many years. A universal life policy remains in good standing as long as the account value is larger than the surrender charge. In the first few years of the policy, though, the account value is unlikely to be larger

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than the surrender charge, so the policy will contain a provision that as long as a certain minimum amount of premium is paid the policy will remain in force during the first few years.

Example 13.3.6 ***For the universal life insurance policy used in Example 13.3.5, the surrender charge is $5,000 in the first year, declining by $500 per year until it reaches $0 after 10 years. (a) If the policy is surrendered after 3 months, how much will the policyholder receive? (b) If the policy's balance has grown to $4,843.25 after 4 years, how much would the policyholder receive if he surrendered at that point?***

(a) After 3 months, the surrender charge is $5,000, which is more than the account value. If the policy lapses or is surrendered, the policyholder would receive nothing, and the entire account value would be forfeited to the insurance company.

(b) After 4 years, the surrender charge will have declined to $5,000 − 4($500) = $3,000. If the policy is surrendered, the policyholder would receive $4,843.25 − $3,000 = $1,843.25.

The interest rate paid, and the mortality costs charged, change over time. At issue, the insurer provides a guaranteed minimum interest rate to be paid, and a table of guaranteed maximum mortality cost rates (based on age) to be charged. These rates will usually be extremely conservative. During the course of the policy, the insurer will set the actual rate to be paid based on the insurer's investment performance and prevailing interest rates in the market, and a mortality cost rate based on a realistic but less overcautious assessment of rates of death. The insurer will provide an illustration of the future cash values of the policy both based on the guaranteed rates and on the rates that are actually being used when the policy is issued. (Both are based on an assumed rate of premium payment.)

Universal life policies tend to be popular when interest rates are high. With high rates, the "current rates" illustrations look very attractive, as high interest on the policy compounds, keeping the interest above the insurance costs and leading to large projected future values. When interest rates are lower, though, the projections don't look nearly as appealing. Over the 1990s and early 2000s, interest rates were quite low, and universal life did not rouse as much interest as it did in earlier, higher interest rate periods. However, as interest rates rise, universal life may come back into prominence. In considering this product it is important to understand how things actually work. Many people who bought universal life policies in the 1980s when interest rates were high and "current rate" projections looked wonderful were very disappointed in how their policies actually performed in the 1990s when interest rates dropped.

An alternative form of this type of policy is ***variable universal life***. Variable universal life works much like regular universal life, except that instead of the account value earning interest, it is invested in portfolios similar to mutual funds. Though the investment choices usually include a wide range of options, variable universal life tends to attract more attention when the stock market is performing well.

Universal life policies are most commonly sold on the individual market. Group offerings, while not unheard of, are not very common.

Other Types of Life Insurance

While the types of life insurance described thus far represent the vast majority of the market, there are some variations. There are whole life policies that are intended to be paid with a limited number of premiums; for example, a variant of a whole life policy might require premium payments for 20 years, and then be considered paid up thereafter. Many other variations exist.

Life insurance policies, whether term, whole life, or universal, commonly offer a ***waiver of premium*** rider. For an additional premium, a waiver of premium rider provides that if the policyholder becomes disabled, the life insurance policy will continue in force without the premium being paid.

Accidental death benefits may be sold either as riders to a regular policy, or as a policy in their own right. This coverage provides a death benefit only if the insured dies in an accident. Accidental death policies are quite popular, mainly because they can provide a large death benefit for a very modest premium, and because people tend to overestimate the likelihood that they will die in a plane crash or car wreck of some similar accident. Accidental death benefit insurance is often offered as part of cafeteria benefit plans because of this popularity. However, the reason this coverage is so inexpensive is that it is very unlikely to pay off. Also, it is not clear how the way that someone dies would affect her family's financial needs following that death. ***Accidental death and dismemberment*** policies provide a benefit in case of accidental death, and also a benefit in case of loss of limb in an accident.

Joint life policies pay a death benefit based on the deaths of two or more people. A ***first-to-die*** policy pays its death benefit when the first of a specified group of insureds dies. A married couple, each looking to provide for the other in the event of death, may find that a first-to-die policy covering them together is less expensive than if each person were to buy an individual policy. A first-to-die policy will, however, be more expensive than one individual policy. ***Last-to-die*** policies pay off when the last of a specified group dies. These policies may be used by a couple concerned about providing for their dependents if both die (but not concerned about either spouse's ability to manage without the other), and also are commonly used to prepare to pay estate taxes.

EXERCISES 13.3

A. Term Insurance

For Exercises 1 to 10 calculate the premium for the policy specified, using the Trustworthy Mutual term insurance rate tables on page 551.

1. $150,000 annually renewable term. 36-year-old standard female nonsmoker. Annual premium.

2. $185,000 20-year level term. 38-year-old female smoker. Quarterly premium.

3. $215,000 annually renewable term. 35-year-old male preferred. Monthly premium.

4. $250,000 annually renewable term. 37-year-old male preferred. Monthly premium.

5. $350,000 20-year level term. 39-year-old female preferred. Semiannual premium.

6. $425,000 annually renewable term. 35-year-old make smoker. Quarterly premium.

7. $325,000 annually renewable term. 38-year-old male standard nonsmoker. Monthly premium.

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8. $125,000 20-year level term. 35-year-old female preferred. Semiannual premium.

9. $175,000 annually renewable term. 35-year-old male smoker. Annual premium. Table B.

10. $100,000 annually renewable term. 39-year-old female standard nonsmoker. 150% table rating. Monthly premium.

B. Whole Life Insurance

Refer to the whole life insurance rate table given below for Exercises 11 to 16. Assume that the same semiannual/quarterly/monthly multiples apply for Trustworthy Mutual's whole life and term policies.

TRUSTWORTHY MUTUAL LIFE INSURANCE COMPANY
ANNUAL RATES PER $1,000
RATES FOR MINIMUM $100,000 MAXIMUM $249,999
PER $1,000 RATES REDUCED BY 3% FOR ISSUE AMOUNTS $250,000 TO $500,000

PARTICIPATING WHOLE LIFE INSURANCE						
	MALE			FEMALE		
Age	*Preferred*	*Standard*	*Smoker*	*Preferred*	*Standard*	*Smoker*
25	$2.03	$2.49	$3.48	$1.91	$2.05	$2.91
26	$2.07	$2.52	$3.51	$1.93	$2.09	$2.95
27	$2.10	$2.55	$3.55	$1.95	$2.13	$2.96
28	$2.14	$2.59	$3.59	$1.97	$2.17	$3.01
29	$2.17	$2.62	$3.64	$2.00	$2.22	$3.03
30	$2.20	$2.65	$3.72	$2.02	$2.24	$3.08

Calculate the insurance premium for each of the following situations:

11. Female age 28, standard nonsmoker, $150,000. Annual premium.

12. Female smoker, age 30, $225,000. Annual premium.

13. Male preferred age 28. $180,000. Quarterly premium.

14. $214,500 policy, female standard nonsmoker age 25. Monthly premium.

15. $350,000 policy, male standard nonsmoker age 26. Semiannual premium.

16. $250,000 policy, male age 27, preferred. Monthly premium.

Refer to the table below for Exercises 17 to 22. This table of nonforfeiture options was provided with the issue of a $50,000 death benefit life insurance policy.

End of Contract Year	**Cash Value (per $1,000)**	**RPU (per $1,000)**	**EXTENDED TERM**	
			Years	**Days**
1	$0	$0	0	0
2	$0	$0	0	0
3	$14	$42	2	100
4	$39	$112	7	32
5	$48	$132	10	63
6	$60	$145	13	48
7	$75	$165	17	19
8	$95	$198	17	321
9	$108	$235	18	96
10	$125	$266	19	105
Age 65	$575	$809	16	33

Calculate the nonforfeiture value of the policy if it is surrendered:

17. At the end of year 5 for cash value.

18. At the end of year 2 for cash value.

19. At the end of year 7 for RPU.

20. At age 65 for RPU.

21. At the end of year 7 for extended term.

22. At the end of year 5 for extended term.

C. Universal Life Insurance

23. A universal life policy has a $300,000 death benefit. The account value is currently $5,792. What is the net amount at risk?

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24. Suppose that the monthly mortality charge is $0.095 per thousand and that the death benefit is $115,000. The interest rate is 5% compounded monthly. Complete the missing values labeled with letters in the table below:

Month	Premium (Less Load)	Mortality Charge	Fees	Interest	End Balance
April					$8,933.28
May	$0.00	(a)	$2.50	(b)	(c)

25. Suppose that the monthly mortality charge is $0.1825 per thousand and the death benefit is $200,000. The interest rate is 5.36% compounded monthly. Complete the missing values in the table below:

Month	Premium (Less Load)	Mortality Charge	Fees	Interest	End Balance
September					$7,205.65
October	$0.00	(a)	$2.50	(b)	(c)

26. The surrender charges for a universal life policy are $4,000 in the first year, declining by $250 per year until they reach zero. The policy has an account value of $3,750.43 at the end of year 3. What are the surrender charges at that time? How much would you receive if you surrendered the policy at that time?

27. The surrender charges for a universal life policy were $4,800 in the first year, declining by $400 per year until they reach zero. At the end of year 7 the policy had an account value of $4,141.55 and it was surrendered. How much did the policyholder receive?

D. Grab Bag

28. Which type of life insurance (term, whole life, or universal life) would you be most likely to choose if you are looking to get the highest possible death benefit at the lowest possible cost right now?

29. Which type of life insurance (term, whole life, or universal life) would you be most likely to choose if you plan to keep the policy for a long time, but want to be able to vary your premium payments, paying more when you have extra cash and less when you don't?

E. Additional Exercise

30. Suppose I have a $100,000 death benefit universal life policy. The interest rate is 6% compounded monthly, and there is a $2.50 monthly policy fee. I made no premium payments either this month or last. At the end of last month my account value was $4,503.29. At the end of this month my account value is $4,511.38. What is the monthly mortality cost per thousand?

Topic	Key Ideas, Formulas, and Techniques	Example(s)
Pure Premium and the Law of Large Numbers, p. 525	• The pure premium of an insurance policy is the predicted amount of claims divided by the number of policies. • Over a large number of policies, the average claims per policy should be very close to the pure premium.	An insurance company offers a policy that pays a flat $5,000 if a policyholder is a victim of identity theft. The company believes that on average 1 out of every 287 policyholders will have a claim on its policies each year. Calculate the pure premium for this policy. (Example 13.1.1)
Calculating Auto Liability Premiums from a Rate Table, p. 528	• Use the rate table to find the appropriate rate based on the insured's information. • If there are adjustment factors (such as for age/sex), obtain these from the factor table and multiply by the table rate.	Using a given rate table above, determine the semiannual rate for a policy for a 17-year-old male with no driver training who lives in Region A and commutes 4 miles per day. (Example 13.1.2)
Deductibles, p. 529	• A deductible is the amount that must be paid by the insured for a covered event before the insurance makes any payment. • Subtract the deductible from the amount of the claim. • If the deductible is larger than the claim, the insurer pays nothing.	Adam has comprehensive coverage on his car. The deductible is $500. The car's value is $13,200, and it is stolen. How much will the insurer pay on this claim? (Example 13.1.4)
Coverage Limits, p. 530	• A coverage limit is the maximum amount the insurer will pay for claims on a policy. • Auto insurance liability policies often state their limits as a maximum per person followed by an overall maximum for liability. • There may also be a third limit that applies to damage to property.	Kelsi has a 25/75/50 motor vehicle liability policy. Kelsi caused a car accident in which she injured two people and caused $18,500 in property damage. A court awarded one of the victims a $35,000 judgment, and $15,000 to the other. How much will her insurance company pay? (Example 13.1.8)
Coinsurance Clauses, p. 531	• If the percent of the property's value that is insured falls below the coinsurance percent, the amount of the claim to be paid is reduced. • Divide the insured value by the actual value, and multiply the results by the amount of the claim. • If the insured percent is more than the coinsurance percent, the claim is covered in full up to the insured amount.	Your warehouse and its contents are worth $2,500,000. You insure the property for $1,500,000. Your policy contains a coinsurance clause. There is a fire, and you incur a $1,200,000 loss. How much will your insurance pay? (Example 13.1.9)

(Continued)

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Topic	Key Ideas, Formulas, and Techniques	Example(s)
Deductibles and Coinsurance with Health Insurance, p. 539	• The deductible may be stated as an amount per person and an overall deductible. • To find what the insurer will pay, subtract the per person deductible from each person's claims and apply the coinsurance percent. • Do not subtract more than the overall deductible however.	Ken is a single father with three children covered under a family health insurance policy. The policy provides 80/20 coinsurance with a \$250/\$500 deductible. Ken submits claims of \$840 for himself, \$100 for one child, and \$670 for another child. He has not submitted any other claims this year. How much will his health insurance pay? (Example 13.2.4)
Adjusted Community Rating, p. 541	• Determine the appropriate demographic or other multipliers for the employer group. • Multiply these factors by the community rate.	Greatest Lakes Health Insurance Corp.'s community rate for its standard HMO plan in the Detroit metropolitan area is \$280 single, \$590 family. A Detroit area school district wants to offer this HMO to its employees. Great Lakes uses adjusted community rating. The rate factor for public schools is 1.15, while the demographic factor for this school district is 0.844. Determine the premium for this school district. (Example 13.2.6)
Experience Rating, p. 541	• Project the claims for the group based on prior year claims. • Multiply by load factors for administrative and other costs. • If the loads do not change from one year to the next, the experience rate may be estimated by multiplying: $\text{Current rates}\left(\dfrac{\text{prior claims projection}}{\text{current claims projection}}\right)$	Last year, the premiums for the Zarofire plan were \$258.10 for a single contract and \$787.50 for a family contract. Those premiums were based on projected claims of \$3,108,103. This year's projection is \$4,174,435. The loads did not change from last year to this year. What will the rates be this year? (Example 13.2.8)
Credibility Tables, p. 542	• Smaller groups may have rates that are only partially based on their own experience. • A credibility factor is based on the group's size. • The group's rate is the credibility factor times their rate based on their own experience plus (100% − the credibility factor) times the community rate.	From experience, Greatest Lakes' health insurance rates for the health insurance plan offered to Burned Beans Coffee Roasters would be \$108 single/\$242 single plus spouse/partner/\$455 family. The community rates are \$175/\$360/\$700. The company has 125 covered employees. What rates will they be charged? (Example 13.2.9)

(Continued)

Topic	Key Ideas, Formulas, and Techniques	Example(s)
Employee Contributions, p. 544	• To determine the employee's portion of group health insurance premiums, follow the formula that the company uses to determine this.	Suppose that the cost of a single health insurance contract is $300 and a family contract is $750 per month. How much would you pay per month if your employer's contribution toward a family contract is the cost of a single contract, plus half of the difference between single and family costs? (Example 13.2.10)
Term and Whole Life Insurance Rates, p. 551	• Obtain the rate per thousand for the insured's age/sex/underwriting classification from a table. • Multiply by the number of thousands of death benefit. • Multiply by any rate adjustments, such as discounts or substandard ratings. • Add the policy fee, if any. • Multiply by the factor for the payment frequency (annual, semiannual, quarterly, monthly).	Amy is a 38-year-old smoker who has applied for a $180,000 annually renewable term policy with Trustworthy Mutual. Because of her high cholesterol, the company's underwriter has rated her at Table B. What would the quarterly premium be for this policy? (Example 13.3.3)
Nonforfeiture Options, p. 553	• If a whole life policy is surrendered, the policyholder receives the cash value. • The policyholder may instead choose to take a reduced paid up policy or extended term insurance. • The values for reduced paid up of extended term are obtained from a table.	Suppose that the nonforfeiture table shown above is for a $150,000 death benefit policy. If the policyowner decides to surrender the policy at the end of the 8th contract year, what would he receive if he chose to surrender the policy for (a) cash, (b) RPU, or (c) extended term? (Example 13.3.4)
Universal Life Insurance, p. 554	• To find the account value for a universal life policy, first calculate the net amount at risk by subtracting the current account value from the death benefit. • Divide by 1000 and multiply by the mortality charge per $1,000. • Calculate interest on the current account balance. • Add the interest to and subtract the mortality and administrative fees from the account value.	Calculate the account value at the end of the third month for the universal life insurance policy shown above. (Example 13.3.5)
Surrender Charges, p. 555	• If a universal life policy is surrendered, the policyholder receives the account value less any surrender charges. • The surrender charges decrease over time according to a schedule determined when the policy is issued.	A universal life insurance policy has a the surrender charge of $5,000 in the first year, declining by $500 per year until it reaches $0. If the policy's balance has grown to $4843.25 after 4 years, how much would the policyholder receive if he surrendered at that point? (Example 13.3.6)

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CHAPTER 14

Evaluating Projected Cash Flows

"Foul cankering rust the hidden treasure frets
But gold that's put to use more gold begets"

—William Shakespeare, "Venus and Adonis"

Learning Objectives

LO 1 Calculate present values when payment continues indefinitely into the future.

LO 2 Use present values to choose between different business or investment opportunities.

LO 3 Calculate the payback period from an investment.

LO 4 Use payback periods to determine whether or not an investment is worthwhile.

Chapter Outline

14.1 The Present Value Method

Deciding what value to place on an investment or on a business as a whole is a challenging proposition. In addition to taking into account all the risks and uncertainties of any opportunity, we must face the fact that most business ventures require substantial up-front investment of time, effort, and capital in the hopes of returning profits in the future. How can we decide whether or not a given business opportunity is financially worthwhile? That is an enormous question that faces everyone, ranging from the multimillionaire real estate developer considering building a new resort complex, to the pizzeria owner deciding whether or not to buy a new oven, to the student wondering whether or not the time, effort, and expense of pursuing a college degree will really ever pay off. There is of course no way to provide a set, guaranteed, one-size-fits-all method for making these decisions. However, in this chapter we will consider how some mathematical tools may help in approaching these sorts of decisions.

Making Financial Projections

Mathematical calculations are only as good as the assumptions on which they are based. Overly optimistic assumptions lead to overly optimistic conclusions, overly cautious assumptions lead to overly cautious conclusions, and overly pessimistic assumptions lead us to abandon all hope. The first step in evaluating a business venture is to *understand* it. No matter how good your mathematical calculations, you can't expect to properly evaluate the prospects of a dairy farm if you don't know a cow from a bull.

Assuming you understand the nature of the business venture you are contemplating, you next need to carefully investigate the prospects for the particular venture itself. The 1990s saw an enormous boom in Internet technology, and vast fortunes were made by visionary business people who saw that potential and seized on it. Nonetheless, there were plenty of failed Internet businesses, even in the midst of this tremendous boom. Businesses with great products or brilliant minds behind them or great business plans lost out to companies with greater products or more brilliant minds or greater business plans.

Finally, it is important to recognize that any projections that you make must inevitably be guesses. Very well educated guesses, hopefully, but guesses nonetheless. It is easy to fall into the trap of having worked through a very solid and well-researched financial projection and then forget that the numbers are predictions. There is always a risk that things will not work out according to plan. The future is more likely to match a well-founded prediction than a bald-faced guess, but we must always be careful to remember that nothing in this life is really guaranteed.

Present Values and Financial Projections

Any type of investment will require some patience to pay off. You may invest in a business opportunity hoping to earn a good return on your investment, but that return will not be immediate. Profits made will be spread out over time in the future. Given the choice between two possible investments, then, you cannot simply add up the amount you expect to earn from each and see which is bigger. Time is a factor. An investment that will pay you back $10,000 each year over the next 3 years is almost certainly preferable to an investment that will pay back $40,000 in one lump sum 20 years from now. It's not worth having to wait an extra 17 years before you get any return on your investment in order to be able to get a bit larger return when it finally arrives. The time value of money is an important factor here.

Mathematically, we can compare investments by considering their present values. Present values allow us to directly compare the two, apples to apples, by comparing their worth to us in today's dollars. What interest rate should we use to determine the present value? That depends on what rate of return we expect from our investments. Suppose that we can safely earn a 5% rate of return from other investment opportunities. Then we would certainly be interested in either investment only if we would earn a higher return—otherwise, why take on the bother or risk? How much higher is a matter of choice, depending on how much risk the investment entails, among other factors. For purposes of this example, let's suppose that we would require a 7% rate of return, and so we will use that to evaluate these investments.

The investment that pays $10,000 annually for 3 years would then be worth:

$$PV = PMTa_{\overline{n}|i}$$
$$PV = \$10{,}000a_{\overline{3}|.07}$$
$$PV = \$10{,}000(2.6243160)$$
$$PV = \$26{,}243$$

On the other hand, the investment that pays $40,000 all at once in 20 years would be worth:

$$FV = PV(1 + i)^n$$
$$\$40{,}000 = PV(1.07)^{20}$$
$$\$40{,}000 = PV(3.8696844)$$
$$PV = \$10{,}337$$

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The mathematics agrees with our commonsense assessment about which investment would have the higher value; using a 7% rate the $10,000 per year for 3 years investment would be worth $26,243, while the higher payoff far in the future is only worth $10,337. Note that this calculation is equivalent to asking how much money we would need to invest at a 7% rate of return to generate either of these future payouts.

Using present values not only allows us to compare these investments, though, it also allows us to decide how much we would be willing to pay for either one. If we are presented with the opportunity to invest in the first one for $20,000, we can see that this would represent a much better return than the 7% that we require, since $20,000 is less than what we would have valued the opportunity at the 7% rate. On the other hand, if we were given the opportunity to invest in the second option for $20,000, we could see that this would represent a much poorer investment than 7% rate that we require, since $20,000 is far more than the present value of this opportunity at 7%.

Example 14.1.1 ***Using present values and an 8.5% rate, would you prefer a business opportunity that could be expected to earn $5,000 per year for 5 years or an investment that would pay $32,000 all at once at the end of 5 years?***

The present value of the $5,000 per year works out to be:

$$PV = PMTa_{\overline{n}|i}$$
$$PV = \$5{,}000a_{\overline{5}|.085}$$
$$PV = \$5{,}000(3.94064208)$$
$$PV = \$19{,}703$$

The present value of the $32,000 lump sum works out to:

$$FV = PV(1 + i)^n$$
$$\$32{,}000 = PV(1.085)^5$$
$$\$32{,}000 = PV(1.50365669)$$
$$PV = \$21{,}281$$

For these present values, the lump sum is large enough to be worth waiting for. Since it has a higher present value, we would prefer it over the $5,000 per year option.

Perpetuities

Suppose that you are considering an investment in part of a small business that you expect would earn you profits of $7,500 per year. The rate of return that you would require is 8%. Using the present value method, what value would you place on this business?

Unfortunately, there is one detail we are missing: what is the term? In order to find the future value, we need to make some assumption about how long the profits will keep on flowing in. Realistically, though, in many cases that is just about impossible to do. If the business will stop earning a profit in the near term, we probably wouldn't be all that interested in investing in it; if the business is likely to keep running profitably well into the future, any sort of prediction about how long it will keep on going would be nothing more than a wild guess.

Suppose that we assume a 20-year term. Using this, we can calculate that the present value would be $73,636. This might be reasonable if you plan to hold onto the investment for only 20 years—except that at the end of the 20 years you would presumably be able to sell your share to someone else for the present value of the expected profits at that point. Just finding the present value of the next 20 years' payouts ignores the fact that the business can be sold at the end of the 20 years. For this reason, we can't ignore the profits farther into the future, even if we don't expect to be the owners of the business at that point.

But how far into the future should we go? Suppose that we use a 50-year term. Then the present value works out to be $91,751. But why stop at 50? Why not 75 years, or 100, or 1,000? In fact, what if we just assume that the business keeps running *indefinitely* into the future? A stream of payments which continues indefinitely into the future is called a ***perpetuity***. You might think that since the payments of a perpetuity go on forever this would lead to an infinite present value, but in fact it doesn't.

To see why, let's go back to one of the alternative formulas for the present value annuity factor from Chapter 4. There, we saw that:

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i}$$

If you plug in very large values for n, the $(1 + i)^{-n}$ in the numerator gets very close to zero. You can verify this for yourself on a calculator; the larger the value of n is, the closer the value of $(1 + i)^{-n}$ gets to zero. The table below illustrates this, using an 8% rate:

n	$(1 + i)^{-n}$
10	0.46319349
30	0.09937733
50	0.02132123
75	0.00311328
100	0.00045459
250	0.0000000044061688
500	0.000000000000000019

So for very large values of n, this part of the formula becomes irrelevant, leaving us for all practical purposes with just 1/i. This leads us to:

FORMULA 14.1.1
Present Value of a Perpetuity

The present value of a stream of payments carrying on indefinitely into the future is

$$PV = \frac{PMT}{i}$$

where
PV represents the PRESENT VALUE
PMT represents the amount of the PAYMENT per period
and
i represents the INTEREST RATE per period

Example 14.1.2 ***Find the present value of a $7,500 per year perpetuity assuming an interest rate of 8%.***

$$PV = \frac{PMT}{i} = \frac{\$7{,}500}{0.08} = \$93{,}750$$

Note that the present value assuming payments never stop is not that much larger than the present value assuming payments stop after 20 or 50 years. This may seem surprising at first, but actually it agrees with common sense. How much are payments that won't be received until 50 or more years in the future worth today? The present value of those far-future profits is so small that they make very little difference in the overall present value.

Of course, assuming that the investment will keep churning out $7,500 per year forever is not realistic. Sooner or later the business will fail, the mountains will crumble, the sun will burn out. But using a perpetuity does not mean that we *literally* expect everything to keep churning on forever. Since the far-future payments contribute so little to the present value, in cases where there is no clear ending point, we can use a perpetuity as a very reasonable approach to determining a present value. The fact that the perpetuity formula is surprisingly simple is another attraction.

This is not to say that a perpetuity is always the appropriate choice, however. Care should be taken to use this tool only in cases where the stream of payments can be expected to continue on into the indefinite future. If there is reason to believe that there will be a foreseeable ending point, a standard annuity is the more appropriate tool.

Example 14.1.3 ***A laundromat owner is considering investing in more efficient washing machines. From her knowledge of her business and research on the new machines, she believes that each machine will save her $245 per year in lower utility***

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bills, and that each machine will last 10 years. Assuming she requires an 8% rate of return on her investments, what would the present value of the savings be?

In this case using a perpetuity would be inappropriate, because we have a specific period of time that we expect the investment to continue paying off. Thus:

$$PV = PMTa_{\overline{n}|i}$$
$$PV = \$245a_{\overline{10}|.08}$$
$$PV = \$245(6.7100814)$$
$$PV = \$1,643.97$$

Knowing this present value can help the laundromat owner decide whether the new washers are a good investment. If the cost of upgrading the washers is less than $1,643.97, then doing so would result in an investment return that is better than the 8% that she normally requires. If the upgrades would cost more than that amount, there still may be reasons to do it, and the investment may still provide a positive return, but it will be below the desired 8%.

More Complicated Projections

In the examples considered above, it was assumed that the payoff from an investment took the form of either a simple lump sum or an annuity. This assumption is not always realistic. As an investor in a business venture you not only hope that the business provides a return on your investment in the form of profits, but you no doubt hope that the business itself will grow and produce increasing profits.

Suppose that you are considering investing in two different businesses, and are trying to figure out which is the better option. One choice is the opportunity to buy in to a profitable local bowling alley for $100,000. Right now, your share of the profits would be $8,000 per year. Another choice is the chance to buy in on a residential solar electric installation business, also for $100,000. Right now, your share of the profits would be $5,000 per year. However, the bowling alley business is a mature business—solid, but not growing especially quickly. You believe that profits can probably grow at a 2% rate in the future. The solar business, though, is growing rapidly, and you believe its profits will grow at a 10% rate. Which is the better choice?

On the one hand, the bowling alley is more profitable right now, but its slower growth rate means that the more rapidly growing solar business will eventually overtake it. The question, then, is which business has the higher present value. Unfortunately, since the payments are increasing, we no longer have a true annuity. (Readers who have covered Chapter 7 may recall that we were faced with a similar challenge when dealing with the impact of inflation.)

Suppose that you require a 12% rate of return from your investments. Then the bowling alley business gives you 2% of this from the growth of the payments themselves, so we calculate the present value of its profits as though they were not increasing, using a 10% rate.[1] We do the same for the solar business, except in this case we use a 12% − 8% = 4% rate.

Assuming that both businesses will continue to operate according to these assumptions for the foreseeable future, we calculate their present values to be:

$$\text{Bowling alley:} \quad PV = \frac{PMT}{i} = \frac{\$8,000}{0.10} = \$80,000$$

$$\text{Solar business:} \quad PV = \frac{PMT}{i} = \frac{\$5,000}{0.04} = \$125,000$$

Since we said that either business would require a $100,000 investment, we can see from these calculations that the bowling alley would not be worth its $100,000 up-front cost. On the other hand, the solar business would actually be worth more than its $100,000 investment.

Note that these present values depend on the required rate of return. If the required rate of return were different, we might put a different value on these businesses.

Example 14.1.4 ***Determine the present values for both the bowling alley's and solar installation business's projected profit, assuming that the required rate of return is 18%.***

[1]As discussed in Chapter 7, this may not be precisely mathematically correct, but in most cases it provides an acceptable approximation.

The rate for the bowling alley would then be 18% − 2% = 16%. The rate for the solar business would be 18% − 8% = 10%.

$$\text{Bowling alley:} \quad PV = \frac{PMT}{i} = \frac{\$8{,}000}{0.16} = \$50{,}000$$

$$\text{Solar business:} \quad PV = \frac{PMT}{i} = \frac{\$5{,}000}{0.10} = \$50{,}000$$

A higher expectation of profits means that future profits are discounted more heavily. So in this case, the much higher rate of return that I demand on my investment means that the higher profits in the near term count more heavily in my thinking. Note, though, that, with a higher required rate of return, the amount I would be willing to put up today for each business is also quite a bit lower. While the 18% required rate of return equalizes the present value of these two opportunities, it also means that neither one of them would be worth the $100,000 up-front investment.

Spreadsheets (covered in Chapter 5) can also be used as a tool for finding present values. While we will not discuss their use here, the Additional Exercises for this section present a few examples where spreadsheets may be the preferable means of evaluation.

Net Present Value

So far, we have assumed that any two business opportunities we are comparing have the same up-front costs, so whichever one offers the higher present value is preferable. What if the two options have different costs?

The ***net present value*** of a series of payments is the difference between the present value of the expected income less the present value of the expected investment. The following example will illustrate how net present value may be used.

Example 14.1.5 ***At a 12% rate of return, we determined that the present value from the bowling alley would be $80,000 and the present value from the solar installation business would be $125,000.***

Suppose that you can buy this share of the bowling alley business for $65,000, while the solar installation business would require a $175,000 up front investment. Which investment option is more attractive?

The bowling alley's investment of $65,000 is less than the present value of its projected profits. This is reflected in a positive net present value of $80,000 − $65,000 = $15,000.

The solar business's investment of $175,000 is greater than the present value of its projected profits. This is reflected in a negative net present value of $125,000 − $175,000 = −$50,000.

On the basis of this assessment, the bowling alley is the more financially attractive investment option.

Calculations and interpretations of net present values can become quite involved. If a business opportunity requires both an initial investment and also additional future investments, for example, we would need to calculate the present value of the investments as well as the present value of the projected income. If the future investments and/or future projected income varies, the present value calculations can require quite a bit of effort.

The net present value can also be a bit misleading. While the solar business had a negative net present value, this does not mean that it is a losing proposition. It only means that with a 12% required rate of return, the projected cash flows do not justify the $175,000 investment. The investment may still be profitable, but just does not produce a 12% rate of return. The actual rate of return that it does produce is referred to as the ***projected rate of return***. Projected rates of return are another useful tool in evaluating business opportunities.

The mathematics of these sorts of evaluation is both challenging and interesting, and of great importance in the business world. A more thorough treatment of these methods lies beyond the scope of this book, however. Students who wish to study this in greater depth may want to pursue reading and/or course work in finance.

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A Few Words of Caution

The intent of this chapter is to give you a *basic* understanding of how some of the mathematical tools we have been using for the time value of money can be used to place a value on future profits from a business. You should absolutely not mistake this for a thorough or comprehensive discussion of a very complicated subject. Readers who are interested in pursuing these ideas further will find that a thorough approach to making financial projections and determining present values for them presents an interesting, but challenging, endeavor.

It also should be understood that the approach of this section takes into account only the projected profits from a business venture. These need not be the only consideration. If the bowling alley is located in an area with a hot commercial real estate market, an investor might be interested in buying the business in the hopes of selling the property for some other use at a high profit; we were only looking at it based on the profits it can make as a bowling alley.

We also did not take into account the risks of a business; the high-growth solar installation business might be at a greater risk of failure than the established bowling alley, though our calculations made no distinction between the two based on this risk. Lastly we also did not take into account anything other than the profits themselves. A passionate bowler, or a passionate advocate for renewable energy, might value one business more highly than the other regardless of the profits either one is expected to earn.

The bottom line here is that, while the examples and techniques presented in this section and further developed in the exercises are illustrative of one approach to putting a value on a business, they are by no means the final word. Take them as an indication, not an authoritative absolute.

EXERCISES 14.1

A. Present Values and Financial Projections

1. Suppose you require an 8% rate of return. Which investment would you prefer, using the present value method: one that pays $50,000 in a lump sum 2 years from now, or one that pays $8,500 per year for the next 7 years?

2. Suppose you require a 15% rate of return. Which investment would you prefer, using the present value method: one that pays $50,000 in a lump sum 2 years from now, or one that pays $8,500 per year for the next 7 years?

3. If you require a 6.5% rate of return, would you rather receive $4,000 in 3 years or $7,500 in 10 years?

4. If you require a 7.25% rate of return, would you be happier with $20,000 five years from now or $50,000 eighteen years from now?

B. Perpetuities

5. Find the present value of payments of $15,000 per year extending indefinitely into the future, assuming an 8% rate.

6. Find the present value of payments of $23,500 per year extending indefinitely into the future, assuming a 6.5% rate.

7. By moving its factory to a new location closer to a railroad line, a manufacturing company believes that it can save $200,000 in transportation costs each year for the foreseeable future. If it requires an 8% rate of return, what is the present value of this cost savings? If the move would cost $4,000,000, is it financially justified by this savings?

8. By investing in an improved telecommunications system, a call center believes it can save $350,000 per year in costs each year for the foreseeable future. If the company requires a 7.25% rate of return, what is the present value of this cost savings? If the new system would cost $2,500,000, is it financially justified by this savings?

C. Projections with Increasing Payments

9. Find the present value of a perpetuity of $5,000 per year, increasing by 4% annually, assuming the required rate of return is 12%.

10. Find the present value of a perpetuity of $7,250 per year, increasing by 5% annually, assuming the required rate of return is 8%.

11. A company is considering building a small hydroelectric plant. The plant would produce a profit of $72,500 in the first year, and the company believes that electric prices will rise in the future, so this profit will grow 5% per year. The company requires an 8% rate of return. The plant can be expected to continue producing power for the foreseeable future. What is the largest cost to build this facility that these projections would justify?

12. According to a guidance counselor, someone with a college degree can be expected to earn $15,000 per year more in her first year after graduation than someone who does not have a degree, and this difference grows by 7% each year. Suppose that you require an 8% rate of return on your investments.

    a. Use a perpetuity to estimate the present value of the projected additional earnings from having a college degree. According to this, is graduating from college financially worth the effort and expense?

    b. Instead of using a perpetuity, suppose that you assume a 40-year working career. Calculate the present value of $15,000 per year for 40 years, assuming the same rate you used in part a. How does this affect your conclusion?

D. Grab Bag

13. Suppose you have the choice of two investments:

    - Part ownership of a landscape company. You project your share of the profits will be $12,000 in the first year, and will increase by 2% annually.
    - Part ownership of a property management company. You project your share of profits will be $12,000 in the first year and will increase by 5% annually.

    You would require a 12% return from either of these investments. The landscape company would require a $125,000 investment; the property management company would require a $150,000 investment. Calculate the net present value of each of these opportunities. On the basis of their net present values, which opportunity is the more attractive?

14. Mohan is considering investing in a business that he projects would earn him profit of $35,000 in the first year. He requires an 11% rate of return. Using perpetuities, calculate the present value of these profits, assuming they are growing at a rate of:

    a. 1%

    b. 3%

    c. 5%

    d. 8%

    e. Does the trend in these values agree with common sense?

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E. Additional Exercises

15. Suppose Tonia is comparing two different investments, each of which is projected to continue to provide profits for the foreseeable future. Option A will pay $38,750 in the first year, and then grow at a 4% annual rate. Option B will pay $52,535 in the first year but grow at a 2.5% annual rate. On the basis of her required rate of return, she determines that both options offer the same present value. What is her required rate of return?

16. *Compare to Exercise 14.* Mohan is considering investing in a business that he projects would earn him profit of $35,000 in the first year. He requires an 11% rate of return. Using perpetuities, calculate the present value of these profits, assuming they are growing at a rate of:

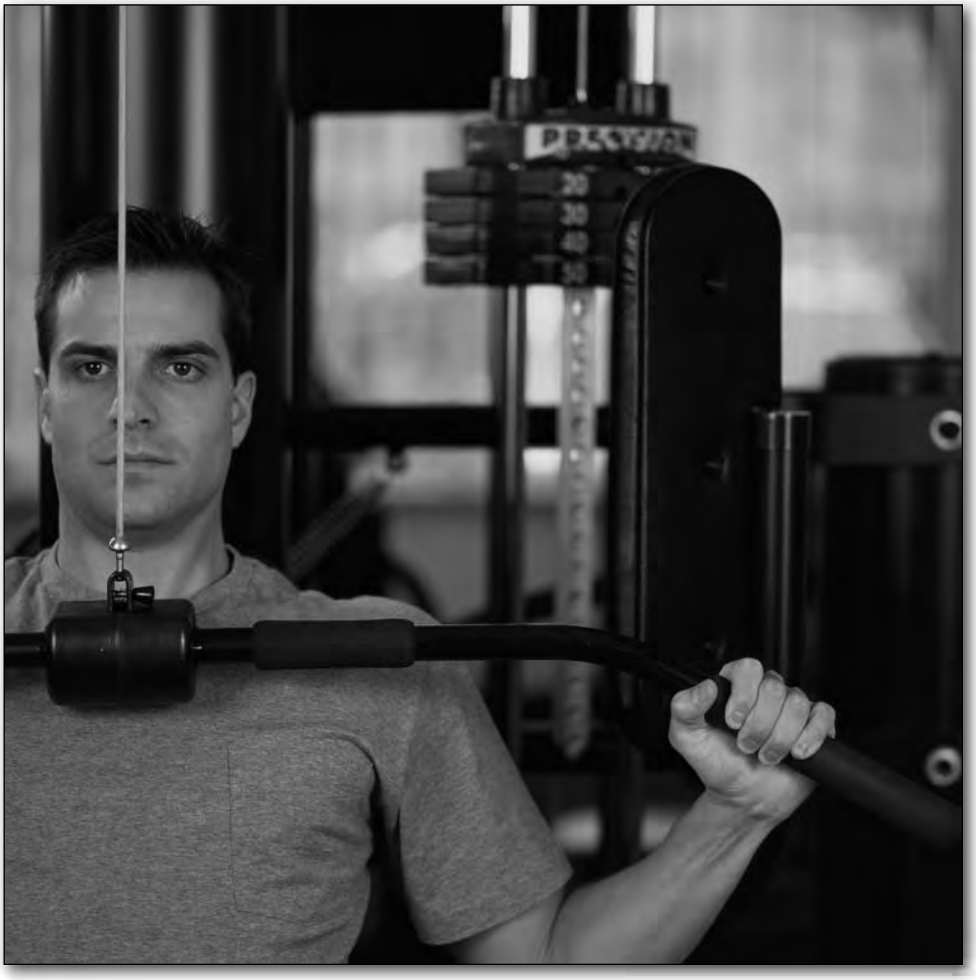
    a. 11%

    b. 15%

    c. Do these values agree with common sense?

14.2 The Payback Period Method

The present value techniques described in Section 14.1 are powerful and widely used methods for evaluating business opportunities. Nonetheless, they come with some serious disadvantages. The present value is highly dependent on what we choose to use as a



Before making any investment, we want to consider the payback!
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required rate of return; two different people, with different rate of return expectations, can come up with very different results and draw very different conclusions as a result. Far worse, these present values are based on projections of profits far into the future, and as we well know even the most well-founded predictions often miss the mark by quite a bit.[2]

In this section, we will consider an alternative method, also widely used, to evaluate the payoff of an investment.

The Payback Period Method

Suppose that the town where Brant lives has just built a new recreation and fitness center, and he's just decided its time to get into better shape. He plans to work out at the fitness center three times a week. To use the center, you either must pay $3.50 each visit, or buy an annual pass for $200. Before working up a sweat at the fitness center, Brant needs to work up a mental sweat to decide which way to pay.

On the one hand, he can work out the math fairly easily. Three visits a week times 52 weeks a year is 156 visits; 156 visits times $3.50 per visit works out to (156)($3.50) = $546, far more than the $200 annual pass. It seems that the annual pass is a no-brainer … but then again, Brant knows that he has a bit of a history of deciding to get serious about working out, going great guns at it for a few weeks, and then just losing steam. If he doesn't stick with it, or if he does keep working out but not quite as often as planned, he might not really get the equivalent of $546 from his annual pass; in fact, he might not even get the equivalent of $200 from his $200 annual pass.

The ***payback period*** of an investment is the amount of time it takes for the profits from the investment to equal the amount invested. In a case like this one, calculating a payback period may be helpful, since it will tell us how long Brant would need to keep up his plans in order for the annual pass to be worthwhile. The $200 cost of the pass can be thought of as an investment, and the profit is the $3.50 per visit that he avoids by having the pass.

In Brant's case, he would be paying 3($3.50) = $10.50 per week. The payback period of the annual pass would be $200/$10.50 = 19.04 weeks, or just over 19 weeks. So, if Brant sticks to his 3 visits a week plan for 19 weeks, the annual pass will have "paid for itself." Knowing this may help him decide whether or not the annual pass is a good idea.

Note that the "periods" of the payback period were weeks, because we expressed the rate of return as $10.50 per week. We could have alternatively used a rate per month or any other reasonable period of time, in which case the payback period would be expressed in that same unit of time. Or, we could even use something other than a standard unit of time. If we looked at the return on his investment as $3.50 per visit instead of $10.50 per week we could calculate the payback period as $200/$3.50 = 57.14 visits. This form might be preferable if Brant is committed to keep working out, but not sure if he can keep up the pace of 3 visits per week.

We can generalize this idea into a formula:

FORMULA 14.2.1
The Payback Period of an Investment

$$P = \frac{I}{R}$$

where
P represents the PAYBACK PERIOD
I represents the AMOUNT INVESTED
and
R represents the INVESTMENT RETURN PER PERIOD

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[2] For an illustration, levitate your fusion-powered hovercar to the space colony library and take a look back to predictions from 50 years ago about what life would be like today. (You may need to ask the reference android for help.) The world in which you live today is almost entirely unlike the world foreseen for today 50 years ago. There is no reason to think that predictions made for 50 years in the future will be any more reliable than predictions made 50 years in the past.

Example 14.2.1 ***Jeanette owns a pizzeria. Right now, she spends $350 each month on advertisements in the local newspaper. The paper offers her the option of paying $2,500 all at once for the same advertising. What would the payback period be for this option?***

Use the formula

$$P = \frac{I}{R} = \frac{\$2,500}{\$350} = 7.14 \text{ months}$$

This result tells Jeanette that after a bit more than 7 months, the annual payment option will have "paid for itself."

In Jeanette's example, the payback period gives a sense of how good the deal really is; we could think of the offer as getting a full year's advertising for a bit more than the price of 7 months' worth. The payback period can also be used to evaluate whether or not an offer is worth taking, as the following example will illustrate.

Example 14.2.2 ***Todd has a no-contract cell phone. He pays no monthly or annual fee, but pays a flat 20 cents per minute used. He is considering signing up for a contract that offers 800 minutes per month and costs $47.99 per month. He knows that he will never actually use the full 800 minutes, but wonders how many minutes he would need to use for it to be worth signing up for the new plan.***

Todd's investment would be $47.99 each month. The return would be avoiding paying the 20 cents per minute that he currently must pay. Thus, his payback period is $47.99/$0.20 = 239.95, or about 240 minutes per month.

In the example of Todd's cell phone, knowing the payback period allows Todd to make a reasoned decision about whether the new plan is worth it. If he believes that he will use at least 240 minutes per month on average, the new plan is worth the $47.99 per month commitment. If he has doubts that he will use that many minutes, he may decide to stick with his current plan. Whichever he decides, knowing the payback period enables him to draw a conclusion.

Notice that in this second example, the "time" of the payback period was not a period of time going by, but the minutes represented the amount of use. When we say that Todd's payback period is 240 minutes, we don't mean that he recovers his investment 240 minutes after signing up for the new plan; we mean that he recovers his investment if he *uses* 240 minutes per month. In our original discussion of Brant's gym pass, we saw that we could have measured his payback period in weeks (comparable to how we measured Jeanette's payback on her advertising costs) or in visits (comparable to how we measured Todd's payback on the new cell phone plan). Whichever type of units we use, though, the payback period can be evaluated in much the same way.

There is some alternative terminology used for this sort of analysis, which can be helpful if we want to emphasize that the payback period is based on something other than elapsed time on the calendar or clock. The payback period is also sometimes called the **break-even point.** Using that terminology, we would say that Jeanette breaks even after 7.14 months, and that Todd breaks even at 240 minutes of monthly use. We will use the terms *payback* and *break-even* interchangeably in this section.

More Involved Payback Calculations

Sometimes the calculation of the cost, and of the payback, can be a bit more involved.

Example 14.2.3 ***Jenny bought a compact fluorescent (CF) lightbulb for $4.95. A comparable standard incandescent bulb would have cost $0.24. The CF bulb will last as long as eight incandescent bulbs, and also uses only about one-quarter the electricity. Her electric rate is $0.12 per kilowatt-hour (kWh), and the fixture in which she is placing the new bulb uses about 14 kilowatt-hours per month with a standard bulb. How long will it take for her investment in the CF bulb to pay for itself?***

Incandescent bulbs that would last as long would have cost 8($0.24) = $1.92. So she has invested an extra $4.95 − $1.92 = $3.03 in the CF bulb.

The incandescent bulb is costing her (14 kWh/month)($0.12 per kWh) = $1.68 per month. The new bulb uses one-fourth the electricity, so its running cost would be $1.68/4 = $0.42 per month. So her return on this investment would be $1.68 − $0.42 = $1.26 per month.

So the payback period would be $3.03/$1.26 = 2.40 months. In other words, the more expensive bulb pays for itself in a bit less than 2½ months.

A strength of the payback period method is that it is easier to explain to other people who are not so familiar with the concept of present value. Another is that it does not require any consideration of rate of return. It allows us to quickly see just how long (whether measured in elapsed time or in use) an investment needs to get in order to justify the investment. The downside, of course, is that it gives no indication of what sort of rate of return actually is being earned, nor does it take into account how much is to be gained beyond the payback period. We know how long Jenny needs to use the CF lightbulb for it to be financially advantageous, and know that it is a fairly short period of time. We did not, however, take any measurement of just how much she will save overall. We might be able to calculate this if we want to, but calculating the overall savings goes beyond simple payback analysis.

Still, knowing the overall potential savings can be a useful addition to payback analysis.

Example 14.2.4 ***The CF lightbulb Jenny bought is rated to last 5 years. If it lasts that long, what will her total savings be by using it instead of an incandescent?***

We saw that Jenny is saving $1.26 per month with the CF bulb. Five years is 60 months, so she would save a total of (60)($1.26) = $75.60 worth of electricity. Since the CF bulb cost an extra $3.03, her total savings would be $75.60 − $3.03 = $72.57.

Using Payback Periods for Comparisons

Payback periods can also be an especially useful tool for comparing a choice of investments opportunities, especially when the long term is hard to predict.

Example 14.2.5 ***Harold and Olivia are taking out a mortgage to buy a house. They will be borrowing $145,000. For a 30-year fixed-rate mortgage, the interest rate would be 8¼%. If they pay 0.5 points, the rate would be reduced to 8%. If they pay 2.5 points, the rate would be 7¾%. (One point equals 1% of the amount borrowed.)[3] Find the payback period for each option.***

0.5-point option: The point would cost (0.5%)($145,000) = $725. They would save ¼% on their rate, or roughly (¼%)($145,000) = $362.50 per year. (This is not exactly correct, because it applies the interest savings to the full original principal. With each payment on the loan, the principal is slightly reduced, and so the savings will actually be a bit less than this. However, since the progress made on paying down the principal of a 30-year mortgage is slow in the early years, this should not be far from the actual savings.)

The payback period would be approximately $725/$362.50 = 2 years.

2.5-point option: The point would cost (2.5%)($145,000) = $3,625. They would save ½% on their rate, or roughly (½%)($145,000) = $725 per year. The payback period is $3,625/$725 = 5 years.

Harold and Olivia may want to take the payback period into account when choosing between these options. As it happens, over the full term of the mortgage, they would actually save much more money with the 2.5-point option (you can verify this by calculating the payments at each rate and then comparing the totals paid over 30 years). However, very few 30-year mortgages actually last 30 years. It is likely that they will refinance their loan, sell the house, or pay off the loan early, even if they are not thinking about doing any of those things right now. The payback period can help them evaluate just how long they need to stay put with the loan to break even on paying the points. They may choose the 0.5-point option over the 2.5-point one, or even forego paying points and accept the higher 8¼%, if they are concerned that things may change in less than the 5-year payback period.

[3] For more information on points, see Chapter 10, Section 10.2.

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Payback Period Where Payments Vary

Formula 14.2.1 and all of the examples above assume that the "profits" continue at a constant rate. This is often realistic. Even though Jennifer does not keep her light on for exactly the same amount of time (and hence the savings from the more efficient lightbulb will not be exactly the same) each month, and even though Harold and Olivia's interest savings will change as the balance on their loan declines, the differences are small enough in the near term to be ignored.

When differences are not too small to be ignored, the calculation of the payback period can require a bit more effort. In those cases, we need to make a chronological list of the savings to figure out when payback is reached.

Example 14.2.6 ***Melina is considering joining her local YMCA at a cost of $840 per year. Right now, she pays $65 a month for a membership at another gym, which she would quit if she joined the Y. Three months from now, her daughter will be going to a summer camp, and as a Y member she will get a discount of $280, and 6 months from now her daughter will be starting an after-school program, for which Y members get a $45 per month discount. Find the payback period for Melina's membership.***

To do this, we list the savings in each month, and also keep a running total of the cumulative savings from the start of the membership.

Month	Savings in Month	Cumulative Savings
1	$65.00	$65.00
2	$65.00	$130.00
3	$345.00	$475.00
4	$65.00	$540.00
5	$65.00	$605.00
6	$110.00	$715.00
7	$110.00	$825.00
8	$110.00	$935.00

We carry the table only as far as month 8 because we can see that the savings exceed the $840 investment between the seventh and eighth months.

The following exercises will provide additional examples of where payback periods may be useful.

EXERCISES 14.2

A. Payback Periods

1. A $3,500 investment is expected to return $500 per year. Calculate its payback period in years.

2. An investment of $525,000 is expected to return $4,000 per month. Calculate its payback period in months.

3. A tae kwon do program charges $125 a month. Students are given the opportunity to instead pay for a full year in advance, in which case the cost is $1,000. What is the payback period, in months, for paying for the year in advance?

4. Cattarauqua Ginseng Enterprises is considering switching to new packaging equipment that would allow the company to package its products more quickly and with less-expensive packaging materials. The new equipment would

cost $72,500, but the company estimates that it would save $14,575 worth of labor and materials costs annually. What is the payback period for this investment?

5. A wholesale club charges a $20 annual membership fee. Members are given the opportunity to upgrade to a "higher" membership level, which instead charges $75 a year but offers a 2% rebate on all purchases made during the year. How much would you have to buy at this wholesale club during the year to break even on the "higher" membership level?

6. Frank's credit card has no annual fee, but charges a 23.99% interest rate. His card issuer offers him the opportunity to switch to a new card with a lower 16.99% rate, but that card has a $35 annual fee. How much of an average balance would Frank have to carry in order for it to be worth switching to the new card?

7. A real estate developer is building a new office building. If he uses a high-efficiency natural gas heating system, he estimates that the system itself will cost $75,000 to install and cost $18,575 a year to operate. A geothermal heat pump would be more expensive to install, costing $160,000 up front, but could be operated for only $5,800 per year. Calculate the payback period for the geothermal system.

8. Gary is shopping for a new car. The model car he wants can be bought as a conventional model costing $18,500 or a hybrid costing $21,700. The conventional model will get 32 miles per gallon overall, while the hybrid will deliver 48 miles per gallon overall.

   Gary drives 15,000 miles per year on average. Assume that gas costs $3.25 per gallon.

   a. Calculate the number of gallons of gas Gary would use in a year with each model.

   b. Calculate how much Gary would spend each year for gas with each model.

   c. What is the payback period for buying the hybrid?

9. A snowplow service charges $15 per visit or $120 for the entire winter. How many visits would it take to make it worthwhile financially for a customer to pay for the entire winter at once?

10. A bookstore allows customers to buy a special discount card for $25. If you purchase the card, you get an immediate $10 discount, and then receive a further 12% discount on all purchases throughout the year. How much would you have to buy at this bookstore to break even on the purchase of a discount card?

B. Payback Periods When Payments Vary

11. An exterminator charges $199 for a "comprehensive" visit and $49 for a "maintenance" visit. Rob and Laura will need to have the exterminator come out for a comprehensive visit 4 months from now and again every 6 months after that; the exterminator comes out for a maintenance visit every month that he is not scheduled for a comprehensive one.

    The exterminator offers an annual plan that would cover all of these visits for $599 per year. What would Rob and Laura's payback period be for this plan?

12. Jude has a medical condition for which he needs to visit his doctor every 6 weeks. With his current health insurance, he pays a $25 copay for each of these visits. He also has three prescriptions that must be filled every 4 weeks, which each carry a $15 copayment, and every 12 weeks he must fill a fourth prescription that carries a $50 copay.

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He currently pays $1,500 per year for his health insurance coverage. His employer offers another plan, which costs $1,950 per year, but would have a $10 doctor's visit copay and a $5 copay for all of his prescriptions.

Calculate the payback period if Jude switches to the higher premium and lower copayment plan. Is it financially worthwhile for him to do so?

C. Grab Bag

13. A snowplow service charges $15 per visit or $120 for the entire winter. If there is a very snowy winter and the plow needs to visit 21 times, how much would you have saved by choosing to pay for the whole season at once?

14. Cattarauqua Ginseng Enterprises is considering switching to new packaging equipment that would allow it to package its products more quickly and with less expensive packaging materials. The new equipment would cost $72,500, but the company estimates that it would save $14,575 worth of labor and materials costs annually. If the equipment can be expected to last 25 years, how much in total savings would switching to the new equipment yield?

15. Suppose that your company contracts its computer support to an outside company. The support company charges $107 an hour, but offers two discount plans. With the Dynamic Discount Plan, you would pay a $2,000 annual fee, but then pay only $60 per hour of tech support. With the Comprehensive Coverage Program, you pay $6,000 annually, but are then billed just $17.50 per hour.

    Calculate the payback period for each of these discount plans. Which has the shorter payback period compared to just paying by the hour? (See Exercises 16 to 17 for a continuation of this exercise).

D. Additional Exercises

16. Suppose that the tech support company from Exercise 15 also offers an Unlimited Limited Plan, which costs $10,000 per year but provides up to 80 hours of free support per year, with additional hours beyond this billed at $20 per hour. Calculate the payback period for this plan compared with paying by the hour.

17. In Exercise 15, you calculated the payback period for two competing options. Is the plan with the shortest payback period necessarily the best choice? Explain your reasoning.

Topic	Key Ideas, Formulas, and Techniques	Examples
The Present Value Method, p. 565	• To compare two different streams of future payments, calculate their present values and choose the higher present value. • Which choice will be preferred depends on the interest rate used for the calculation.	Using present values and an 8.5% rate, would you prefer a business opportunity that could be expected to earn \$5,000 per year for 5 years or an investment that would pay \$32,000 all at once at the end of 5 years? (Example 14.1.1)
Perpetuities, p. 566	• The present value of a stream of payments that continues indefinitely is given by: $PV = \frac{PMT}{i}$	Find the present value of a \$7,500 per year perpetuity assuming an interest rate of 8%. (Example 14.1.2)
The Present Value Method with Increasing Payments, p. 568	• Calculate the present value, using a rate equal to the required rate of return less the projected growth rate.	An investment in a bowling alley is expected to return \$8,000 in the first year, growing at a 2% annual rate. An investment in a solar installation business is projected to return \$5,000 in the first year, growing at an 8% rate. Which option has the higher present value assuming an 18% required rate of return? (Example 14.1.4)
Net Present Value, p. 569	• To find the net present value, subtract the present value of the investment payments from the present value of the projected income.	At a 12% rate of return, we determined that the present value from the bowling alley would be \$80,000 and the present value from the solar installation business would be \$125,000. Suppose that you can buy this share of the bowling alley business for \$65,000, while the solar installation business would require a \$175,000 up-front investment. Which investment option is more attractive? (Example 14.1.5)
Payback Period, p. 572	• $P = \frac{I}{R}$ • Payback period is the time required for investment to provide a return equal to its costs.	Todd has a no-contract cell phone. He pays no monthly or annual fee, but pays a flat 20 cents per minute used. He is considering signing up for a contract that offers 800 minutes per month and costs \$47.99 per month. He knows that he will never actually use the full 800 minutes, but wonders how many minutes he would need to use for it to be worth signing up for the new plan. (Example 14.2.2)

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CHAPTER 15

Payroll and Inventory

"It has not yet been recorded that any human being has gained a very large or permanent contentment from meditation upon the fact that he is better off than others."

—Sinclair Lewis, "Main Street"

Learning Objectives

LO 1 Calculate gross pay based on salary, hourly wages, piece rate, or commission.

LO 2 Calculate net pay.

LO 3 Value inventory at cost based on average cost, FIFO, or LIFO.

LO 4 Use inventory valuation to calculate cost of goods sold.

LO 5 Estimate inventory at cost based on retail value of inventory.

Chapter Outline

15.1 Payroll

In this chapter, we will consider the calculations that support two very important financial tasks faced by a business: keeping track of its ***payroll*** and its ***inventory***. In this first section, 15.1, we will address payroll, the correct calculation and payment of wages and salaries to employees. In Section 15.2 we will cover the calculations used to determine the value of a business's inventory. While both of these might sound straightforward enough, in actuality there are a number of different techniques that may be used for each of these tasks.

There are two main calculations that must be done to determine payroll. An employee's ***gross pay*** is the amount she earns before any deductions for benefits or taxes. ***Net pay***, also called ***take-home pay***, is the amount she actually receives in her paycheck after all deductions have been taken. We will discuss calculating gross pay first.

There are four main ways that an employee's financial compensation can be structured:

Type	*How It Works*
Salary	Employee is paid on the basis of a fixed annual rate. The amount he is paid does not depend on the actual hours worked.
Hourly	Employee is paid on the basis of a rate per hour worked.
Commission	Employee is paid on the basis of a percent of his sales.
Piecework	Employee is paid on the basis of the number of items he produces.

The calculations for gross pay depend on the way that the employee's wages are structured. We will consider each of these in turn.

Gross Pay Based on Salary

The simplest case for calculating gross pay is when an employee is salaried. The only thing to consider in this case is how to convert the annual salary into an amount per pay period. This gross pay per period depends on how often the company pays its workers. The two most common pay periods in the United States are ***semimonthly*** (pay period is a half-month; for example, payday might be on the 15th and on the last day of every month) and ***biweekly*** (pay period is 2 weeks; for example, payday might be every other Friday).

If employees are paid semimonthly, they will receive 24 paychecks each year. To calculate an employee's gross pay on this schedule, simply divide the annual salary by 24.

Example 15.1.1 ***Stephany earns a $26,500 annual salary. She is paid semimonthly. Calculate her gross semimonthly pay.***

$$\text{Gross semimonthly pay} = \frac{\text{Salary}}{24} = \frac{\$26{,}500}{24} = \$1{,}104.17$$

If employees are paid biweekly, they will receive 26 paychecks per year (in most years) so to calculate biweekly gross pay we divide the annual pay by 26.

Example 15.1.2 ***Jason earns a $25,900 annual salary. He is paid biweekly. Calculate his gross biweekly pay.***

$$\text{Gross biweekly pay} = \frac{\text{Salary}}{26} = \frac{\$25{,}900}{26} = \$996.15$$

Unfortunately, a year does not contain exactly 52 weeks, and occasionally one of the paydays falls on an "extra" day, meaning that there may be 27 paydays on a biweekly schedule. This creates an issue for businesses that pay their salaried employees biweekly. Some companies calculate biweekly pay by dividing by 26 regardless. This means, though, that in years with an extra payday salaried employees wind up receiving 27/26 of their intended annual salary—a significant additional expense for the business. Other companies divide annual salaries by 27 if there are 27 paydays in the year. This results in everyone getting the correct total salary, but doing this requires good communication with employees, since at the start of a 27 payment year salaried employees will see an apparent drop in their gross pay.

Salaried employees may also be paid on a weekly or a monthly pay schedule. The calculation of gross pay in those cases is essentially the same as for biweekly or semimonthly; the only difference is that the annual pay is divided by 52 for weekly or 12 for monthly.

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Correct calculation of employees' pay is an important responsibility of any business. © Ryan McVay/Getty Images/DIL

Gross Pay for Hourly Employees

For employees paid at an hourly rate, gross pay is calculated by multiplying the hours worked by the hourly rate.

This simple calculation can be complicated by ***overtime pay***. In the United States, the ***Fair Labor Standards Act*** requires that most hourly employees be paid ***time and a half*** for any time worked beyond 40 hours in a given week. (There are some exceptions; certain types of hourly employees may be exempt from overtime pay.) Salaried employees are not generally eligible to receive overtime pay, regardless of the hours worked in a week, though a few states have passed laws that may require some amount of overtime pay for salaried workers.

To calculate the gross pay for a worker who has put in overtime hours, we can determine how many hours exceed the 40 hours weekly limit, and calculate the pay for those overtime hours separately. Or, as an alternative, we can calculate pay on the basis of the ***straight time*** (i.e., regular hourly) rate and then add on extra pay at one-half the regular rate for any overtime hours. The following example will illustrate both methods.

Example 15.1.3 Dylan is paid a regular wage of $12.50 per hour. His hours worked in for the past 2 weeks as reported on his time card are:

Week Ending	*Mon*	*Tues*	*Wed*	*Thurs*	*Fri*	*Sat*	*Sun*
October 20	8	8	10	7.5	9	3	0
October 27	6	4	10	4	0	0	0

Calculate Dylan's gross wages for this two-week pay period.

In the first week, Dylan worked a total of 45.5 hours, so 5.5 of those hours are overtime. In the second week he worked 24 hours, so none of those are overtime. In total he has 69.5 hours worked, consisting of 64 hours of straight time and 5.5 hours of overtime.

The overtime rate based on Dylan's regular wage would be 1.5($12.50) = $18.75 per hour. His gross pay for this pay period is thus:

Straight time: (64 hours)($12.50/hour) = $800

Overtime: (5.5 hours)($18.75/hour) = $103.13

Total gross pay: $800 + $103.13 = $903.13.

As an alternative way of calculating this, we could instead note that he gets an extra half of his hourly wage for overtime hours worked. So he earns an extra $12.50/2 = $6.25 per hour for overtime. (This amount is called the ***overtime differential****). We could instead have calculated his gross pay by applying his straight time rate to his total hours worked, and then add on the overtime differential for his overtime hours:*

Wages at straight time rate: (69.5 hours)($12.50/hour) = $868.75

Overtime differential pay: (5.5 hours)($6.25/hour) = $34.38

Total gross pay: $868.75 + $34.38 = $903.13

The two different methods presented here always give the same results.

In some situations, a state law, union contract, or company policy may require overtime to be paid when the standard over 40 hours threshold would not require it. For example, a company may offer the overtime rate if an employee works on a holiday, or at an unattractive time (such as a middle-of-the-night "third shift"). Or, the company might offer a different ***holiday differential*** or ***shift differential***.

Example 15.1.4 ***Giragossian Mechanical Corp pays its hourly employees double time for working on holidays and a 35% shift differential for third shift. Gennaro worked 30 hours last week, of which 6 were on Christmas Day and 12 were third shift. His hourly rate is $9.20. Calculate his gross pay for the week.***

It is easiest in this case to calculate his wages on the basis of straight time and then add on the extra wages for the differentials. Since the holiday rate is double time, he gets and extra $9.20 per hour for holiday work. The third-shift differential gives him an extra (35%)($9.20) = $3.22 per hour for third shift.

Wages at straight time rate: (30 hours)($9.20/hour) = $276.00

Holiday differential pay: (6 hours)($9.20/hour) = $55.20

Shift differential pay: (12 hours)($3.22/hour) = $38.64

Total gross pay: $276.00 + $55.20 + $38.64 = $369.84

Gross Pay Based on Piece Rate

Some workers are paid based on the amount they produce rather than on the time spent producing it. ***Piece rates*** are not common in the United States today, but they are still in use in some industries.

A piece rate may be a flat amount per item produced or may be based on a ***differential pay scale*** that pays different amounts, depending on the number of items produced. The first few may be paid at a certain rate, with additional items paid at a higher (or lower) rate per item.

Example 15.1.5 ***Fraiser's Tasty Land Acres Farms pays strawberry pickers on the basis of the number of quarts of strawberries they pick. The pay scale, based on daily production is:***

Quarts	Rate per Quart
0–25	$0.65
26–75	$0.70
Over 75	$0.75

Mickey picked 82 quarts of strawberries today. What is her gross pay for the day?

Her first 25 quarts will be paid at the $0.65 rate, the next 50 at the $0.70 rate, and the last 7 at the highest $0.75 rate. So her total daily gross pay is:

25($0.65) + (50)($0.70) + 7($0.75) = $16.25 + $35.00 + $5.25 = $56.50

The pay scale given in this example could easily be misunderstood. Someone could easily assume from reading it that, since Mickey picked 82 quarts, she would be paid 75 cents per quart for all 82 quarts. Even though the way it was stated in the example is not unusual, it would be clearer to rephrase it as:

Quarts	Rate per Quart
First 25	$0.65
Next 50	$0.70
Additional quarts	$0.75

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Gross Pay Based on Commission

Some workers, particularly those involved in sales, are paid a ***commission*** based on the amount that they sell. At its simplest, an employee paid ***straight commission*** receives some percentage of sales. For example, an employee paid 8% commission on sales who makes $5,000 in sales in a week would be paid (8%)($5,000) = $400 gross for the week.

There are several variations on this approach, however.

Some commissioned employees are paid a ***draw***, which is essentially an advance against the commission that they will earn. When the employee is paid, the draw is subtracted from the commission. (If the commission earned is less than the draw, the employee either will have to repay it or carry it forward against future commissions.)

Example 15.1.6 Brianna works at a jewelry store and is paid a commission of 5.25% on sales. In the month of April, she was given a $500 draw against her commission. Her sales in April totaled $34,900. Calculate her gross pay for April.

Brianna's commissions would be (5.25%)($34,900) = $1,832.25. Subtracting her draw from this, she would be paid $1,832.25 − $500 = $1,332.25 gross.

With a variable commission schedule, the percent commission rate varies, depending on the amount of sales.

Example 15.1.7 A financial services company pays its representatives a commission on their monthly mutual fund sales. The commission schedule is:

Monthly Sales	*Commission Percent*
0–$10,000	1.25%
$10,000–$25,000	1.75%
$25,000+	2.15%

Ivan made $18,250 in mutual fund sales last month. Calculate his commission from these sales for the month.

The first $10,000 is paid at the 1.25% rate; the remaining $8,250 is paid at the 1.75% rate.

(1.25%)($10,000) + (1.75%)($8,250) = $125 + $144.38 = $269.38.

Another variation on commissions is ***salary plus commission***, or ***base plus commission***. Some employees are paid a salary (or hourly wage) with the opportunity to additionally receive a commission based on their sales. Often, commissions are paid only on sales above a certain minimum amount.

Example 15.1.8 Bex works as a salesperson at a car dealership that pays her an $18,000 annual salary, plus a 0.75% commission on sales above $175,000 in each month. In August she made $304,000 in sales. Calculate her gross pay for August.

Her salary for the month is $18,000/12 = $1,500.

She will be paid a commission on $304,000 − $175,000 = $129,000 of sales. The commission amounts to (0.75%)($129,000) = $967.50.

Bex's total gross pay for August would be $1,500 + $967.50 = $2,467.50.

Calculating Net Pay

While a worker's gross pay is the starting point of his paycheck, anyone who has ever had a job is painfully aware that quite a bit may be subtracted from that gross to get to the net pay. ***Payroll deductions*** are amounts subtracted from gross pay, and such deductions can be made for a host of different reasons.

Taxes: Taxes represent one of the largest, if not the largest, payroll deductions for workers. ***Income tax withholding*** is money that an employer deducts from gross pay for federal, state, or local income taxes. The employer is required to calculate the amount to withhold according to the information each employee provides on his W-4 form and a withholding schedule published by the IRS and state or local tax agencies. This money is then forwarded to the appropriate tax collection agencies. In this section, we will simply state the amount of withholding without discussing the details of its calculation.[1]

In addition to income tax withholding, employers are required to deduct ***payroll taxes*** for Social Security and Medicare, collectively referred to as ***FICA***. These taxes are currently 6.2% for Social Security and 1.45% for Medicare, applied to income after pretax benefits are subtracted. The Social Security tax applies only up to a certain maximum income (the maximum increases each year—in 2006 the amount was $94,200), but this maximum is large enough that this is an issue for only very highly paid employees.

Insurance Benefit Deductions: Many employers offer their workers the opportunity to obtain ***health insurance*** and other insurance benefits such as ***dental, life***, or ***disability insurance*** through their work. While a few employers bear the entire cost of these benefits, in today's world this has become quite rare. Usually, employees are required to pay at least a portion of the cost of these benefits, and deductions are made from gross pay to cover the employee's share of the costs. In most cases the amount the employee pays for these benefits is tax free; the costs are subtracted from income *before* income tax is calculated. For this reason, these are often called ***pretax deductions***.

Companies use a wide variety of formulas to determine the employee's share of these benefits. Some commonly used formulas are discussed in Chapter 13, Section 13.2.

Flexible spending accounts (FSAs) are another common pretax deduction. An FSA is a special account that is funded by a pretax payroll deduction, which can be used to pay for medical expenses or child care expenses. By avoiding paying taxes on the money deposited to these accounts, an employee may be able to save quite a bit versus paying these expenses with after-tax money.

Other benefits: Some companies also offer their employees the opportunity to obtain other benefits through payroll deduction. Employees may be able to purchase auto or homeowner's or long-term-care insurance, or purchase prepaid legal services plans or real-estate service discounts or other similar programs. Some employers may even offer veterinary insurance—health insurance for your pets! The deductions for these types of programs, though, are generally not tax-deductible; these costs are subtracted *after* taxes are calculated. For this reason these are often called ***after-tax benefits***. The entire cost of these sorts of benefits is usually paid by the employees who choose to take advantage of them. Even though the costs may not be reduced by any employer subsidy, the group rates for these plans may be more favorable than the rates that you would have to pay for similar benefits if you bought them as an individual.

Retirement Programs: Many employers offer retirement plans as part of their benefits offerings. The costs for some retirement plans may be borne entirely by the employer, while others may be funded primarily by the employee. ***401(k) plans*** are a type of retirement savings plan to which a worker can make pretax contributions through payroll deductions. The company may, or may not, offer to match employees' 401(k) contributions; even if the company does not, these programs are attractive because they both allow workers to avoid current income taxes on their contributions and also allow the money invested in them to grow tax-deferred until retirement.

Usually an employee is free to decide how much, if at all, to contribute to a 401(k) plan, though recent legislation appears to be moving in the direction of making some level of 401(k) participation mandatory. Employees may specify that they want to contribute a certain percent of their gross pay, or may contribute a fixed dollar amount each pay period.[2]

As a result of technical distinctions in the tax code, some employers offer plans that are similar to 401(k)'s (such as 403(b) plans offered by nonprofit employers) but are called by different names. Some employers also offer the opportunity to put money into other types of investment programs through payroll deduction, though these programs may not offer the same tax advantages as 401(k)'s.

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[1]The calculation of income tax withholding amounts is covered in detail in Chapter 9, Section 9.2.

[2]Retirement plans are discussed in more detail in Chapter 7.

Other Payroll Deductions. A company may offer other payroll deduction programs. Many companies encourage their workers to contribute to charitable programs like the United Way through payroll deductions, or offer the opportunity to buy U.S. savings bonds.

In recent years there has been significant growth in ***529 plans***, state-sponsored programs similar to 401(k)'s but used to save for a child's college education. Contributions to 529 plans are made on an after-tax basis, however.

Another fairly common payroll deduction is ***wage garnishment***. This is a court-ordered deduction made from a worker's pay to cover a legal judgment, such as child support.

Regardless of the nature and amount of these deductions, all must be subtracted from an employee's gross pay to find net, or take-home, pay.

Example 15.1.9 ***Dave earns a $37,500 annual salary, and is paid semimonthly. His pretax benefit deductions are $88.29 semimonthly, and he also has $12.29 in after-tax deductions. His federal income tax withholding is $121.01 semimonthly and he also has $35.09 withheld for state income tax, and $93.92 for FICA. He contributes 7% of his gross salary to his 401(k) plan and puts a total of $2,500 each year into a dependent care FSA.***

Calculate Dave's gross and net semimonthly pay.

Dave's gross pay is $37,500/24 = $1,562.50.

To determine his net pay, we need to add up all of his deductions. There are quite a few to keep track of; to stay organized we will use a table:

Item	*Calculation*	*Amount*
Benefits	$88.29 + $12.29	$100.58
Income taxes	$121.01 + $35.09	$156.10
FICA	Given	$93.92
401(k)	$\frac{(7\%)(\$37{,}500)}{24}$	$109.38
FSA	$2,500/24	$104.17
Total	Add up deduction amounts	$564.15

Dave's net, or take-home, pay is the result of subtracting all of these deductions from his gross pay. So his net pay is $1,562.50 − $564.15 = $998.35.

Cafeteria Plans

Because different employees have different benefit needs, some employers offer programs called ***cafeteria plans***, so named because they allow each person to choose the benefits that fit his needs. Each year the company offers an ***open-enrollment period*** when workers can choose their benefits for the coming year. The company may grant each worker a certain benefit allowance, and then provide a menu of options with different costs. Each employee can select the benefits that he wants from the menu and apply the benefit allowance to the stated cost. If the cost of selected benefits exceeds the benefit allowance, the additional cost is deducted from the employee's salary.

Below is an example of a cafeteria plan offering:

OSWEGATCHIE PERAMBULATOR COMPANY
2008 CAFETERIA PLAN EMPLOYEE BENEFIT SELECTION FORM

Benefit Allowance Per Pay Period	$72.55	
I. Health Insurance	**HMO**	**PPO**
a. Employee only	$34.34	55.75
b. Employee plus spouse/partner	$37.90	$38.90
c. Employee plus children	$75.89	$123.21
d. Family	$96.15	$156.10

II. Dental Plan	
Single	$18.55
Family	$33.55
III. Life Insurance	
1 × Salary	$0.00
Additional multiples of salary (up to 5 ×)	$4.75 per additional multiple
IV. Disability	
Basic Disability (60% of income)	$0.00
Additional Disability (70% of income)	$4.75
V. Legal Services Plan	$9.35
VI. Additional Vacation Days	
Each additional day (up to 5)	$7.05
Total deductions for selected coverages	
Less benefit allowance	−$72.55

Net Payroll Deduction

Other Benefit Selections (Not included in Cafeteria Plan)

VII. Company-Sponsored 401(k) Program
You may contribute 2% up to 16% of your salary to this program.
Contributions are made on a pretax basis and are eligible for 50% match up to 8%.

VIII. Flexible Spending Accounts
You may contribute up to $750 for medical expenses and up to $5,000 for dependent care expenses annually.
Administrative costs for this plan are paid by the company.

Note that the "costs" for benefits offered in this plan may not be the full actual cost of the benefit. Companies offering these types of plans often set the "costs" of cafeteria benefits to employees by a formula that includes the company bearing some of the benefits costs in addition to the employee's benefit allowance.

Example 15.1.10 ***Using the cafeteria plan shown above, calculate the benefit payroll deduction for an employee who selects HMO insurance coverage for himself and his children, family dental coverage, basic disability coverage, and 3 × salary for life insurance, and no other benefits.***

From the table, the HMO cost would be $75.89 and the dental plan's cost is $33.55, and there is no charge for the basic disability. Since he is selecting 2 additional multiples of salary for life insurance, that will cost 2($4.75) = $9.50.

The total costs for all selected benefits is $75.89 + $33.55 + $9.50 = $118.94.

Subtracting the benefit allowance, we arrive at $118.94 − $72.55 = $46.39.

Because of the expense of administering them, cafeteria plans are more common among large companies than smaller businesses. Also, smaller businesses seldom feel that they are in a position to offer many of the more exotic benefits that a cafeteria plan may include.

Employer Payroll Taxes

Just as an employee's take home pay does not equal his gross, neither does an employer's costs for an employee equal the employee's gross pay.

Employers are responsible for paying certain taxes on their payrolls. These taxes do not directly affect any employee's paycheck, and many employees are not even aware of them. They do represent a significant cost and responsibility to employers, however.

The two most significant employer payroll taxes are ***FICA*** and ***unemployment taxes***.

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We have already discussed FICA from the employee's side. FICA includes Social Security and Medicare taxes, and the employee's share totals 7.65% of gross income less pretax benefit costs. The employer, however, also must pay FICA taxes in an amount equal to the employee's share. (The self-employed must pay a ***self-employment tax*** of 15.3%, equal to the combined employer and employee shares.)

Unemployment taxes are taxes to fund federal and state unemployment benefits. It is typical for an employer to pay a combined total of 5.4% of the first $7,000 of each employee's wages for these taxes. In some cases, though, an employer may pay more than this, depending on the amount of unemployment claims filed by its workers.

Example 15.1.11 Arturo's body shop has three employees, who earned $32,750, $28,595 and $44,016 (after pretax benefit deductions) last year. Assuming he pays the typical unemployment tax rate, how much did Arturo have to pay for FICA and unemployment taxes last year?

FICA is 7.65% of these salaries: (7.65%)($32,750 + $28,595 + $44,016) = (7.65%)($105,361) = $8,060.12.

Unemployment tax applies only to the first $7,000 of income; since each of his employees earned more than that $7,000, he would pay the tax on 3($7,000) = $21,000 of income: (5.4%)($21,000) = $1,134.

Employers may have other costs based on their total payrolls. Costs for employee benefits, liability insurance, and worker's compensation insurance, among other costs, must also be taken into an account in considering how much a worker costs the company.

EXERCISES 15.1

A. Gross Pay: Salary

1. Trent earns a salary of $39,835 annually. Calculate his gross biweekly pay.

2. Luna earns $54,836 annual salary. Calculate her gross semimonthly pay.

B. Gross Pay: Hourly

3. Sam earns $7.35 an hour. Last week she worked 36½ hours. Calculate her gross weekly pay.

4. Gideon makes $18.95 an hour, and is paid biweekly. Last week he worked 32 hours; this week he worked 17.5 hours. Calculate his gross pay for this 2-week period.

5. Jorge makes $15.92 an hour. His time card for the last 2 weeks is given below. Calculate his gross pay for this 2-week period.

Week Ending	*Sun*	*Mon*	*Tues*	*Wed*	*Thurs*	*Fri*	*Sat*
March 12	0	8	10	11.5	7.5	8	0
March 19	0	9	7	6	6.5	4	0

6. Cyndi makes $22.45 an hour and is paid biweekly. Her time card for the last 2 weeks is given below. Calculate her gross pay for this pay period.

Week Ending	*Sun*	*Mon*	*Tues*	*Wed*	*Thurs*	*Fri*	*Sat*
January 24	0	5	3	0	7	7	5
January 31	0	8	14	12	11	7	0

7. Tom is paid $11.36 an hour for straight time, and receives a 75% weekend differential for Saturdays and Sundays. His time card for the last 2 weeks is given below. Calculate his gross pay for this 2-week period.

Week Ending	*Sun*	*Mon*	*Tues*	*Wed*	*Thurs*	*Fri*	*Sat*
January 11	0	0	8	5	8	8	8
January 18	0	0	0	6	9	11	2

8. Suppose you are paid $12.44 an hour straight time, but you earn double-time for holidays. Your time card for the last 2 weeks is shown below. Calculate your gross pay for this 2-week period. (Don't forget that July 4 is a holiday!)

Week Ending	*Sun*	*Mon*	*Tues*	*Wed*	*Thurs*	*Fri*	*Sat*
July 5	0	5	7	9	3	9	0
June 18	0	8	8.5	8.5	9.5	0	8

C. Gross Pay: Piece Rate

9. Kriss has a job doing customer surveys at a local shopping mall. She is paid a base weekly salary of $200, and she also is paid on the basis of the number of surveys she completes in the week using the following scale:

Surveys	*Rate*
0–20	$1.40
21–50	$3.40
Over 50	$4.50

Calculate her gross pay for the week if she completes 72 customer surveys.

10. A telemarketer for a credit card company is paid $25 per shift, plus a piece rate based on the number of his calls that result in new credit card applications. The rate scale is:

Applications	*Rate*
0–5	$4.25
6–10	$5.15
Over 10	$7.25

Calculate his gross pay for a shift in which he solicited 10 applications.

D. Gross Pay: Commission

11. Suppose I am paid a 14% commission on weekly sales up to $2000, and 18% on weekly sales above this amount. (a) Calculate my commissions for a week in which I made $4785 in sales. (b) If I received a draw of $400 for that week, calculate my gross pay for the week.

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12. Steve is paid 10% commission on sales. In the month of October, he received a $1,250 draw and made sales totaling $18,350. Calculate his gross pay for October.

13. A financial services company pays its representatives a commission based on the investment management fees that they generate. The commission schedule is:

Monthly Fees	***Commission Percent***
0–$2,500	40%
$2,500–$10,000	45%
$10,000+	52%

Bill generated $13,450 in investment management fees last month. Calculate his monthly commission.

14. A financial services company pays its representatives a salary plus a commission based on the investment management fees that they generate. The commission schedule is:

Monthly Fees	***Commission Percent***
0–$2,000	16%
$2,000–$5,000	25%
$5,000+	37%

Chad generated $8,450 in investment management fees last month. He is also paid a $17,500 annual salary. Calculate his gross monthly pay for last month.

15. A radio station pays its advertising executives a $21,500 annual salary plus a commission of 17.5% on sales over $25,000 on a month.
    a. Laurie made sales of $24,943 in May. Calculate her gross pay for that month.
    b. Laurie made sales of $78,900 in June. Calculate her gross monthly pay for that month.

E. Net Pay

16. Jonathan's gross pay for the last 2 weeks was $993.15. He had deductions of $67.55 for health insurance, $59.75 for federal and $12.16 for state income taxes, $65.08 for FICA, and $75 for retirement plan contributions. Calculate Jonathan's net pay.

17. Enrico earns a $31,909 annual salary and is paid semimonthly. He has semimonthly deductions of $92.57 for health and dental insurance, $50 for 401(k) contributions, and he contributes $2,750 annually to a child care FSA. His income tax withholding is $101.05 for federal and $33.03 for state. His FICA is $90.80. Calculate his net semimonthly pay.

18–23. Calculate the net biweekly pay for each of the employees listed in the table at the top of page 591.

F. Cafeteria Plans

For Exercises 24 to 26, use the Oswegatchie Perambulator Corp. cafeteria plan selection form from pages 586–587.

24. Calculate the benefit payroll deduction for someone who selects family PPO coverage, family dental, additional disability coverage, life insurance for 1 × salary, the legal services plan, and 4 extra vacation days.

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PRODUCTION AND DELIVERY DEPARTMENT
PAY PERIOD ENDING AUGUST 5

	Exercise 18	*Exercise 19*	*Exercise 20*	*Exercise 21*	*Exercise 22*	*Exercise 23*
Employee	Kentsman	Chao	Marini	Comstock	Jobe	Wilden
Gross wages	$1,075.93	$1,213.06	$1,195.08	$995.93	$1,302.91	$1,147.90
Pretax benefits	$84.15	$84.15	$32.55	$0	$32.55	$32.55
Federal tax	$38.73	$49.74	$50.79	$50.79	$84.69	$63.13
State tax	$16.14	$18.20	$17.93	$14.94	$19.54	$17.22
FICA	$72.58	$86.36	$88.93	$64.76	$92.70	$78.30
Retirement	$43.04	$0	$0	$149.39	$58.63	$91.83
Other deductions	$10	$0	$0	$0	$25	$300
Net pay						

25. Calculate the benefit payroll deduction for someone who selects single HMO coverage, single dental, 1 × salary life insurance, basic disability, and no other benefits.

26. Calculate the benefit payroll deduction for someone who selects employee plus children HMO, family dental, 5 × salary life insurance, basic disability, and no other benefits.

G. Employer Taxes

27. Hank's pay for the year is $52,403.92 after pretax benefit deductions. Using the rates given in the text, calculate the FICA and unemployment taxes paid by his employer, based on Hank's income.

H. Grab Bag

28–33. Calculate the (a) gross pay, (b) FICA, and (c) net pay for each of the hourly employees listed in the table below.

STAN'S NORTHSIDE SYRACUSE STYLE PIZZA
HOURLY EMPLOYEES
PAY PERIOD ENDING FEBRUARY 17

	Exercise 28	*Exercise 29*	*Exercise 30*	*Exercise 31*	*Exercise 32*	*Exercise 33*
Employee	Mike P.	Mike R.	Lina	Chelsea	Sebastian	Melissa
Hourly rate	$9.25	$14.40	$12.30	$11.40	$10.65	$14.25
Week 1 hours	32	28	38	48	44	34
Week 2 hours	43	51	17	32	30	28
Gross wages	a)	a)	a)	a)	a)	a)
Benefits	$0.00	$0.00	$0.00	$0	$84.44	$84.44
Federal tax	$14.80	$20.16	$23.37	$27.36	$23.43	$24.23
State tax	$3.70	$5.04	$5.84	$6.84	$5.86	$6.06
FICA	b)	b)	b)	b)	b)	b)
Other deductions	$0	$75	$0	$0	$5	$0
Net pay	c)	c)	c)	c)	c)	c)

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34. Brian is the electronics manager for a department store. He earns a $37,400 annual salary, plus a commission of 3.5% on his department's monthly sales over $45,000. In January, his department's sales totaled $58,703. Calculate his gross pay for that month.

35. Tionesta works for a cell phone company at a kiosk at the mall. She earns $7.25 an hour, and she can receive a bonus based on the number of new subscribers she signs up each week, based on the table given below:

New Subscribers	***Bonus for Each***
First 5	$0
Next 10	$25
Additional	$50

Last week, Tionesta worked 35 hours and signed up 14 new subscribers. Calculate her gross pay for the week.

I. Additional Exercises

36. Based on the commission program described in Exercise 15, what is the most you could earn in a year when your sales totaled $375,000? What is least?

37. Suppose that you work for a company that offers a cafeteria plan. If the "menu price" of the benefits you select totals less than your benefit allowance, you can receive the difference as additional pay. Why might it not be a good idea for you to do this?

38. In many of our examples, a worker was paid a higher rate per item (or a higher commission rate on sales) if they produced (or sold) more. Why do you suppose that a business would use this sort of scale instead of just paying a flat rate?

15.2 Inventory

The term ***inventory*** refers to the goods that a business owns for the purpose of selling to customers. To effectively operate a business, management must be able to keep track of the goods that it has in inventory, and assign a value to them. Determining the value of inventory is necessary both to determine how much value the company has on hand in the form of goods, and also to determine the cost of the merchandise that has been sold. (The use of inventory to determine cost of goods sold is discussed both in this section and, in a bit more detail, in Chapter 12.) In this section, we will consider several commonly used techniques for placing a value on (known as ***valuing***) inventory.

For most purposes, it is most useful for a business to know the value of its inventory ***at cost***. In this section we will be primarily discussing at-cost valuation; toward the end of the section we will consider alternatives.

Specific Identification

The logically simplest method of valuing inventory is ***specific identification***, keeping track of the cost of each item of inventory individually. The total value of the inventory then is simply the sum of the values of the items in inventory.

Specific identification is a reasonable method to use when each item is readily distinguishable from the other items, and has a high enough value to be kept track of separately. A car dealer, for example, might use specific identification, since the cars he has in inventory differ from each other in make, model, accessories, color, VIN (vehicle identification number), etc., and each car is expensive enough that the dealer would keep cost and sales records matched up to each individual car bought or sold.

There are many other situations, though, where specific identification is not a reasonable approach to take. A hardware store, for example, would probably not be expected to track of the specific cost of each box of screws that it sells. While the store would need to keep track of how many boxes it buys and sells, and the costs and sale prices of these boxes, the store is unlikely to keep a separate record for each individual box of screws. One 5-pound box of 2½-inch wood screws is pretty much the same as another, and the value of each box is not large enough to justify tracking individually. With most merchandise, while it is important to keep track of the purchases, sales, and inventory of each particular *type* of item, we would not keep track of the individual items to be able to use specific identification.

Average Cost Method

With the ***average cost method***, a business considers the cost of each item it has in stock to be the average cost of all items of that type. This average is calculated in the usual way: divide the total cost of all of the items by the total number of items.

Example 15.2.1 ***Stassler Hardware ordered 500 Kandelrite brand 14-watt compact fluorescent light bulbs. The total cost of this order was $750. Realizing that this order was not large enough, the company placed an additional order for 900 more at a cost of $1,278. Using the average cost method, what is the store's cost per bulb?***

$$\text{Cost per bulb} = \frac{\$750 + \$1,278}{500 + 900} = \frac{\$2,028}{1400} = \$1.45 \text{ per bulb}$$

Note that this figure is rounded. If the store wants greater precision in its cost calculations, this could be carried out to more decimal places.

Note that the first order of bulbs cost $750/500 = $1.50 each, while the second order cost $1,278/900 = $1.42 each. Even if the store is able to accurately track which order each individual bulb came from, though, it does not make much sense to consider some bulbs as costing $1.50 and others as costing $1.42 since they are identical. The average cost takes the logical step of considering all of these identical bulbs as having the same cost.

Of course, a hardware store does not just place an order or two for light bulbs and then hold onto them forever. The store will be continually selling the bulbs and ordering new ones to replenish their stock. If the new bulbs come at a different price, this will affect the average cost.

Example 15.2.2 ***The light bulbs sold well, and Stassler Hardware soon placed a new order for another 1,200 bulbs at a cost of $1,560. When this order arrived, the store had 475 bulbs left from the original orders. What is the value of its inventory of this particular bulb once the new order arrives, based on the average cost method?***

The first step of this calculation is to determine the store's total cost for the 475 light bulbs still in stock.

There are two ways to do this calculation; each will produce a slightly different result because of rounding.

One method is to use the $1.45 per bulb average cost that we calculated previously and multiply that by 475 bulbs. This would give a total cost of $688.75 for those bulbs. There is a small amount of inaccuracy in this figure, though, due to the fact that the $1.45 per bulb cost was rounded.

A second, more precise method, would be to calculate the cost of these bulbs as a percent of the total cost of the original orders. The original orders included 1,400 bulbs, so it makes sense to consider that the remaining bulbs' cost is 475/1,400 of the total.

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According to this logic, the cost of the 475 bulbs would be

$$\left(\frac{475}{1{,}400}\right)\$2{,}028 = \$688.07$$

Since this second calculation avoids the issue of rounding, it is more precise, and so we will use this value. It should be noted, though, that the difference between the values is so small that it hardly makes any difference, and in practice either method might be reasonably used.

Once we have a cost for the remaining bulbs, the total value of the light bulb inventory is the sum of the cost of the "old" bulbs and the cost of the "new" ones.

$$\text{Inventory value} = \$688.07 + \$1{,}560 = \$2{,}248.07.$$

The question did not ask for this, but we could also calculate the new average cost per bulb. There are 475 + 1,200 = 1,675 bulbs in stock, so

$$\text{Average cost} = \frac{\$2{,}248.07}{1{,}675} = \$1.34 \text{ per bulb}$$

In summary:

Inventory Valuation by the Average Cost Method

The value of the inventory of a given item is:

$$\textbf{Inventory value} = \left(\frac{\textbf{Number of items left}}{\textbf{Number of items as of last purchase}}\right)(\textbf{Total value as of last purchase})$$

FIFO

With the average cost method, when bulbs are sold there is no assumption made about which bulbs come from which order. Whenever the merchant gets an order of light bulbs they all go "into the same pot."

An alternative method of valuing inventory assumes that the oldest merchandise is always sold first. This method is known as ***first in, first out***, better known by the acronym ***FIFO***.

Let's revisit Stassler Hardware's light bulbs, this time using FIFO to value its inventory.

Example 15.2.3 ***Stassler Hardware ordered 500 Kandelrite brand 14-watt compact fluorescent light bulbs. The total cost of this order was $750. Realizing that this order was not large enough, the company placed an additional order for 900 more at a cost of $1,278. The light bulbs sold well, and Stassler Hardware soon placed a new order for another 1,200 bulbs at a cost of $1,560. When this order arrived, the store had 475 bulbs left from the original orders. What is the value of its inventory of this particular bulb once the new order arrives, based on the FIFO method?***

To calculate inventory values with FIFO, it is convenient to have a list of the bulb orders in table form:

Lot	Number of Bulbs	Value
1	500	$750
2	900	$1,278
3	1,200	$1,560

Now, the problem states that of the original orders 475 bulbs remained. Since the original orders totaled 1,400 bulbs, this means that 1,400 − 475 = 925 bulbs had been sold. We can adjust our table to show where these bulbs came from, and how many of the bulbs from each lot remained.

The number sold exceeds the total of the first lot of bulbs, so we assume that all 500 of those bulbs were sold. This leaves 925 − 500 = 425 left to account for. These are assumed *to have come from the second lot, leaving 900 − 425 = 475 bulbs from that lot not yet sold.*

Lot	Number of Bulbs	Value	Number Sold	Number Left
1	500	$750	500	0
2	900	$1,278	425	475
3	1,200	$1,560	0	1,200

Now, to determine the inventory's value. The current inventory contains 475 bulbs from the second order, which had a total of 900 bulbs, so the cost we attribute to that lot is (475/900) of the total for that lot. So the value of bulbs from that lot is:

$$\frac{475}{900}(\$1{,}278) = \$674.50.$$

The current inventory also includes the entire third lot, so we include its entire cost. The total value of the current inventory is thus:

$$\text{Inventory value} = \$674.50 + \$1{,}560 = \$2{,}234.50.$$

In summary:

Inventory Valuation by FIFO:

The value of the inventory of a given item is found as follows:

1. **Make a list of previous purchases' quantity and cost.**
2. **Assume that sales came first from the oldest lot, and move forward until all sales are accounted for.**
3. **The inventory value is the cost of the remaining items.**

LIFO

An alternative, less commonly used, valuation method is ***last in***, ***first out***, or ***LIFO***. LIFO works like FIFO, except that we assume that the *newest* merchandise is sold first. LIFO can be calculated in the same way as FIFO, except that instead of working from the start we work from the end.

We can rework our light bulb example by using LIFO as well.

Example 15.2.4 ***Rework Example 15.2.3 using LIFO instead of FIFO.***

As before, of the original order 475 bulbs remained. Since the original orders totaled 1,400 bulbs, this means that 1,400 − 475 = 925 bulbs had been sold. We can adjust our table to show where these bulbs came from, and how many of the bulbs from each lot remained. However, since we are using LIFO, we assume that the bulbs came from the last order first. So, of the 1,200 bulbs from that order, 1,200 − 925 = 275 remain, along with all the bulbs from the earlier lots.

Lot	Number of Bulbs	Value	Number Sold	Number Left
1	500	$750	0	500
2	900	$1,278	0	900
3	1,200	$1,560	925	275

The value of the remaining bulbs from the last order is:

$$\frac{275}{1200}(\$1{,}560) = \$357.50.$$

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The overall value of the inventory is therefore:

Inventory value = \$357.50 + \$1,278 + \$750 = \$2,385.50

In summary:

Inventory Valuation by LIFO:

The value of the inventory of a given item is found as follows:

1. **Make a list of previous purchases' quantity and cost.**
2. **Assume that sales came first from the newest lot, and move backward until all sales are accounted for.**
3. **The inventory value is the cost of the remaining items.**

The use of LIFO presents some logical problems. With the average cost method and with FIFO we did not claim that we *really* knew which lots the sold light bulbs came from, but the assumptions that underlay both of those methods seem pretty reasonable. With LIFO, though, we are assuming that the 925 sold bulbs came from the third order, even though we know for a fact that those could not possibly have been the actual bulbs sold. This is not necessarily a problem, since any of these methods are based on hypothetical assumptions. Still, the hypothetical assumptions for average cost and FIFO seem far more reasonable. Despite this, LIFO is very commonly used in the United States; Exercise 28 may offer a hint as to why a business might prefer this method.

Perpetual versus Periodic Inventory Valuation

In all of our discussion and in all of our examples so far, we have assumed that a company will revalue its inventory of an item every time a new purchase of that type of item is made. This approach to inventory valuation called ***perpetual***. An alternative approach would be to place a value on the company's inventory only when the company actually does its books, typically at the end of each quarter. This approach is called ***periodic***.

The timing of when we value inventory might not seem to be much of a mathematical issue. However, it can make a difference in the value that actually is placed on the inventory. It will be easiest to see this by means of an example.

Suppose we are doing inventory for an office supply store. During the first quarter of the year, the company received one shipment of copier paper. The history of the store's inventory during the first quarter is shown in the table below:

Lot	Number of Boxes	Value (at Cost)	Boxes Sold before Next Order
Beginning Inventory	217	\$3,742	130
Shipment 1	80	\$1,278	92

Suppose we set out to determine the inventory value as of the end of the quarter, using the *perpetual* average cost method (the way we did in Examples 15.1.1 and 15.1.2).

Right before the first shipment arrived, the store had an inventory of $217 - 130 = 87$ boxes of paper. We would value these at:

$$\left(\frac{87}{217}\right)\$3,742 = \$1,500.25$$

With the new shipment, the store's inventory increases to $87 + 80 = 167$ boxes, and the value of this inventory would be $\$1,500.25 + \$1,278 = \$2,778.25$.

At the end of the quarter, the store would be left with $167 - 92 = 75$ boxes. We would value these at:

$$\left(\frac{75}{167}\right)\$2,778.25 = \$1,247.72$$

So, using the perpetual average cost method those 75 boxes have a value of $1,247.72.

Now, let's consider the *periodic* average cost method. Rather than recalculate the inventory's value after each new order, we just consider where things stand at the end of the quarter. Over the course of the entire quarter, there were 217 + 80 = 297 boxes of this copier paper that the store had in its inventory at some point. The total cost of these boxes was $3,742 + $1,278 = $5,020. The store sold 130 + 92 = 222 boxes, leaving 297 − 222 = 75 boxes in inventory at quarter's end. The value we would then place on the 75 boxes would be:

$$\left(\frac{75}{297}\right)\$5{,}020 = \$1{,}267.68$$

Notice that the periodic approach gives a different inventory value than the perpetual approach. Which one is correct? In fact, either one could be considered the correct value for the company's copier paper inventory. As we have seen throughout this section, the value placed on inventory depends on the method used, and there is no single method that must be used in every situation. If the company uses perpetual valuation, the correct answer is $1,247.72; if the company uses periodic valuation, the correct value is $1,267.68.

Historically, periodic valuation has been by far the more commonly used method. The reason for this, though, has mainly been a matter of convenience. With perpetual valuation, a business needs to recalculate its inventory value whenever a new shipment arrives, and must know precisely the number of items it has in stock when the new order arrives. If doing this requires counting boxes in the warehouse and by-hand calculation and recording the perpetual approach is, frankly, more trouble than its worth.

Today, though, most businesses keep track of their purchases and sales, track their inventory, and even do their books electronically. Determining our office supply store's copier paper inventory more likely involves a trip to the computer than a trip to the warehouse, and the inventory valuation is likely to be done automatically by the store's electronic bookkeeping system. The use of technology robs periodic valuation of much of its appeal, and as a result businesses are increasingly using the perpetual approach.

For this reason, in this section we have focused on the perpetual approach, and except where specifically noted otherwise *all of the exercises for this section should be done in that way*.

We have used the average cost method to consider the difference between perpetual and periodic valuation. What if we used FIFO or LIFO instead? With FIFO it makes no difference whether inventory is valued periodically or perpetually. With LIFO, though, the two approaches may lead to differing results, just as they did with average cost.

Calculating Cost of Goods Sold Based on Inventory

A business must periodically determine its profits (or losses); most businesses do this quarterly, or at least annually. When calculating the business's profit or loss, one component of that calculation must be the cost of the merchandise that was sold during the period.

Regardless of the method used to value inventory, we can use the at-cost value of the inventory to determine this. Suppose that your business began the quarter with inventory that had a cost of $15,000 and during the quarter you purchased additional inventory at a cost of $35,000. This brings the total at-cost value of all inventory you had during the quarter up to a total of $50,000. Naturally, you sold some of this merchandise, and at quarter's end you had inventory valued at a cost of $20,000.

What was the cost of the merchandise that you sold? A moment's thought reveals that, if $20,000 remained from a $50,000 total, the cost of the goods sold must be $30,000. In other words, we can calculate the cost of goods sold in a given period by:

$$\text{Cost of goods sold} = \text{Starting inventory} + \text{Purchases} - \text{Ending inventory}$$

Example 15.2.5 ***Suppose that the Plum Street Pharmacy began the third quarter with inventory valued at $545,636. The pharmacy made purchases totaling $1,275,936, and its end-of-quarter inventory was $659,800. Calculate the cost of goods sold in the third quarter.***

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Cost of goods sold = Starting inventory + Purchases − Ending inventory

Cost of goods sold = $545,636 + $1,275,936 − $659,800

Cost of goods sold = $1,161,772

The method of inventory valuation does not matter here; regardless of how the inventory was calculated, the cost of goods sold can be determined in the same way. However, different inventory methods place different values on inventory. If a business wants to minimize its calculated profits (and hence minimize its calculated taxes), the business may have reason to select the valuation method that tends to make its cost of goods sold appear highest.

Valuing Inventory at Retail

Sometimes a business may choose to evaluate its inventory on the basis of retail prices rather than on cost. Valuing inventory at retail would not require Stassler Hardware to make any assumptions about which lot each light bulb came from. The store's management simply needs to count the number of light bulbs it has and multiply this by the retail price. To determine the overall business's inventory at retail, the management would simply need to add up the total retail prices for all the merchandise in inventory.

Knowing the inventory value at retail can also provide a way of estimating the inventory at cost. Any business's management should have a reasonable idea of what percent of retail sales the merchandise's cost represents, *on average*. Applying this percentage to the inventory's value at retail provides an estimate of the value at cost.

Example 15.2.6 ***Stassler Hardware has 1,675 Kandelrite brand 14-watt compact fluorescent light bulbs in stock, which sell for a retail price of $2.75. The store's management knows that on average the cost of the light bulbs it sells is 55% of the retail price. Use these facts to estimate the inventory value at cost.***

The total inventory of these bulbs at cost would be (1,675 bulbs)($2.75/bulb) = $4,606.25.

Assuming that the cost is 55% of the retail price, this inventory would be valued at:

(55%)($4,604.25) = $2,533.44

While this example calculates the inventory value for only one particular item, a similar calculation could be used to estimate the inventory value at cost for the company's entire inventory. For a company with a very large number of different products, using this method could be much simpler than having to examine the company's orders for hundreds or thousands of different products. Unfortunately this method provides only an *estimate* of the inventory value at cost.

A similar approach can be applied to estimate the cost of goods sold.

Example 15.2.7 ***The Plum Street Pharmacy had retail sales totaling $1,475,023 in the third quarter. Their cost of goods sold normally averages approximately 78%. Use these facts to estimate the cost of the goods sold in the third quarter.***

(78%)($1,475,023) = $1,150,517.94

Cost Basis

Investors who buy and sell stocks (and other similar investments) are required to pay ***capital gains taxes*** (a type of income tax) on the profits (called ***capital gains***) they make from their investments. If a stock investment is made all at once, and the entire investment is sold at once, calculating the capital gain is simple a matter of finding the difference between the sale price and the amount paid for the shares sold (called the ***cost basis***). If, however, the purchases are made at different prices at different times, and the stock is not sold all at once, determining the capital gain depends on which shares of stock are assumed to be sold.

Under most circumstances, capital gains are calculated on a FIFO basis.

Example 15.2.8 ***Sarah recently sold 250 shares of Tesla Nitrates Corp. for a total sale price of $8,745. These shares came from an investment in the company that she acquired by buying 100 shares of stock in 1999 for $1,755, then 200 shares in 2003, for which she paid $5,578, and finally 450 shares that she bought in 2005 for $24,935. Determine the cost basis for the shares she sold and the capital gain from the sale.***

To calculate her cost basis, we first need to determine which shares she sold from her "inventory" of shares of Tesla Nitrates stock. We do this in the same way as with inventory.

Lot	Number of Shares	Cost	Number Sold	Number Left
1	100	$1,755	100	0
2	200	$5,578	150	50
3	450	$24,935	0	450

In this case, though, we are not interested in the cost of her remaining shares, we are interest in her cost for the shares she sold. She sold the entire first lot and 150/200 of her second lot. Her cost basis for the shares she sold was then:

$$\text{Cost basis} = \$1{,}755 + \left(\frac{150}{200}\right)\$5{,}578 = \$1{,}755 + \$4{,}183.50 = \$5{,}938.50$$

Her capital gain is then:

$$\text{Capital gain} = \$8{,}745 - \$5{,}938.50 = \$2{,}806.50$$

EXERCISES 15.2

A. Average Cost

1. A dairy bought 450 gallons of organic milk from one farmer for $1,135, and 675 gallons from a second farmer for $2,403. Based on these purchases, calculate the dairy's average cost per gallon of organic milk.

2. A municipal electric utility charges its business customers 5.2 cents per kilowatt hour for the first 2,800 kilowatt-hours used in a month, and 8.3 cents per kilowatt-hour for any use above this amount. If your company uses 4,372 kilowatt-hours in May, what would your average cost per kilowatt hour be for that month?

3. Burned Beans Coffee Shop ordered 85 pounds of Charred Roast coffee beans at a cost of $5.35 per pound, for a total cost of $454.75. They placed a new order for 57 pounds at $6.10 per pound for a total cost of $347.70. When the new order arrived they had 28 pounds left from the original order. Calculate the value of their inventory of this type of coffee bean using the average cost method.

4. Stassler Hardware ordered 25 model TJX-35 chain saws. The total cost of this order was $1,813. After a few weeks, the store manager placed a new order for another 20 at a cost of $1,575. When this order arrived, they had 8 saws left

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from the original order. What was the value of their inventory of this particular saw when the new order arrived, using the average cost method?

5. At the start of the month, a bakery had 800 pounds of premium cake flour at a cost of $635. Later, the bakery placed an order for 650 pounds of the same type of flour at a cost of $472.50. When the new order arrived, the bakery had 250 pounds of this flour left. At the end of the month, the bakery had 148 pounds of flour left.
    a. Calculate the value of their flour inventory when the order arrived, using the average cost method.
    b. Calculate the value of their flour inventory at the end of the month, using the average cost method.

6. Smith's County Line Heating Supply purchased 10,000 gallons of heating oil for $18,825 from one supplier. The next day the company bought an additional 14,000 gallons from another supplier for $24,360. The company already had 8,250 gallons in inventory at a cost of $12,350 when they made these purchases. They did not make any sales in the time between these two purchases.
    a. Calculate the value (at cost) of the company's heating oil inventory immediately after these purchases, using the average cost method.
    b. Suppose that the company sold 16,000 gallons of heating oil from this inventory. Calculate the value (at cost) of the remaining inventory using the average cost method.
    c. Suppose that the company then purchased 20,000 additional gallons of heating oil for $38,953. Calculate the value (at cost) of the company's heating oil inventory after this purchase, using the average cost method.
    d. Suppose that the company sells 18,000 gallons from this inventory. Calculate the value (at cost) of the remaining inventory, using the average cost method.

7. The bookstore at Northeastern Pennsylvania Community College purchased 153 copies of a biology textbook at a cost of $73.50 each. From enrollment figures, the bookstore realized that it did not have enough books and so it rush-ordered 80 additional copies at a cost of $98.35 each. This order arrived at the end of the first week of classes, at which time the bookstore had sold 120 copies of the book. At the end of the third week of classes it had 27 copies of the book left in stock.
    a. Calculate the value (at cost) of the inventory of this book at the end of the first week of classes, using the average cost method.
    b. Calculate the value (at cost) of the inventory of this book at the end of the third week of classes, using the average cost method.
    c. Calculate the average cost per book (i) at the end of the first week and (ii) at the end of the third week of classes, using the average cost method.

B. FIFO

8. What does the acronym FIFO stand for?

9. Burned Beans Coffee Shop ordered 85 pounds of Charred Roast coffee beans at a cost of $5.35 per pound. It placed a new order for 57 pounds at $6.10 per pound. When the new order arrived, it had 28 pounds left from the original order. Calculate the value of its inventory of this type of coffee bean, using FIFO.

10. Stassler Hardware ordered 25 model TJX-35 chain saws. The total cost of this order was $1,813. After a few weeks, the store manager placed a new order for another 20 at a cost of $1,575. When this order arrived, the store had 8 saws left from the original order. What was the value of its inventory of this particular saw when the new order arrived, based on FIFO?

11. At the start of the month, a bakery had 800 pounds of premium cake flour at a cost of $635. Later, the bakery placed an order for 650 pounds of the same type of flour at a cost of $472.50. When the new order arrived, the bakery had 250 pounds of this flour left. At the end of the month, the bakery had 148 pounds of flour left.
    a. Calculate the value of the flour inventory when the order arrived, using FIFO.
    b. Calculate the value of the flour inventory at the end of the month, using FIFO.

12. Smith's County Line Heating Supply purchased 10,000 gallons of heating oil for $18,825 from one supplier. The next day the company bought an additional 14,000 gallons from another supplier for $24,360. The company already had 8,250 gallons in inventory at a cost of $12,350 when it made these purchases. It did not make any sales in the time between these two purchases.
    a. Calculate the value (at cost) of the company's heating oil inventory immediately after these purchases, using FIFO.
    b. Suppose that the company sold 16,000 gallons of heating oil from this inventory. Calculate the value (at cost) of the remaining inventory, using FIFO.
    c. Suppose that the company then purchased 20,000 additional gallons of heating oil for $38,953. Calculate the value (at cost) of the company's heating oil inventory after this purchase, using FIFO.
    d. Suppose that the company sells 18,000 gallons from this inventory. Calculate the value (at cost) of the remaining inventory, using FIFO.

13. The bookstore at Northeastern Pennsylvania Community College purchased 153 copies of a biology textbook at a cost of $73.50 each. From enrollment figures, the bookstore realized that it did not have enough books, and so it rush-ordered 80 additional copies at a cost of $98.35 each. This order arrived at the end of the first week of classes, at which time it had sold 120 copies of the book. At the end of the third week of classes it had 27 copies of the book left in stock.
    a. Calculate the value (at cost) of the inventory of this book at the end of the first week of classes, using FIFO.
    b. Calculate the value (at cost) of the inventory of this book at the end of the third week of classes, using FIFO.
    c. Calculate the average cost per book (i) at the end of the first week and (ii) at the end of the third week of classes, using FIFO.

C. LIFO

14. What does the acronym LIFO stand for?

15. Burned Beans Coffee Shop ordered 85 pounds of Charred Roast coffee beans at a cost of $5.35 per pound. It placed a new order for 57 pounds at $6.10 per pound. When the new order arrived it had 28 pounds left from the original order. Calculate the value of the inventory of this type of coffee bean, using LIFO.

16. Stassler Hardware ordered 25 model TJX-35 chain saws. The total cost of this order was $1,813. After a few weeks, the store manager placed a new order for another 20 at a cost of $1,575. When this order arrived, the store had 8 saws left from the original order. What was the value of its inventory of this particular saw when the new order arrived, based on LIFO?

17. At the start of the month, a bakery had 800 pounds of premium cake flour at a cost of $635. Later, the bakery placed an order for 650 pounds of the same type of flour at a cost of $472.50. When the new order arrived, the bakery had 250 pounds of this flour left. At the end of the month, the bakery had 148 pounds of flour left.
    a. Calculate the value of the flour inventory when the order arrived, using LIFO.

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b. Calculate the value of the flour inventory at the end of the month, using LIFO.

18. Smith's County Line Heating Supply purchased 10,000 gallons of heating oil for $18,825 from one supplier. The next day the company bought an additional 14,000 gallons from another supplier for $24,360. The company already had 8,250 gallons in inventory at a cost of $12,350 when it made these purchases. It did not make any sales in the time between the two purchases.
    a. Calculate the value (at cost) of the company's heating oil inventory immediately after these purchases, using LIFO.
    b. Suppose that the company sold 16,000 gallons of heating oil from this inventory. Calculate the value (at cost) of the remaining inventory, using LIFO.
    c. Suppose that the company then purchased 20,000 additional gallons of heating oil for $38,953. Calculate the value (at cost) of the company's heating oil inventory after this purchase, using LIFO.
    d. Suppose that the company sells 18,000 gallons from this inventory. Calculate the value (at cost) of the remaining inventory, using LIFO.

19. The bookstore at Northeastern Pennsylvania Community College purchased 153 copies of a biology textbook at a cost of $73.50 each. From enrollment figures, the bookstore realized that it did not have enough books, and so it rush-ordered 80 additional copies at a cost of $98.35 each. This order arrived at the end of the first week of classes, at which time the bookstore had sold 120 copies of the book. At the end of the third week of classes, it had 27 copies of the book left in stock.
    a. Calculate the value (at cost) of the inventory of this book at the end of the first week of classes, using LIFO.
    b. Calculate the value (at cost) of the inventory of this book at the end of the third week of classes, using LIFO.
    c. Calculate the average cost per book (i) at the end of the first week and (ii) at the end of the third week of classes, using LIFO.

D. Cost of Goods Sold

20. Calculate the cost of the textbooks sold by Northeastern Pennsylvania Community College's bookstore if the inventory is valued, using:
    a. average cost (see Exercise 7)
    b. FIFO (see Exercise 13)
    c. LIFO (see Exercise 19)

21. At the start of the quarter, Tokamak Home Generators had inventory valued at $75,925. During the quarter, the company made purchases amounting to $135,926, and at the end of the quarter its inventory was valued at $91,592. Find the cost of goods sold in the quarter.

E. Estimating Inventory Cost Based on Retail

22. A small appliance store's inventory, valued at retail, is worth $107,300. Typically, its cost of goods sold is 62% of retail. Estimate the inventory value at cost.

23. An auto dealership has inventory of $18,735,926 valued at list price. The dealer's actual cost is 91% of list. Estimate the inventory value at cost.

F. Cost Basis

24. Sandra recently sold 500 shares of stock for $3,350. She had bought stock of this company in the following lots:

Lot	*Number of Shares*	*Cost*
1	200	$1,000
2	200	$1,450
3	300	$4,900

Calculate her cost basis for the shares she sold, and her capital gain (or loss) on the sale.

25. Suppose that you sold 500 shares of stock in Elmira Glassworks for $10,000. Before this sale, you had been purchasing shares of this company's stock for some time. Your holdings of this stock were obtained from the following purchases:

Lot	*Number of Shares*	*Cost*
1	250	$11,700
2	150	$3,000
3	300	$2,400
4	500	$5,500

Calculate the cost basis of the shares you sold and your capital gain (or loss) on this sale.

G. Grab Bag

26. A sporting goods store began selling a new model of tennis racket during the second quarter of the year. Its inventory of this tennis racket is shown in the table below:

Lot	*Number of Rackets*	*Cost*	*Number in Stock after Shipment*
Beginning inventory	10	$825	10
Shipment	15	$1,025	18
Ending inventory	n/a	n/a	7

Calculate the value of the ending inventory based on:

a. The average cost method

b. FIFO

c. LIFO

27. *This question is a continuation of Exercise 26*. For the sporting goods store in Exercise 26, calculate the cost of tennis rackets sold if the inventory is valued by:

a. The average cost method

b. FIFO

c. LIFO

H. Additional Exercises

28. If prices are rising and you want to make your company's profits appear as small as possible, would you prefer to use average cost, LIFO, or FIFO to value your inventory?

29. Rework Exercise 26, using periodic valuation instead of perpetual.

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CHAPTER 15 SUMMARY

Topic	Key Ideas, Formulas, and Techniques	Examples
Gross Pay Based on Salary, p. 581	• Divide annual salary by number of pay periods per year. • Semimonthly is 24 times per year. • Biweekly is 26 times per year in most years.	Jason earns a $25,900 annual salary. He is paid biweekly. Calculate his gross biweekly pay. (Example 15.1.2)
Gross Pay Based on Hourly Rate, p. 582	• Hours up to 40 per week are straight time; hours that exceed 40 per week are overtime. • Multiply straight time hours by hourly rate. • Multiply overtime hours by 1.5 times the hourly rate. • Total straight and overtime pay. • Some hours worked may also receive holiday pay or shift differential.	Dylan is paid $12.50 per hour. In the first week he worked 45.5 hours and in the second he worked 24. Calculate his pay for this pay period. (Example 15.1.3)
Gross Pay Based on Piece Rate, p. 583	• Multiply items produced by the piece rate. • If rate varies with the number of pieces produced, calculate the pay for items at each rate level separately, then add to get the total.	Fraiser's Tasty Land Acres Farms pays strawberry pickers on the basis of the number of quarts of strawberries they pick. The pay scale, based on daily production, is: **Quarts** — **Rate per Quart** 0–25 — $0.65 26–75 — $0.70 Over 75 — $0.75 Mickey picked 82 quarts of strawberries today. What is her gross pay for the day? (Example 15.1.5)
Gross Pay Based on Commission, p. 584	• Multiply sales by commission rate. • If commission rate varies by sales level, calculate commission at each rate level separately, then add to get the total. • If a draw was taken, subtract it from the commission.	A financial services company pays its representatives a commission on their monthly mutual fund sales. The commission schedule is: **Monthly Sales** — **Commission Percent** 0–$10,000 — 1.25% $10,000–$25,000 — 1.75% $25,000+ — 2.15% Ivan made $18,250 in mutual fund sales last month. Calculate his commission from these sales for the month. (Example 15.1.7)

(Continued)

Topic	Key Ideas, Formulas, and Techniques	Examples
Net Pay, p. 584	• Calculate gross pay. • Subtract all deductions from gross pay to find net pay.	Dave earns a $37,500 annual salary, and is paid semimonthly. His pretax benefit deductions are $88.29 semimonthly, and he also has $12.29 in after-tax deductions. His federal income tax withholding is $121.01 semimonthly and he also has $35.09 withheld for state income tax, and $93.92 for FICA. He contributes 7% of his gross salary to his 401(k) plan and puts a total of $2,500 each year into a dependent-care FSA. Calculate Dave's gross and net semimonthly pay. (Example 15.1.9)
Payroll Taxes, p. 584	• The employer must pay 7.65% of each employee's pay (after pre-tax deductions) for FICA. • Unemployment taxes also must be paid. • In many cases, the unemployment tax is 5.4% of the first $7,000 of salary, though this may vary.	Arturo's body shop has three employees, who earned $32,750, $28,595 and $44,016 (after pretax benefit deductions) last year. Assuming he pays the typical unemployment tax rate, how much did Arturo have to pay for FICA and unemployment taxes last year? (Example 15.1.11)
Average Cost Method for Inventory, p. 593	• Inventory is valued on the basis of the average cost of items in stock. • Each item sold is assumed to have cost this average.	Stassler Hardware had an inventory of 1,400 light bulbs at a $2,028 cost. The light bulbs sold well, and Stassler Hardware soon placed a new order for another 1,200 bulbs at a cost of $1,560. When this order arrived, it had 475 bulbs left from the original orders. What is the value of the inventory of this particular bulb once the new order arrives, based on the average cost method? (Example 15.2.2)
FIFO, p. 594	• Inventory value is determined by assuming that the oldest items are sold before newer ones. • Create a table listing all purchases and their costs. • Remove items from inventory oldest first. • Inventory value is the cost of the remaining items.	Recalculate the value of Stassler Hardware's light bulb inventory, using FIFO. (Example 15.2.3)
LIFO, p. 595	• Inventory value is determined by assuming that the newest items are sold before older ones. • Create a table listing all purchases and their costs. • Remove items from inventory newest first. • Inventory value is the cost of the remaining items.	Recalculate the value of Stassler Hardware's light bulb inventory, using LIFO. (Example 15.2.4)

(Continued)

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Topic	Key Ideas, Formulas, and Techniques	Examples
Calculating Cost of Goods Sold, p. 597	• Cost of goods sold = beginning inventory value + purchases − ending inventory value	Suppose that Plum Street Pharmacy began the third quarter with inventory valued at $545,636. It made purchases totaling $1,275,936, and its end-of-quarter inventory was $659,800. Calculate the cost of goods sold in the third quarter. (Example 15.2.5)
The Retail Method of Inventory Valuation, p. 598	• Multiply the total retail price of inventory by the assumed cost percent.	Plum Street Pharmacy had retail sales totaling $1,475,023 in the third quarter. Its cost of goods sold normally averages approximately 78%. Use these facts to estimate the cost of the goods sold in the third quarter. (Example 15.2.7)
Cost Basis and Capital Gains, p. 598	• Cost basis is the calculated cost of an investment. • Cost basis for stock investments when some shares have been sold is done on a FIFO basis. • Capital gain = amount from sale − cost basis.	Sarah recently sold 250 shares of Tesla Nitrates Corp. for a total sale price of $8,745. These shares came from an investment in the company that she acquired by buying 100 shares of stock in 1999 for $1,755, 200 shares in 2003 for which she paid $5,578, and 450 shares that she bought in 2005 for $24,935. Determine the cost basis for the shares she sold and the capital gain from the sale. (Example 15.2.8)

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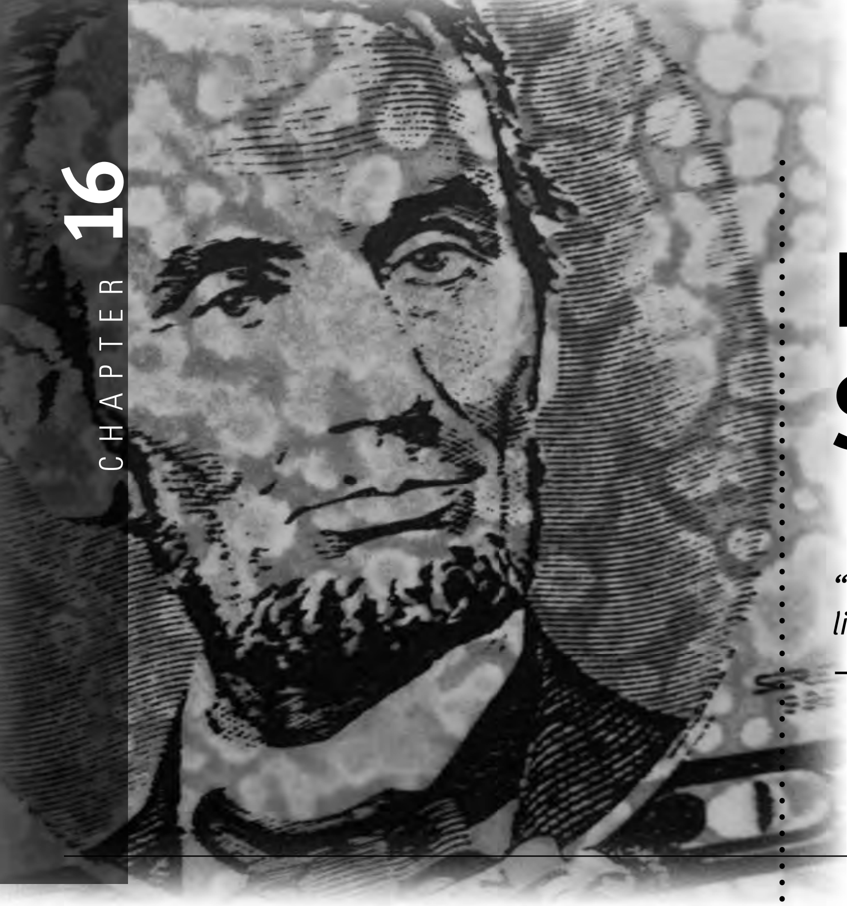
CHAPTER 16



Business Statistics

"There are three kinds of lies: lies, damned lies, and statistics."

—Benjamin Disraeli (often attributed to Mark Twain)

Learning Objectives

LO 1 Summarize statistical data using graphs and charts, making an appropriate choice of the type of chart or graph to use and generating it with Excel.

LO 2 Calculate the mean and median of a set of data, and recognize which measure of average is the more appropriate choice.

LO 3 Interpret indexes and use them appropriately in making comparisons.

LO 4 Calculate and interpret standard deviations appropriately.

Chapter Outline

16.1 Charts and Graphs

Statistics can be defined as the science (and sometimes the art) of turning raw data into useful information. In business, many decisions need to be based on facts about the economy, competitive environment, and a host of other factors about which we need to obtain and interpret data. If we are considering launching a new product, for example, we need to collect data about competing products' features and prices, and consumers' need for and expectations of such a product, so that we can make appropriate projections and decisions about how to design and sell this product.

To be able to make effective use of this data, though, we need to process it. If we collect data by surveying potential customers, for example, we won't be able to use that data effectively if we leave it in the form of a large pile of completed surveys stacked in a box.

To make any use of the data we collect, we need to be able to convert this raw data into a summary form that can be readily understood.

Statistics is a broad and deep area of mathematics; many colleges require an entire course in statistics for business majors, as well as for students of other disciplines such as the social sciences. In this chapter, we will introduce some of the most widely employed tools of statistics for business.

Data can often be most easily understood by using a picture representation. ***Charts*** and ***graphs*** are pictorial displays used to put data into an accessible form. In this section, we will consider several basic types of charts and graphs. Since it is difficult to create professional-appearing and accurately drawn charts or graphs by hand, they are almost always created by using computer software. In this section, we will describe the steps necessary to create basic charts and graphs with Microsoft Excel, since this is a commonly used and widely available program. Other statistical programs are also widely available, though, and the user's manuals for these programs will usually include instructions for creating basic charts and graphs.

Pie Charts

Pie charts, also known as ***circle charts*** or ***circle graphs*** are a familiar tool for summarizing statistical data. They are particularly useful for describing how different categories contribute to a larger whole. A pie chart consists of a circle and each category is given a wedge whose area is the same percent of the overall circle's area as the category it represents is of the whole. Loosely speaking, the size of each category's "piece of the pie" is determined by its contribution to the overall total.

For example, suppose that you are working for a company that plans to introduce a new brand of windshield washer fluid. Since there are already plenty of brands of washer fluid on the market, you would naturally want to have a good idea of who your strongest competitors are. One way of measuring this would be too look at what percent of overall washer fluid sales go to each brand in the area where you plan to sell your product. Each competitor's sales can be thought of as a portion of the total market sales of washer fluid. This situation is well-suited to a pie chart display.

We will first give the instructions to produce a basic pie chart with Excel, and then illustrate the results for our wiper fluid example. Note that the steps may vary slightly, depending on the version of Excel you are using; if you have difficulty using these instructions, check with your course instructor or your owner's manual.

How to Create a Pie Chart in Excel

1. List the categories in Column A of a spreadsheet.
2. List the amounts for each category in Column B. The amounts may be amounts or percents.
3. Select from menus: Insert → Chart → Pie.
4. Click on the chart style of your choice (the default is a basic pie chart).
5. Click on NEXT.
6. Use your mouse to highlight the area of the spreadsheet where your categories and amounts are stored.
7. Next to "Series in" select "Columns."
8. Click on NEXT.
9. If you wish, you can next enter a title or other formatting, using the options on the next screen.
10. Click on NEXT and then FINISH.

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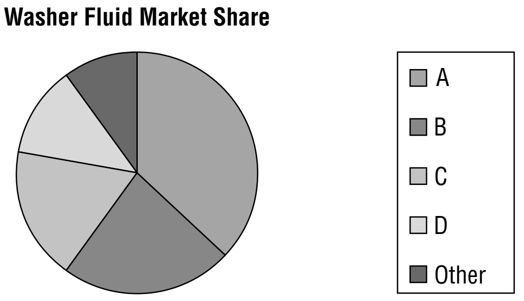
Example 16.1.1 ***Suppose that the wiper fluid market in your area is divided up as follows: Brand A has 37%, Brand B 23%, Brand C 18%, Brand D 12%, and other brands make up 10% of the market. Construct a pie chart to illustrate this.***

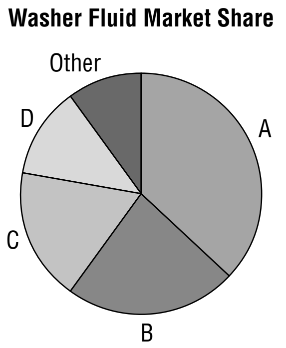
Following the instructions, we first enter the data into a spreadsheet. The result looks like this:

	A	B
1	A	37%
2	B	23%
3	C	18%
4	D	12%
5	Other	10%

Following the instruction Steps 2 to 10, entering a title in Step 9, we get:



If we want, we can adjust the pie chart to, for example, have a more stylish look by selecting a type other than the basic one in Step 4. We could improve this pie chart in particular by connecting the categories more closely to their wedges by eliminating the key on the side and instead making the letters appear next to their wedges. To do this, in Step 9 click on the tab for Legend and unclick "show legend"; this gets rid of the key on the side. Then click on the Data Labels tab and click on the box for Category Name. The result will look like this:



You could also adjust this to show the percent or amount for each category as well by choices from the Data Labels tab.

While pie charts can be a very useful display tool, they must be used with some caution. If there are many different categories, a pie chart can easily become cluttered and hard to read. Also, pie charts should not be used if the categories are not really pieces of some overall total. Example 16.1.2 (in the next subsection) will provide an example of a situation where a pie chart would not be appropriate.

Bar Charts

Like pie charts, ***bar charts*** (or ***bar graphs***) are useful for comparing different categories, particularly when the categories are not thought of as being parts of some whole. With a bar chart, each category is listed and a bar is placed next to it. The height of the bar is proportional to the total number of items that fall into that category. ***Column charts*** are identical to bar charts, except that columns are placed on top of the categories, rather than bars next to them.

Suppose you are managing a hotel and at checkout you ask your guests to fill out a survey, where one of the questions asks for their reasons for choosing to stay at your hotel. Each guest may choose to select one or more choices from a list of possibilities: location, reputation, happiness with experience on a prior visit, amenities, friend's recommendation, and price.

While the totals for each of these responses *could* be thought of as a percent of the overall responses, it may not be entirely appropriate to look at them in this way. It may be more useful to consider each of the categories separately, comparing how often each one is cited, rather than looking at each as a percent of the total responses.

As with pie charts, we will list the steps needed to create a bar chart in Excel, and then illustrate with an example.

How to Create a Bar Chart (Or Column Chart) in Excel

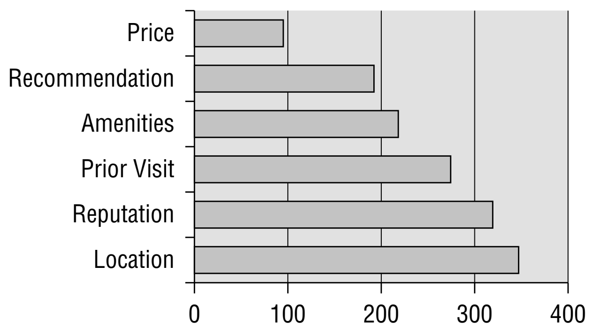
1. List the categories in column A of a spreadsheet.
2. List the amounts for each category in column B. The amounts may be amounts or percents.
3. Select from menus: Insert → Chart → Bar (or Column).
4. Click on the chart style of your choice (the default is a basic bar chart).
5. Click on NEXT.
6. Use your mouse to highlight the area of the spreadsheet where your categories and amounts are stored.
7. Next to "Series in" select "Columns."
8. Click on NEXT.
9. If you wish, you can next enter a title or other formatting by using the options on the next screen.
10. Click on NEXT and the FINISH.

Example 16.1.2 ***During the month of September, 347 guests at your hotel indicated that location was a reason for choosing your hotel, 319 said reputation, 274 said they were happy with their experience on a prior visit, 218 listed amenities, 192 listed a friend's recommendation, and 95 said price. Create bar and column charts to illustrate this.***

Following the instructions given above, we first enter the data into a spreadsheet like this:

	A	B
1	Location	347
2	Reputation	319
3	Prior Visit	274
4	Amenities	218
5	Recommendation	192
6	Price	95

Then, we follow the remaining steps with the following result:



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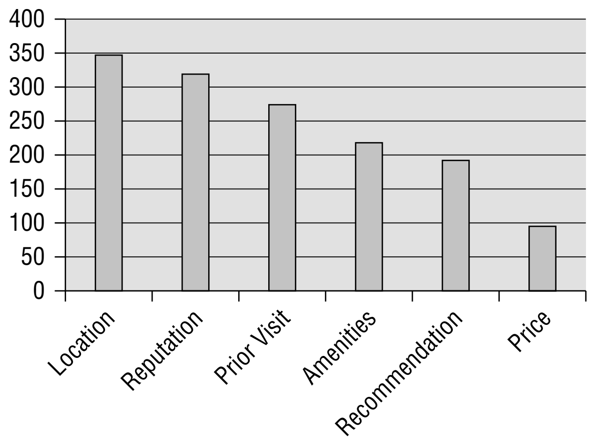
If we follow the same steps to create a column chart, the result would be something like this:



Line Graphs

Pie charts and bar charts are useful when we need to compare data that is broken into different categories. Sometimes, though, we are looking at how a quantity changes over time.

Line graphs are a tool for displaying changes over time. With a line graph, the horizontal scale is used to represent time, and the vertical scale is used to indicate the value of a quantity. Markers are placed at the appropriate height above each point of time to show the value of the quantity at that time. It is also common practice to "connect the dots" to fill in the gaps between points in time that are not part of our data.

How to Create a Line Graph in Excel

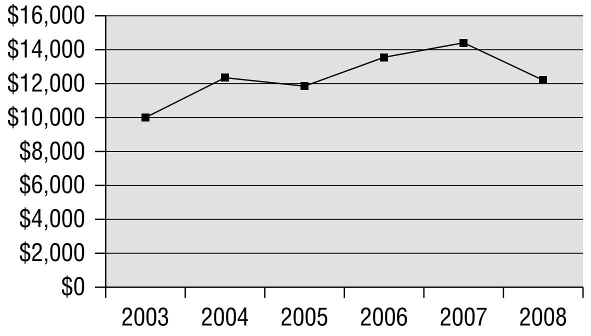
1. List the times in column A of a spreadsheet.
2. List the amounts for each time in column B.
3. Select from menus: Insert → Chart → Line.
4. Click on the chart style of your choice (the default is a basic bar chart).
5. Click on NEXT.
6. Use your mouse to highlight the area of the spreadsheet where your amounts (but *not* your times) are stored.
7. Next to "Series in" select "Columns."
8. Click on the "Series" tab and click on the field for "Category (X-axis) labels."
9. Use your mouse to highlight the area of the spreadsheet where your times are stored.
10. Click on NEXT.
11. If you wish, you can next enter a title or other formatting, using the options on the next screen. (You may want to deselect "Show legend.")
12. Click on NEXT and the FINISH.

Example 16.1.3 ***Suppose that the investment managers at Second Law Capital Management want to illustrate how investments made with their company have performed over the years. Suppose that a $10,000 investment made at the end of 2003 would have had a value of $12,350 at the end of 2004, and then $11,795, $13,552, $14,401 and $12,199 at the end of each successive year. Create a line chart to illustrate this.***

Following the instructions given above, we first enter the data into a spreadsheet like this:

	A	B
1	2003	$10,000
2	2004	$12,350
3	2005	$11,795
4	2006	$13,552
5	2007	$14,401
6	2008	$12,199

Then, we follow the remaining steps with the following result:



Other Charts and Graphs

Histograms are a tool to illustrate the distribution of numerical values. A histogram closely resembles a column graph; the main difference is that with a histogram, instead of having bars for distinct categories, we instead have bars that represent the number of items whose numeric value falls within a given range. For example, a call center manager might use a histogram like the one in Figure 16.1 to summarize the wait times for calls to the center.

Histograms are not easy to create in Microsoft Excel, and so we will not discuss how to generate them in this text. However, it is worth seeing an example of one, as you may encounter them at some point in your life or career.

Other charts and graphs may be useful in other specialized situations. A few moments spent exploring the Chart menu in Excel will provide an opportunity to see some of the other options that exist. As our purpose here is only to provide a very loose introduction, however, we will confine our discussion to the basic types that we have already discussed. Students who are likely to have a professional need to a more in-depth exposure to charts and graphs should strongly consider a full semester course in statistics, if one is not already required for their majors.

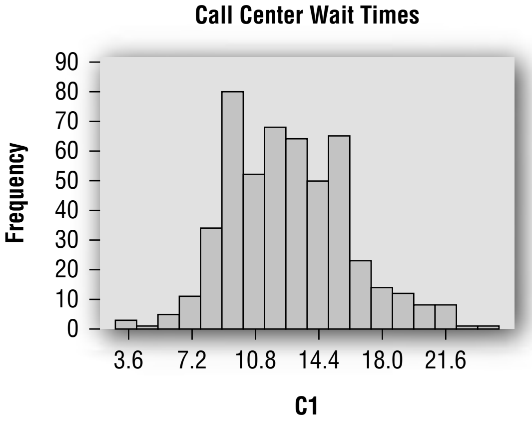


FIGURE 16.1
Sample Histogram

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EXERCISES 16.1

A. Creating Charts and Graphs

1. Montag Fire Equipment has five regional offices. In the fourth quarter of the year, the sales for each of these offices was:

Office	***Sales***
Atlanta	$3,510,000
Portland	$7,500,000
Chicago	$5,300,000
Phoenix	$1,750,000
San Jose	$9,500,000

Create a pie chart to illustrate the contribution of each office to the company's overall fourth-quarter sales.

2. The Chatakoin County supervisor is running for reelection. She is particularly proud of the efforts she has made control costs and keep county property taxes low. The county property tax rate in Chatakoin County is $0.83 per $1,000 market value. The tax rates in neighboring counties are:

County	***Tax Rate Per Thousand***
Birchcrest	$0.94
Lafayette	$0.89
Drummond	$1.05
Welland	$1.14
Canaseraga	$0.96

Create a column chart showing the comparison of county tax rates that she might want to use in a campaign flyer.

3. Speedywheels Bicycle Manufacturing is launching an advertising campaign to emphasize the high quality of its bicycles. A major biking magazine recently completed an assessment of quality and reliability for major bike manufacturers, and gave Speedywheels' bicycles a rating of 945 points. The ratings for other manufacturers were:

Competitor	***Rating***
A	582
B	703
C	771
D	406
E	811

Create a bar chart that the company could use in its ad campaign.

4. Frank's Falafel Hut started out with a single franchised location in 2002, and has been rapidly growing, adding more and more new locations each year. The number of franchised locations at the end of each year is shown below:

End of Year	***Number of Franchises***
2002	1
2003	4
2004	9
2005	16
2006	25
2007	36

To highlight the company's success in a prospectus for potential franchise owners, Frank wants to include a line graph showing this growth. Create a line graph he could use for this.

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B. Grab Bag

5. For each of the situations given below, decide whether a pie chart, bar chart, or line graph would be the most appropriate choice to use.

a. A gas station chain company wants to show customers where the cost of gasoline comes from by breaking down the price into the costs for crude oil, refining, taxes, overhead, and profit.

b. The economic development commission of a state wants to illustrate how workers compensation costs in the state compare to neighboring states.

c. An electric utility wants to illustrate the proportions that different types of electric generation (coal, natural gas, nuclear, wind, biomass, etc.) contribute to the company's overall electric generation.

d. A wind turbine manufacturer wants to illustrate the growth of wind power generation from 2000 to 2008.

e. A real estate agent wants to illustrate the difference in average housing prices among several suburbs of a major city.

6. For each of the situations described below, decide whether or not the choice of graph is appropriate. If it is, say so. If not, state which of the three main types of charts we have discussed (pie, bar, line) would be the better choice.

   a. Anita is planning to use a pie chart to illustrate how her company's sales have grown over the years.

   b. Baab's Tree Service is planning to use a line graph to illustrate how the company's charges for stump removal compare to its competitors.

   c. A company wants to illustrate how much each of its products contributes to the company's overall annual sales. It is planning to use a pie chart.

   d. A utility company wants to use a bar chart to show its customers how its rates for electricity have changed over the years.

   e. An industrial economist plans to use a pie chart to a compare average hourly wages in different countries.

C. Additional Exercise

7. In Exercise 2, you created a column chart showing tax rates in different counties. (If you have not done Exercise 2, do it now.) This chart served to illustrate the difference in the county tax rates, with its shortest column showing how Chatakoin County's tax rate is the lowest.

   If you move your mouse over to the left side of the chart where the dollar amounts are shown along the vertical axis and double click, this should open a box labeled "format axis." Click on the tab for scale, and change the minimum value to 0.80. This will create a new chart, where the bottom of each column begins at 0.80 instead of 0.00.

   Which version of the chart do you think that the county supervisor would want to use in her campaign flyer? Why?

16.2 Measures of Average

Graphs and charts can be used to help summarize data in a visually appealing way, but they are by no means the only way of summarizing data. It is also common practice to calculate numerical values that can be used as a way to summarize and help draw conclusions from data. ***Averages*** provide a familiar example.

If you are considering a career as a pharmacist and researching the salaries that pharmacists earn in your area, you would not expect to find a listing of the salaries of hundreds of individual pharmacists. Nor would you be able to easily make much sense from such a list beyond maybe a general feel for how high or low the numbers on the list seemed. A histogram might help make some sense of this data, but most people would also be looking for information about the average pharmacist's salary. ***Statistical measures*** are numerical values like averages which are used to summarize a large amount of data with a single number.[1]

Mean and Median

While the term "average" may be a familiar example of a statistical measure, matters are not quite as simple as we might assume. There are actually several commonly used measures that may be meant when we use the term average, each of which has its pros and cons in any given situation. The term ***measure of central tendency*** is a general term for any of these "averages".

The most familiar measure of central tendency, and what most people usually think of when they hear the word "average," is the ***mean***. The mean of a set of numerical values is found by adding all of the values up and then dividing by the number of values. For example, suppose that you are considering opening a coffee shop and are doing some market research about what other coffee shops in town charge for a large regular coffee. You investigate five competitors, and find that their prices are:

Competitor	*Price*
A	$1.49
B	$1.59
C	$1.25
D	$3.75
E	$1.50

The mean price in town then would be:

$$\text{Mean} = \frac{\$1.49 + \$1.59 + \$1.25 + \$3.75 + \$1.50}{5} = \$1.92$$

While $1.92 is the mathematical "average" in the usual sense, if we look at the prices themselves we can see that it is not a particularly good estimate of what is "average" in the sense of being typical; $1.92 is significantly higher than the prices charged by four of the five competing coffee shops, and so if you based a pricing decision on the assumption that the market would regard $1.92 as an "average" price you would be mistaken. The mean is being brought up by one shop that charges a much higher price than the others. This shop may or may not be successful with its higher pricing, but its pricing certainly is not *typical*. Its impact on the mean makes this "average" a misleading measurement of what is typical.

An alternative measure of "average" is the ***median***. The median of a set of values is calculated by listing the values in increasing (or decreasing) order, and then choosing the value in the middle of the list (if there are an odd number of values) or the mean of the middle two values of the list (if there are an even number).

Putting our coffee prices in order,

$1.25, $1.49, $1.50, $1.59, $3.75

we can see that the median price is $1.50. Looking over the prices, we see that $1.50 is a pretty good measure of a typical, middle-of-the-road price. The strength of the median is that it effectively ignores the size of the highest and lowest values. If Competitor E raised its price from $3.75 a cup to $5,000 a cup, the mean would shoot up ridiculously high, but the median would not change at all.

[1]Technically speaking, if the value is calculated on the basis of *all* relevant individuals, it is called a **parameter.** The average salary mentioned here would be a parameter if it is calculated from the salaries of all pharmacists. If the value is based on some sampling of individuals, it is called a **statistic.** The average mentioned would be a statistic if it is calculated from, say, "500 pharmacists surveyed" or some other collection of pharmacists short of all of them. The distinction between parameters and statistics is important, but for our purposes here it will not be necessary to distinguish between the two. We will use the term **statistical measure** regardless of whether it is based on the whole group or just a portion of the whole.

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There are other measures of central tendency used in certain special situations. Some examples include the ***mode*** (the value that occurs most often) and the ***midrange*** (the mean of the highest and lowest values). Since the mean and median are overwhelmingly the most commonly used, however, we will confine our discussion to those two measures.

People are often sloppy in their terminology, using the term "average" to indicate either the mean or the median. Averages reported in the media may be either means or medians, and it is often unclear which measure is actually being used. As the above discussion has demonstrated, and as many of the exercises will further show, there is often a significant difference between the two.

Example 16.2.1 A cell phone manufacturer is testing the time that a new phone can be used before the battery must be recharged. The company takes six phones, brings them up to full charge, and then tests to see how long the charges last with the phone in use. The times for each of the phones (in minutes) are 408, 348, 386, 420, 468, and 400. Find the mean and median time.

$$\text{Mean} = \frac{408 + 348 + 386 + 420 + 468 + 400}{6} = 405 \text{ minutes}$$

To find the median, we put the times in order: 348, 386, 400, 408, 420, 468. Since there is an even number of times, there is no one single value in the middle, so we take the two middle values and find their mean:

$$\text{Median} = \frac{400 + 408}{2} = 404 \text{ minutes}$$

In this case, the mean and median are quite close. (This often, though certainly not always, happens.)

These two different "averages" are summarized in the table below.

"Average"	*How to Calculate It*	*What It Tells You*
Mean	Add up all the values, then divide by the number of values.	The average in the usual sense of the term. Unusually high or unusually low values have an impact on the mean.
Median	Put the values in order, then choose the middle value. If there are an even number of values, take the mean of the two middle values.	The middle value. Unusually high or unusually low values are thrown out, and have no impact on the median.

Weighted Averages

A weighted average is an average in which certain items are counted more heavily (i.e., given greater "weight" than others.)

We have seen weighted averages in many areas of this text. For example,

- In Chapter 6, we used them to estimate the predicted performance of an investment portfolio (each asset's performance was weighted according to the percentage of the overall portfolio that it represented),
- In Chapter 10, we used them to find a credit card's average daily balance (each balance was weighted according to the number of days that it was in place).
- In Chapter 15, the average cost method for inventory uses a weighted average as well.

Weighted averages are very useful in business when an overall average value is needed, but some items are considered more important to that value than others, for example, because of their size in relation to the overall business picture.

Suppose, for instance, that we revisit the coffee price example that we were considering earlier. We had obtained the prices for five competing coffee shops, and wanted to get a sense of the average price in the local market. We did not consider in either of our "average" calculations, though, how much business each of the coffee shops actually does. If competitor A sells more coffee than the others, we might want to consider its price as more important than the others. Suppose, for example, that we know the weekly sales for each of these competitors. We might then want to give each coffee shop a different weight based on the number of cups of coffee sold.

Competitor	***Price***	***Estimated Weekly Sales (Cups)***
A	\$1.49	850
B	\$1.59	550
C	\$1.25	700
D	\$3.75	500
E	\$1.50	1000

To calculate the weighted average price, we multiply each of the prices by its "weight"; in this case the weight is the estimated sales. Then, we divide by the total of the weights. This gives us a weighted average of:

$$\text{Weighted average} = \frac{850(\$1.49) + 550(\$1.59) + 700(\$1.25) + 500(\$3.75) + 1{,}000(\$1.50)}{850 + 550 + 700 + 500 + 1{,}000}$$

$$= \frac{\$1{,}266.50 + \$874.50 + \$875.00 + \$1875.00 + \$1{,}500.00}{850 + 550 + 700 + 500 + 1{,}000}$$

$$= \frac{\$6{,}391.00}{3{,}600}$$

$$= \$1.78$$

Example 16.2.2 ***Suppose that a computer services company has five part-time workers. Each worker's hourly wage and number of hours worked in a typical week are listed in the table below. Calculate (a) the mean of the hourly rates, (b) the median, and (c) the weighted average, weighted by hours worked.***

Employee	***Hourly Wage***	***Typical Weekly Hours***
A	\$23.50	25
B	\$27.50	34
C	\$18.00	28
D	\$38.00	10
E	\$45.00	8

(a) The mean would be:

$$\text{Mean} = \frac{\$23.50 + \$27.50 + \$18.00 + \$38.00 + \$45.00}{5} = \frac{\$152}{5} = \$30.40$$

(b) We put the hourly wages in order: \$18.00, \$23.50, \$27.50, \$38.00, \$45.00. The mean is the middle value of this list: \$27.50.

(c) To find the weighted average:

$$\text{Weighted average} = \frac{25(\$23.50) + 34(\$27.50) + 28(\$18.00) + 10(\$38.00) + 8(\$45.00)}{25 + 34 + 28 + 10 + 8}$$

$$= \frac{\$587.50 + \$935.00 + \$504.00 + \$380.00 + \$360.00}{25 + 34 + 28 + 10 + 8}$$

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$$= \frac{\$2,766.50}{105}$$
$$= \$26.34$$

Notice that when we compare the three "averages" in this example, the weighted average is lower than the mean or median. Neither the mean nor the median takes into account the fact that the workers earning the higher wages are working fewer hours; this is taken into account with the weighted average, and so it is lower. In this case, the weighted average is the better representation of the average rate per hour that the company pays.

Indexes[2]

An ***index*** is another type of (usually weighted) average. An index is a numerical value, calculated according to some set formula, intended to numerically summarize some overall situation. One familiar example is a ***market index***, used to reflect the value of the stock market (or some other financial market). The Dow Jones Industrial Average (the Dow) and the Standard and Poor's 500 (S&P 500) are two U.S. stock market indexes whose values are reported in the daily news. The S&P 500 is based on an average of the prices of 500 large companies' stocks, weighted by the overall market value of each company. The Dow Jones Industrial Average is based on 30 of the largest publicly traded companies' stock prices. It is also a weighted average, though over the many years that this index has been kept, the formula has become quite complicated; it has been adjusted many times to reflect changes in which companies are included in the average.

Market indices also exist for foreign markets, or for companies engaged in certain types of businesses, or for companies that meet some other criterion. There are indexes specifically for telecommunications companies or retailers, and indexes for companies that meet certain guidelines for "socially responsibility."

Indexes differ from averages, though, in that the formulas used to calculate them need not correspond to any of the familiar average types. For example, suppose you are constructing a very simple index to track the performance of five stocks. The stocks, and their prices, are:

Company	Stock Price
A	$25
B	$45
C	$30
D	$10
E	$5

You could construct an index by taking the straight average of the prices of these stocks, in which case your index would now stand at 23.00. Or, your index formula could be to add up all the stock prices and use that number, in which case your index would now stand at 115.00. Or you could use a formula that weights Company A and Company B twice as heavily as the others, calculates a weighted average, and then divides the result by 0.0175. (Never mind asking *why* you might be using this formula; you could use it if you wanted to.) In that case your index would be 1510.20. Any formula that calculates a numerical value based on these stocks' prices could be used as a formula for an index of these prices.

Because the value of an index depends on the formula used to calculate it, and the formulas can vary so wildly, the value of a market index looked at in isolation does not necessarily tell you much. If the GlobalInvestexx CleanTech 75 Index stands at 24,535.72, does that mean that stocks of the companies it tracks are doing better or worse than the stocks in the Dow Jones Industrial Average, which stands at 12,000? Without knowing anything about how those indexes are calculated, it is impossible to say. One may be higher than the other because of differences in the stocks, or one may be higher than the other simply because of how the formula is calculated.

[2] The technically correct plural of "index" is "indices." However, "indexes" is now in common use, and so we will follow modern usage and feel free to use either term.

However, index formulas can still be used as a basis for comparison. If you know that the market closed at "10,952 points, up 241," by the relative size of the increase you know that the stocks included in the index, and presumably the rest of the market as well, had a very good day. Also, if you know that 5 years ago the index sat at 11,275 points, you also know that the stocks in the index, and presumably the rest of the market as well, have not had such a great past 5 years.

Example 16.2.3 ***Suppose that the 1 year ago the GlobalInvestexx CleanTech 75 index stood at 21,736.14 points. Today the index stands at 24,535.72. What does this tell you about the stocks that are tracked by this index?***

Since the value of the index has risen, we can assume that the stocks that it tracks have on average risen over the past year. In fact, we can calculate the percent increase by dividing.

$$\frac{24{,}535.72}{21{,}736.14} = 1.1288$$

The index is now 112.88% of where it was a year ago, a 12.88% increase. We can take this as an indication that the "clean tech" stocks that this index is intended to track have risen in value by about 12.88% on average. We must be careful in interpreting this, though. This index is a type of average; while this does indicate that stocks of clean tech companies have risen around 12.88% on average, this does not tell us what the price change would be for any individual stock, or for a portfolio of clean tech stocks that does not follow the weightings used in this index.

Another type of index is a ***comparative index***. Suppose that you earn a salary of $35,000, living in Fort Wayne, Indiana. You get a job offer that would require you to move to Boston, Massachusetts. You know that the cost of living in Boston is much higher than in Fort Wayne and so you know you will need to make more money to maintain the same standard of living. But how much more? A comparative index would provide a way to get a handle on how the costs of living compare in these two different locations. A comparative index would be based on the prices of a range of different goods and services at each location, weighted according to some formula meant to reflect how much of a typical person's budget each item eats up. (For example, the cost of cable television would count for less than the cost of food, since[3] people spend more for food than for cable.)

Suppose that you look up a cost-of-living index for Boston and find that it is 1,184.29. This, by itself, tells you nothing. However, if you also know that the index for Fort Wayne is 939.25, you can use these together by dividing the indexes. Boston is more expensive than Fort Wayne. How much more expensive overall? Overall, the cost of living in Boston is 1184.29/939.25 times as much as in Fort Wayne.

Example 16.2.4 ***Using the index numbers given above, what salary would be needed in Boston to match $35,000 in Fort Wayne.***

$$(\$35{,}000)\left(\frac{1{,}184.29}{939.25}\right) = \$44{,}131$$

Comparative indexes can be helpful, but they have to be taken with a grain of salt. The items included in an index may not always reflect how any given person divides up

Indexes can be helpful when comparing costs of living or doing business in different locations. © Photodisc/Getty Images/DIL

[3] most

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his budget. An index number for the cost of living in most cities would include the cost of owning and operating a car. The costs (parking, insurance, etc.) of operating a car in Manhattan are extremely high, but many people living there don't own cars. They may find it easier and more cost-effective to get around by using cabs and public transportation. So, how much should the cost of owning and operating a car be counted in an index for Manhattan? The compiler of the index cannot possibly know whether or not someone using that index would choose to keep her car when moving from Raleigh to Manhattan.

Along the same lines, in some areas the cost of owning real estate is much higher than renting (or vice versa), whereas in others the costs are much closer aligned. If you are a renter, you would want the index numbers to reflect the costs of rent, whereas if you are an owner, you would want the index to reflect the cost of owning. These, and many other related factors, can make comparative indexes troublesome if taken too literally. Still, they do provide a way to quickly make rough comparisons.

Cost-of-living index numbers can be readily found at the library or over the Internet. Some Internet sites can create specialized indexes to reflect your circumstances, by automatically calculating an index based on the answers given to questions on an interactive quiz. As with any Internet resources, though, you should be careful in assessing just how credible the source of the information is before relying on it too heavily.

Expected Frequency and Expected Value

Suppose that you are managing a company that manufactures mobile phones. You know that despite your best quality control efforts some of the phones will turn out to be defective. In order to effectively manage your business, you would want to have some sense of how many of your phones will turn out to be defective, to help determine what sort of warranty you are going to offer and how much you should budget for the costs of returns and repairs.

Of course, there is no way to know in advance how many phones will turn out to have defects, but still you could make a reasonable attempt to put a good estimate on this.

One logical approach would be to look at what your company's experience has been with similar phones made in the past. For example, suppose that in the past year your company has manufactured 184,300 similar phones, and 5,724 have been returned for repair or replacement. You should be able to use this information as the basis of an educated guess about our new phones.

The ***relative frequency*** (often simply referred to* as the ***frequency***) of an event happening is the fraction (or percent) of the time that this event actually has happened. In the example we are considering, we would say that the relative frequency of defective phones is 5,724 out of 184,300, or 5,724/184,300, or 3.11%.

The ***expected relative frequency*** of an event occurring is the fraction (or percent) of the time that we predict that a future event will happen. The expected relative frequency may be only a guess, since we cannot know the future, but even so it should be an educated guess based on the best information available. In our mobile phone example, it might be reasonable to take 3.11% as our expected relative frequency, since that is what we have seen with similar products in the past. If you have good reason to believe that our quality control has improved, using a slightly lower percent might be justified. On the other hand, if you want to be conservative and make sure that you do not underestimate the defective phones (and hence not adequately budget for the cost of repairs and replacement) you might want to use a somewhat higher percent, such as 3.25% or even higher.

The ***expected frequency*** of an event occurring is the number of times we expect it to actually occur. Expected frequency is very similar to expected relative frequency, except that frequency is a number while relative frequency is a fraction or percent. In our mobile phone example, the expected frequency would be the *number* of defective phones that you expect. Expected frequency can be found by multiplying:

$$\text{Expected frequency} = (\text{Expected relative frequency})(\text{Total number of items})$$

*incorrectly

So, if your company manufactured 82,000 of the new phones, and you decided to use a 3.25% expected relative frequency, the expected frequency of defective phones would be:

$$\text{Expected frequency} = (3.25\%)(82{,}000) = 2{,}665$$

Expected value is similar to expected frequency, but not exactly the same. The expected value of a quantity is the average amount you expect that quantity to represent on a per item basis. This expected value can be calculated by multiplying:

$$\text{Expected value} = (\text{Expected relative frequency})(\text{Amount per occurrence})$$

Suppose, for example, that you are estimating from past experience that the average cost to your company of a warranty claim for a mobile phone is $127.40. The expected value (per phone) of warranty costs would then be:

$$\text{Expected value} = (3.25\%)(\$127.40) = \$4.14$$

The term "expected value" is not always used. It is usual to replace the word *value* with whatever the value is supposed to represent. In this example, you might refer to this expected value as the "expected warranty cost per phone" or some similar description.

Example 16.2.5 ***A minor league baseball events promoter is offering a promotion at an upcoming game. One fan will be randomly chosen from the stands and given the opportunity to bat against the team's ace pitcher. If the fan hits a home run before accumulating three strikes, he will win $50,000.***

From past experience of similar events, the promoter believes that such a fan will win the prize one out of every 1,250 times the promotion is run. Calculate (a) the expected relative frequency of the fan winning and (b) the expected prize for this promotion.

(a) The expected relative frequency would be 1/1,250 or 0.08%.

(b) The expected prize would be

$$\frac{1}{1{,}250}(\$50{,}000) = \$40$$

The word *expected* can be misleading. When we say that the expected warranty cost per phone is $4.14, we certainly don't mean that we actually expect each phone to generate $4.14 in warranty costs. Likewise, we certainly don't mean that we actually expect the baseball promotion to cost $40 in prize money. In fact, it is very unlikely that an individual phone would generate $4.14 in warranty costs, and it is *impossible* that a fan could receive $40 in prize money from the baseball contest.

The word *expected* is used here in a technical way. Loosely speaking, it is an average. The expected cost per phone means that we expect that the average warranty claims cost per phone will be $4.14, not that any individual phone will generate that cost. The expected prize means that we expect that, if we run similar contests repeatedly, the average amount of winnings per contest will work out to be $40, not that $40 will be paid out in any individual contest. We do not take these values as applying to any individual occurrence, any more than we would say that since the average American family has 2.3 children we therefore expect that Bob and Sally Smith will have exactly 2.3 children.

Expected values are useful when interpreted as predicted averages. Knowing that the expected cost for warranty repairs is $4.14 would tell you that you need to build $4.14 into the price of each phone to generate the money that you will need overall to cover your warranty repair and replacement costs. Knowing that the expected prize of the baseball contest is $40 would help an insurance company to know how much to charge the baseball promoter for a policy that would pay the $50,000 prize if it is won.[4]

[4] Yes, this is actually common practice. Usually if a promotion like this is offered, rather than take the risk of having to pay a large prize, the contest sponsor actually will purchase a type of insurance to cover the event.

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EXERCISES 16.2

Unless otherwise specified, round your answers to the same number of decimal places used for the original data.

A. Mean and Median

1. Suppose that a cereal company needs to determine the number of grams of sugar contained in a serving of their granola. The company takes one serving from each of five boxes of their granola, and finds sugar contents of 16.3 g, 16.1 g, 17.3 g, 16.4 g, and 17.2 g.
   a. What is the mean grams per serving based on these five servings?
   b. What is the median grams per serving based on these five servings?

2. Over the last 7 weeks, the number of videos rented at a local rental store has been 352, 308, 417, 408, 363, 305, and 319. Calculate (a) the mean and (b) the median number of rentals per week over this period.

3. Jared is a life insurance salesman. Over the last 6 months, his monthly sales (based on initial premiums for the policies he has sold) have amounted to $4,875, $3,070, $916, $3,382, $7,455, and $8,332. Calculate his (a) mean and (b) median monthly sales.

4. The price per gallon of gasoline at six local stations is $2.59, $2.63, $2.61, $2.59, $2.58, and $2.72. Calculate the (a) mean and (b) median price per gallon among these stations.

B. Weighted Averages

5. Tickets for a minor league hockey team are sold at three different price levels. General admission tickets cost $8.50, reserved seating tickets cost $12.75, and premier seating tickets cost $18.00. Last night, the team sold 2,204 general admission tickets, 802 reserved seating tickets, and 608 premier seating tickets. Calculate the weighted average ticket price based on the number of tickets sold.

6. Piet has a line of credit on which he pays interest based on his average monthly balance. The average used is a weighted average based on the number of days his account stays at each balance. Last month he had a balance of $1,750.00 for 13 days, $2,250.00 for 16 days, and $2,700.00 for 2 days. Calculate his average monthly balance.

7. Montag Fire Equipment has five regional offices. The table given below shows that the sales generated by each of its regional offices and the sales per employee at that office.

Office	*Sales*	*Sales per Employee*
Atlanta	$3,510,000	$125,357
Portland	$7,500,000	$178,571
Chicago	$5,300,000	$240,909
Phoenix	$1,750,000	$97,222
San Jose	$9,500,000	$287,879

Calculate the company's overall average sales per employee.

8. Brendan's Doghouse Flowershop pays a different rate of electricity based on the time of day. For peak hours the rate is $0.118 per kilowatt-hour, while for off-peak use, the rate drops to $0.062 per kilowatt-hour. Last month, the shop used 3,895 peak kilowatt-hours and 1,975 off-peak kilowatt-hours. Calculate the shop's overall average cost per kilowatt-hour.

C. Indexes

9. The GlobalInvestexx Canadian stock index, which is designed to reflect the overall stock price performance of Canadian companies, stands at 1,404.53 right now. One year ago, this index stood at 1,233.59.
   a. By what percent did this index's value grow over the past year?
   b. If I own a portfolio of Canadian stocks, what does this index tell you about how my portfolio did in the past year?
   c. If Zarofire Systems is one of the stocks included in this index, what can you say about what happened to the price of its stock in the past year?

10. If the GlobalInvestexx Mexican stock index stands at 4,800 and the GlobalInvestexx Australian stock index stands at 9,600, what does this tell you about the value of Australian stocks compared to Mexican stocks?

11. A personal finance magazine compiles an index of real estate prices in different markets around the country. On this index, St. Louis has a value of 114.98 and Seattle has a value of 184.92. If a house in St. Louis costs $162,500, how much would you expect a comparable house in Seattle to cost on the basis of these index values?

12. A personal finance magazine compiles an index of real estate prices in different markets around the country. On this index, Houston has a value of 102.39 and Buffalo has a value of 91.15. If a house in Houston costs $174,300, how much would you expect a comparable house in Buffalo to cost on the basis of these index values?

13. A cost-of-living index lists Cleveland at 103.45 and Des Moines at 97.33. If you are making $47,500 a year in Cleveland, how much does this index suggest you would need to earn in Des Moines to have the same buying power?

14. A cost of living index lists Albany at 114.43 and Pittsburgh at 104.35. Right now, Sara is earning $37,340 at a job in Pittsburgh. If she is offered a new job in Albany paying $40,000, is this really a raise for her? Justify your answer.

D. Expected Values

15. An auto insurer expects, from past experience, that 4 out of every 100 auto liability policies it sells will have a claim in the next year.
   a. If the company sells 19,253 policies, what is the expected number of claims from these policies?
   b. If the average claim on these policies is expected to be $22,350 per claim, what is the expected claims cost per contract?

16. A grocery chain is offering a promotion where every 2,000th customer checking out will get his purchase for free. The average customer's purchase is $98.35.
   a. If the store has 6,000 customers each week, how many prizes should it expect to give away each week?
   b. What is the expected cost per customer of this promotion?

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17. A bakery knows that on average 1 out of every 8 loaves of bread baked will not be sold while still fresh.
    a. If the bakery produces 1,075 loaves of bread each week, how many loaves are expected to not sell while still fresh?
    b. Each loaf sells for $3.50 when fresh. Loaves that do not sell fresh are sold at a discount, fetching an average price of $1.00 per loaf. What is the average cost per loaf for spoilage?

E. Grab Bag

18. If you are considering moving to a new town and buying a house, which average would give you the better sense of what housing prices are like there: the mean or the median? Why?

19. Suppose that Wally's Widget World has 450 widgets in stock. Fifty of them were purchased at a discounted cost of $2.00 each, while the remaining 400 were purchased for $8.00 each. Calculate the average cost per widget for the widgets.

20. An electronics company offers a large-screen television for sale at $899, less a $300 mail-in rebate. Based on past experience, the company expects that only 62% of the people who buy this TV will actually submit the claim for the rebate. What is the expected cost to the company for the rebate for each TV sold? What is the company's actual expected income per television sold?

21. (Continued from Exercise 20). Explain why a company might prefer to offer a rebate rather than just offering their product for sale at a lower price.

F. Additional Exercise

22. Speedywheels Bicycles estimates that 40% of the cost of each of its bicycles comes from manufacturing costs, 25% from raw materials, and 35% from other overhead. The company's budget for next year projects that manufacturing costs will rise by 5.3%, raw material costs will rise by 8.5%, and other overhead costs will drop by 1.3%. If these projections are correct, by what percent will the cost of a Speedywheels bicycle rise next year.

16.3 Measures of Variation

Measures of average are very important for a business trying to make appropriate management decisions based on statistical data. Averages, though, can't tell the whole story. It is often important not just to measure the average of some amount, but also to measure just how widely the values *vary*.

Suppose, for example, that you are working for an electric company evaluating different possible locations for a wind turbine. You are considering two different possible sites for the turbine. The wind will generate the electricity, but the wind does not blow with the same strength in different locations, and so you decide to do some testing of the two sites.

Of course, the wind does not always blow with the same strength at different times, either. At each of the two sites, you measure the wind speed in miles per hour (mph) at five different times[5] and obtain the following results (listed from slowest to fastest at each location):

[5] In reality, you would want far more than just five measurements to base your decision on. In this discussion we are just using five measurements in order to demonstrate how statistical measures could be used in this sort of situation, while keeping the work of calculating the measures manageable. If we had several thousand measurements at each location, the calculations would obviously require much more effort, but they would be done in the same way.

Location A	Location B
14 mph	2 mph
15 mph	9 mph
16 mph	16 mph
17 mph	23 mph
18 mph	30 mph

Which location is the better place to site your turbine? One way to approach this question is to look at the average wind speed at each location, with the idea that the site with the higher average speed would be expected to produce more electricity. If we calculate the mean speed at each location, though, we can see that they are tied, with each site having a mean wind speed of 16 mph. What about the median? We can quickly see from the data that each location also has a median speed of 16 mph. The two sites are equivalent whichever measure of average we use. If you based your decision only on the average, you would not see any reason to prefer one location over another.

However, looking at the data, we can see that the two locations are definitely not the same. At Location A, the wind speed varies, but not by all that much, while at Location B there appears to be much more variation. Neither the mean nor the median captures this difference between the two sites, nor would we expect them to: they are designed to measure the “average” of a set of data, not how much variation there is in the set. Statistical measures that are used to measure this are called ***measures of dispersion***, or of ***variation***.

Measures of Variation

The simplest of these measures is the ***range***. The range of a set of data is the difference between the highest and lowest values. For Location A, the range is $18 - 14 = 4$ mph, while for Location B the range is $30 - 2 = 28$ mph. Comparing these two ranges reveals the far greater variation experienced at Location B. While range can be a useful tool, it has its limitations. The range does not take into account what occurs *between* the extreme high and low values. (See Exercise 4 for an example where the range falls short as a measure of variation.)

The ***standard deviation*** is an alternative, commonly used measure of variation. It has the advantage of including all pieces of the data in its calculation, though it suffers from the disadvantage of being much more complicated than the range in both its calculation and interpretation. The standard deviation is probably best introduced through an example of its calculation; we will discuss its interpretation afterward.

Here is how we would calculate the standard deviation for the wind speeds at Location A. We first look at how much each individual speed measurement differs from the mean of 16 mph. To do this, we subtract the mean from each value. For example, for the first measurement we subtract 14 mph $-$ 16 mph to get -2 mph, reflecting the fact that the 14-mph measurement was 2 mph “below average.” These differences are called the ***deviations***. The wind speeds from Location A together with their deviations are shown in the table below.

Speed	Deviation
14	−2
15	−1
16	0
17	1
18	2

The speeds that fall below the mean have negative deviations, while the ones above the mean have positive deviations. We are interested in measuring the total amount of variation, though, regardless of whether the variation comes from above- or below-average

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values. To get rid of the positive/negative distinction, we next square the deviations (since the squares of both positive and negative numbers come out positive):

Speed	*Deviation*	*Squared Deviation*
14	−2	4
15	−1	1
16	0	0
17	1	1
18	2	4

The next step is to add up these squared deviations and divide by $n - 1$, where n is the number of values in the data set. The idea here is more or less to "average" the squared deviations, though we divide by one less than the number of values rather than the number of values. (The reasons for using $n - 1$ instead of n are technical and fall outside the scope of this section.)[6]

Summing the column of squared deviations gives:

Speed	*Deviation*	*Squared Deviation*
14	−2	4
15	−1	1
16	0	0
17	1	1
18	2	4
Total		10

Dividing this by $5 - 1 = 4$, we get $10/4 = 2.5$. This value is known as the ***variance***.

Our last step is intended to undo a side effect of how we eliminated the positives and negatives. Squaring not only makes everything positive, it also changes the magnitude of the numbers. So, as a last step to bring things back into proportion, we take the square root of the variance to get:

$$\text{Standard deviation} = \sqrt{2.5} = 1.58$$

Before discussing the interpretation of this value, let's get a bit more practice with the calculation.

Example 16.3.1 ***Calculate the standard deviation of the wind speeds for Location B.***

We build a table just as we did for Location A:

Speed	*Deviation*	*Squared Deviation*
2	−14	196
9	−7	49
16	0	0
23	7	49
30	14	196
Total		490

Following through with the remaining steps, we get:

$$\text{Variance} = 490/(5 - 1) = 490/4 = 122.5$$
$$\text{Standard deviation} = 11.07$$

[6]Actually, the rule is that you divide by n when your data set represents the entire population being studied. In this and in most other cases we are only working from some representative sample–the wind measured at 5 moments rather than at every moment–and so in this book we will always divide by $n - 1$.

Interpreting Standard Deviation

If we compare the standard deviation from Location A with the one from Location B, we can get a sense of how to interpret the standard deviation. The measured speeds at Location B were much more variable, and the standard deviation reflects that. More variation means larger deviations, which makes for larger squared deviations, which makes for a larger total of the squared deviations, which leads to a larger standard deviation. On the other hand, at Location A, where there was not much variation in the wind speeds, the deviations were small, and following through the process, we found these small deviations led to a small standard deviation.

Speaking generally, we can thus say that the size of the standard deviation is an indicator of the amount of variation in a set of data. The key to interpreting standard deviation is this: *the higher the standard deviation, the greater the variation; the lower the standard deviation, the lower the variation.*

Example 16.3.2 ***Suppose that you are considering investing in two different mutual funds. Over the past 10 years, the annual returns of the Hopewell American Growth Fund have had a standard deviation of 11.25%. The annual returns of the Hopewell Adrenaline Aggressive Growth Fund have had a standard deviation of 28.4%. Which investment has seen the greatest variation in its annual rates of return?***

The Adrenaline Aggressive Growth fund has a larger standard deviation. We can conclude from this that it has had a much greater degree of variation.

We must be careful, though, when comparing standard deviations, that the items that we are comparing are on similar scales. In Example 16.2.3 they were (since both were percents) and in all of the remaining examples and exercises of this section this will also be true. However, the Additional Exercises at the end of this section provide an example where differing scales can make the comparison of standard deviations misleading.

It is usually easiest to interpret standard deviations in comparisons. Interpreting the standard deviation in absolute terms is a bit trickier. For example, suppose you are told that the mean exam score in your business math class was 78.2 and the standard deviation was 8.3. How can you interpret this?

Interpreting a standard deviation by itself is not as clear as interpreting one in comparison. From everyday situations, we all have at least a rough sense of what an "average" is. We are less likely to have a good everyday sense of what a standard deviation is, and unfortunately standard deviation does not possess any simple, commonsense interpretation.

However, there are a few tools that may be helpful. One is ***Chebyshev's Theorem***, which gives a minimum percent of the scores from any collection of data that must fall within a certain number of standard deviations from the average. Chebyshev's theorem guarantees that for any collection of data, *at least* ¾ (75%) of the data must fall within 2 standard deviations of the mean, and *at least* ⁸⁄₉ *(88.9%)* must fall within 3 standard deviations. It does not put any minimum on how many values must fall within 1 standard deviation, though.

Example 16.3.3 ***If the mean is 78.2 and the standard deviation is 8.3, use Chebyshev's theorem to interpret what this tells you about where the class's scores fell.***

Two standard deviations would be 2(8.3) = 16.6.

Two standard deviations below the mean would be 78.2 − 16.6 = 61.6; two standard deviations above the mean would be 78.2 + 16.6 = 94.8. Chebyshev's theorem tells us that at least ¾ of the class scored between 61.6 and 94.8.

Following the same approach with three standard deviations, Chebyshev's Theorem tells us that ⁸⁄₉ of the class scored between 53.3 and 103.1.

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Knowing this may help give some sense of how the class's scores were spread out. Keep in mind, though, that this gives a *minimum*. It is possible that more than ¾ of the class fell into the 61.6 to 94.8 range, or even that all of the class did.

This example may be a bit disappointing. The ranges that we have to work with here are awfully broad − 61.6 to 94.8 is a pretty wide range, and 53.3 to 103.1 is even worse. Plus, Chebyshev's theorem does not tell us what proportion of the class fell into these ranges, only that the proportion must be at least ¾ or ⁸⁄₉. Chebyshev's theorem tells us something about how to interpret that 8.3 point standard deviation, but it doesn't tell us all that much.

In certain special situations, we can say something much stronger than what Chebyshev's theorem tells us. It is often, though by no means always, the case that data will fall into a ***normal distribution***. The normal distribution is recognizable by the familiar bell-shaped curve that results when you create a histogram for normally distributed data.

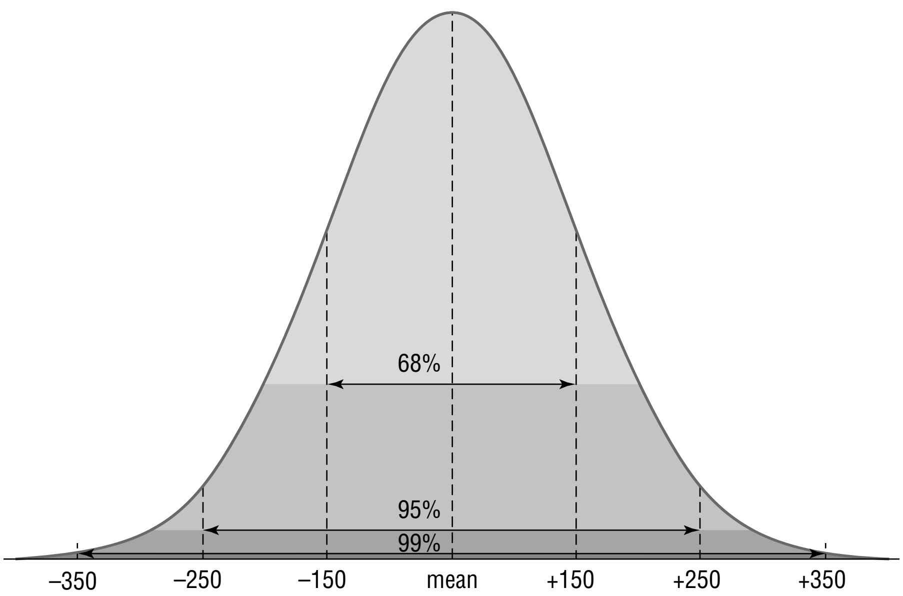
If—and only if—we know that our data is approximately normally distributed, we can make use of the ***Empirical Rule***. The empirical rule says that approximately 68% of the data falls within 1 standard deviation of the mean, approximately 95% falls within 2, and better than 99% falls within 3. Not only are these percents higher than with Chebyshev's theorem, they are approximations, not just minimums.

Example 16.3.4 ***Suppose that your teacher tells you that the class's test scores were normally distributed. If the mean is 78.2 and the standard deviation is 8.3, use the Empirical Rule to interpret what this tells you about where the class's scores fell.***

The empirical rule says that approximately 68% of the class must have scored within 1 standard deviation of the mean. So 68% of the class scored between 78.2 − 8.3 = 69.9 and 78.2 + 8.3 = 86.5.

The ranges for 2 and 3 standard deviations are the same as they were in Example 16.3.3. However, we know quite a bit more, since we can use the Empirical Rule. Approximately 95% of the class scored between 61.6 and 94.8; essentially the entire class scored between 53.3 and 103.1.

It may be helpful to visualize what the empirical rule tells us with the following:



Our two tools for interpreting standard deviation are summarized in the table below:

Percentage of the Data that Falls within:	If the Data is Distributed: Normally (Bell Curve)	Any Way at All
1 standard deviation of the mean	Approximately 68%	Can't say anything
2 standard deviations of the mean	Approximately 95%	At least 75%
3 standard deviations of the mean	Approximately 99%	At least 88.9%

EXERCISES 16.3

A. Measures of Variation

1. Suppose that a company is testing the durability of its light bulbs. The company tests five bulbs to see how long they will last and finds that the hours each bulb lasts are: 835, 842, 808, 798 and 907.

   a. Calculate the mean number of hours these bulbs lasted.

   b. Complete the table below:

Hours	Deviation	Deviation Squared
835		
842		
808		
798		
907		
Total		

   c. Use the values you calculated in the table for part b to calculate the standard deviation of the hours the bulbs lasted.

2. Suppose that a bakery is testing how consistent the weights of its bread loaves are. The company tests six loaves and measures their weights. The weights are 681.3 g, 682.2 g, 685.1g, 680.0 g, 680.0 g, and 679.2 g.

   a. Calculate the mean weight of the bread loaves.

   b. Complete the table below:

Weight	Deviation	Deviation Squared
Total		

   c. Use the values you calculated in the table for part b to calculate the standard deviation of the bread loaves' weights.

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3. Suppose that a grocery store manager is trying to assess how long customers have to wait in line to check out. He randomly selects 20 customers and times how long it takes them to complete checking out. The slowest time to check out was 19 minutes, the quickest was 4 minutes. What is the range for these checkout times?

4. Suppose that a consumer testing agency has collected data on the octane level of two different brands of gasoline. The agency collected and tested 8 samples of standard gasoline from each company's stations. The results are shown in the table below:

Brand X	*Brand Y*
87.0	87.0
87.0	87.5
87.1	86.4
88.7	88.1
87.1	87.2
87.1	88.0
87.1	86.5
87.0	86.9

   a. Calculate the range for octane levels of each brand.
   b. By looking at the values for each brand, state which brand appears to have the more consistent octane levels?
   c. Explain how this example illustrates the need to use standard deviation as a measure of variation.

B. Interpreting Standard Deviation

5. Suppose that the light bulb company from Exercise 1 also tested five of its leading competitor's bulbs and found that the standard deviation of the competitors' bulbs was 74.8 hours. Does this suggest that the competitors' bulbs are more, or less, consistent in how long they last?

6. If you are a working as a quality control manager for a manufacturing company, would you want to see high standard deviations or low standard deviations when you take measurements of the products produced by your company? Why?

7. A college, most of whose students are in the 18 to 24 age range, is trying to attract a more diverse student population. Would the college want to see the standard deviation of its students' ages increasing or decreasing? Why?

8. Suppose that a radio station finds that the average age of its listeners is 32.7, with a standard deviation of 9.3 years.
   a. How many years would 2 standard deviations be?
   b. Give the range of ages that would represent "within 2 standard deviations of the mean."
   c. What percent of the station's listeners fall within this age range?

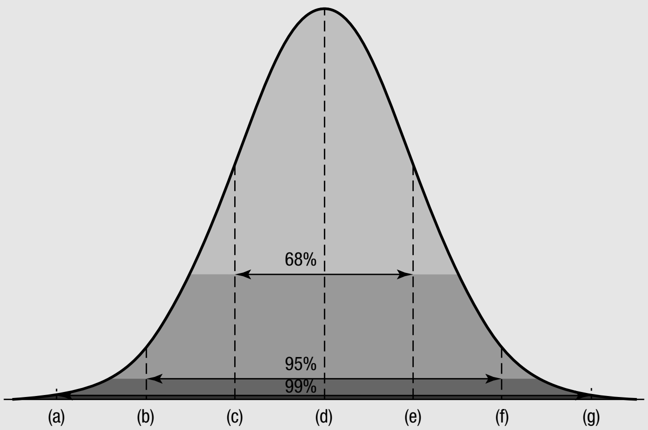
9. Suppose that a bottled water company's quality control staff has determined that the mean volume of water in a 1-gallon bottle is 128.31 ounces, with a standard deviation of 0.52 ounces.
   a. How many ounces would 3 standard deviations be?
   b. Give the range of volumes that would represent "within 2 standard deviations of the mean."

c. What percent of the bottles have volumes that fall within this range?

10. If the radio station from Exercise 8 knew that its listeners' ages were normally distributed, how would that change your answers to Exercise 8?

11. If the bottled water company from Exercise 9 knew that its bottle volumes were normally distributed, how would that change your answers to Exercise 9?

12. Suppose that the highway gas mileage for a particular model of car is normally distributed with a mean of 36.2 mpg with a standard deviation of 0.8. Fill in the missing numbers on the graph below:



C. Grab Bag

13. A grass seed company is testing the germination rates for its grass seed mix. In a test of seven bags of seed, the company found germination rates of 87.2%, 88.1%, 87.7%, 87.5%, 87.7%, 89.1%, and 86.6%.
    a. Calculate the mean germination rate based on these samples.
    b. Calculate the standard deviation of the germination rates of these samples. (Avoid rounding until the end; express your final answer as a percent carried to two decimal places.)
    c. The company's quality control standards specify that germination rates should average 87.5% with a standard deviation of 1.0%. How do the results of this test stand up against these standards?

14. Suppose that the mean number of minutes that a customer waits on hold before talking to a live customer service representative is 12.5 with a standard deviation of 2.5.
    a. The call center manager wants to know how many customers have to wait between 5 and 20 minutes. If the wait times are not normally distributed, is it possible that 90% of all customer calls fall within these waiting times?
    b. Is it possible if the wait times are normally distributed?

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D. Additional Exercises

15. In Exercise 5, we asked if a comparison of standard deviations *suggests* that one company's light bulbs have more consistent durability than the others. Why can't we say for sure which company's bulbs are more consistent?

16. Suppose that you are doing a comparison study of retail prices at different pharmacies in a metropolitan area. You collect prices for 500 count bottles of aspirin, and also for a 3-month prescription for an expensive medication. The results are shown below:

Pharmacy	***Aspirin Price***	***Prescription Cost***
A	$1.79	$871.79
B	$4.79	$874.79
C	$5.79	$875.79
D	$7.79	$877.79

a. Without doing any calculations, which of these two products would you say has the greatest variation in prices?

b. Calculate the standard deviation for each product's prices.

c. What does this tell you about comparing standard deviations?

Topics	Key Ideas, Formulas, and Techniques	Examples
Charts and Graphs, p. 608	• Charts and graphs are used to summarize and illustrate statistical data. • Pie charts are used to illustrate how different parts make up a whole. • Bar and column charts compare different categories. • Line charts illustrate how a quantity changes over time. • Follow instructions given in text to create these by using Excel.	See the examples in Section 16.1.
Mean and Median, p. 617	• The mean is calculated by adding up all the values and dividing by the number of values. • Mean is usually what people mean by "average." • Median is found by listing the values in order and selecting the middle value, or if there are two middle values, by finding their mean. • Mean is affected by unusually high or low values, median is not.	A cell phone manufacturer is testing the time that a new phone can be used before the battery must be recharged. The company takes six phones, brings them up to full charge, and then tests to see how long the charges last with the phone in use. The times for each of the phones (in minutes) are 408, 348, 386, 420, 468, and 400. Find the mean and median time. (Example 16.2.1)
Weighted Averages, p. 618	• A weighted average counts some values more heavily than others. • Multiply each value by its weight and add up the total. • Divide by the total of the weights.	Suppose that a computer services company has five part-time workers. Each worker's hourly wage and number of hours worked in a typical week are listed in the table below. Calculate (a) the mean of the hourly rates, (b) the median, and (c) the weighted average, weighted by hours worked. (Example 16.2.2)
Indexes, p. 620	• Indexes are used to measure the overall cost or value of something. • To use comparative indexes, multiply by the ratio of the indexes.	Suppose that you look up a cost-of-living index for Boston and find that it is 1,184.29. You also know that the index for Fort Wayne is 939.25 .What salary would be needed in Boston to match $35,000 in Fort Wayne. (Example 16.2.4)
Expected Relative Frequency and Expected Value, p. 622	• The expected relative frequency is the percent or fraction of the time that a particular event is predicted to occur. • The expected frequency is the number of times the event is predicted to happen. • Expected frequency = (expected relative frequency)(number of chances). • Expected cost = (expected relative frequency)(cost per occurrence).	A minor league baseball events promoter is offering a promotion at an upcoming game. One fan will be randomly chosen from the stands and given the opportunity to bat against the team's ace pitcher. If the fan hits a home run before accumulating three strikes, he will win $50,000. From past experience of similar events, the promoter believes that such a fan will win the prize 1 out of every 1,250 times the promotion is run. Calculate (a) the expected relative frequency of the fan winning and (b) the expected prize for this promotion. (Example 16.2.5)

(Continued)

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Topics	Key Ideas, Formulas, and Techniques	Examples
Measures of Variation, p. 626	• Range = highest value − lowest value. • Standard deviation is calculated by completing the table as shown in this section.	Calculate the standard deviation of the wind speeds of 2, 9, 16, 23, and 30 mph. (Example 16.3.1)
Interpreting Standard Deviation, p. 629	• The higher the standard deviation, the greater the variation. • The lower the standard deviation, the less the variation.	Suppose that you are considering investing in two different mutual funds. Over the past 10 years, the annual returns of the Hopewell American Growth Fund have had a standard deviation of 11.25%. The annual returns of the Hopewell Adrenaline Aggressive Growth Fund have had a standard deviation of 28.4%. Which investment has seen the greatest variation in its annual rates of return? (Example 16.3.2)
Chebyshev's Theorem, p. 629	• At least ¾ of the data must fall within 2 standard deviations of the mean. • At least 88.9% of the data must fall within 3 standard deviations of the mean.	If the mean is 78.2 and the standard deviation is 8.3, use Chebyshev's theorem to interpret what this tells you about where the class's scores fell. (Example 16.3.3)
The Empirical Rule, p. 630	• Approximately 68% of the data falls within 1 standard deviation of the mean. • For 2 standard deviations the value is 95%, for 3, it's 99%. • Can be used only if the data is normally distributed (the bell curve).	Suppose that your teacher tells you that the class's test scores were normally distributed. If the mean is 78.2 and the standard deviation is 8.3, use the Empirical Rule to interpret what this tells you about where the class's scores fell. (Example 16.3.4)

Appendix A

Answers to Odd-Numbered Exercises

Chapter 1 Section 1.1

1. $125
3. $68.47
5. $2,300
7. $285.26
9. a) $2500, b) 2 years, c) Jin's parents, d) Jin
11. a) 0.12, b) 0.153, c) 0.08, d) 0.0435, e) 0.0075, f) 1.25, g) 0.095, h) 0.1975, i) 0.05625, j) 0.204375, k) 0.00875, l) 3.75125
13. $41.05
15. $9,143
17. $176
19. $1,561.61
21. $1,568.48
23. $23,713.86
25. $1.25
27. $3,363.05
29. $2,244
31. $2,039.44
33. $6,658.13
35. $16,609.16
37. $3,880
39. a) $5,800; b) $5,832; c) Jerry earned interest on $5,400 for the second year, Tom earned interest only on his original $5,000.
41. $52.50

Chapter 1 Section 1.2

1. $60
3. $3,293.42
5. $920,713.33
7. $1,486.30
9. $4,422.93
11. $8,640
13. $59.86
15. $2,265.39
17. $5,301.75
19. $2.99
21. $8,509.32
23. $1,093.75
25. $95.13
27. $26,750,000
29. $23,222.95
31. $4,053.51
33. $50.63
35. a) $1.11, b) $1.10
37. £3,041.57

Chapter 1 Section 1.3

1. $350
3. $34,000
5. $640,350.88
7. a) 4.53%, b) 10.11%, c) 10.43%, d) 104.3%, e) 18.25%, f) 10%
9. 30%
11. 13.90%
13. 307 days
15. 312 days
17. 91 days
19. 8.58%
21. 232 days
23. $1,072,386.06
25. 13.01%
27. 66 days
29. 480 days
31. a) 63.16%, usury, b) 3.95%, not usury, c) 24.44%, not usury, d) 30.42%, usury
33. 136.60%

Chapter 1 Section 1.4

1. a) January 17, 2004, b) $12,000, c) April 17, 2004, d) $12,800, e) 3 months or 91 days
3. a) July 1, 2006, b) $5,350, c) September 29, 2006, d) $5,500, e) 90 days
5. 172 days
7. 203 days
9. November 22, 2005
11. May 17, 2005
13. 968 days
15. 167 days
17. September 16, 2002
19. August 16, 2006
21. $10,223.56
23. a) 163 days, b) 14.00%
25. August 8, 2005
27. 149 days
29. 45 days
31. February 10, 2008
33. March 30, 2008
35. $150,546.55

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37. $590.88; July 25
39. October 28, 2009
41. $183.08
43. September 15, 2011
45. 14.17%
47. March 2, 2010
49. August 13, 2100 (2100 is not a leap year)

Chapter 1 Section 1.5

1. $8.93
3. $2,935.97
5. $891.41
7. a) 5.02%, b) 3%, c) 30.11%, d) 182.5%, e) 6%
9. a) 0.02466% daily, 0.75% monthly;
   b) 0.03699% daily, 1.125% monthly;
   c) 0.06575% daily, 2% monthly
11. a) 0.05477%, b) $5.97
13. $33.64
15. 24.00%
17. $5,263.93

Chapter 1 Exercises

1. $9,467.46
3. $2,363.39
5. $203,659.49
7. 482 days
9. $6,410.70
11. August 7, 2006; $5,773.03
13. a) $7,695.83, b) $7,590.41
15. September 5, 2007
17. 440 days
19. $3,831.45
21. August 11
23. a) 0.0475, b) 0.10875, c) 0.135625,
    d) 0.0534375
25. $25,314.38
27. $276.75
29. $1,770.48
31. $69.04
33. 919 days
35. $6,208.93
37. 9.41%
39. $48,500

Chapter 2 Section 2.1

1. a) $800, b) $750, c) $50
3. a) $20,000, b) $18,000, c) $2,000
5. a) $1,308.55, b) $1,275, c) $33.55
7. a) $493.15, b) $9,506.85
9. $94,166.67
11. $4,000
13. 4.05%
15. 9 days
17. 133.33%
19. 4.95%
21. $849,109.59
23. 21.17%
25. $9,959.19
27. $26,715.07; $17,473,284.93
29. May 26, 2007

Chapter 2 Section 2.2

1. a) $875, b) $900, c) $25, d) $875, e) $875,
   f) $900, g) $25, h) $900
3. a) $352.45, b) $363.79, c) $11.34, d) $352.45,
   e) $352.45, f) $363.79, g) $11.34, h) $363.79
5. a) 7.85%, b) 7.37%
7. a) 869.05%, b) 651.79%
9. 13.64%
11. 9.64%
13. 66.67%
15. a) $18.82, b) 86.96%, c) 89.08%
17. 22.81%; 23.04%
19. 4.51%
21. 44.48%
23. $1,757,250; better off, since the equivalent simple
    interest rate for the first offer is 9.73%
25. The tax preparer's simple interest rate is 29.64%,
    which is higher.
27. July 31, 2007
29. 180%

Chapter 2 Section 2.3

1. a) $3,086.82, b) $3,034.86
3. $2,842.24
5. $10,248.59
7. $5,863.80
9. a) 7.81%, b) 9.43%, c) 8.45%
11. a) 12.12%, b) 17.49%, c) 16%
13. a) 9.92%, b) 13.34%, c) 8.71%
15. a) −23.02%, b) 13.59%, c) 6.75%
17. $13,390.53
19. $5,045.05
21. $2,016.13
23. 7.46%; 3.92%
25. a) $35,863.01, b) $35,709.73, c) 8.22%,
    d) 5.22%
27. 13.29%
29. $6,141.25; 0.25%
31. 7%
33. 3.75%

Chapter 2 Exercises

1. $511.01; $17,828.99
3. $12,772.89
5. $19,360.98; 6.48%
7. 17.14%; 17.65%
9. $23,392.69
11. 293 days
13. $13,277.47

15. 89.50%; 91.75%
17. July 26; $1,499.95
19. 7%
21. $142.84
23. $15,369.08
25. $9,821.56
27. $2,958.60
29. a) $25,000, b) $32,500, c) $7,500, d)$7,500
31. 56 days
33. 24.31%; 25.04%
35. 14.20%; −14.97%

Chapter 3 Section 3.1

1.

Year	Start	Interest	End
1	$2,500.00	$175.00	$2,675.00
2	$2,675.00	$187.25	$2,862.25
3	$2,862.25	$200.36	$3,062.61
4	$3,062.61	$214.38	$3,276.99

3.

Year	Start	Interest	End
1	$4,250.00	$276.25	$4,526.25
2	$4,526.25	$294.21	$4,820.46
3	$4,820.46	$313.33	$5,133.79
4	$5,133.79	$333.70	$5,467.49
5	$5,467.49	$355.39	$5,822.88

5. $4,205.94
7. $11,836.82; $28,022.05; $157,047.10
9. $24,194,748
11. $3,506.75
13. $3,052.66
15. All answers are approximate: a) 72 years, b) 36 years, c) 18 years, d) 12 years, e) 9 years, f) 6 years
17. All answers are approximate: a) 72%, b) 36%, c) 18%, d) 12%, e) 9%, f) 6%
19. Approximately 7.2%
21. $10,042.36; good approximation
23.

	5	10	20	40
3%	$1,159.27	$1,343.92	$1,806.11	$3,262.04
6%	$1,338.23	$1,790.85	$3,207.14	$10,285.72
9%	$1,538.62	$2,367.36	$5,604.41	$31,409.42
12%	$1,762.34	$3,105.85	$9,646.29	$93,050.97

25. 7.2%; 20 years
27. $261.56
29. $3,984.62
31. $85.36
33. $99,378.27
35. $0. The interest does not compound in this problem.
37. 9.39%

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Chapter 3 Section 3.2

1.

Quarter	*Start*	*Interest*	*End*
1	$1,450.00	$21.75	$1,471.75
2	$1,471.75	$22.08	$1,493.83
3	$1,493.83	$22.41	$1,516.24
4	$1,516.24	$22.74	$1,538.98

3. a) i = 0.05/365, n = 2,920; b) i = 0.0719/4, n = 8; c) i = 0.0303/365, n = 365; d) 0.0475/12, n = 348
5. a) i = 0.0547/365, n = 3,650, $54,959.12; b) i = 0.025, n = 4, $3,311.44; c) i = 0.07/360, n = 720, $112,150.17; d) i = 0.0925/365, n = 13,870, $67,201.92
7. $4,102.22
9. $841.75
11. $19,267.05
13. $1,523,339.50
15. $2,423.06
17. 2 cents
19. a) $1,636.78, b) $3,595.17, c) $1,093.95, d) $8,427.56, e) $9,237.57, f) $1,245.33, g) $157,439.10, h) $3,999.61
21. $2,658.73
23. $13,786.06
25. $3,668.96
27. $3,366.59
29. Approximately 12.3 years
31. $18,998.51
33. a) $1,521.67, b) $1,787.33
35. $4,626.22
37. $3,538.18

Chapter 3 Section 3.3

1. Brockport
3. Bank C
5. First Cattaraugus
7. a) 9%, b) 9.20%, c) 9.31%, d) 9.38%, e) 9.42%
9. 14.13%
11. 7.64%
13. First 9.21%; Second 9.30% (highest); Third 9.12%; Fourth 9.21%; Fifth 9.04% (lowest)
15. Penfield Mutual
17. $432.89
19. $1,004.62 (rate is effective so compounding frequency doesn't matter)
21. $48,440.58
23. $57,704.53; $7,704.53
25. $399,347.05
27. $223.37
29. $6,980.00
31. 1,945,425
33. $3.30
35. 6.55% daily = 6.77% effective; 6.68% quarterly = 6.85% effective; choose the higher effective rate.
37. $27,761.14
39. 5% quarterly = 5.09% effective; 4.93% daily = 5.05% effective, which would not be better.
41. Knapp Creek
43. 6.45%
45. Dylan had slightly more.
47. 6.18%

Chapter 3 Section 3.4

1. Nominal = $4,037.56; APY = $4,037.93. Do not match exactly, but should be close.
3. a) 5.54%, b) $5,834.79, c) $5,834.17, d) Not an exact match, but should be close
5. a) $53,306.11, b) 6.15%, c) $53,314.51, d) (a) does
7. a) $538.20; b) 10.24%, $538.21; c) (a) does
9. 6.49%; $5,143.93; $5,144.33

Chapter 3 Section 3.5

1. 12.40%
3. 6.71%
5. 53.09%; 18.57%; 8.89%
7. 6.72%
9. 5.22%
11. 549 days
13. 8.41%
15. 10.2 years
17. 7.07%

Chapter 3 Exercises

1. a) $5,126.94, b) $1,126.94
3. SNB Bank
5. Derby
7. 13.54%
9. Approximately $1,377.49
11. a) $882.75, b) $7,839.06
13. $635.15
15. Springwater
17. $7,311.58
19. No. Her account earned a 4.91% effective rate.
21. 8.31%
23. $550,998,029
25. 12%

Chapter 4 Section 4.1

1. Annuity
3. Amount varies from week to week. Not an annuity.
5. Annuity

7. Amount varies because of different number of days per month. Not an annuity.
9. Annuity
11. Present value
13. Future value
15. Future value
17. Present value
19. Annuity due. The payments start right away.
21. Timing not specified so assume ordinary.
23. Annuity due. Payment is made at the beginning of the month.
25. Timing not specified, so assume ordinary.
27. Future value of an ordinary annuity.
29. Present value of an ordinary annuity.

Chapter 4 Section 4.2

1.

Year	*Start*	*Interest*	*Deposit*	*End*
1	\$0.00	\$0.00	\$1,359.55	\$1,359.55
2	\$1,359.55	\$77.49	\$1,359.55	\$2,796.59
3	\$2,796.59	\$159.41	\$1,359.55	\$4,315.55
4	\$4,315.55	\$245.99	\$1,359.55	\$5,921.09

Year	*Payment*	*Years' Interest*	*FV*
1	\$1,359.55	3	\$1,605.54
2	\$1,359.55	2	\$1,518.96
3	\$1,359.55	1	\$1,437.04
4	\$1,359.55	0	\$1,359.55
		Total	\$5,921.09

3. a) $\frac{(1 + .065)^{54} - 1}{.065}$

b)

(1 + .065) ^54=	29.98325786
−1=	28.98325786
/.065=	445.89627485

5. a) 54.12223267, b) \$54,122.23
7. a) 25, b) 0.075, c) 67.97786150, d) \$203,933.58
9. \$364,402.25
11. \$6,986,993.30

13. a) $\frac{\left(1 + \frac{0.082}{12}\right)^{72} - 1}{\frac{0.082}{12}}$

b)

(1 + .082/12)^72=	1.632849452
− 1=	0.632849452
/(.082/12)=	92.61211497

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15. a) 72, b) 0.0594/4, c) 127.28876734, d) $152,746.52
17. a) 324, b) 0.0875/12, c) 1306.53830990, d) $176,761.57
19. $45,750.23
21. $1,054.41
23. $2,646.00
25. $15,148.26
27. a) $188,338.38, b) $1,643.25
29. $164,328.61
31. $160,304.79
33. $468,990.49
35. $2,024.43; $10,253.16; $69,951.09
37. Ron $436,375.98; Don $295,490.33
39. $233,768.53
41. 15,809
43. Annual: $42,041.35; semi: $43,079.38; quarterly: $43,622.00; monthly: $43,993.02; weekly $44,135.98; daily $44,179.59. Sooner money is deposited the better, though not much is gained by more frequent deposits than monthly.

Chapter 4 Section 4.3

1. $219,130.94
3. $762.91
5. $326.04
7. a) $41,576.92, b) $1,105.16
9. $171.05
11. $745.76
13. $56.60
15. $240,341.84
17. $871,799.37

Chapter 4 Section 4.4

1. a) $1,003,766.12, b) $790.85, c)$790.85
3. $129,448.11; $171,621.15
5. $1,320.78
7. a) 131.56359231, b) 71.76576624
9. 12.84926350
11. 35.52283787
13. a) 120, b) 0.09/12 = 0.0075, c) 78.94169267, d) $11,841.25
15. a) 20, b) 0.05/4 = 0.0125, c) 17.59931613, d) $30,806.19
17. a) 84, b) 0.1259/12, c) 55.64890489, d) $40,921.42
19. $13,718.75
21. $389.64
23. $301.95
25. $5,791,301.20
27. $954,323.93
29. a) $1,299.58, b) $1,736.44
31. 13.31785901
33. $44,029.50
35. $338.03
37. #4: $966.00; #5: $1,321.20. Match is not exact because of rounding.
39. **MONTHLY PAYMENT PER $1,000 BORROWED**

	INTEREST RATE			
Years	***6.00%***	***7.00%***	***8.00%***	***9.00%***
3	$30.42	$30.88	$31.34	$31.80
4	$23.49	$23.95	$24.41	$24.89
5	$19.33	$19.80	$20.28	$20.76

Chapter 4 Section 4.5

1.

Month	***Payment***	***To Interest***	***To Principle***	***End Balance***
1	$250.00	$175.88	$74.12	$23,375.88
2	$250.00	$175.32	$74.68	$23,301.20
3	$250.00	$174.76	$75.24	$23,225.96
4	$250.00	$174.19	$75.81	$23,150.15
5	$250.00	$173.63	$76.37	$23,073.78
6	$250.00	$173.05	$76.95	$22,996.83

3. $140.81; interest = $54.27; principle = $86.54
5. a) $59.59 interest, $2.91 principle; b) $59.59 interest, $190.41 principle; c) 196 months at $62.50, 18 months at $250
7. $3,496.06
9. $10,013.24
11. a) $787.20, $25,341.12; b) $134,661.32; c) $907.94; d) no, because she has extended the term to 24 years for all of her debts
13. $157.27; $4,533.12
15. Interest $34,456.25; principle $25,278.75; remaining $1,824,721.25
17. $731,245.64

Chapter 4 Section 4.6

1. a) $21,140.69, b) $153,156.82
3. $104,688.63
5. $1,442,015; $1,422,515
7. $90,754.11; $998.84
9. $155,055.62
11. $1,719,547

Chapter 4 Section 4.7

1. $5,565,041
3. $199,809
5. $166,607
7. $18,361,161
9. $85.86
11. a) $817.76, b) nothing!

13. $26,816
15. $170,609

Chapter 4 Exercises

1. $101,072.23; $110,000
3. $225,852
5. $1,901.97
7. $119,002
9. $267.93
11. $71,406.12
13. $259,584; $117,456
15. $193,482; $270.94; $85,009
17. $114.96; $91.58 interest, $23.38 principle
19. $78.58
21. $908.26
23. $5,154.41
25. a) $420,440, b) $2,868.46
27. $473.59;

Month	Payment	To Interest	To Principle	End Balance
1	$473.59	$210.00	$263.59	$29,736.41
2	$473.59	$208.15	$265.44	$29,470.97
3	$473.59	$206.30	$267.29	$29,203.68

29. $30,902

Chapter 5 Section 5.1

1. Result should match.
3. $5,070.66

5.

Model	Unit Price	Number of Sales	$ Sales
A-780	$14,500	128	$1,856,000
B-1000	$17,900	75	$1,342,500
C-2750	$25,800	12	$309,600
D-365	$46,000	34	$1,564,000
Total		249	$5,072,100

7.

Model	Unit Price	Number of Sales	$ Sales	Commission %	Commission $
A-780	$14,500	6	$87,000	2.50%	$2,175.00
B-1000	$17,900	4	$71,600	2.50%	$1,790.00
C-2750	$25,800	0	$0	1.75%	$0.00
D-365	$46,000	2	$92,000	1.75%	$1,610.00
Total		12	$250,600		$5,575.00

9. Spreadsheet should resemble Example 5.3.3; ending account value is $22,626.43
11. $10,283.93; $14.55; $62.61
13. $36,041.51

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15. a) Shown with hours from b

Person	Hourly Rate	Billable Hours	Billable $
Dewey	$175.00	3.5	$612.50
Cheatham	$215.00	0	$0.00
Howe	$275.00	11.5	$3,162.50
Yu	$85.00	6	$510.00
Bette	$85.00	1.5	$127.50
Totals		22.5	$4,412.50

b) $4,412.50, c) $1,018.75, d) $15,820.00

Chapter 5 Section 5.2

1. a) $1,090,943.20, b) $1,090,943.24
3. $17,746.30; $83,225,57; $905,943.24
5. $55,373.09
7. a) $221,982.92, b) $248,629.98, c) $332,716.33
9. $648,930.22
11. a) $37,150.97, b) $49,529.20, c) $56,055.87, d) $75,461.12, e) $102,696.02
13. $5,416,364
15. $199,808.79
17. a) $297,330.36, b) $374,368.97, c) $731,004.37
19. $3,073,726.51; $2,382,526.51
21. $3,163,233.00
23. a) $124,887.82, b) $132,231.11
25. $450,132.89

Chapter 5 Section 5.3

1. b) $354.60, c) $2,935.44
3. a) $962.29, b) $6,774.96, c) $961.80, d) $46,032.91
5. a) Forever, b) 49 months, c) 20 months
7. 21 quarters (5 years, 3 months); $169.15; $25,169.15
9. $313.24
11. $1,134.49; 99 months; forever
13. 36 months
15. Forever; 167 months

Chapter 5 Section 5.4

1. 32 years
3. 14.84%
5. 9.28%
7. 55 quarters (13¾ years)
9. 8.57%
11. 28 years; 14.80%
13. 18.97%

Chapter 5 Exercises

1.

	DesMoines	*Omaha*	*Denver*	*Boise*	*Spokane*	*Total*
3Q sales	$1,845,275	$2,087,416	$1,878,080	$3,567,029	$2,502,135	$11,879,935
Target	$2,000,000	$2,000,000	$2,000,000	$2,000,000	$2,000,000	$10,000,000
+/− target	−$154,725	$87,416	−$121,920	$1,567,029	$502,135	$1,879,935

	DesMoines	*Omaha*	*Denver*	*Boise*	*Spokane*	*Total*
4Q sales	$2,534,026	$1,546,032	$1,994,032	$3,075,075	$2,403,716	$11,552,881
Target	$2,000,000	$2,000,000	$2,000,000	$3,000,000	$2,500,000	$11,500,000
+/− target	$534,026	−$453,968	−$5,968	$75,075	−$96,284	$52,881

3. 88 months
5. $793,145.48
7. $458,754.13
9. 8.32%
11. 31 quarters (7¾ years); $165.79; $9,570.79
13. Forever
15. 12.59%

Chapter 6 Section 6.1

1. $215.88
3. a) $12.52, b) $18,091.40
5. a) Cindy $1,458, Jim $1,215, Shawn $972; b) $1,215
7. 2.12%
9. Axerixia 3.45%, Zovaxacquin 5.07%, so Zovaxacquin is actually higher
11. a) $27,000, b) 15.77%
13. −37.76%
15. 8.90%
17. 4.90%
19. a) CD 6.45%, stock 12.80%; b) $5,665.80; c) impossible to determine
21. 6.17%
23. −90.45%
25. −24.21%
27. a) $0.325/share, b) $154.38
29. 7.48%
31. $91.94
33. 20.95%

Chapter 6 Section 6.2

1. $37.50
3. $262.50; $10,000
5. Discount of $234.81
7. 6.92%
9. 6.02%
11. a) $10,351.60; 6.04%
13. a) 6.22%, b) 6.06%
15. a) Both discount, b) SomeOtherCo, c) SomeOtherCo, d) AnyCorp

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17. Bad news
19. a) $8,048.83, b) $7,731.95, c) $6,576.53
21. Bad news
23. 6.35%
25. 7.63%
27. $407,367.99
29. Rose
31. 6.52%
33. a) $805.59; b) $26.25 semiannually, $1,000 at maturity; c) discount, d) higher
35. 41.52% (issuer is most likely in very bad financial shape!)
37. $10,161.17

Chapter 6 Section 6.3

1. a) Long, b) short, c) long, d) short
3. a) Spot, b) neither
5. a) Made, b) $1,440, c) $2,300, d) 62.61%, e) 202.23%
7. a) Lost, b) $11,580, c) $7,835, d) −147.80%, e) −1,798.21%
9. a) Lost, b) $3,200, c) yes, d) nothing
11. Short September OJ
13. Call
15. Buy calls
17. a) Sell 2,000 shares, b) $10,360, c) $10,360 loss, d) $10,360 loss, e) $5,360 loss
19. c) −100%, d) −100%, e) −51.7%
21. a) $2,602.50, b) $2,545.50, c) $750 loss, d) 1,167.35
23. Puts
25. a) Sell 800 shares, b) $3,760, c) $3,760 loss, d) $3,760 loss, e) $4,240 profit
27. 109.21%; 486.10%
29. a) Lost, b) $2,900, c) −55.61%, d) −178.05%
31. 57,972.68%

Chapter 6 Section 6.4

1. Best = North, worst = West
3. $5,625; $4,010; $7,750; $4,075
5. $5,262
7. 6.4%; 10.05%
9. Impossible to know in advance.
11. $41.02
13. 22.987 shares
15. a) $3,102.98, b) 12.17%
17. 2.15%
19. Less; −1.09%
21. a, b, e are false; c, d are true.
23. 6.29%
25. $3,574.47; 4.48%
27. 9.29%

Chapter 7 Section 7.1

1. a) $15,270.50, b) $22,905.75
3. $17,783.15
5. a) $0, b) $2,303.60, c) $4,261.66
7. a) $1,000, b) $5,275, c) $7,475
9. $9,239.30
11. $10,350
13. a) 6%, b) 12%, c) 18%, d) 22%
15. DB. She does not have the advantage of a long time for her DC balance to grow.
17. $19,505.99
19. a) $103,584.84, b) $64,423.64, c) Brad's is much higher, d) they would be the same

Chapter 7 Section 7.2

1. a) $119,147.85; b) save $1,750, worth $77,446.10, c) save $0, worth $119,147.85
3. $322,153.66 (answers may vary slightly due to rounding)
5. 2.67%
7. 8.02%
9. a) $1,354.35; b) $0; c) 401(k) $200,908, Roth $318,902; d) it would make the 401(k) much more attractive
11. $4,260.23

Chapter 7 Section 7.3

1. $315,699
3. $6.75
5. 5%
7. a) $2,415,186, b) $9,531.40
9. $5,531.84
11. $353.64
13. $258.35
15. False. No matter how reasonable her assumptions and correct her calculations, nothing is guaranteed in this life.
17. 2% when inflation is 0%
19. $7,692.68

Chapter 8 Section 8.1

1. a) $43.91, b) $2.43, c) $73.30, d) $225.44, e) $43.22
3. $11.51
5. $39.45; $9.86
7. a) 185.23%, b) 57.32%, c) 7.90%
9. $363.39; $36.34
11. 25%
13. a) $69.96; b) $6.65, c) $4.39
15. $39.96
17. $11.92; $67.54
19. a) 26.32%, b) 9.40%, c) 80.04%
21. 35.09%
23. 338 jobs
25. 693 megawatts

27. $51.29 per thousand
29. 1,071,061 kilowatt-hours; 136,025 kwh saved
31. 28.57%
33. 17.65%
35. 8.70%
37. a) $274.13, b) $205.60
39. a) $11.40, b) 23.08%
41. $11.92
43. 73,885 visitors
45. 33.36%
47. 22.01%
49. a) $25,988.24; b) $267.92, $27,059.92; c) 4.12%

Chapter 8 Section 8.2

1. a) 21.92%, b) 50.00%, c) 41.60%, d) 34.82%
3. $256.42; 34.19%
5. 35.83%
7. $3.23; $14.72
9. $6,346,015
11. 10.90%
13. a) 13.04%, b) −29.87%
15. −5.0%
17. $22.44
19. a) $14.80, $9.25; b) $81.04, $17.02; c) $22.10, $5.07; d) $147.50, $46.49
21. a) $70.22, $7.72, 12.35%; b) $23.75, $8.31, 53.82%; c) $744.99, $305.97, 69.69%
23. $1.88
25. $44.44
27. $72.35; $25.32
29. 31.91%; 5.12%
31. −1.6%
33. $9.82; 75.60%
35. $1.30/pound
37. $67.61
39. It can't be determined without knowing expenses.
41. 26.8%
43. 0.57%

Chapter 8 Section 8.3

1. a) $132,302.20, b) $71,614.64, c) $15,095.40, d) $963,704.96
3. $881.27; $17,625.46
5. $41,587.87
7. $138.95
9. a) $3.39, b) 35.40%, c) 54.79% (answers may vary slightly due to rounding)
11. 12.16%
13. $53.97
15. a) 44.75%, b) 44.75%
17. a) $31.48 or $40.48, b) $40.48
19. a) 1/29, b)2/10, c)2/1, d) 2/16
21. a) $3,662.87, b)$3,736.70, c) $3,736.70
23. Gross margin 25%; markup 33.33%
25. $1.68
27. a) $5,560.00, b) $17,625.00, c) $23,185.00, d) $4,405.15, e) $18,779.85, f) $19,229.85
29. 36.04%
31. a) $73.69, b) $53.43, c) 26.20%, d) 20.76%
33. 40%
35. 16.55%

Chapter 8 Section 8.4

1. $371,338
3. $21,584.92; $14,226.22
5. 12.42%
7. $5,000; $22,500
9. a) $2,500, b) $1,458.33, c) $11,041.67
11. a) $12,900, b) $12,900, c)

Year	*Depreciation Amount*	*Depreciated Value*
2007	$12,900.00	$51,600.00
2008	$20,640.00	$30,960.00
2009	$12,384.00	$18,576.00
2010	$7,430.40	$11,145.60
2011	$7,430.40	$3,715.20
2012	$3,715.20	$0.00

13. $813,250
15. a) $59,066.39, b) $59,612.50
17. $26,625
19.

End of Year	*Depreciated Value*
2006	$6910
2007	$5750
2008	$4590
2009	$3430
2010	$2270
2011	$1400

21.

Years of Use	*Depreciated Value*
1	$26,400
2	$20,800
3	$15,200
4	$9600
5	$4000

23. Cannot be determined. Depreciated value is not the same as market value.

Chapter 9 Section 9.1

1. $11.79
3. $2.10
5. $256.13
7. $3.72; $52.01
9. $289.77

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11. a) $97.43, b) $6.09, c) $103.52
13. $11,162.79
15. $455.64
17. $2,845.24
19. $22.39
21. Self $801.00; dealer $789.95
23. $69.95
25. $19.50
27. $2,857.14
29. $10.35
31. 7.25%

Chapter 9 Section 9.2

1. $55,453; 4
3. $43,616; 1
5. a) $7,562.95, b) 15%, c) 13.64%, d) 9.58%
7. $7,461.50
9. $2,968.30
11. a) $6,431.25, b) $655.00, c) $6,151.75
13. a) $1,496.35, b) $380.76, c) $937.67, d) $111.20
15. $668.07; $887.21
17. a) $7,042.43, b) $6,750.55, c) $291.88 refund
19. a) Owe $4,237.81, b) $1,459.83 refund
21. $3,018.38
23. Social Security $5,840.40; Medicare $1,823.65
25. $4,176.90
27. a) $183.44, b) $100.39
29. $5,906.50
31. 15%; 7.93%
33. a) $146.13, b) $122.41, c) $1,331.55
35. a) $265.51, b) $157.33, c) $1,633.74
37. a) $173.44, b) $165.47, c) $1,824.04
39. a) $58.22, b) $116.41, c) $1,347.09
41. $5,840.40; 0.03%

Chapter 9 Section 9.3

1. $750.61
3. $8,079.75
5. $3,224.00
7. Probably 25%
9. a) Approximately $68,550, b) probably not
11. a) $208,773, b) yes
13. a) $20.44/thousand, b) $2.044/hundred, c) 2.0443%, d) 20.44 mills
15. 5.09% increase
17. a) $65,263; b) $9.60/thousand, slight difference due to rounding; c) $3,740,000, no change
19. $78,645.40
21. South is lower. Tax on a $100,000 property would be $3,172 in North and $2,337.20 in South.
23. $3,476.69
25. $14.82/thousand residential; $11.85/thousand commercial

Chapter 9 Section 9.4

1. $1.60; $2.40
3. $6,806.25
5. $171,211.25
7. $145.00
9. a) $2,280,500, b) $627,445, c) $126,225
11. a) $923,355.90, b) $248,355.90, c) $0
13. $0
15. $22,000
17. All the same
19. a) $4,140,000, b) $0
21. $3.80
23. $2.79
25. $4,222,222.22; 23.68%
27. a) No estate tax, $486 capital gains tax; b) no estate tax, capital gains tax is $8,946, c) $8,500 increase

Chapter 10 Section 10.1

1.

Date	Activity		Balance	Days at Balance	(Balance)(Days)
12 April	Start		$1755.28	2	$3,510.56
14 April	Charged	$128.53	$1,883.81	15	$28,257.15
29 April	Paid	500	$1,383.81	8	$11,070.48
7 May	Charged	62.45	$1,446.26	2	$2,892.52
9 May	Charged	197.65	$1,643.91	3	$4,931.73
12 May				30	$50,662.44
		ADB = 50662.44/30 = $1,688.75			

3. $525.66
5. $26.36
7. a) $4.75, b) $1,305.04
9. No.
11. Yes. Prior month's balance was not paid in full.
13. $1.37; $76.98
15. $196.60
17. a) Myrtle, b) Myrtle, c) Myrtle, d) Rew, e) Rew
19. a) More than $375, b) more than $668.90
21. a) More than $285.71, b) more than $2,980.13, c) less than $443.95, d) never
23. a) $0, b) $267.98
25. a) Kashong, b) Dresden, c) Crooked Lake, d) more than $2,333.33, e) between $421.94 and $2,333.33
27. $531.25
29. No. Payment was made after the end of the grace period.
31. $13.79
33. $109.38; probably not
35. B if she's sure she will not carry a balance; C otherwise
37. $21,250

Chapter 10 Section 10.2

1. $38,463
3. $16,191
5. $65,710; −$32,775
7. $1,214.86; 3 years
9. $380.25
11. $913.05
13. $1,784.98
15. False. There may be other requirements (credit history, employment, etc.)
17. False. Not all lenders use these exact tests.
19. Fails both tests, Does not qualify.
21. Yes, yes, and yes.
23. $19,215
25. False. No-closing-cost options and other programs are available.
27. $1,812.13

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29. a) $2,087.07; b) $854.40, $783.46; c) $25,538.40; d) 29.42 months, or approximately 2½ years
31. $9,850
33. Yes. Since his down payment is 17.6%, which is less than 20%, he may be required to pay PMI.
35. $121,976.46; −$12,900
37. $1,097.48
39. a) $1,297.30, b) yes
41. a) $24,465, b) no
43. 47.7 months; possibly, but probably not
45. $55,767.30

Chapter 10 Section 10.3

1. $75.63
3. $95.30
5. 80,200
7. $12.69 interest; $62.94 principle
9. a) $205.42, b) $2,383.79
11. 43,639
13. a) 14.98%, b) 16.05%
15. a) 17.98%, b) 18.46%
17. Spreadsheet rate. The approximation formula just gives an approximation.
19. $416.67
21. a) $1,083.33, b) 12.40%, c) 10.59%
23. Amortization 9.70%; approximation 10.14%
25. a) $840, b) $835.59

Chapter 10 Section 10.4

1. a) $434.89, b) $298.68
3. a) $349.06, b) $191.56
5. $779.76
7. None
9. a) $324.17, b) $366.84, c) 12,500 plan would cost less
11. $630.74
13. −$55.20 (the leasing company would probably not accept this deal)
15. $21,658

Chapter 11 Section 11.1

1. a) £0.5300, b) 3.7509 riyal, c) 10.9290 pesos, d) 37.693 baht, e) 2.1496 reais
3. a) 15,676, b) A$ 722.76, c) 62,571.43 rand, d) 1,746.51 rupees, e) 1,467,980 rubles
5. a) $1.4114, b) $0.2282, c) $0.3090, d) $0.3233, e) $0.1254
7. a) $66,041.50, b) $241.45, c) $24.24, d) $43.89, e) $1.63
9. a) C$587.98, b) £2,593.50, c) 2,204.10 pesos
11. US $ equivalent
13. a) $179.44, b) 379.09, c) ¥16,322, d) $303.32
15. $22.64
17. £39.95
19. $1,042.50
21. 20,910,038 rupiah
23. a) $634,900, b) $761,000, c) $30,450, d) $128,600, e) $633,800
25. $7.94
27. $7,015.58
29. One example: gasoline prices per gallon
31. a) $215,475, b) $214,700, c) $775
33. $3.07

Chapter 12 Section 12.1

1. $5,400
3. a) $47,904, b) ($36,719)
5. Gross profit $337,000; net income $137,800
7. $2,542,290
9. $769,640
11. $1,772,650
13. a) $3,559,633, b) $1,696,814, c) $739,214
15. a) $57,500, b) ($5,000), c) ($11,000), d) 100.00%, e) 32.35%, f) 67.65%, g) 73.53%, h) −5.88%, i) 7.06%, j) −12.94%

17.

	2006	2007	Amount	Percent
Sales	$275,043	$265,951	(9,092)	−3.31%
−cost of goods sold	$192,343	$144,043	(48,300)	−25.11%
Gross profit	$82,700	$121,908	39,208	47.41%
−expenses	$51,000	$72,500	21,500	42.16%
Net income	$31,700	$49,408	17,708	55.86%

19. Vertical
21. $406,800
23. $148,000
25. $258,800
27. $194,825
29. $53,975
31. 54.16%
33. 13.27%
35. −100%

Chapter 12 Section 12.2

1. $140,000
3. $107,400
5. $620,193
7. a) $77,900, b) $729,943
9. $350,000; cannot determine

11.

Assets 2007	Amount	Percent
Current assets:		
Cash	$85,000	3.01%
Accounts receivable	$255,000	9.02%
Total current assets	$340,000	12.03%
Property, plant, and equipment:		
Buildings	$362,000	12.81%
Land	$250,000	8.84%
Other plant and equipment	$1,875,000	66.32%
Total P, P, and E	$2,487,000	87.97%
Total assets	$2,827,000	100.00%

Liabilities		
Current liabilities:		
Accounts payable	$76,500	2.71%
Other	$62,500	2.21%
Total current liabilities	$139,000	4.92%
Total long-term liabilities	$1,675,000	59.25%
Total liabilities	$1,814,000	64.17%

Equity		
Contributed capital	$250,000	8.84%
Retained earnings	$763,000	26.99%
Total equity	$1,013,000	35.83%
Total liabilities and equity	$2,827,000	100.00%

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13.

	2007	2006	Amount	Percent
Assets	$4,508,000	$3,950,000	$558,000	14.13%
Liabilities	$3,299,995	$3,454,000	−$154,005	−4.46%
Equity	$1,208,005	$496,000	$712,005	143.55%
Total L&E	$4,508,000	$3,950,000	$558,000	14.13%

15. a) ($210,975), b) can't say, assets are not listed on the balance sheet at their actual market value

Chapter 12 Section 12.3

1. a) 14.89%, b) 56.57%, c) Stan is not managing them as well as his competition
3. $22,342.69
5. ROA 12.39%, ROE 24.94%
7. Could be doing better, since others are earning a higher return
9. 0.73; Cohen
11. 7 days; largely a cash business
13. Current 2.82; quick 2.05
15. Company B
17. 1.01
19. EPS $1.36; book $8.34
21. EPS −$2.41; book $15.74
23. PE 12.90; PB 2.10
25. PE undefined, "NMF"; PB 3.72
27. 20.68
29. $61,471,460
31. a) High, b) high, c) high, d) low
33. Not necessarily. It depends whether the profits have grown or dropped.
35. Quick ratio
37. Price to book
39. Inventory turnover
41. 2.27

Chapter 13 Section 13.1

1. $200.00
3. $463.13
5. Not enough people to take advantage of the law of large numbers. Could make it work with reinsurance though.
7. $153.86
9. $292.00
11. $253.17
13. 125%
15. $307.16
17. Policy cancelled
19. $2,050
21. $0
23. $130,000
25. All of them
27. $95,000; $25,000
29. $225,700; $25,000

31. Entire claim would be covered.
33. $714,286
35. $1725
37. $0.40
39. $360,000; $0
41. a) $153.86, b) $172.32
43. $50.17 semiannually
45. $2,925

Chapter 13 Section 13.2

1. $260
3. $1,308
5. $7,720
7. $40
9. $25,500
11. S $201.27, D $402.54, F $655.42
13. $6,911.10
15. S $333.50, F $858.20
17. S $227.90, F $483.55
19. No. If even a couple of employees become seriously ill and run up large claims, the company's costs could soar. The group is not large enough to rely on the law of large numbers.
21. S $55.95, F $255.92
23. S $25.60, F $150.59
25. $952
27. Traditional S $0, F $308.97; POS S $0, F $206.46
29. The last one, because it makes the employees pay more of, and therefore feel, the cost difference. This gives employees more incentive to choose the lower cost plan.
31. If employees have to pay too much, some may choose to not take the coverage. The healthiest people are the most likely to do this, leaving the insurer with almost the same claims costs, but less premium to cover them.

Chapter 13 Section 13.3

1. $161.50
3. $29.21
5. $221.32
7. $48.49
9. $510.63
11. $325.50
13. $102.08
15. $444.88
17. $2,400
19. $8250 paid-up death benefit
21. 17 years and 19 days of continued term coverage
23. $294,208
25. a) $35.18, b) $32.19, c) $7,200.16
27. $2,141.55
29. Universal

Chapter 14 Section 14.1

1. $8,500 per year
3. $7500 in 10 years
5. $187,500
7. $2,500,000; no
9. $62,500
11. $2,416,667
13. Landscape company −$5,000; property management $21,429; property management is more attractive
15. 8.22%

Chapter 14 Section 14.2

1. 7 years
3. 8 months
5. $2,750
7. 6.65 years
9. 8 visits
11. Just over 9 months
13. $195
15. Dynamic = 42.55 hours; comprehensive = 67.04 hours
17. Not necessarily. If the company knows that it will use several hundred hours of tech support, the comprehensive plan would be less expensive and thus a better choice.

Chapter 15 Section 15.1

1. $1,532.12
3. $268.28
5. $1,273.60
7. $823.60
9. $429.00
11. a) $781.30, b) $381.30
13. $6,169.00
15. a) $1,791.67, b) $11,224.17
17. $847.51
19. $974.61
21. $716.05
23. $564.87
25. −$19.66 (would receive of $19.66 extra pay)
27. $4,386.90
29. a) $1,216.80, b) $93.09, c) $1,023.51
31. a) $957.60, b) $73.26, c) $850.14
33. a) $883.50, b) $61.12, c) $707.65
35. $478.75
37. Each cafeteria plan "dollar" buys more than $1 worth of benefits.

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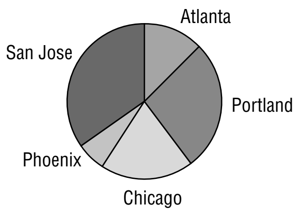
Chapter 15 Section 15.2

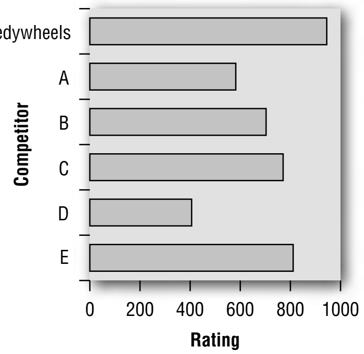
1. $3.14 per gallon
3. $497.50
5. a) $670.94, b) $110.33
7. a) $10,293.50; b) $2,459.51; c) $91.09, $91.09
9. $497.50
11. a) $670.94, b) $107.58
13. a) $10,293.50; b) $2,655.45; c) $91.09, $98.35
15. $497.50
17. a) $670.94, b) $117.48
19. a) $10,293.50; b) $1,984.50; c) $91.09, $73.50
21. $120,259
23. $17,049,693
25. $15,500; $5,500 loss
27. a) $1,355.14, b) $1,371.67, c) $1,329.17
29. a) $518, b) $478.33, c) $577.50

Chapter 16 Section 16.1

1.



3.



5. a) pie, b) bar, c) pie, d) line, e) bar
7. The revised chart. It accentuates the difference between the counties' tax rates; it makes it appear as though Chatakoin County's tax rate is extremely low compared to the others.

Chapter 16 Section 16.2

1. a) 16.7 g, b) 16.4 g
3. a) $4,672, b) $4,128.50
5. $11.04
7. $216,295/employee
9. a) 13.86%, b) probably grew by a similar percent, but not necessarily the same, c) nothing; index tells you nothing about any individual stock
11. $261,345
13. $44,690
15. a) 770 claims, b) $894.00
17. a) 134 loaves, b) $0.31
19. $7.33
21. The company can advertise a price of "$599 after rebate" when, in fact, they expect to be getting $713 per TV.

Chapter 16 Section 16.3

1. a) 838

b)

Hours	*Deviation*	*Deviation Squared*
835	−3	9
842	+4	16
808	−30	900
798	−40	1,600
907	+69	4,761
	Total:	7,286

c) 42.7 hours

3. 15 minutes
5. Less consistent
7. Increasing; more variation in student ages
9. a) 1.56 ounces, b) between 126.75 and 129.87 ounces, c) at least 88.9%
11. c) Would be approximately 99%
13. a) 87.7%, b)0.78%, c) The mean is higher, which is good, and the standard deviation is lower, which is also good. The results of these samples appear to exceed the company's standards.
15. We are basing this on only a few lightbulbs; we might have been lucky and made a particularly good selection.

The Metric System

Overview

*The **metric system** is a decimalized system of measurement in which each unit of measure is 10 times as large as the previous unit. Its units of measure are based on the meter, the gram, and the liter. While the United States and the United Kingdom both continue to use measurement systems based on old English units (referred to here as* U.S. measures*), most of the rest of the world uses the metric system.*

Metric Units of Measure

Length

The standard unit of length in the metric system is the meter (abbreviated as m*). Other units of length and their equivalents in meters are as follows:*

1 millimeter = 0.001 meter; abbreviated as 1 mm
1 centimeter = 0.01 meter; abbreviated as 1 cm
1 decimeter = 0.1 meter; abbreviated as 1 dm
1 kilometer = 1,000 meters; abbreviated as 1 km

One meter is slightly longer than 1 yard or 3 feet. A centimeter is slightly less than half an inch. A millimeter is about the thickness of a dime.

Volume

The standard unit of volume in the metric system is the liter (abbreviated as l). One liter is equal to 1,000 cubic centimeters in volume. Other units of volume and their equivalents in liters are as follows:

1 milliliter = 0.001 liter; abbreviated as 1 ml
1 centiliter = 0.01 liter; abbreviated as 1 cl
1 deciliter = 0.1 liter; abbreviated as 1 dl
1 kiloliter = 1,000 liters; abbreviated as 1 kl

One liter is slightly more than 1 quart. One teaspoon equals about 5 milliliters.

Mass

The standard unit of mass in the metric system is the gram (abbreviated as g). Other units of mass and their equivalents in grams are as follows:

1 milligram = 0.001 gram; abbreviated as 1 mg
1 centigram = 0.01 gram; abbreviated as 1 cg
1 decigram = 0.1 gram; abbreviated as 1 dg
1 kilogram = 1,000 grams; abbreviated as 1 kg

One gram is approximately the mass of a paper clip. One kilogram is approximately the mass of a liter of water.

U.S. Units of Measure

Length

In the United States, length is measured in four units: inch, foot, yard, and mile.

Inch; abbreviated as in
1 foot = 12 inches; abbreviated as ft
1 yard = 3 feet; abbreviated as yd
1 mile = 5,280 feet; abbreviated as mi

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Volume and Mass

In the United States, volume and mass are measured in the following units.

Ounce (liquid or dry); abbreviated as oz
1 (liquid) cup = 8 fluid ounces; abbreviated as c
1 (liquid) pint = 16 fluid ounces; abbreviated as pt
1 (dry) pint = 33.6003125 cubic inches; abbreviated as pt
1 (liquid) quart = 2 pints or 32 fluid ounces; abbreviated as qt
1 (dry) quart = 2 pints; abbreviated as qt
1 (liquid) gallon = 4 quarts or 128 fluid ounces; abbreviated as gal
1 (dry) gallon = 4 quarts; abbreviated as gal
1 peck = 2 (dry) gallons

Converting between U.S. and Metric Measures

TO CONVERT FROM U.S. TO METRIC			TO CONVERT FROM METRIC TO U.S.		
U.S.	***Metric***	***Multiply by***	***Metric***	***U.S.***	***Multiply by***
Length			**Length**		
Inches (in)	Meters (m)	.025	Meters (m)	Inches (in)	39.37
Feet (ft)	Meters (m)	.31	Meters (m)	Feet (ft)	3.28
Yards (yd)	Meters (m)	.91	Meters (m)	Yards (yd)	1.1
Miles	Kilometers (km)	1.6	Kilometers (km)	Miles	.62
Mass			**Mass**		
Ounces (oz)		28	Grams (g)	Ounces (oz)	.035
Pounds (lb)		454	Grams (g)	Pounds (lb)	.0022
Pounds (lb)		.45	Kilograms (kg)	Pounds (lb)	2.2
Volume			**Volume**		
Pints	Liters (l)	.47	Liters (l)	Pints	2.1
Quarts	Liters	.95	Liters	Quarts	1.06
Gallons (gal)	Liters	3.8	Liters	Gallons (gal)	.26

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An ALL NEW Approach to Business Math

The Mathematics of Money:
Math for Business and Personal Finance Decisions
by Timothy J. Biehler
Finger Lakes Community College

Timothy Biehler is an Assistant Professor at Finger Lakes Community College, where he has been teaching since 1999 and where he received the State University of New York Chancellor's Award for Excellence in Teaching. Before moving into academics, Tim worked for seven years as an actuary in the life and health insurance industry. He served as Director of Strategic Planning for Health Services Medical Corp. of Central New York, Syracuse, where he earlier served as Rating and Underwriting Manager. Tim's B.A. in math and philosophy and his M.A. in math are from the State University of New York at Buffalo, where he was Phi Beta Kappa and a Woodburn Graduate Fellow.

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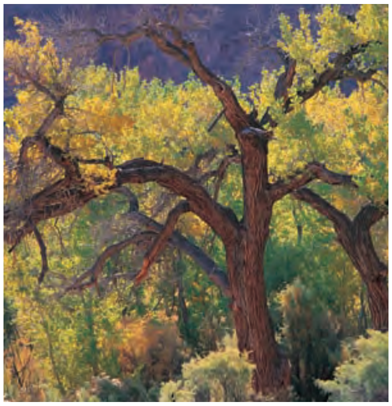

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