Determinants and Their Applications in Mathematical Physics

Robert Vein
Paul Dale

Springer

Applied Mathematical Sciences
Volume 134

Springer
New York
Berlin
Heidelberg
Barcelona
Budapest
Hong Kong
London
Milan
Paris
Singapore
Tokyo

Applied Mathematical Sciences

1. *John:* Partial Differential Equations, 4th ed.
2. *Sirovich:* Techniques of Asymptotic Analysis.
3. *Hale:* Theory of Functional Differential Equations, 2nd ed.
4. *Percus:* Combinatorial Methods.
5. *von Mises/Friedrichs:* Fluid Dynamics.
6. *Freiberger/Grenander:* A Short Course in Computational Probability and Statistics.
7. *Pipkin:* Lectures on Viscoelasticity Theory.
8. *Giacoglia:* Perturbation Methods in Non-linear Systems.
9. *Friedrichs:* Spectral Theory of Operators in Hilbert Space.
10. *Stroud:* Numerical Quadrature and Solution of Ordinary Differential Equations.
11. *Wolovich:* Linear Multivariable Systems.
12. *Berkovitz:* Optimal Control Theory.
13. *Bluman/Cole:* Similarity Methods for Differential Equations.
14. *Yoshizawa:* Stability Theory and the Existence of Periodic Solution and Almost Periodic Solutions.
15. *Braun:* Differential Equations and Their Applications, 3rd ed.
16. *Lefschetz:* Applications of Algebraic Topology.
17. *Collatz/Wetterling:* Optimization Problems.
18. *Grenander:* Pattern Synthesis: Lectures in Pattern Theory, Vol. I.
19. *Marsden/McCracken:* Hopf Bifurcation and Its Applications.
20. *Driver:* Ordinary and Delay Differential Equations.
21. *Courant/Friedrichs:* Supersonic Flow and Shock Waves.
22. *Rouche/Habets/Laloy:* Stability Theory by Liapunov's Direct Method.
23. *Lamperti:* Stochastic Processes: A Survey of the Mathematical Theory.
24. *Grenander:* Pattern Analysis: Lectures in Pattern Theory, Vol. II.
25. *Davies:* Integral Transforms and Their Applications, 2nd ed.
26. *Kushner/Clark:* Stochastic Approximation Methods for Constrained and Unconstrained Systems.
27. *de Boor:* A Practical Guide to Splines.
28. *Keilson:* Markov Chain Models—Rarity and Exponentiality.
29. *de Veubeke:* A Course in Elasticity.
30. *Shiatycki:* Geometric Quantization and Quantum Mechanics.
31. *Reid:* Sturmian Theory for Ordinary Differential Equations.
32. *Meis/Markowitz:* Numerical Solution of Partial Differential Equations.
33. *Grenander:* Regular Structures: Lectures in Pattern Theory, Vol. III.
34. *Kevorkian/Cole:* Perturbation Methods in Applied Mathematics.
35. *Carr:* Applications of Centre Manifold Theory.
36. *Bengtsson/Ghil/Källén:* Dynamic Meteorology: Data Assimilation Methods.
37. *Saperstone:* Semidynamical Systems in Infinite Dimensional Spaces.
38. *Lichtenberg/Lieberman:* Regular and Chaotic Dynamics, 2nd ed.
39. *Piccini/Stampacchia/Vidossich:* Ordinary Differential Equations in $\mathbf{R}^n$.
40. *Naylor/Sell:* Linear Operator Theory in Engineering and Science.
41. *Sparrow:* The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors.
42. *Guckenheimer/Holmes:* Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.
43. *Ockendon/Taylor:* Inviscid Fluid Flows.
44. *Pazy:* Semigroups of Linear Operators and Applications to Partial Differential Equations.
45. *Glashoff/Gustafson:* Linear Operations and Approximation: An Introduction to the Theoretical Analysis and Numerical Treatment of Semi-Infinite Programs.
46. *Wilcox:* Scattering Theory for Diffraction Gratings.
47. *Hale et al:* An Introduction to Infinite Dimensional Dynamical Systems—Geometric Theory.
48. *Murray:* Asymptotic Analysis.
49. *Ladyzhenskaya:* The Boundary-Value Problems of Mathematical Physics.
50. *Wilcox:* Sound Propagation in Stratified Fluids.
51. *Golubitsky/Schaeffer:* Bifurcation and Groups in Bifurcation Theory, Vol. I.
52. *Chipot:* Variational Inequalities and Flow in Porous Media.
53. *Majda:* Compressible Fluid Flow and System of Conservation Laws in Several Space Variables.
54. *Wasow:* Linear Turning Point Theory.
55. *Yosida:* Operational Calculus: A Theory of Hyperfunctions.
56. *Chang/Howes:* Nonlinear Singular Perturbation Phenomena: Theory and Applications.
57. *Reinhardt:* Analysis of Approximation Methods for Differential and Integral Equations.
58. *Dwoyer/Hussaini/Voigt (eds):* Theoretical Approaches to Turbulence.
59. *Sanders/Verhulst:* Averaging Methods in Nonlinear Dynamical Systems.
60. *Ghil/Childress:* Topics in Geophysical Dynamics: Atmospheric Dynamics, Dynamo Theory and Climate Dynamics.

(continued following index)

Robert Vein Paul Dale

Determinants and Their Applications in Mathematical Physics

Springer

Robert Vein
Paul Dale
Aston University
Department of Computer Science and
Applied Mathematics
Aston Triangle
Birmingham B47ET
UK

Editors

Mathematics Subject Classification (1991): 15A15, 33A65, 34A34, 83C99, 35Q20

Library of Congress Cataloging-in-Publication Data
Vein, Robert.
Determinants and their applications in mathematical physics/
by Robert Vein and Paul Dale
p. cm.—(Applied mathematical sciences ; 134)
Includes bibliographical references and index.
ISBN 0-387-98558-1 (alk. paper)
1. Determinants. 2. Mathematical physics. I. Dale, Paul.
II. Title. III. Series: Applied mathematical sciences
(Springer-Verlag New York Inc.); v. 134.
QA1.A647
[QC20.7.D45]
510 s—dc21
[530.15] 98-7731

ISBN 0-387-98558-1 Springer-Verlag New York Berlin Heidelberg SPIN 10681036

Preface

The last treatise on the theory of determinants, by T. Muir, revised and enlarged by W.H. Metzler, was published by Dover Publications Inc. in 1960. It is an unabridged and corrected republication of the edition originally published by Longman, Green and Co. in 1933 and contains a preface by Metzler dated 1928. The Table of Contents of this treatise is given in Appendix 13.

A small number of other books devoted entirely to determinants have been published in English, but they contain little if anything of importance that was not known to Muir and Metzler. A few have appeared in German and Japanese. In contrast, the shelves of every mathematics library groan under the weight of books on linear algebra, some of which contain short chapters on determinants but usually only on those aspects of the subject which are applicable to the chapters on matrices. There appears to be tacit agreement among authorities on linear algebra that determinant theory is important only as a branch of matrix theory. In sections devoted entirely to the establishment of a determinantal relation, many authors define a determinant by first defining a matrix $\mathbf{M}$ and then adding the words: "Let $\det \mathbf{M}$ be the determinant of the matrix $\mathbf{M}$" as though determinants have no separate existence. This belief has no basis in history. The origins of determinants can be traced back to Leibniz (1646–1716) and their properties were developed by Vandermonde (1735–1796), Laplace (1749–1827), Cauchy (1789–1857) and Jacobi (1804–1851) whereas matrices were not introduced until the year of Cauchy's death, by Cayley (1821–1895). In this book, most determinants are defined directly.

It may well be perfectly legitimate to regard determinant theory as a branch of matrix theory, but it is such a large branch and has such large and independent roots, like a branch of a banyan tree, that it is capable of leading an independent life. Chemistry is a branch of physics, but it is sufficiently extensive and profound to deserve its traditional role as an independent subject. Similarly, the theory of determinants is sufficiently extensive and profound to justify independent study and an independent book.

This book contains a number of features which cannot be found in any other book. Prominent among these are the extensive applications of scaled cofactors and column vectors and the inclusion of a large number of relations containing derivatives. Older books give their readers the impression that the theory of determinants is almost entirely algebraic in nature. If the elements in an arbitrary determinant A are functions of a continuous variable x, then A possesses a derivative with respect to x. The formula for this derivative has been known for generations, but its application to the solution of nonlinear differential equations is a recent development.

The first five chapters are purely mathematical in nature and contain old and new proofs of several old theorems together with a number of theorems, identities, and conjectures which have not hitherto been published. Some theorems, both old and new, have been given two independent proofs on the assumption that the reader will find the methods as interesting and important as the results.

Chapter 6 is devoted to the applications of determinants in mathematical physics and is a unique feature in a book for the simple reason that these applications were almost unknown before 1970, only slowly became known during the following few years, and did not become widely known until about 1980. They naturally first appeared in journals on mathematical physics of which the most outstanding from the determinantal point of view is the *Journal of the Physical Society of Japan*. A rapid scan of Section 15A15 in the *Index of Mathematical Reviews* will reveal that most pure mathematicians appear to be unaware of or uninterested in the outstanding contributions to the theory and application of determinants made in the course of research into problems in mathematical physics. These usually appear in Section 35Q of the Index. Pure mathematicians are strongly recommended to make themselves acquainted with these applications, for they will undoubtedly gain inspiration from them. They will find plenty of scope for purely analytical research and may well be able to refine the techniques employed by mathematical physicists, prove a number of conjectures, and advance the subject still further. Further comments on these applications can be found in the introduction to Chapter 6.

There appears to be no general agreement on notation among writers on determinants. We use the notion $A_n = |a_{ij}|_n$ and $B_n = |b_{ij}|_n$, where i and j are row and column parameters, respectively. The suffix n denotes the order of the determinant and is usually reserved for that purpose. Rejecter minors of A_n are denoted by $M_{ij}^{(n)}$, etc., retainer minors are denoted by N_{ij}, etc., simple cofactors are denoted by $A_{ij}^{(n)}$, etc., and scaled cofactors are denoted by A_n^{ij}, etc. The n may be omitted from any passage if all the determinants which appear in it have the same order. The letter D, sometimes with a suffix x, t, etc., is reserved for use as a differential operator. The letters h, i, j, k, m, p, q, r, and s are usually used as integer parameters. The letter l is not used in order to avoid confusion with the unit integer. Complex numbers appear in some sections and pose the problem of conflicting priorities. The notation $\omega^2 = -1$ has been adopted since the letters i and j are indispensable as row and column parameters, respectively, in passages where a large number of such parameters are required. Matrices are seldom required, but where they are indispensable, they appear in boldface symbols such as $\mathbf{A}$ and $\mathbf{B}$ with the simple convention $A = \det \mathbf{A}$, $B = \det \mathbf{B}$, etc. The boldface symbols $\mathbf{R}$ and $\mathbf{C}$, with suffixes, are reserved for use as row and column vectors, respectively. Determinants, their elements, their rejecter and retainer minors, their simple and scaled cofactors, their row and column vectors, and their derivatives have all been expressed in a notation which we believe is simple and clear and we wish to see this notation adopted universally.

The Appendix consists mainly of nondeterminantal relations which have been removed from the main text to allow the analysis to proceed without interruption.

The Bibliography contains references not only to all the authors mentioned in the text but also to many other contributors to the theory of determinants and related subjects. The authors have been arranged in alphabetical order and reference to *Mathematical Reviews*, *Zentralblatt für Mathematik*, and *Physics Abstracts* have been included to enable the reader who has no easy access to journals and books to obtain more details of their contents than is suggested by their brief titles.

The true title of this book is *The Analytic Theory of Determinants with Applications to the Solutions of Certain Nonlinear Equations of Mathematical Physics*, which satisfies the requirements of accuracy but lacks the virtue of brevity. Chapter 1 begins with a brief note on Grassmann algebra and then proceeds to define a determinant by means of a Grassmann identity. Later, the Laplace expansion and a few other relations are established by Grassmann methods. However, for those readers who find this form of algebra too abstract for their tastes or training, classical proofs are also given. Most of the contents of this book can be described as complicated applications of classical algebra and differentiation.

In a book containing so many symbols, misprints are inevitable, but we hope they are obvious and will not obstruct our readers' progress for long. All reports of errors will be warmly appreciated.

We are indebted to our colleague, Dr. Barry Martin, for general advice on computers and for invaluable assistance in algebraic computing with the

Maple system on a Macintosh computer, especially in the expansion and factorization of determinants. We are also indebted by Lynn Burton for the most excellent construction and typing of a complicated manuscript in Microsoft Word programming language Formula on a Macintosh computer in camera-ready form.

Birmingham, U.K. P.R. Vein
P. Dale

Contents

1

Determinants, First Minors, and Cofactors

1.1 Grassmann Exterior Algebra

Let V be a finite-dimensional vector space over a field F. Then, it is known that for each non-negative integer m, it is possible to construct a vector space $\Lambda^m V$. In particular, $\Lambda^0 V = F$, $\Lambda V = V$, and for $m \geq 2$, each vector in $\Lambda^m V$ is a linear combination, with coefficients in F, of the products of m vectors from V.

If $\mathbf{x}_i \in V$, $1 \leq i \leq m$, we shall denote their vector product by $\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_m$. Each such vector product satisfies the following identities:

i. $\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_{r-1}(a\mathbf{x} + b\mathbf{y})\mathbf{x}_{r+1} \cdots \mathbf{x}_n = a\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_{r-1}\mathbf{x}\mathbf{x}_{r+1} \cdots \mathbf{x}_n$ $+b\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_{r-1}\mathbf{y} \cdots \mathbf{x}_{r+1} \cdots \mathbf{x}_n$, where $a, b \in F$ and $\mathbf{x}, \mathbf{y} \in V$.

ii. If any two of the $\mathbf{x}$'s in the product $\mathbf{x}_1\mathbf{x}_2 \cdots \mathbf{x}_n$ are interchanged, then the product changes sign, which implies that the product is zero if two or more of the $\mathbf{x}$'s are equal.

1.2 Determinants

Let $\dim V = n$ and let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be a set of base vectors for V. Then, if $\mathbf{x}_i \in V$, $1 \leq i \leq n$, we can write

$$\mathbf{x}_i = \sum_{k=1}^{n} a_{ik}\mathbf{e}_k, \quad a_{ik} \in F. \tag{1.2.1}$$

It follows from (i) and (ii) that

$$\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n = \sum_{k_1=1}^{n}\cdots\sum_{k_n=1}^{n} a_{1k_1}a_{2k_2}\cdots a_{nk_n}\mathbf{e}_{k_1}\mathbf{e}_{k_2}\cdots\mathbf{e}_{k_n}. \tag{1.2.2}$$

When two or more of the k's are equal, $\mathbf{e}_{k_1}\mathbf{e}_{k_2}\cdots\mathbf{e}_{k_n} = 0$. When the k's are distinct, the product $\mathbf{e}_{k_1}\mathbf{e}_{k_2}\cdots\mathbf{e}_{k_n}$ can be transformed into $\pm\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n$ by interchanging the dummy variables k_r in a suitable manner. The sign of each term is unique and is given by the formula

$$\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n = \left[\sum^{(n!\text{ terms})} \sigma_n a_{1k_1}a_{2k_2}\cdots a_{nk_n}\right]\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n, \tag{1.2.3}$$

where

$$\sigma_n = \operatorname{sgn}\left\{\begin{matrix} 1 & 2 & 3 & 4 & \cdots & (n-1) & n \\ k_1 & k_2 & k_3 & k_4 & \cdots & k_{n-1} & k_n \end{matrix}\right\} \tag{1.2.4}$$

and where the sum extends over all $n!$ permutations of the numbers k_r, $1 \le r \le n$. Notes on permutation symbols and their signs are given in Appendix A.2.

The coefficient of $\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n$ in (1.2.3) contains all n^2 elements a_{ij}, $1 \le i,j \le n$, which can be displayed in a square array. The coefficient is called a determinant of order n.

Definition.

$$A_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}_n = \sum^{(n!\text{ terms})} \sigma_n a_{1k_1}a_{2k_2}\cdots a_{nk_n}. \tag{1.2.5}$$

The array can be abbreviated to $|a_{ij}|_n$. The corresponding matrix is denoted by $[a_{ij}]_n$. Equation (1.2.3) now becomes

$$\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n = |a_{ij}|_n\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n. \tag{1.2.6}$$

Exercise. If $\begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$ is a fixed permutation, show that

$$\begin{aligned} A_n = |a_{ij}|_n &= \sum_{k_1,\ldots,k_n}^{n!\text{ terms}} \operatorname{sgn}\begin{pmatrix} j_1 & j_2 & \cdots & j_n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix} a_{j_1k_1}a_{j_2k_2}\cdots a_{j_nk_n} \\ &= \sum_{k_1,\ldots,k_n}^{n!\text{ terms}} \operatorname{sgn}\begin{pmatrix} j_1 & j_2 & \cdots & j_n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix} a_{k_1j_1}a_{k_2j_2}\cdots a_{k_nj_n}. \end{aligned}$$

1.3 First Minors and Cofactors

Referring to (1.2.1), put

$$\begin{aligned}\mathbf{y}_i &= \mathbf{x}_i - a_{ij}\mathbf{e}_j \\ &= (a_{i1}\mathbf{e}_1 + \cdots + a_{i,j-1}\mathbf{e}_{j-1}) + (a_{i,j+1}\mathbf{e}_{j+1} + \cdots + a_{in}\mathbf{e}_n) \quad (1.3.1)\\ &= \sum_{k=1}^{n-1} a'_{ik}\mathbf{e}'_k, \quad (1.3.2)\end{aligned}$$

where

$$\begin{aligned}\mathbf{e}'_k &= \mathbf{e}_k \qquad 1 \le k \le j-1 \\ &= \mathbf{e}_{k+1}, \quad j \le k \le n-1 \quad (1.3.3)\\ a'_{ik} &= a_{ik} \qquad 1 \le k \le j-1 \\ &= a_{i,k+1}, \quad j \le k \le n-1. \quad (1.3.4)\end{aligned}$$

Note that each a'_{ik} is a function of j.

It follows from Identity (ii) that

$$\mathbf{y}_1\mathbf{y}_2\cdots\mathbf{y}_n = 0 \quad (1.3.5)$$

since each $\mathbf{y}_r$ is a linear combination of $(n-1)$ vectors $\mathbf{e}_k$ so that each of the $(n-1)^n$ terms in the expansion of the product on the left contains at least two identical $\mathbf{e}$'s. Referring to (1.3.1) and Identities (i) and (ii),

$$\begin{aligned}&\mathbf{x}_1\cdots\mathbf{x}_{i-1}\mathbf{e}_j\mathbf{x}_{i+1}\cdots\mathbf{x}_n \\ &\quad= (\mathbf{y}_1 + a_{1j}\mathbf{e}_j)(\mathbf{y}_2 + a_{2j}\mathbf{e}_j)\cdots(\mathbf{y}_{i-1} + a_{i-1,j}\mathbf{e}_j) \\ &\qquad \mathbf{e}_j(\mathbf{y}_{i+1} + a_{i+1,j}\mathbf{e}_j)\cdots(\mathbf{y}_n + a_{nj}\mathbf{e}_j) \\ &\quad= \mathbf{y}_1\cdots\mathbf{y}_{i-1}\mathbf{e}_j\mathbf{y}_{i+1}\cdots\mathbf{y}_n \quad (1.3.6)\\ &\quad= (-1)^{n-i}(\mathbf{y}_1\cdots\mathbf{y}_{i-1}\mathbf{y}_{i+1}\cdots\mathbf{y}_n)\mathbf{e}_j. \quad (1.3.7)\end{aligned}$$

From (1.3.2) it follows that

$$\mathbf{y}_1\cdots\mathbf{y}_{i-1}\mathbf{y}_{i+1}\cdots\mathbf{y}_n = M_{ij}(\mathbf{e}'_1\mathbf{e}'_2\cdots\mathbf{e}'_{n-1}), \quad (1.3.8)$$

where

$$M_{ij} = \sum \sigma_{n-1} a'_{1k_1} a'_{2k_2} \cdots a'_{i-1,k_{i-1}} a'_{i+1,k_{i+1}} \cdots a'_{n-1,k_{n-1}} \quad (1.3.9)$$

and where the sum extends over the $(n-1)!$ permutations of the numbers $1, 2, \ldots, (n-1)$. Comparing M_{ij} with A_n, it is seen that M_{ij} is the determinant of order $(n-1)$ which is obtained from A_n by deleting row i and column j, that is, the row and column which contain the element a_{ij}. M_{ij} is therefore associated with a_{ij} and is known as a first minor of A_n.

Hence, referring to (1.3.3),

$$\begin{aligned}&\mathbf{x}_1\cdots\mathbf{x}_{i-1}\mathbf{e}_j\mathbf{x}_{i+1}\cdots\mathbf{x}_n \\ &\quad= (-1)^{n-i}M_{ij}(\mathbf{e}'_1\mathbf{e}'_2\cdots\mathbf{e}'_{n-1})\mathbf{e}_j\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{n-i} M_{ij} (\mathbf{e}'_1 \cdots \mathbf{e}'_{j-1})(\mathbf{e}'_j \cdots \mathbf{e}'_{n-1})\mathbf{e}_j \\
&= (-1)^{n-i} M_{ij} (\mathbf{e}_1 \cdots \mathbf{e}_{j-1})(\mathbf{e}_{j+1} \cdots \mathbf{e}_n)\mathbf{e}_j \\
&= (-1)^{i+j} M_{ij} (\mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n). \qquad (1.3.10)
\end{aligned}
$$

Now, $\mathbf{e}_j$ can be regarded as a particular case of $\mathbf{x}_i$ as defined in (1.2.1):

$$\mathbf{e}_j = \sum_{k=1}^{n} a_{ik} \mathbf{e}_k,$$

where

$$a_{ik} = \delta_{jk}.$$

Hence, replacing $\mathbf{x}_i$ by $\mathbf{e}_j$ in (1.2.3),

$$\mathbf{x}_1 \cdots \mathbf{x}_{i-1} \mathbf{e}_j \mathbf{x}_{i+1} \cdots \mathbf{x}_n = A_{ij} (\mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n), \qquad (1.3.11)$$

where

$$A_{ij} = \sum \sigma_n a_{1k_1} a_{2k_2} \cdots a_{ik_i} \cdots a_{nk_n},$$

where

$$
\begin{aligned}
a_{ik_i} &= 0 \qquad k_i \neq j \\
&= 1 \qquad k_i = j.
\end{aligned}
$$

Referring to the definition of a determinant in (1.2.4), it is seen that A_{ij} is the determinant obtained from $|a_{ij}|_n$ by replacing row i by the row

$$[0 \ldots 0 \; 1 \; 0 \ldots 0],$$

where the element 1 is in column j. A_{ij} is known as the cofactor of the element a_{ij} in A_n.

Comparing (1.3.10) and (1.3.11),

$$A_{ij} = (-1)^{i+j} M_{ij}. \qquad (1.3.12)$$

Minors and cofactors should be written $M_{ij}^{(n)}$ and $A_{ij}^{(n)}$ but the parameter n can be omitted where there is no risk of confusion.

Returning to (1.2.1) and applying (1.3.11),

$$
\begin{aligned}
\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_n &= \mathbf{x}_1 \cdots \mathbf{x}_{i-1} \left(\sum_{k=1}^{n} a_{ik} \mathbf{e}_k \right) \mathbf{x}_{i+1} \cdots \mathbf{x}_n \\
&= \sum_{k=1}^{n} a_{ik} (\mathbf{x}_1 \cdots \mathbf{x}_{i-1} \mathbf{e}_k \mathbf{x}_{i+1} \cdots \mathbf{x}_n) \\
&= \left[\sum_{k=1}^{n} a_{ik} A_{ik} \right] \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n. \qquad (1.3.13)
\end{aligned}
$$

Comparing this result with (1.2.5),

$$|a_{ij}|_n = \sum_{k=1}^{n} a_{ik} A_{ik} \tag{1.3.14}$$

which is the expansion of $|a_{ij}|_n$ by elements from row i and their cofactors.

From (1.3.1) and noting (1.3.5),

$$\begin{aligned}
\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n &= (\mathbf{y}_1 + a_{1j}\mathbf{e}_j)(\mathbf{y}_2 + a_{2j}\mathbf{e}_j)\cdots(\mathbf{y}_n + a_{nj}\mathbf{e}_j)\\
&= a_{1j}\mathbf{e}_j\mathbf{y}_2\mathbf{y}_3\cdots\mathbf{y}_n + a_{2j}\mathbf{y}_1\mathbf{e}_j\mathbf{y}_3\cdots\mathbf{y}_n\\
&\quad + \cdots + a_{nj}\mathbf{y}_1\mathbf{y}_2\cdots\mathbf{y}_{n-1}\mathbf{e}_j\\
&= (a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj})\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n\\
&= \left[\sum_{k=1}^{n} a_{kj}A_{kj}\right]\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n. \qquad (1.3.15)
\end{aligned}$$

Comparing this relation with (1.2.5),

$$|a_{ij}|_n = \sum_{k=1}^{n} a_{kj} A_{kj} \tag{1.3.16}$$

which is the expansion of $|a_{ij}|_n$ by elements from column j and their cofactors.

1.4 The Product of Two Determinants — 1

Put

$$\begin{aligned}
\mathbf{x}_i &= \sum_{k=1}^{n} a_{ik}\mathbf{y}_k,\\
\mathbf{y}_k &= \sum_{j=1}^{n} b_{kj}\mathbf{e}_j.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n &= |a_{ij}|_n\mathbf{y}_1\mathbf{y}_2\cdots\mathbf{y}_n,\\
\mathbf{y}_1\mathbf{y}_2\cdots\mathbf{y}_n &= |b_{ij}|_n\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n.
\end{aligned}$$

Hence,

$$\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n = |a_{ij}|_n|b_{ij}|_n\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n. \tag{1.4.1}$$

But,

$$\mathbf{x}_i = \sum_{k=1}^{n} a_{ik}\sum_{j=1}^{n} b_{kj}\mathbf{e}_j$$

$$= \sum_{j=1}^{n} c_{ij}\mathbf{e}_j,$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \tag{1.4.2}$$

Hence,

$$\mathbf{x}_1\mathbf{x}_2\cdots\mathbf{x}_n = |c_{ij}|_n \mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n. \tag{1.4.3}$$

Comparing (1.4.1) and (1.4.3),

$$|a_{ij}|_n |b_{ij}|_n = |c_{ij}|_n. \tag{1.4.4}$$

Another proof of (1.4.4) is given in Section 3.3.5 by applying the Laplace expansion in reverse.

The Laplace expansion formula is proved by both a Grassmann and a classical method in Chapter 3 after the definitions of second and higher rejector and retainor minors and cofactors.

2

A Summary of Basic Determinant Theory

2.1 Introduction

This chapter consists entirely of a summary of basic determinant theory, a prerequisite for the understanding of later chapters. It is assumed that the reader is familiar with these relations, although not necessarily with the notation used to describe them, and few proofs are given. If further proofs are required, they can be found in numerous undergraduate textbooks.

Several of the relations, including Cramer's formula and the formula for the derivative of a determinant, are expressed in terms of column vectors, a notation which is invaluable in the description of several analytical processes.

2.2 Row and Column Vectors

Let row i (the ith row) and column j (the jth column) of the determinant $A_n = |a_{ij}|_n$ be denoted by the boldface symbols $\mathbf{R}_i$ and $\mathbf{C}_j$ respectively:

$$\begin{aligned}\mathbf{R}_i &= \bigl[a_{i1}\ a_{i2}\ a_{i3} \cdots a_{in}\bigr],\\ \mathbf{C}_j &= \bigl[a_{1j}\ a_{2j}\ a_{3j} \cdots a_{nj}\bigr]^T\end{aligned} \tag{2.2.1}$$

where T denotes the transpose. We may now write

$$A_n = \begin{vmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \\ \vdots \\ \mathbf{R}_n \end{vmatrix} = \left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3 \cdots \mathbf{C}_n\right|. \qquad (2.2.2)$$

The column vector notation is clearly more economical in space and will be used exclusively in this and later chapters. However, many properties of particular determinants can be proved by performing a sequence of row and column operations and in these applications, the symbols $\mathbf{R}_i$ and $\mathbf{C}_j$ appear with equal frequency.

If every element in $\mathbf{C}_j$ is multiplied by the scalar k, the resulting vector is denoted by $k\mathbf{C}_j$:

$$k\mathbf{C}_j = \left[ka_{1j}\ ka_{2j}\ ka_{3j} \cdots ka_{nj}\right]^T.$$

If $k = 0$, this vector is said to be zero or null and is denoted by the boldface symbol $\mathbf{O}$.

If a_{ij} is a function of x, then the derivative of $\mathbf{C}_j$ with respect to x is denoted by $\mathbf{C}'_j$ and is given by the formula

$$\mathbf{C}'_j = \left[a'_{1j}\ a'_{2j}\ a'_{3j} \cdots a'_{nj}\right]^T.$$

2.3 Elementary Formulas

2.3.1 Basic Properties

The arbitrary determinant

$$A = |a_{ij}|_n = \left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3 \cdots \mathbf{C}_n\right|,$$

where the suffix n has been omitted from A_n, has the properties listed below. Any property stated for columns can be modified to apply to rows.

a. The value of a determinant is unaltered by transposing the elements across the principal diagonal. In symbols,

$$|a_{ji}|_n = |a_{ij}|_n.$$

b. The value of a determinant is unaltered by transposing the elements across the secondary diagonal. In symbols

$$|a_{n+1-j,n+1-i}|_n = |a_{ij}|_n.$$

c. If any two columns of A are interchanged and the resulting determinant is denoted by B, then $B = -A$.

Example.

$$\left|\mathbf{C}_1\ \mathbf{C}_3\ \mathbf{C}_4\ \mathbf{C}_2\right| = -\left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_4\ \mathbf{C}_3\right| = \left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3\ \mathbf{C}_4\right|.$$

Applying this property repeatedly,

i.

$$\left|\mathbf{C}_m\ \mathbf{C}_{m+1}\cdots\mathbf{C}_n\ \mathbf{C}_1\ \mathbf{C}_2\cdots\mathbf{C}_{m-1}\right| = (-1)^{(m-1)(n-1)}A,$$
$$1 < m < n.$$

The columns in the determinant on the left are a cyclic permutation of those in A.

ii. $\left|\mathbf{C}_n\ \mathbf{C}_{n-1}\ \mathbf{C}_{n-2}\cdots\mathbf{C}_2\ \mathbf{C}_1\right| = (-1)^{n(n-1)/2}A.$

d. Any determinant which contains two or more identical columns is zero.

$$\left|\mathbf{C}_1\cdots\mathbf{C}_j\cdots\mathbf{C}_j\cdots\mathbf{C}_n\right| = 0.$$

e. If every element in any one column of A is multiplied by a scalar k and the resulting determinant is denoted by B, then $B = kA$.

$$B = \left|\mathbf{C}_1\ \mathbf{C}_2\cdots(k\mathbf{C}_j)\cdots\mathbf{C}_n\right| = kA.$$

Applying this property repeatedly,

$$|ka_{ij}|_n = \left|(k\mathbf{C}_1)\ (k\mathbf{C}_2)\ (k\mathbf{C}_3)\cdots(k\mathbf{C}_n)\right|$$
$$= k^n|a_{ij}|_n.$$

This formula contrasts with the corresponding matrix formula, namely

$$[ka_{ij}]_n = k[a_{ij}]_n.$$

Other formulas of a similar nature include the following:

i. $|(-1)^{i+j}a_{ij}|_n = |a_{ij}|_n,$
ii. $|ia_{ij}|_n = |ja_{ij}|_n = n!|a_{ij}|_n,$
iii. $|x^{i+j-r}a_{ij}|_n = x^{n(n+1-r)}|a_{ij}|_n.$

f. Any determinant in which one column is a scalar multiple of another column is zero.

$$\left|\mathbf{C}_1\cdots\mathbf{C}_j\cdots(k\mathbf{C}_j)\cdots\mathbf{C}_n\right| = 0.$$

g. If any one column of a determinant consists of a sum of m subcolumns, then the determinant can be expressed as the sum of m determinants, each of which contains one of the subcolumns.

$$\left|\mathbf{C}_1\cdots\left(\sum_{s=1}^m\mathbf{C}_{js}\right)\cdots\mathbf{C}_n\right| = \sum_{s=1}^m\left|\mathbf{C}_1\cdots\mathbf{C}_{js}\cdots\mathbf{C}_n\right|.$$

Applying this property repeatedly,

$$\left|\left(\sum_{s=1}^m\mathbf{C}_{1s}\right)\cdots\left(\sum_{s=1}^m\mathbf{C}_{js}\right)\cdots\left(\sum_{s=1}^m\mathbf{C}_{ns}\right)\right|$$

$$= \sum_{k_1=1}^{m} \sum_{k_2=1}^{m} \cdots \sum_{k_n=1}^{m} \left| \mathbf{C}_{1k_1} \cdots \mathbf{C}_{jk_j} \cdots \mathbf{C}_{nk_n} \right|_n.$$

The function on the right is the sum of m^n determinants. This identity can be expressed in the form

$$\left| \sum_{k=1}^{m} a_{ij}^{(k)} \right|_n = \sum_{k_1,k_2,\ldots,k_n=1}^{m} \left| a_{ij}^{(k_j)} \right|_n.$$

h. *Column Operations.* The value of a determinant is unaltered by adding to any one column a linear combination of all the other columns. Thus, if

$$\mathbf{C}'_j = \mathbf{C}_j + \sum_{r=1}^{n} k_r \ \mathbf{C}_r \qquad k_j = 0,$$

$$= \sum_{r=1}^{n} k_r \ \mathbf{C}_r, \qquad k_j = 1,$$

then

$$\left| \mathbf{C}_1 \ \mathbf{C}_2 \cdots \mathbf{C}'_j \cdots \mathbf{C}_n \right| = \left| \mathbf{C}_1 \ \mathbf{C}_2 \cdots \mathbf{C}_j \cdots \mathbf{C}_n \right|.$$

$\mathbf{C}'_j$ should be regarded as a new column j and will not be confused with the derivative of $\mathbf{C}_j$. The process of replacing $\mathbf{C}_j$ by $\mathbf{C}'_j$ is called a column operation and is extensively applied to transform and evaluate determinants. Row and column operations are of particular importance in reducing the order of a determinant.

Exercise. If the determinant $A_n = |a_{ij}|_n$ is rotated through 90° in the clockwise direction so that a_{11} is displaced to the position $(1, n)$, a_{1n} is displaced to the position (n, n), etc., and the resulting determinant is denoted by $B_n = |b_{ij}|_n$, prove that

$$b_{ij} = a_{j,n-i}$$
$$B_n = (-1)^{n(n-1)/2} A_n.$$

2.3.2 Matrix-Type Products Related to Row and Column Operations

The row operations

$$\mathbf{R}'_i = \sum_{j=i}^{3} u_{ij} \mathbf{R}_j, \quad u_{ii} = 1, \quad 1 \le i \le 3; \quad u_{ij} = 0, \quad i > j, \tag{2.3.1}$$

namely

$$\begin{aligned} \mathbf{R}'_1 &= \mathbf{R}_1 + u_{12}\mathbf{R}_2 + u_{13}\mathbf{R}_3 \\ \mathbf{R}'_2 &= \qquad\quad \mathbf{R}_2 + u_{23}\mathbf{R}_3 \\ \mathbf{R}'_3 &= \qquad\qquad\qquad \mathbf{R}_3, \end{aligned}$$

can be expressed in the form

$$\begin{bmatrix} \mathbf{R}'_1 \\ \mathbf{R}'_2 \\ \mathbf{R}'_3 \end{bmatrix} = \begin{bmatrix} 1 & u_{12} & u_{13} \\ & 1 & u_{23} \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{bmatrix}.$$

Denote the upper triangular matrix by $\mathbf{U}_3$. These operations, when performed in the given order on an arbitrary determinant $A_3 = |a_{ij}|_3$, have the same effect as *pre*multiplication of A_3 by the unit determinant U_3. In each case, the result is

$$A_3 = \begin{vmatrix} a_{11} + u_{12}a_{21} + u_{13}a_{31} & a_{12} + u_{12}a_{22} + u_{13}a_{32} & a_{13} + u_{12}a_{23} + u_{13}a_{33} \\ a_{21} + u_{23}a_{31} & a_{22} + u_{23}a_{32} & a_{23} + u_{23}a_{33} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{2.3.2}$$

Similarly, the column operations

$$\mathbf{C}'_i = \sum_{j=i}^{3} u_{ij}\mathbf{C}_j, \quad u_{ii} = 1, \quad 1 \le i \le 3; \quad u_{ij} = 0, \quad i > j, \tag{2.3.3}$$

when performed in the given order on A_3, have the same effect as *post*multiplication of A_3 by U_3^T. In each case, the result is

$$A_3 = \begin{vmatrix} a_{11} + u_{12}a_{12} + u_{13}a_{13} & a_{12} + u_{23}a_{13} & a_{13} \\ a_{21} + u_{12}a_{22} + u_{13}a_{23} & a_{22} + u_{23}a_{23} & a_{23} \\ a_{31} + u_{12}a_{32} + u_{13}a_{33} & a_{32} + u_{23}a_{33} & a_{33} \end{vmatrix}. \tag{2.3.4}$$

The row operations

$$\mathbf{R}'_i = \sum_{j=1}^{i} v_{ij}\mathbf{R}_j, \quad v_{ii} = 1, \quad 1 \le i \le 3; \quad v_{ij} = 0, \quad i < j, \tag{2.3.5}$$

can be expressed in the form

$$\begin{bmatrix} \mathbf{R}'_1 \\ \mathbf{R}'_2 \\ \mathbf{R}'_3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ v_{21} & 1 & \\ v_{31} & v_{32} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{bmatrix}.$$

Denote the lower triangular matrix by $\mathbf{V}_3$. These operations, when performed *in reverse order* on A_3, have the same effect as *pre*multiplication of A_3 by the unit determinant V_3.

Similarly, the column operations

$$\mathbf{C}_i' = \sum_{j=1}^{i} v_{ij}\mathbf{C}_j, \quad v_{ii}=1, \quad 1 \le i \le 3, \quad v_{ij}=0, \quad i>j, \tag{2.3.6}$$

when performed on A_3 in *reverse order*, have the same effect as *post*multiplication of A_3 by V_3^T.

2.3.3 First Minors and Cofactors; Row and Column Expansions

To each element a_{ij} in the determinant $A = |a_{ij}|_n$, there is associated a subdeterminant of order $(n-1)$ which is obtained from A by deleting row i and column j. This subdeterminant is known as a first minor of A and is denoted by M_{ij}. The first cofactor A_{ij} is then defined as a signed first minor:

$$A_{ij} = (-1)^{i+j} M_{ij}. \tag{2.3.7}$$

It is customary to omit the adjective *first* and to refer simply to minors and cofactors and it is convenient to regard M_{ij} and A_{ij} as quantities which belong to a_{ij} in order to give meaning to the phrase "an element and its cofactor."

The expansion of A by elements from row i and their cofactors is

$$A = \sum_{j=1}^{n} a_{ij}A_{ij}, \quad 1 \le i \le n. \tag{2.3.8}$$

The expansion of A by elements from column j and their cofactors is obtained by summing over i instead of j:

$$A = \sum_{i=1}^{n} a_{ij}A_{ij}, \quad 1 \le j \le n. \tag{2.3.9}$$

Since A_{ij} belongs to but is independent of a_{ij}, an alternative definition of A_{ij} is

$$A_{ij} = \frac{\partial A}{\partial a_{ij}}\,. \tag{2.3.10}$$

Partial derivatives of this type are applied in Section 4.5.2 on symmetric Toeplitz determinants.

2.3.4 Alien Cofactors; The Sum Formula

The theorem on alien cofactors states that

$$\sum_{j=1}^{n} a_{ij}A_{kj} = 0, \quad 1 \le i \le n, \quad 1 \le k \le n, \quad k \ne i. \tag{2.3.11}$$

The elements come from row i of A, but the cofactors belong to the elements in row k and are said to be alien to the elements. The identity is merely an expansion by elements from row k of the determinant in which row $k =$ row i and which is therefore zero.

The identity can be combined with the expansion formula for A with the aid of the Kronecker delta function δ_{ik} (Appendix A.1) to form a single identity which may be called the sum formula for elements and cofactors:

$$\sum_{j=1}^{n} a_{ij} A_{kj} = \delta_{ik} A, \quad 1 \le i \le n, \quad 1 \le k \le n. \tag{2.3.12}$$

It follows that

$$\sum_{j=1}^{n} A_{ij} \mathbf{C}_j = [0 \ldots 0 \; A \; 0 \ldots 0]^T, \quad 1 \le i \le n,$$

where the element A is in row i of the column vector and all the other elements are zero. If $A = 0$, then

$$\sum_{j=1}^{n} A_{ij} \mathbf{C}_j = 0, \quad 1 \le i \le n, \tag{2.3.13}$$

that is, the columns are linearly dependent. Conversely, if the columns are linearly dependent, then $A = 0$.

2.3.5 Cramer's Formula

The set of equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad 1 \le i \le n,$$

can be expressed in column vector notation as follows:

$$\sum_{j=1}^{n} \mathbf{C}_j x_j = \mathbf{B},$$

where

$$\mathbf{B} = \begin{bmatrix} b_1 \; b_2 \; b_3 \cdots b_n \end{bmatrix}^T.$$

If $A = |a_{ij}|_n \neq 0$, then the unique solution of the equations can also be expressed in column vector notation. Let

$$A = \left| \mathbf{C}_1 \; \mathbf{C}_2 \cdots \mathbf{C}_j \cdots \mathbf{C}_n \right|.$$

Then

$$x_j = \frac{1}{A} \left| \mathbf{C}_1 \; \mathbf{C}_2 \cdots \mathbf{C}_{j-1} \; \mathbf{B} \; \mathbf{C}_{j+1} \cdots \mathbf{C}_n \right|$$

$$= \frac{1}{A} \sum_{i=1}^{n} b_i A_{ij}. \tag{2.3.14}$$

The solution of the triangular set of equations

$$\sum_{j=1}^{i} a_{ij} x_j = b_i, \quad i = 1, 2, 3, \dots$$

(the upper limit in the sum is i, not n as in the previous set) is given by the formula

$$x_i = \frac{(-1)^{i+1}}{a_{11} a_{22} \cdots a_{ii}} \begin{vmatrix} b_1 & a_{11} & & & & \\ b_2 & a_{21} & a_{22} & & & \\ b_3 & a_{31} & a_{32} & a_{33} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{i-1} & a_{i-1,1} & a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,i-1} \\ b_i & a_{i1} & a_{i2} & a_{i3} & \cdots & a_{i,i-1} \end{vmatrix}_i . \tag{2.3.15}$$

The determinant is a Hessenbergian (Section 4.6).

Cramer's formula is of great theoretical interest and importance in solving sets of equations with algebraic coefficients but is unsuitable for reasons of economy for the solution of large sets of equations with numerical coefficients. It demands far more computation than the unavoidable minimum. Some matrix methods are far more efficient. Analytical applications of Cramer's formula appear in Section 5.1.2 on the generalized geometric series, Section 5.5.1 on a continued fraction, and Section 5.7.2 on the Hirota operator.

Exercise. If

$$f_i^{(n)} = \sum_{j=1}^{n} a_{ij} x_j + a_{in}, \quad 1 \le i \le n,$$

and

$$f_i^{(n)} = 0, \quad 1 \le i \le n, \quad i \ne r,$$

prove that

$$f_r^{(n)} = \frac{A_n x_r}{A_{rn}^{(n)}}, \quad 1 \le r < n,$$

$$f_n^{(n)} = \frac{A_n (x_n + 1)}{A_{n-1}},$$

where

$$A_n = |a_{ij}|_n,$$

provided

$$A_{rn}^{(n)} \ne 0, \quad 1 \le i \le n.$$

2.3.6 The Cofactors of a Zero Determinant

If $A = 0$, then

$$A_{p_1q_1}A_{p_2q_2} = A_{p_2q_1}A_{p_1q_2}, \tag{2.3.16}$$

that is,

$$\begin{vmatrix} A_{p_1q_1} & A_{p_1q_2} \\ A_{p_2q_1} & A_{p_2q_2} \end{vmatrix} = 0, \quad 1 \le p_1, p_2, q_1, q_2 \le n.$$

It follows that

$$\begin{vmatrix} A_{p_1q_1} & A_{p_1q_2} & A_{p_1q_3} \\ A_{p_2q_1} & A_{p_2q_2} & A_{p_2q_3} \\ A_{p_3q_1} & A_{p_3q_2} & A_{p_3q_2} \end{vmatrix} = 0$$

since the second-order cofactors of the elements in the last (or any) row are all zero. Continuing in this way,

$$\begin{vmatrix} A_{p_1q_1} & A_{p_1q_2} & \cdots & A_{p_1q_r} \\ A_{p_2q_1} & A_{p_2q_2} & \cdots & A_{p_2q_r} \\ \cdots & \cdots & \cdots & \cdots \\ A_{p_rq_1} & A_{p_rq_2} & \cdots & A_{p_rq_r} \end{vmatrix}_r = 0, \quad 2 \le r \le n. \tag{2.3.17}$$

This identity is applied in Section 3.6.1 on the Jacobi identity.

2.3.7 The Derivative of a Determinant

If the elements of A are functions of x, then the derivative of A with respect to x is equal to the sum of the n determinants obtained by differentiating the columns of A one at a time:

$$\begin{aligned} A' &= \sum_{j=1}^{n} \left| \mathbf{C}_1 \ \mathbf{C}_2 \cdots \mathbf{C}'_j \cdots \mathbf{C}_n \right| \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} a'_{ij} A_{ij}. \end{aligned} \tag{2.3.18}$$

3

Intermediate Determinant Theory

3.1 Cyclic Dislocations and Generalizations

Define column vectors $\mathbf{C}_j$ and $\mathbf{C}_j^*$ as follows:

$$\mathbf{C}_j = \left[a_{1j}\ a_{2j}\ a_{3j} \cdots a_{nj}\right]^T$$
$$\mathbf{C}_j^* = \left[a_{1j}^*\ a_{2j}^*\ a_{3j}^* \cdots a_{nj}^*\right]^T$$

where

$$a_{ij}^* = \sum_{r=1}^{n}(1-\delta_{ir})\lambda_{ir}a_{rj},$$

that is, the element a_{ij}^* in $\mathbf{C}_j^*$ is a linear combination of all the elements in $\mathbf{C}_j$ except a_{ij}, the coefficients λ_{ir} being independent of j but otherwise arbitrary.

Theorem 3.1.

$$\sum_{j=1}^{n}\left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_j^* \cdots \mathbf{C}_n\right| = 0.$$

PROOF.

$$\begin{aligned}\left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_j^* \cdots \mathbf{C}_n\right| &= \sum_{i=1}^{n} a_{ij}^* A_{ij}\\ &= \sum_{i=1}^{n} A_{ij} \sum_{r=1}^{n}(1-\delta_{ir})\lambda_{ir}a_{rj}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{j=1}^{n}\left|\mathbf{C}_1\ \mathbf{C}_2\cdots\mathbf{C}_j^*\ \cdots\ \mathbf{C}_n\right| &= \sum_{i=1}^{n}\sum_{r=1}^{n}(1-\delta_{ir})\lambda_{ir}\sum_{j=1}^{n}a_{rj}A_{ij}\\ &= A_n\sum_{i=1}^{n}\sum_{r=1}^{n}(1-\delta_{ir})\lambda_{ir}\delta_{ir}\\ &= 0\end{aligned}$$

which completes the proof. □

If

$$\begin{aligned}\lambda_{1n} &= 1,\\ \lambda_{ir} &= \begin{cases}1, & r=i-1, \qquad i>1\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

that is,

$$[\lambda_{ir}]_n = \begin{bmatrix}0 & & & & & 1\\ 1 & 0 & & & & 0\\ & 1 & 0 & & & 0\\ & & 1 & 0 & & 0\\ & & & \cdots & \cdots & \cdots\\ & & & & 1 & 0\end{bmatrix}_n,$$

then $\mathbf{C}_j^*$ is the column vector obtained from $\mathbf{C}_j$ by dislocating or displacing the elements one place downward in a cyclic manner, the last element in $\mathbf{C}_j$ appearing as the first element in $\mathbf{C}_j^*$, that is,

$$\mathbf{C}_j^* = \left[a_{nj}\ a_{1j}\ a_{2j}\cdots a_{n-1,j}\right]^T.$$

In this particular case, Theorem 3.1 can be expressed in words as follows:

Theorem 3.1a. *Given an arbitrary determinant A_n, form n other determinants by dislocating the elements in the jth column one place downward in a cyclic manner, $1 \le j \le n$. Then, the sum of the n determinants so formed is zero.*

If

$$\lambda_{ir} = \begin{cases}i-1, & r=i-1, \qquad i>1\\ 0, & \text{otherwise,}\end{cases}$$

then

$$\begin{aligned}a_{ij}^* &= (i-1)a_{i-1,j},\\ \mathbf{C}_j^* &= \left[0\ a_{1j}\ 2a_{2j}\ 3a_{3j}\cdots(n-1)a_{n-1,j}\right]^T.\end{aligned}$$

This particular case is applied in Section 4.9.2 on the derivatives of a Turanian with Appell elements and another particular case is applied in Section 5.1.3 on expressing orthogonal polynomials as determinants.

Exercises

1. Let δ^r denote an operator which, when applied to $\mathbf{C}_j$, has the effect of dislocating the elements r positions downward in a cyclic manner so that the lowest set of r elements are expelled from the bottom and reappear at the top without change of order.

$$\begin{aligned}\delta^r \mathbf{C}_j &= \left[a_{n-r+1,j}\ a_{n-r+2,j} \cdots a_{nj}\ a_{1j}\ a_{2j} \cdots a_{n-r,j}\right]^T, \\ &\qquad 1 \le r \le n-1, \\ \delta^0 \mathbf{C}_j &= \delta^n \mathbf{C}_j = \mathbf{C}_j.\end{aligned}$$

Prove that

$$\sum_{j=1}^{n} \left|\mathbf{C}_1 \cdots \delta^r \mathbf{C}_j \cdots \mathbf{C}_n\right| = \begin{cases} 0, & 1 \le r \le n-1 \\ nA, & r = 0, n. \end{cases}$$

2. Prove that

$$\sum_{r=1}^{n} \left|\mathbf{C}_1 \cdots \delta^r \mathbf{C}_j \cdots \mathbf{C}_n\right| = s_j S_j,$$

where

$$\begin{aligned} s_j &= \sum_{i=1}^{n} a_{ij}, \\ S_j &= \sum_{i=1}^{n} A_{ij}. \end{aligned}$$

Hence, prove that an arbitrary determinant $A_n = |a_{ij}|_n$ can be expressed in the form

$$A_n = \frac{1}{n} \sum_{j=1}^{n} s_j S_j. \qquad \text{(Trahan)}$$

3.2 Second and Higher Minors and Cofactors

3.2.1 Rejecter and Retainer Minors

It is required to generalize the concept of first minors as defined in Chapter 1.

Let $A_n = |a_{ij}|_n$, and let $\{i_s\}$ and $\{j_s\}$, $1 \le s \le r \le n$, denote two independent sets of r distinct numbers, $1 \le i_s$ and $j_s \le n$. Now let $M^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ denote the subdeterminant of order $(n-r)$ which is obtained from A_n by *rejecting* rows $i_1, i_2, \ldots, i_r$ and columns $j_1, j_2, \ldots, j_r$. $M^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ is known as an rth minor of A_n. It may conveniently be called a *rejecter* minor. The numbers i_s and j_s are known respectively as row and column parameters.

Now, let $N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ denote the subdeterminant of order r which is obtained from A_n by *retaining* rows $i_1, i_2, \ldots, i_r$ and columns $j_1, j_2, \ldots, j_r$ and rejecting the other rows and columns. $N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ may conveniently be called a *retainer* minor.

Examples.

$$M^{(5)}_{13,25} = \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \\ a_{51} & a_{53} & a_{54} \end{vmatrix} = N_{245,134},$$

$$M^{(5)}_{245,134} = \begin{vmatrix} a_{12} & a_{15} \\ a_{32} & a_{35} \end{vmatrix} = N_{13,25}.$$

The minors $M^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ and $N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ are said to be mutually complementary in A_n, that is, each is the complement of the other in A_n. This relationship can be expressed in the form

$$\begin{aligned} M^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r} &= \operatorname{comp} N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}, \\ N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r} &= \operatorname{comp} M^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}. \end{aligned} \tag{3.2.1}$$

The order and structure of rejecter minors depends on the value of n but the order and structure of retainer minors are independent of n provided only that n is sufficiently large. For this reason, the parameter n has been omitted from N.

Examples.

$$\begin{aligned} N_{ip} &= \left| a_{ip} \right|_1 = a_{ip}, \quad n \geq 1, \\ N_{ij,pq} &= \begin{vmatrix} a_{ip} & a_{iq} \\ a_{jp} & a_{jq} \end{vmatrix}, \quad n \geq 2, \\ N_{ijk,pqr} &= \begin{vmatrix} a_{ip} & a_{iq} & a_{ir} \\ a_{jp} & a_{jq} & a_{jr} \\ a_{kp} & a_{kq} & a_{kr} \end{vmatrix}, \quad n \geq 3. \end{aligned}$$

Both rejecter and retainer minors arise in the construction of the Laplace expansion of a determinant (Section 3.3).

Exercise. Prove that

$$\begin{vmatrix} N_{ij,pq} & N_{ij,pr} \\ N_{ik,pq} & N_{ik,pr} \end{vmatrix} = N_{ip} N_{ijk,pqr}.$$

3.2.2 Second and Higher Cofactors

The first cofactor $A^{(n)}_{ij}$ is defined in Chapter 1 and appears in Chapter 2. It is now required to generalize that concept.

In the definition of rejecter and retainer minors, no restriction is made concerning the relative magnitudes of either the row parameters i_s or the column parameters j_s. Now, let each set of parameters be arranged in ascending order of magnitude, that is,

$$i_s < i_{s+1}, \; j_s < j_{s+1}, \quad 1 \le s \le r-1.$$

Then, the rth cofactor of A_n, denoted by $A^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}$ is defined as a signed rth rejecter minor:

$$A^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r} = (-1)^k M^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}, \tag{3.2.2}$$

where k is the sum of the parameters:

$$k = \sum_{s=1}^{r} (i_s + j_s).$$

However, the concept of a cofactor is more general than that of a signed minor. The definition can be extended to zero values and to all positive and negative integer values of the parameters by adopting two conventions:

i. The cofactor changes sign when any two row parameters or any two column parameters are interchanged. It follows without further assumptions that the cofactor is zero when either the row parameters or the column parameters are not distinct.

ii. The cofactor is zero when any row or column parameter is less than 1 or greater than n.

Illustration.

$$A^{(4)}_{12,23} = -A^{(4)}_{21,23} = -A^{(4)}_{12,32} = A^{(4)}_{21,32} = M^{(4)}_{12,23} = N_{34,14},$$
$$A^{(6)}_{135,235} = -A^{(6)}_{135,253} = A^{(6)}_{135,523} = A^{(6)}_{315,253} = -M^{(6)}_{135,235} = -N_{246,146},$$

$$A^{(n)}_{i_2 i_1 i_3; j_1 j_2 j_3} = -A^{(n)}_{i_1 i_2 i_3; j_1 j_2 j_3} = A^{(n)}_{i_1 i_2 i_3; j_1 j_3 j_2},$$
$$A^{(n)}_{i_1 i_2 i_3; j_1 j_2 (n-p)} = 0 \quad \text{if } p < 0 \text{ or } p \ge n \text{ or } p = n - j_1 \text{ or } p = n - j_2.$$

3.2.3 The Expansion of Cofactors in Terms of Higher Cofactors

Since the first cofactor $A^{(n)}_{ip}$ is itself a determinant of order $(n-1)$, it can be expanded by the $(n-1)$ elements from any row or column and their first cofactors. But, first, cofactors of $A^{(n)}_{ip}$ are second cofactors of A_n. Hence, it is possible to expand $A_{ip}^{(n)}$ by elements from any row or column and second cofactors $A_{ij,pq}^{(n)}$. The formula for row expansions is

$$A_{ip}^{(n)} = \sum_{q=1}^{n} a_{jq} A_{ij,pq}^{(n)}, \quad 1 \le j \le n, \quad j \ne i. \tag{3.2.3}$$

The term in which $q = p$ is zero by the first convention for cofactors. Hence, the sum contains $(n-1)$ nonzero terms, as expected. The $(n-1)$ values of j for which the expansion is valid correspond to the $(n-1)$ possible ways of expanding a subdeterminant of order $(n-1)$ by elements from one row and their cofactors.

Omitting the parameter n and referring to (2.3.10), it follows that if $i < j$ and $p < q$, then

$$\begin{aligned} A_{ij,pq} &= \frac{\partial A_{ip}}{\partial a_{jq}} \\ &= \frac{\partial^2 A}{\partial a_{ip} \partial a_{jq}} \end{aligned} \tag{3.2.4}$$

which can be regarded as an alternative definition of the second cofactor $A_{ij,pq}$.

Similarly,

$$A_{ij,pq}^{(n)} = \sum_{r=1}^{n} a_{kr} A_{ijk,pqr}^{(n)}, \quad 1 \le k \le n, \quad k \ne i \text{ or } j. \tag{3.2.5}$$

Omitting the parameter n, it follows that if $i < j < k$ and $p < q < r$, then

$$\begin{aligned} A_{ijk,pqr} &= \frac{\partial A_{ij,pq}}{\partial a_{kr}} \\ &= \frac{\partial^3 A}{\partial a_{ip} \partial a_{jq} \partial a_{kr}} \end{aligned} \tag{3.2.6}$$

which can be regarded as an alternative definition of the third cofactor $A_{ijk,pqr}$.

Higher cofactors can be defined in a similar manner. Partial derivatives of this type appear in Section 3.3.2 on the Laplace expansion, in Section 3.6.2 on the Jacobi identity, and in Section 5.4.1 on the Matsuno determinant.

The expansion of an rth cofactor, a subdeterminant of order $(n-r)$, can be expressed in the form

$$A_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}^{(n)} = \sum_{q=1}^{n} a_{pq} A_{i_1 i_2 \ldots i_r p; j_1 j_2 \ldots j_r q}^{(n)}, \tag{3.2.7}$$

$$1 \le p \le n, \quad p \ne i_s, \quad 1 \le s \le r.$$

The r terms in which $q = j_s$, $1 \le s \le r$, are zero by the first convention for cofactors. Hence, the sum contains $(n-r)$ nonzero terms, as expected.

The $(n-r)$ values of p for which the expansion is valid correspond to the $(n-r)$ possible ways of expanding a subdeterminant of order $(n-r)$ by elements from one row and their cofactors.

If one of the column parameters of an rth cofactor of A_{n+1} is $(n+1)$, the cofactor does not contain the element $a_{n+1,n+1}$. If none of the row parameters is $(n+1)$, then the rth cofactor can be expanded by elements from its last row and their first cofactors. But first cofactors of an rth cofactor of A_{n+1} are $(r+1)$th cofactors of A_{n+1} which, in this case, are rth cofactors of A_n. Hence, in this case, an rth cofactor of A_{n+1} can be expanded in terms of the first n elements in the last row and rth cofactors of A_n. This expansion is

$$A^{(n+1)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_{r-1}(n+1)} = -\sum_{q=1}^{n} a_{n+1,q} A^{(n)}_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_{r-1} q}. \tag{3.2.8}$$

The corresponding column expansion is

$$A^{(n+1)}_{i_1 i_2 \ldots i_{r-1}(n+1); j_1 j_2 \ldots j_r} = -\sum_{p=1}^{n} a_{p,n+1} A^{(n)}_{i_1 i_2 \ldots i_{r-1} p; j_1 j_2 \ldots j_r}. \tag{3.2.9}$$

Exercise. Prove that

$$\frac{\partial^2 A}{\partial a_{ip} \partial a_{jq}} = -\frac{\partial^2 A}{\partial a_{iq} \partial a_{jp}},$$
$$\frac{\partial^3 A}{\partial a_{ip} \partial a_{jq} \partial a_{kr}} = \frac{\partial^3 A}{\partial a_{kp} \partial a_{iq} \partial a_{jr}} = \frac{\partial^3 A}{\partial a_{jp} \partial a_{kq} \partial a_{ir}}$$

without restrictions on the relative magnitudes of the parameters.

3.2.4 Alien Second and Higher Cofactors; Sum Formulas

The $(n-2)$ elements a_{hq}, $1 \le q \le n$, $q \ne h$ or p, appear in the second cofactor $A^{(n)}_{ij,pq}$ if $h \ne i$ or j. Hence,

$$\sum_{q=1}^{n} a_{hq} A^{(n)}_{ij,pq} = 0, \quad h \ne i \text{ or } j,$$

since the sum represents a determinant of order $(n-1)$ with two identical rows. This formula is a generalization of the theorem on alien cofactors given in Chapter 2. The value of the sum of $1 \le h \le n$ is given by the sum formula for elements and cofactors, namely

$$\sum_{q=1}^{n} a_{hq} A^{(n)}_{ij,pq} = \begin{cases} A^{(n)}_{ip}, & h = j \ne i \\ -A^{(n)}_{jp}, & h = i \ne j \\ 0, & \text{otherwise} \end{cases} \tag{3.2.10}$$

which can be abbreviated with the aid of the Kronecker delta function [Appendix A]:

$$\sum_{q=1}^{n} a_{hq} A_{ij,pq}^{(n)} = A_{ip}^{(n)} \delta_{hj} - A_{jp}^{(n)} \delta_{hi}.$$

Similarly,

$$\begin{aligned}
\sum_{r=1}^{n} a_{hr} A_{ijk,pqr}^{(n)} &= A_{ij,pq}^{(n)} \delta_{hk} + A_{jk,pq}^{(n)} \delta_{hi} + A_{ki,pq}^{(n)} \delta_{hj}, \\
\sum_{s=1}^{n} a_{hs} A_{ijkm,pqrs}^{(n)} &= A_{ijk,pqr}^{(n)} \delta_{hm} - A_{jkm,pqr}^{(n)} \delta_{hi} \\
&\quad + A_{kmi,pqr}^{(n)} \delta_{hj} - A_{mij,pqr}^{(n)} \delta_{hk} \qquad (3.2.11)
\end{aligned}$$

etc.

Exercise. Show that these expressions can be expressed as sums as follows:

$$\begin{aligned}
\sum_{q=1}^{n} a_{hq} A_{ij,pq}^{(n)} &= \sum_{u,v} \operatorname{sgn} \begin{Bmatrix} u & v \\ i & j \end{Bmatrix} A_{up}^{(n)} \delta_{hv}, \\
\sum_{r=1}^{n} a_{hr} A_{ijk,pqr}^{(n)} &= \sum_{u,v,w} \operatorname{sgn} \begin{Bmatrix} u & v & w \\ i & j & k \end{Bmatrix} A_{uv,pq}^{(n)} \delta_{hw}, \\
\sum_{s=1}^{n} a_{hs} A_{ijkm,pqrs}^{(n)} &= \sum_{u,v,w,x} \operatorname{sgn} \begin{Bmatrix} u & v & w & x \\ i & j & k & m \end{Bmatrix} A_{uvw,pqr}^{(n)} \delta_{hx},
\end{aligned}$$

etc., where, in each case, the sums are carried out over all possible cyclic permutations of the lower parameters in the permutation symbols. A brief note on cyclic permutations is given in Appendix A.2.

3.2.5 Scaled Cofactors

Cofactors $A_{ip}^{(n)}$, $A_{ij,pq}^{(n)}$, $A_{ijk,pqr}^{(n)}$, etc., with both row and column parameters written as subscripts have been defined in Section 3.2.2. They may conveniently be called simple cofactors. Scaled cofactors A_n^{ip}, $A_n^{ij,pq}$, $A_n^{ijk,pqr}$, etc., with row and column parameters written as superscripts are defined as follows:

$$\begin{aligned}
A_n^{ip} &= \frac{A_{ip}^{(n)}}{A_n}, \\
A_n^{ij,pq} &= \frac{A_{ij,pq}^{(n)}}{A_n}, \\
A_n^{ijk,pqr} &= \frac{A_{ijk,pqr}^{(n)}}{A_n}, \qquad (3.2.12)
\end{aligned}$$

etc. In simple algebraic relations such as Cramer's formula, the advantage of using scaled rather than simple cofactors is usually negligible. The Jacobi identity (Section 3.6) can be expressed in terms of unscaled or scaled cofactors, but the scaled form is simpler. In differential relations, the advantage can be considerable. For example, the sum formula

$$\sum_{j=1}^{n} a_{ij} A_{kj}^{(n)} = A_n \delta_{ki}$$

when differentiated gives rise to three terms:

$$\sum_{j=1}^{n} \left[a'_{ij} A_{kj}^{(n)} + a_{ij} (A_{kj}^{(n)})' \right] = A'_n \delta_{ki}.$$

When the cofactor is scaled, the sum formula becomes

$$\sum_{j=1}^{n} a_{ij} A_n^{kj} = \delta_{ki} \tag{3.2.13}$$

which is only slightly simpler than the original, but when it is differentiated, it gives rise to only two terms:

$$\sum_{j=1}^{n} \left[a'_{ij} A_n^{kj} + a_{ij} (A_n^{kj})' \right] = 0. \tag{3.2.14}$$

The advantage of using scaled rather than unscaled or simple cofactors will be fully appreciated in the solution of differential equations (Chapter 6).

Referring to the partial derivative formulas in (2.3.10) and Section 3.2.3,

$$\begin{aligned} \frac{\partial A^{ip}}{\partial a_{jq}} &= \frac{\partial}{\partial a_{jq}} \left(\frac{A_{ip}}{A} \right) \\ &= \frac{1}{A^2} \left[A \frac{\partial A^{ip}}{\partial a_{jq}} - A_{ip} \frac{\partial A}{\partial a_{jq}} \right] \\ &= \frac{1}{A^2} \left[A\, A_{ij,pq} - A_{ip}\, A_{jq} \right] \\ &= A^{ij,pq} - A^{ip} A^{jq}. \end{aligned} \tag{3.2.15}$$

Hence,

$$\left(A^{jq} + \frac{\partial}{\partial a_{jq}} \right) A^{ip} = A^{ij,pq}. \tag{3.2.16}$$

Similarly,

$$\left(A^{kr} + \frac{\partial}{\partial a_{kr}} \right) A^{ij,pq} = A^{ijk,pqr}. \tag{3.2.17}$$

The expressions in brackets can be regarded as operators which, when applied to a scaled cofactor, yield another scaled cofactor. Formula (3.2.15) is applied in Section 3.6.2 on the Jacobi identity. Formulas (3.2.16) and (3.2.17) are applied in Section 5.4.1 on the Matsuno determinant.

3.3 The Laplace Expansion

3.3.1 A Grassmann Proof

The following analysis applies Grassmann algebra and is similar in nature to that applied in the definition of a determinant.

Let i_s and j_s, $1 \le s \le r$, $r \le n$, denote r integers such that

$$\begin{aligned} &1 \le i_1 < i_2 < \cdots < i_r \le n, \\ &1 \le j_1 < j_2 < \cdots < j_r \le n \end{aligned}$$

and let

$$\begin{aligned} \mathbf{x}_i &= \sum_{k=1}^{n} a_{ij}\mathbf{e}_k, \quad 1 \le i \le n, \\ \mathbf{y}_i &= \sum_{t=1}^{r} a_{ij_t}\mathbf{e}_{j_t}, \quad 1 \le i \le n, \\ \mathbf{z}_i &= \mathbf{x}_i - \mathbf{y}_i. \end{aligned}$$

Then, any vector product is which the number of $\mathbf{y}$'s is greater than r or the number of $\mathbf{z}$'s is greater than $(n-r)$ is zero.

Hence,

$$\begin{aligned} \mathbf{x}_1 \cdots \mathbf{x}_n &= (\mathbf{y}_1 + \mathbf{z}_1)(\mathbf{y}_2 + \mathbf{z}_2) \cdots (\mathbf{y}_n + \mathbf{z}_n) \\ &= \sum_{i_1 \dots i_r} \mathbf{z}_1 \cdots \mathbf{y}_{i_1} \cdots \mathbf{y}_{i_2} \cdots \mathbf{y}_{i_r} \cdots \mathbf{z}_n, \end{aligned} \tag{3.3.1}$$

where the vector product on the right is obtained from $(\mathbf{z}_1 \cdots \mathbf{z}_n)$ by replacing $\mathbf{z}_{i_s}$ by $\mathbf{y}_{i_s}$, $1 \le s \le r$, and the sum extends over all $\binom{n}{r}$ combinations of the numbers $1, 2, \dots, n$ taken r at a time. The $\mathbf{y}$'s in the vector product can be separated from the $\mathbf{z}$'s by making a suitable sequence of interchanges and applying Identity (ii). The result is

$$\mathbf{z}_1 \cdots \mathbf{y}_{i_1} \cdots \mathbf{y}_{i_2} \cdots \mathbf{y}_{i_r} \cdots \mathbf{z}_n = (-1)^p \big(\mathbf{y}_{i_1} \cdots \mathbf{y}_{i_r}\big)\big(\mathbf{z}_1 \overset{*}{\cdots} \mathbf{z}_n\big), \tag{3.3.2}$$

where

$$p = \sum_{s=1}^{n} i_s - \tfrac{1}{2}r(r+1) \tag{3.3.3}$$

and the symbol $*$ denotes that those vectors with suffixes $i_1, i_2, \dots, i_r$ are omitted.

Recalling the definitions of rejecter minors M, retainer minors N, and cofactors A, each with row and column parameters, it is found that

$$\mathbf{y}_{i_1} \cdots \mathbf{y}_{i_r} = N_{i_1 \ldots i_r; j_1 \ldots j_r} \big(\mathbf{e}_{j_1} \cdots \mathbf{e}_{j_r}\big),$$
$$\mathbf{z}_1 \overset{*}{\cdots} \mathbf{z}_n = M_{i_1 \ldots i_r; j_1 \ldots j_r} \big(\mathbf{e}_1 \overset{*}{\cdots} \mathbf{e}_n\big),$$

where, in this case, the symbol $*$ denotes that those vectors with suffixes $j_1, j_2, \ldots, j_r$ are omitted. Hence,

$$\begin{aligned} &\mathbf{x}_1 \cdots \mathbf{x}_n \\ &\quad = \sum_{i_1 \ldots i_r} (-1)^p N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}\, M_{i_1 i_2 \ldots, i_r; j_1 j_2 \ldots j_r} \big(\mathbf{e}_{j_1} \cdots \mathbf{e}_{j_r}\big)\big(\mathbf{e}_1 \overset{*}{\cdots} \mathbf{e}_n\big). \end{aligned}$$

By applying in reverse order the sequence of interchanges used to obtain (3.3.2), it is found that

$$\big(\mathbf{e}_{j_1} \cdots \mathbf{e}_{j_r}\big)\big(\mathbf{e}_1 \overset{*}{\cdots} \mathbf{e}_n\big) = (-1)^q (\mathbf{e}_1 \cdots \mathbf{e}_n),$$

where

$$q = \sum_{s=1}^{n} j_s - \tfrac{1}{2} r(r+1).$$

Hence,

$$\begin{aligned} \mathbf{x}_1 \cdots \mathbf{x}_n &= \left[\sum_{i_1 \ldots i_r} (-1)^{p+q} N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r} M_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}\right] \mathbf{e}_1 \cdots \mathbf{e}_n \\ &= \left[\sum_{i_1 \ldots i_r} N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r} A_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}\right] \mathbf{e}_1 \cdots \mathbf{e}_n. \end{aligned}$$

Comparing this formula with (1.2.5) in the section on the definition of a determinant, it is seen that

$$A_n = |a_{ij}|_n = \sum_{i_1 \ldots i_r} N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}\, A_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}, \tag{3.3.4}$$

which is the general form of the Laplace expansion of A_n in which the sum extends over the row parameters. By a similar argument, it can be shown that A_n is also equal to the same expression in which the sum extends over the column parameters.

When $r = 1$, the Laplace expansion degenerates into a simple expansion by elements from column j or row i and their first cofactors:

$$\begin{aligned} A_n &= \sum_{i \text{ or } j} N_{ij} A_{ij}, \\ &= \sum_{i \text{ or } j} a_{ij} A_{ij}. \end{aligned}$$

When $r = 2$,

$$A_n = \sum N_{ir,js} A_{ir,js}, \quad \text{summed over } i, r \text{ or } j, s,$$
$$= \sum \begin{vmatrix} a_{ij} & a_{is} \\ a_{rj} & a_{rs} \end{vmatrix} A_{ir,js}.$$

3.3.2 A Classical Proof

The following proof of the Laplace expansion formula given in (3.3.4) is independent of Grassmann algebra.

Let

$$A = |a_{ij}|_n.$$

Then referring to the partial derivative formulas in Section 3.2.3,

$$A_{i_1 j_1} = \frac{\partial A}{\partial a_{i_1 j_1}} \tag{3.3.5}$$
$$A_{i_1 i_2; j_1 j_2} = \frac{\partial A_{i_1 j_1}}{\partial a_{i_2 j_2}}, \quad i_1 < i_2 \text{ and } j_1 < j_2,$$
$$= \frac{\partial^2 A}{\partial a_{i_1 j_1} \partial a_{i_2 j_2}}. \tag{3.3.6}$$

Continuing in this way,

$$A_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r} = \frac{\partial^r A}{\partial a_{i_1 j_1} \partial a_{i_2 j_2} \cdots \partial a_{i_r j_r}}, \tag{3.3.7}$$

provided that $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_r$.

Expanding A by elements from column j_1 and their cofactors and referring to (3.3.5),

$$A = \sum_{i_1=1}^{n} a_{i_1 j_1} A_{i_1 j_1}$$
$$= \sum_{i_1=1}^{n} a_{i_1 j_1} \frac{\partial A}{\partial a_{i_1 j_1}}$$
$$= \sum_{i_2=1}^{n} a_{i_2 j_2} \frac{\partial A}{\partial a_{i_2 j_2}} \tag{3.3.8}$$
$$\frac{\partial A}{\partial a_{i_1 j_1}} = \sum_{i_2=1}^{n} a_{i_2 j_2} \frac{\partial^2 A}{\partial a_{i_1 j_1} \partial a_{i_2 j_2}}$$
$$= \sum_{i_2=1}^{n} a_{i_2 j_2} A_{i_1 i_2; j_1 j_2}, \quad i_1 < i_2 \text{ and } j_1 < j_2. \tag{3.3.9}$$

Substituting the first line of (3.3.9 and the second line of (3.3.8),

$$\begin{aligned} A &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} a_{i_1 j_1} a_{i_2 j_2} \frac{\partial^2 A}{\partial a_{i_1 j_1} \partial a_{i_2 j_2}} \\ &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} a_{i_1 j_1} a_{i_2 j_2} A_{i_1 i_2; j_1 j_2}, \quad i_1 < i_2 \text{ and } j_1 < j_2. \end{aligned} \tag{3.3.10}$$

Continuing in this way and applying (3.3.7) in reverse,

$$\begin{aligned} A &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_r=1}^{n} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_r j_r} \frac{\partial^r A}{\partial a_{i_1 j_1} \partial a_{i_2 j_2} \cdots \partial a_{i_r j_r}} \\ &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_r=1}^{n} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_r j_r} A_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}, \end{aligned} \tag{3.3.11}$$

subject to the inequalities associated with (3.3.7) which require that the i_s and j_s shall be in ascending order of magnitude.

In this multiple sum, those rth cofactors in which the dummy variables are not distinct are zero so that the corresponding terms in the sum are zero. The remaining terms can be divided into a number of groups according to the relative magnitudes of the dummies. Since r distinct dummies can be arranged in a linear sequence in $r!$ ways, the number of groups is $r!$. Hence,

$$A = \sum^{(r!\text{ terms})} G_{k_1 k_2 \ldots, k_r},$$

where

$$\begin{aligned} G_{k_1 k_2 \ldots k_r} = \sum_{i \le i_{k_1} < i_{k_2} < \cdots < i_{k_r} \le n} & a_{i_{k_1} j_{k_1}} \; a_{i_{k_2} j_{k_2}} \\ & \cdots a_{i_{k_r} j_{k_r}} A_{i_{k_1} i_{k_2} \cdots i_{k_r}; j_{k_1} j_{k_2} \cdots j_{k_r}}. \end{aligned} \tag{3.3.12}$$

In one of these $r!$ terms, the dummies $i_1, i_2, \ldots, i_r$ are in ascending order of magnitude, that is, $i_s < i_{s+1}$, $1 \le s \le r-1$. However, the dummies in the other $(r!-1)$ terms can be interchanged in such a way that the inequalities are valid for those terms too. Hence, applying those properties of rth cofactors which concern changes in sign,

$$A = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left[\sum \sigma_r a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_r j_r} \right] A_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r},$$

where

$$\sigma_r = \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & \cdots & r \\ i_1 & i_2 & i_3 & \cdots & i_r \end{Bmatrix}. \tag{3.3.13}$$

(Appendix A.2). But,

$$\sum \sigma_r a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_r j_r} = N_{i_1 i_2 \ldots i_r; j_1 j_2 \ldots j_r}.$$

The expansion formula (3.3.4) follows.

Illustrations

1. When $r = 2$, the Laplace expansion formula can be proved as follows: Changing the notation in the second line of (3.3.10),

$$A = \sum_{p=1}^{n} \sum_{q=1}^{n} a_{ip} a_{jq} A_{ij;pq}, \quad i < j.$$

This double sum contains n^2 terms, but the n terms in which $q = p$ are zero by the definition of a second cofactor. Hence,

$$A = \sum_{p<q} a_{ip} a_{jq} A_{ij,pq} + \sum_{q<p} a_{ip} a_{jq} A_{ij;pq}.$$

In the second double sum, interchange the dummies p and q and refer once again to the definition of a second cofactor:

$$\begin{aligned} A &= \sum_{p<q} \begin{vmatrix} a_{ip} & a_{iq} \\ a_{jp} & a_{jq} \end{vmatrix} A_{ij;pq} \\ &= \sum_{p<q} N_{ij;pq} A_{ij;pq}, \quad i < j, \end{aligned}$$

which proves the Laplace expansion formula from rows i and j. When $(n, i, j) = (4, 1, 2)$, this formula becomes

$$\begin{aligned} A = N_{12,12} A_{12,12} &+ N_{12,13} A_{12,13} + N_{12,14} A_{12,14} \\ &+ N_{12,23} A_{12,23} + N_{12,24} A_{12,24} \\ &+ N_{12,34} A_{12,34}. \end{aligned}$$

2. When $r = 3$, begin with the formula

$$A = \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{r=1}^{n} a_{ip} a_{jq} a_{kr} A_{ijk,pqr}, \quad i < j < k,$$

which is obtained from the second line of (3.3.11) with a change in notation. The triple sum contains n^3 terms, but those in which p, q, and r are not distinct are zero. Those which remain can be divided into $3! = 6$ groups according to the relative magnitudes of p, q, and r:

$$A = \left[\sum_{p<q<r} + \sum_{p<r<q} + \sum_{q<r<p} + \sum_{q<p<r} + \sum_{r<p<q} + \sum_{r<q<p} \right] a_{ip} a_{jq} a_{kr} A_{ijk,pqr}.$$

Now, interchange the dummies wherever necessary in order that $p < q < r$ in all sums. The result is

$$\begin{aligned} A &= \sum_{p<q<r} \big[a_{ip}a_{jq}a_{kr} - a_{ip}a_{jr}a_{kq} + a_{iq}a_{jr}a_{kp} \\ &\qquad - a_{iq}a_{jp}a_{kr} + a_{ir}a_{jp}a_{kq} - a_{ir}a_{jq}a_{kp}\big] A_{ijk,pqr} \\ &= \sum_{p<q<r} \begin{vmatrix} a_{ip} & a_{iq} & a_{ir} \\ a_{jp} & a_{jq} & a_{jr} \\ a_{kp} & a_{kq} & a_{kr} \end{vmatrix} A_{ijk,pqr} \\ &= \sum_{p<q<r} N_{ijk,pqr} A_{ijk,pqr}, \quad i<j<k, \end{aligned}$$

which proves the Laplace expansion formula from rows i, j, and k.

3.3.3 Determinants Containing Blocks of Zero Elements

Let $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$, and $\mathbf{O}$ denote matrices of order n, where $\mathbf{O}$ is null and let

$$A_{2n} = \begin{vmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{vmatrix}_{2n}.$$

The Laplace expansion of A_{2n} taking minors from the first or last n rows or the first or last n columns consists, in general, of the sum of $\binom{2n}{n}$ nonzero products. If one of the submatrices is null, all but one of the products are zero.

Lemma.

a. $\begin{vmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{O} & \mathbf{S} \end{vmatrix}_{2n} = PS,$

b. $\begin{vmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{vmatrix}_{2n} = (-1)^n QR$

PROOF. The only nonzero term in the Laplace expansion of the first determinant is

$$N_{12\ldots n;12\ldots n} A_{12\ldots n;12\ldots n}.$$

The retainer minor is signless and equal to P. The sign of the cofactor is $(-1)^k$, where k is the sum of the row and column parameters.

$$k = 2\sum_{r=1}^{n} r = n(n+1),$$

which is even. Hence, the cofactor is equal to $+S$. Part (a) of the lemma follows.

The only nonzero term in the Laplace expansion of the second determinant is

$$N_{n+1,n+2,\ldots,2n;12\ldots n} A_{n+1,n+2,\ldots,2n;12\ldots n}.$$

The retainer minor is signless and equal to R. The sign of the cofactor is $(-1)^k$, where

$$k = \sum_{r=1}^{n} (n + 2r) = 2n^2 + n.$$

Hence, the cofactor is equal to $(-1)^n Q$. Part (b) of the lemma follows. □

Similar arguments can be applied to more general determinants. Let $\mathbf{X}_{pq}$, $\mathbf{Y}_{pq}$, $\mathbf{Z}_{pq}$, and $\mathbf{O}_{pq}$ denote matrices with p rows and q columns, where $\mathbf{O}_{pq}$ is null and let

$$A_n = \begin{vmatrix} \mathbf{X}_{pq} & \mathbf{Y}_{ps} \\ \mathbf{O}_{rq} & \mathbf{Z}_{rs} \end{vmatrix}_n, \tag{3.3.14}$$

where $p + r = q + s = n$. The restriction $p \geq q$, which implies $r \leq s$, can be imposed without loss of generality. If A_n is expanded by the Laplace method taking minors from the first q columns or the last r rows, some of the minors are zero. Let U_m and V_m denote determinants of order m. Then, A_n has the following properties:

a. If $r + q > n$, then $A_n = 0$.
b. If $r + q = n$, then $p + s = n$, $q = p$, $s = r$, and $A_n = X_{pp} Z_{rr}$.
c. If $r + q < n$, then, in general,

$$\begin{aligned} A_n &= \text{sum of } \tbinom{p}{q} \text{ nonzero products each of the form } U_q V_s \\ &= \text{sum of } \tbinom{s}{r} \text{ nonzero products each of the form } U_r V_r. \end{aligned}$$

Property (a) is applied in the following examples.

Example 3.2. If $r + s = n$, then

$$U_{2n} = \begin{vmatrix} \mathbf{E}_{n,2r} & \mathbf{F}_{ns} & \mathbf{O}_{ns} \\ \mathbf{E}_{n,2r} & \mathbf{O}_{ns} & \mathbf{F}_{ns} \end{vmatrix}_{2n} = 0.$$

PROOF. It is clearly possible to perform n row operations in a single step and s column operations in a single step. Regard U_{2n} as having two "rows" and three "columns" and perform the operations

$$\begin{aligned} \mathbf{R}_1' &= \mathbf{R}_1 - \mathbf{R}_2, \\ \mathbf{C}_2' &= \mathbf{C}_2 + \mathbf{C}_3. \end{aligned}$$

The result is

$$\begin{aligned} U_{2n} &= \begin{vmatrix} \mathbf{O}_{n,2r} & \mathbf{F}_{ns} & -\mathbf{F}_{ns} \\ \mathbf{E}_{n,2r} & \mathbf{O}_{ns} & \mathbf{F}_{ns} \end{vmatrix}_{2n} \\ &= \begin{vmatrix} \mathbf{O}_{n,2r} & \mathbf{O}_{ns} & -\mathbf{F}_{ns} \\ \mathbf{E}_{n,2r} & \mathbf{F}_{ns} & \mathbf{F}_{ns} \end{vmatrix}_{2n} \\ &= 0 \end{aligned}$$

since the last determinant contains an $n \times (2r + s)$ block of zero elements and $n + 2r + s > 2n$. □

Example 3.3. Let

$$V_{2n} = \begin{vmatrix} \mathbf{E}_{ip} & \mathbf{F}_{iq} & \mathbf{G}_{iq} \\ \mathbf{E}_{ip} & \mathbf{G}_{iq} & \mathbf{F}_{iq} \\ \mathbf{O}_{jp} & \mathbf{H}_{jq} & \mathbf{K}_{jq} \end{vmatrix}_{2n},$$

where $2i + j = p + 2q = 2n$. Then, $V_{2n} = 0$ under each of the following independent conditions:

i. $j + p > 2n$,
ii. $p > i$,
iii. $\mathbf{H}_{jq} + \mathbf{K}_{jq} = \mathbf{O}_{jq}$.

PROOF. Case (i) follows immediately from Property (a). To prove case (ii) perform row operations

$$V_{2n} = \begin{vmatrix} \mathbf{E}_{ip} & \mathbf{F}_{iq} & \mathbf{G}_{iq} \\ \mathbf{O}_{ip} & (\mathbf{G}_{iq} - \mathbf{F}_{iq}) & (\mathbf{F}_{iq} - \mathbf{G}_{iq}) \\ \mathbf{O}_{jp} & \mathbf{H}_{jq} & \mathbf{K}_{jq} \end{vmatrix}_{2n}.$$

This determinant contains an $(i + j) \times p$ block of zero elements. But, $i + j + p > 2i + j = 2n$. Case (ii) follows.

To prove case (iii), perform column operations on the last determinant:

$$V_{2n} = \begin{vmatrix} \mathbf{E}_{ip} & (\mathbf{F}_{iq} + \mathbf{G}_{iq}) & \mathbf{G}_{iq} \\ \mathbf{O}_{ip} & \mathbf{O}_{iq} & (\mathbf{F}_{iq} - \mathbf{G}_{iq}) \\ \mathbf{O}_{jp} & \mathbf{O}_{jq} & \mathbf{K}_{jq} \end{vmatrix}_{2n}.$$

This determinant contains an $(i + j) \times (p + q)$ block of zero elements. However, since $2(i+j) > 2n$ and $2(p+q) > 2n$, it follows that $i+j+p+q > 2n$. Case (iii) follows. □

3.3.4 *The Laplace Sum Formula*

The simple sum formula for elements and their cofactors (Section 2.3.4), which incorporates the theorem on alien cofactors, can be generalized for the case $r = 2$ as follows:

$$\sum_{p<q} N_{ij,pq} A_{rs,pq} = \delta_{ij,rs} A,$$

where $\delta_{ij,rs}$ is the generalized Kronecker delta function (Appendix A.1). The proof follows from the fact that if $r \neq i$, the sum represents a determinant in which row r = row i, and if, in addition, $s \neq j$, then, in addition, row s = row j. In either case, the determinant is zero.

Exercises

1. If $n = 4$, prove that

$$\sum_{p<q} N_{23,pq} A_{24,pq} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{vmatrix} = 0$$

(row 4 = row 3), by expanding the determinant from rows 2 and 3.

2. Generalize the sum formula for the case $r = 3$.

3.3.5 The Product of Two Determinants — 2

Let

$$\begin{aligned} A_n &= |a_{ij}|_n \\ B_n &= |b_{ij}|_n. \end{aligned}$$

Then

$$A_n B_n = |c_{ij}|_n,$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

A similar formula is valid for the product of two matrices. A proof has already been given by a Grassmann method in Section 1.4. The following proof applies the Laplace expansion formula and row operations but is independent of Grassmann algebra.

Applying in reverse a Laplace expansion of the type which appears in Section 3.3.3,

$$A_n B_n = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & & & & \\ a_{21} & a_{22} & \dots & a_{2n} & & & & \\ \dots & \dots & \dots & \dots & & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & & & & \\ -1 & & & & b_{11} & b_{12} & \dots & b_{1n} \\ & -1 & & & b_{21} & b_{22} & \dots & b_{2n} \\ & & \dots & & \dots & \dots & \dots & \dots \\ & & & -1 & b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix}_{2n}. \tag{3.3.15}$$

Reduce all the elements in the first n rows and the first n columns, at present occupied by the a_{ij}, to zero by means of the row operations

$$\mathbf{R}'_i = \mathbf{R}_i + \sum_{j=1}^{n} a_{ij} \mathbf{R}_{n+j}, \quad 1 \le i \le n. \tag{3.3.16}$$

The result is:

$$A_nB_n = \begin{vmatrix} & & & & c_{11} & c_{12} & \dots & c_{1n} \\ & & & & c_{21} & c_{22} & \dots & c_{2n} \\ & & & & \dots & \dots & \dots & \dots \\ & & & & c_{n1} & c_{n2} & \dots & c_{nn} \\ -1 & & & & b_{11} & b_{12} & \dots & b_{1n} \\ & -1 & & & b_{21} & b_{22} & \dots & b_{2n} \\ & & \dots & & \dots & \dots & \dots & \dots \\ & & & -1 & b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix}_{2n}. \tag{3.3.17}$$

The product formula follows by means of a Laplace expansion. c_{ij} is most easily remembered as a scalar product:

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \bullet \begin{bmatrix} b_{1j} \\ b_{2j} \\ \dots \\ b_{nj} \end{bmatrix}. \tag{3.3.18}$$

Let $\mathbf{R}_i$ denote the ith row of A_n and let $\mathbf{C}_j$ denote the jth column of B_n. Then,

$$c_{ij} = \mathbf{R}_i \bullet \mathbf{C}_j.$$

Hence

$$\begin{aligned} A_nB_n &= |\mathbf{R}_i \bullet \mathbf{C}_j|_n \\ &= \begin{vmatrix} \mathbf{R}_1 \bullet \mathbf{C}_1 & \mathbf{R}_1 \bullet \mathbf{C}_2 & \cdots & \mathbf{R}_1 \bullet \mathbf{C}_n \\ \mathbf{R}_2 \bullet \mathbf{C}_1 & \mathbf{R}_2 \bullet \mathbf{C}_2 & \cdots & \mathbf{R}_2 \bullet \mathbf{C}_n \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ \mathbf{R}_n \bullet \mathbf{C}_1 & \mathbf{R}_n \bullet \mathbf{C}_2 & \cdots & \mathbf{R}_n \bullet \mathbf{C}_n \end{vmatrix}_n. \end{aligned} \tag{3.3.19}$$

Exercise. If $A_n = |a_{ij}|_n$, $B_n = |b_{ij}|_n$, and $C_n = |c_{ij}|_n$, prove that

$$A_nB_nC_n = |d_{ij}|_n,$$

where

$$d_{ij} = \sum_{r=1}^{n} \sum_{s=1}^{n} a_{ir}b_{rs}c_{sj}.$$

A similar formula is valid for the product of three matrices.

3.4 Double-Sum Relations for Scaled Cofactors

The following four double-sum relations are labeled (A)–(D) for easy reference in later sections, especially Chapter 6 on mathematical physics, where they are applied several times. The first two are formulas for the derivatives

A' and $(A^{ij})'$ and the other two are identities:

$$\frac{A'}{A} = (\log A)' = \sum_{r=1}^{n} \sum_{s=1}^{n} a'_{rs} A^{rs}, \tag{A}$$

$$(A^{ij})' = -\sum_{r=1}^{n} \sum_{s=1}^{n} a'_{rs} A^{is} A^{rj}, \tag{B}$$

$$\sum_{r=1}^{n} \sum_{s=1}^{n} (f_r + g_s) a_{rs} A^{rs} = \sum_{r=1}^{n} (f_r + g_r), \tag{C}$$

$$\sum_{r=1}^{n} \sum_{s=1}^{n} (f_r + g_s) a_{rs} A^{is} A^{rj} = (f_i + g_j) A^{ij}. \tag{D}$$

PROOF. (A) follows immediately from the formula for A' in terms of unscaled cofactors in Section 2.3.7. The sum formula given in Section 2.3.4 can be expressed in the form

$$\sum_{s=1}^{n} a_{rs} A^{is} = \delta_{ri}, \tag{3.4.1}$$

which, when differentiated, gives rise to only two terms:

$$\sum_{s=1}^{n} a'_{rs} A^{is} = -\sum_{s=1}^{n} a_{rs} (A^{is})'. \tag{3.4.2}$$

Hence, beginning with the right side of (B),

$$\begin{aligned}
\sum_{r=1}^{n} \sum_{s=1}^{n} a'_{rs} A^{is} A^{rj} &= \sum_{r} A^{rj} \sum_{s} a'_{rs} A^{is} \\
&= -\sum_{r} A^{rj} \sum_{s} a_{rs} (A^{is})' \\
&= -\sum_{s} (A^{is})' \sum_{r} a_{rs} A^{rj} \\
&= -\sum_{s} (A^{is})' \delta_{sj} \\
&= -(A^{ij})'
\end{aligned}$$

which proves (B).

$$\begin{aligned}
&\sum_{r} \sum_{s} (f_r + g_s) a_{rs} A^{is} A^{rj} \\
&\qquad = \sum_{r} f_r A^{rj} \sum_{s} a_{rs} A^{is} + \sum_{s} g_s A^{is} \sum_{r} a_{rs} A^{rj}
\end{aligned}$$

$$= \sum_r f_r A^{rj} \delta_{ri} + \sum_s g_s A^{is} \delta_{sj}$$
$$= f_i A^{ij} + g_j A^{ij}$$

which proves (D). The proof of (C) is similar but simpler. □

Exercises

Prove that

1. $\sum_{r=1}^{n} \sum_{s=1}^{n} [r^k - (r-1)^k + s^k - (s-1)^k] a_{rs} A^{rs} = 2n^k.$

2. $a'_{ij} = -\sum_{r=1}^{n} \sum_{s=1}^{n} a_{is} a_{rj} (A^{rs})'.$

3. $\sum_{r=1}^{n} \sum_{s=1}^{n} (f_r + g_s) a_{is} a_{rj} A^{rs} = (f_i + g_j) a_{ij}.$

Note that (2) and (3) can be obtained formally from (B) and (D), respectively, by interchanging the symbols a and A and either raising or lowering all their parameters.

3.5 The Adjoint Determinant

3.5.1 Definition

The adjoint of a matrix $\mathbf{A} = [a_{ij}]_n$ is denoted by adj $\mathbf{A}$ and is defined by

$$\operatorname{adj} \mathbf{A} = [A_{ji}]_n.$$

The adjoint or adjugate or a determinant $A = |a_{ij}|_n = \det \mathbf{A}$ is denoted by adj A and is defined by

$$\begin{aligned} \operatorname{adj} A &= |A_{ji}|_n = |A_{ij}|_n \\ &= \det(\operatorname{adj} \mathbf{A}). \end{aligned} \tag{3.5.1}$$

3.5.2 The Cauchy Identity

The following theorem due to Cauchy is valid for all determinants.

Theorem.

$$\operatorname{adj} A = A^{n-1}.$$

The proof is similar to that of the matrix relation

$$\mathbf{A} \operatorname{adj} \mathbf{A} = A\mathbf{I}.$$

PROOF.

$$\begin{aligned} A \operatorname{adj} A &= |a_{ij}|_n |A_{ji}|_n \\ &= |b_{ij}|_n, \end{aligned}$$

where, referring to Section 3.3.5 on the product of two determinants,

$$\begin{aligned} b_{ij} &= \sum_{r=1}^{n} a_{ir} A_{jr} \\ &= \delta_{ij} A. \end{aligned}$$

Hence,

$$\begin{aligned} |b_{ij}|_n &= \operatorname{diag}|A \ A \ \ldots \ A|_n \\ &= A^n. \end{aligned}$$

The theorem follows immediately if $A \neq 0$. If $A = 0$, then, applying (2.3.16) with a change in notation, $|A_{ij}|_n = 0$, that is, $\operatorname{adj} A = 0$. Hence, the Cauchy identity is valid for all A. □

3.5.3 An Identity Involving a Hybrid Determinant

Let $A_n = |a_{ij}|_n$ and $B_n = |b_{ij}|_n$, and let H_{ij} denote the hybrid determinant formed by replacing the jth row of A_n by the ith row of B_n. Then,

$$H_{ij} = \sum_{s=1}^{n} b_{is} A_{js}. \tag{3.5.2}$$

Theorem.

$$|a_{ij} x_i + b_{ij}|_n = A_n \left| \delta_{ij} x_i + \frac{H_{ij}}{A_n} \right|_n, \quad A_n \neq 0.$$

In the determinant on the right, the x_i appear only in the principal diagonal.

PROOF. Applying the Cauchy identity in the form

$$|A_{ji}|_n = A_n^{n-1}$$

and the formula for the product of two determinants (Section 1.4),

$$\begin{aligned} |a_{ij} x_i + b_{ij}|_n A_n^{n-1} &= |a_{ij} x_i + b_{ij}|_n |A_{ji}|_n \\ &= |c_{ij}|_n, \end{aligned}$$

where

$$\begin{aligned} c_{ij} &= \sum_{s=1}^{n} (a_{is} x_i + b_{is}) A_{js} \\ &= x_i \sum_{s=1}^{n} a_{is} A_{js} + \sum_{s=1}^{n} b_{is} A_{js} \\ &= \delta_{ij} A_n x_i + H_{ij}. \end{aligned}$$

Hence, removing the factor A_n from each row,

$$|c_{ij}|_n = A_n^n \left|\delta_{ij}x_i + \frac{H_{ij}}{A_n}\right|_n$$

which yields the stated result.

This theorem is applied in Section 6.7.4 on the K dV equation. □

3.6 The Jacobi Identity and Variants

3.6.1 The Jacobi Identity — 1

Given an arbitrary determinant $A = |a_{ij}|_n$, the rejecter minor $M_{p_1p_2\ldots p_r;q_1q_2\ldots q_r}$ of order $(n-r)$ and the retainer minor $N_{p_1p_2\ldots p_r;q_1q_2\ldots q_r}$ of order r are defined in Section 3.2.1.

Define the retainer minor J of order r as follows:

$$\begin{aligned} J = J_{p_1p_2\ldots p_r;q_1q_2\ldots q_r} &= \operatorname{adj} N_{p_1p_2\ldots p_r;q_1q_2\ldots q_r} \\ &= \begin{vmatrix} A_{p_1q_1} & A_{p_2q_1} & \cdots & A_{p_rq_1} \\ A_{p_1q_2} & A_{p_2q_2} & \cdots & A_{p_rq_2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{p_1q_r} & A_{p_2q_r} & \cdots & A_{p_rq_r} \end{vmatrix}_r . \end{aligned} \tag{3.6.1}$$

J is a minor of adj A. For example,

$$\begin{aligned} J_{23,24} &= \operatorname{adj} N_{23,24} \\ &= \operatorname{adj} \begin{vmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{vmatrix} \\ &= \begin{vmatrix} A_{22} & A_{32} \\ A_{24} & A_{34} \end{vmatrix}. \end{aligned}$$

The Jacobi identity on the minors of adj A is given by the following theorem:

Theorem.

$$J_{p_1p_2\ldots p_r;q_1q_2\ldots q_r} = A^{r-1} M_{p_1p_2\ldots p_r;q_1q_2\ldots q_r}, \quad 1 \le r \le n-1.$$

Referring to the section on the cofactors of a zero determinant in Section 2.3.7, it is seen that if $A = 0$, $r > 1$, then $J = 0$. The right-hand side of the above identity is also zero. Hence, in this particular case, the theorem is valid but trivial. When $r = 1$, the theorem degenerates into the definition of $A_{p_1q_1}$ and is again trivial. It therefore remains to prove the theorem when $A \ne 0$, $r > 1$.

The proof proceeds in two stages. In the first stage, the theorem is proved in the particular case in which

$$p_s = q_s = s, \quad 1 \le s \le r.$$

It is required to prove that

$$\begin{aligned} J_{12\ldots r;12\ldots r} &= A^{r-1} M_{12\ldots r;12\ldots r} \\ &= A^{r-1} A_{12\ldots r;12\ldots r}. \end{aligned}$$

The replacement of the minor by its corresponding cofactor is permitted since the sum of the parameters is even. In some detail, the simplified theorem states that

$$\begin{vmatrix} A_{11} & A_{21} & \ldots & A_{r1} \\ A_{12} & A_{22} & \ldots & A_{r2} \\ \ldots & \ldots & \ldots & \ldots \\ A_{1r} & A_{2r} & \ldots & A_{rr} \end{vmatrix}_r = A^{r-1} \begin{vmatrix} a_{r+1,r+1} & a_{r+1,r+2} & \cdots & a_{r+1,n} \\ a_{r+2,r+1} & a_{r+2,r+2} & \cdots & a_{r+2,n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n,r+1} & a_{n,r+2} & \cdots & a_{nn} \end{vmatrix}_{n-r}. \tag{3.6.2}$$

PROOF. Raise the order of $J_{12\ldots r;12\ldots r}$ from r to n by applying the Laplace expansion formula in reverse as follows:

$$J_{12\ldots r;12\ldots r} = \begin{vmatrix} A_{11} & \ldots & A_{r1} & & & \\ \vdots & & \vdots & & & \\ A_{1r} & \ldots & A_{rr} & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{1,r+1} & \ldots & A_{r,r+1} & 1 & & \\ \vdots & & \vdots & & \ddots & \\ A_{1n} & \ldots & A_{rn} & & & 1 \end{vmatrix}_n \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} r \text{ rows} \\ \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} (n-r) \text{ rows} \end{matrix}. \tag{3.6.3}$$

Multiply the left-hand side by A, the right-hand side by $|a_{ij}|_n$, apply the formula for the product of two determinants, the sum formula for elements and cofactors, and, finally, the Laplace expansion formula again

$$\begin{aligned} AJ_{12\ldots r;12\ldots r} &= \begin{vmatrix} A & & & \vdots & a_{1,r+1} & \ldots & a_{1n} \\ & \ddots & & \vdots & \vdots & & \vdots \\ & & A & \vdots & a_{r,r+1} & \ldots & a_{rn} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & \vdots & a_{r+1,r+1} & \ldots & a_{r+1,n} \\ & & & \vdots & \vdots & & \vdots \\ & & & \vdots & a_{n,r+1} & \ldots & a_{nn} \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} r \text{ rows} \\ \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} (n-r) \text{ rows} \end{matrix} \\ &= A^r \begin{vmatrix} a_{r+1,r+1} & \cdots & a_{r+1,n} \\ \vdots & & \vdots \\ a_{n,r+1} & \cdots & a_{nn} \end{vmatrix}_{n-r} \\ &= A^r A_{12\ldots r;12\ldots r}. \end{aligned}$$

The first stage of the proof follows.

The second stage proceeds as follows. Interchange pairs of rows and then pairs of columns of adj A until the elements of J as defined in (3.6.1) appear as a block in the top left-hand corner. Denote the result by $(\operatorname{adj} A)^*$. Then,

$$(\operatorname{adj} A)^* = \sigma \operatorname{adj} A,$$

where

$$\begin{aligned}\sigma &= (-1)^{(p_1-1)+(p_2-2)+\cdots+(p_r-r)+(q_1-1)+(q_2-2)+\cdots+(q_r-r)}\\ &= (-1)^{(p_1+p_2+\cdots+p_r)+(q_1+q_2+\cdots+q_r)}.\end{aligned}$$

Now replace each A_{ij} in $(\operatorname{adj} A)^*$ by a_{ij}, transpose, and denote the result by $|a_{ij}|^*$. Then,

$$|a_{ij}|^* = \sigma|a_{ij}| = \sigma A.$$

Raise the order of J from r to n in a manner similar to that shown in (3.6.3), augmenting the first r columns until they are identical with the first r columns of $(\operatorname{adj} A)^*$, denote the result by J^*, and form the product $|a_{ij}|^* J^*$. The theorem then appears. □

Illustration. Let $(n, r) = (4, 2)$ and let

$$J = J_{23,24} = \begin{vmatrix} A_{22} & A_{32} \\ A_{24} & A_{34} \end{vmatrix}.$$

Then

$$\begin{aligned}(\operatorname{adj} A)^* &= \begin{vmatrix} A_{22} & A_{32} & A_{12} & A_{42} \\ A_{24} & A_{34} & A_{14} & A_{44} \\ A_{21} & A_{31} & A_{11} & A_{41} \\ A_{23} & A_{33} & A_{13} & A_{43} \end{vmatrix}\\ &= \sigma \operatorname{adj} A,\end{aligned}$$

where

$$\sigma = (-1)^{2+3+2+4} = -1$$

and

$$\begin{aligned}|a_{ij}|^* &= \begin{vmatrix} a_{22} & a_{24} & a_{21} & a_{23} \\ a_{32} & a_{34} & a_{31} & a_{33} \\ a_{12} & a_{14} & a_{11} & a_{13} \\ a_{42} & a_{44} & a_{41} & a_{43} \end{vmatrix}\\ &= \sigma|a_{ij}| = \sigma A.\end{aligned}$$

The first two columns of J^* are identical with the first two columns of $(\operatorname{adj} A)^*$:

$$\begin{aligned}J = J^* &= \begin{vmatrix} A_{22} & A_{32} & & \\ A_{24} & A_{34} & & \\ A_{21} & A_{31} & 1 & \\ A_{23} & A_{33} & & 1 \end{vmatrix},\\ \sigma AJ &= |a_{ij}|^* J^*\end{aligned}$$

$$
\begin{aligned}
&= \begin{vmatrix} A & & a_{21} & a_{23} \\ & A & a_{31} & a_{33} \\ & & a_{11} & a_{13} \\ & & a_{41} & a_{43} \end{vmatrix}\\
&= A^2 \begin{vmatrix} a_{11} & a_{13} \\ a_{41} & a_{43} \end{vmatrix}\\
&= A^2 M_{23,24}\\
&= \sigma A^2 A_{23,24}.
\end{aligned}
$$

Hence, transposing J,

$$
J = \begin{vmatrix} A_{22} & A_{24} \\ A_{32} & A_{34} \end{vmatrix} = A\,A_{23,24}
$$

which completes the illustration.

Restoring the parameter n, the Jacobi identity with $r = 2, 3$ can be expressed as follows:

$$
r = 2: \qquad \begin{vmatrix} A_{ip}^{(n)} & A_{iq}^{(n)} \\ A_{jp}^{(n)} & A_{jq}^{(n)} \end{vmatrix} = A_n A_{ij,pq}^{(n)}. \tag{3.6.4}
$$

$$
r = 3: \qquad \begin{vmatrix} A_{ip}^{(n)} & A_{iq}^{(n)} & A_{ir}^{(n)} \\ A_{jp}^{(n)} & A_{jq}^{(n)} & A_{jr}^{(n)} \\ A_{kp}^{(n)} & A_{kq}^{(n)} & A_{kr}^{(n)} \end{vmatrix} = A_n^2 A_{ijk,pqr}^{(n)}. \tag{3.6.5}
$$

3.6.2 The Jacobi Identity — 2

The Jacobi identity for small values of r can be proved neatly by a technique involving partial derivatives with respect to the elements of A. The general result can then be proved by induction.

Theorem 3.4. *For an arbitrary determinant A_n of order n,*

$$
\begin{vmatrix} A_n^{ij} & A_n^{iq} \\ A_n^{pj} & A_n^{pq} \end{vmatrix} = A_n^{ip,jq},
$$

where the cofactors are scaled.

PROOF. The technique is to evaluate $\partial A^{ij}/\partial a_{pq}$ by two different methods and to equate the results. From (3.2.15),

$$
\frac{\partial A^{ij}}{\partial a_{pq}} = \frac{1}{A^2}\left[A A_{ip,jq} - A_{ij} A_{pq}\right]. \tag{3.6.6}
$$

Applying double-sum identity (B) in Section 3.4,

$$
\frac{\partial A^{ij}}{\partial a_{pq}} = -\sum_r \sum_s \frac{\partial a_{rs}}{\partial a_{pq}} A^{is} A^{rj}
$$

$$= -\sum_r \sum_s \delta_{rp}\delta_{sq} A^{is} A^{rj}$$
$$= -A^{iq}A^{pj}$$
$$= -\frac{1}{A^2}[A_{iq}A_{pj}]. \tag{3.6.7}$$

Hence,

$$\begin{vmatrix} A_{ij} & A_{iq} \\ A_{pj} & A_{pq} \end{vmatrix} = AA_{ip,jq}, \tag{3.6.8}$$

which, when the parameter n is restored, is equivalent to (3.6.4). The formula given in the theorem follows by scaling the cofactors. □

Theorem 3.5.

$$\begin{vmatrix} A^{ij} & A^{iq} & A^{iv} \\ A^{pj} & A^{pq} & A^{pv} \\ A^{uj} & A^{uq} & A^{uv} \end{vmatrix} = A^{ipu,jqv},$$

where the cofactors are scaled.

PROOF. From (3.2.4) and Theorem 3.4,

$$\frac{\partial^2 A}{\partial a_{pq}\partial a_{uv}} = A_{pu,qv}$$
$$= AA^{pu,qv}$$
$$= A\begin{vmatrix} A^{pq} & A^{pv} \\ A^{uq} & A^{uv} \end{vmatrix}. \tag{3.6.9}$$

Hence, referring to (3.6.7) and the formula for the derivative of a determinant (Section 2.3.7),

$$\frac{\partial^3 A}{\partial a_{ij}\partial a_{pq}\partial a_{uv}}$$
$$= \frac{\partial A}{\partial a_{ij}}\begin{vmatrix} A^{pq} & A^{pv} \\ A^{uq} & A^{uv} \end{vmatrix} + A\begin{vmatrix} \frac{\partial A^{pq}}{\partial a_{ij}} & A^{pv} \\ \frac{\partial A^{uq}}{\partial a_{ij}} & A^{uv} \end{vmatrix} + A\begin{vmatrix} A^{pq} & \frac{\partial A^{pv}}{\partial a_{ij}} \\ A^{uq} & \frac{\partial A^{uv}}{\partial a_{ij}} \end{vmatrix}$$
$$= A_{ij}\begin{vmatrix} A^{pq} & A^{pv} \\ A^{uq} & A^{uv} \end{vmatrix} - AA^{iq}\begin{vmatrix} A^{pj} & A^{pv} \\ A^{uj} & A^{uv} \end{vmatrix} - AA^{iv}\begin{vmatrix} A^{pq} & A^{pj} \\ A^{uq} & A^{uj} \end{vmatrix}$$
$$= \frac{1}{A^2}\left[A_{ij}\begin{vmatrix} A_{pq} & A_{pv} \\ A_{uq} & A_{uv} \end{vmatrix} - A_{iq}\begin{vmatrix} A_{pj} & A_{pv} \\ A_{uj} & A_{uv} \end{vmatrix}\right.$$
$$\left. + A_{iv}\begin{vmatrix} A_{pj} & A_{pq} \\ A_{uj} & A_{uq} \end{vmatrix}\right]$$
$$= \frac{1}{A^2}\begin{vmatrix} A_{ij} & A_{iq} & A_{iv} \\ A_{pj} & A_{pq} & A_{pv} \\ A_{uj} & A_{uq} & A_{uv} \end{vmatrix}. \tag{3.6.10}$$

But also,

$$\frac{\partial^3 A}{\partial a_{ij}\partial a_{pq}\partial a_{uv}} = A_{ipu,jqv}. \tag{3.6.11}$$

Hence,

$$\begin{vmatrix} A_{ij} & A_{iq} & A_{iv} \\ A_{pj} & A_{pq} & A_{pv} \\ A_{uj} & A_{uq} & A_{uv} \end{vmatrix} = A^2 A_{ipujqv}, \tag{3.6.12}$$

which, when the parameter n is restored, is equivalent to (3.6.5). The formula given in the theorem follows by scaling the cofactors. Note that those Jacobi identities which contain scaled cofactors lack the factors A, A^2, etc., on the right-hand side. This simplification is significant in applications involving derivatives. □

Exercises

1. Prove that

$$\sum_{\text{ep}\{p,q,r\}} A_{pt} A_{qr,st} = 0,$$

where the symbol ep$\{p, q, r\}$ denotes that the sum is carried out over all even permutations of $\{p, q, r\}$, including the identity permutation (Appendix A.2).

2. Prove that

$$\begin{vmatrix} A^{ps} & A^{pi,js} \\ A^{rq} & A^{ri,jq} \end{vmatrix} = \begin{vmatrix} A^{rj} & A^{rp,qj} \\ A^{is} & A^{ip,qs} \end{vmatrix} = \begin{vmatrix} A^{iq} & A^{ir,sq} \\ A^{pj} & A^{pr,sj} \end{vmatrix}.$$

3. Prove the Jacobi identity for general values of r by induction.

3.6.3 Variants

Theorem 3.6.

$$\begin{vmatrix} A^{(n)}_{ip} & A^{(n+1)}_{i,n+1} \\ A^{(n)}_{jp} & A^{(n+1)}_{j,n+1} \end{vmatrix} - A_n A^{(n+1)}_{ij;p,n+1} = 0, \tag{A}$$

$$\begin{vmatrix} A^{(n)}_{ip} & A^{(n)}_{iq} \\ A^{(n+1)}_{n+1,p} & A^{(n+1)}_{n+1,q} \end{vmatrix} - A_n A^{(n+1)}_{i,n+1;pq} = 0, \tag{B}$$

$$\begin{vmatrix} A^{(n)}_{rr} & A^{(n+1)}_{rr} \\ A^{(n)}_{nr} & A^{(n+1)}_{nr} \end{vmatrix} - A^{(n+1)}_{n+1,r} A^{(n+1)}_{rn;r,n+1} = 0. \tag{C}$$

These three identities are consequences of the Jacobi identity but are distinct from it since the elements in each of the second-order determinants are cofactors of two different orders, namely $n-1$ and n.

PROOF. Denote the left side of variant (A) by E. Then, applying the Jacobi identity,

$$\begin{aligned}
A_{n+1}E &= A_{n+1}\begin{vmatrix} A_{ip}^{(n)} & A_{i,n+1}^{(n+1)} \\ A_{jp}^{(n)} & A_{j,n+1}^{(n+1)} \end{vmatrix} - A_n \begin{vmatrix} A_{ip}^{(n+1)} & A_{i,n+1}^{(n+1)} \\ A_{jp}^{(n+1)} & A_{j,n+1}^{(n+1)} \end{vmatrix} \\
&= A_{i,n+1}^{(n+1)}F_j - A_{j,n+1}^{(n+1)}F_i, \qquad (3.6.13)
\end{aligned}$$

where

$$\begin{aligned}
F_i &= A_n A_{ip}^{(n+1)} - A_{n+1}A_{ip}^{(n)} \\
&= \left[\begin{vmatrix} A_{ip}^{(n+1)} & A_{i,n+1}^{(n+1)} \\ A_{n+1,p}^{(n+1)} & A_{n+1,n+1}^{(n+1)} \end{vmatrix} - A_{n+1}A_{ip}^{(n)}\right] + A_{i,n+1}^{(n+1)}A_{n+1,p}^{(n+1)} \\
&= A_{i,n+1}^{(n+1)}A_{n+1,p}^{(n+1)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
A_{n+1}E &= \left[A_{i,n+1}^{(n+1)}A_{j,n+1}^{(n+1)} - A_{j,n+1}^{(n+1)}A_{i,n+1}^{(n+1)}\right]A_{n+1,p}^{(n+1)} \\
&= 0. \qquad (3.6.14)
\end{aligned}$$

The result follows and variant (B) is proved in a similar manner. Variant (A) appears in Section 4.8.5 on Turanians and is applied in Section 6.5.1 on Toda equations.

The proof of (C) applies a particular case of (A) and the Jacobi identity. In (A), put $(i, j, p) = (r, n, r)$:

$$\begin{vmatrix} A_{rr}^{(n)} & A_{r,n+1}^{(n+1)} \\ A_{nr}^{(n)} & A_{n,n+1}^{(n+1)} \end{vmatrix} - A_n A_{rn;r,n+1}^{(n+1)} = 0. \qquad (A_1)$$

Denote the left side of (C) by P

$$\begin{aligned}
A_nP &= A_n \begin{vmatrix} A_{rr}^{(n)} & A_{rr}^{(n+1)} \\ A_{nr}^{(n)} & A_{nr}^{(n+1)} \end{vmatrix} - A_{n+1,r}^{(n+1)} \begin{vmatrix} A_{rr}^{(n)} & A_{r,n+1}^{(n+1)} \\ A_{nr}^{(n)} & A_{n,n+1}^{(n+1)} \end{vmatrix} \\
&= \begin{vmatrix} A_{rr}^{(n)} & A_{rr}^{(n+1)} & A_{r,n+1}^{(n+1)} \\ A_{nr}^{(n)} & A_{nr}^{(n+1)} & A_{n,n+1}^{(n+1)} \\ \bullet & A_{n+1,r}^{(n+1)} & A_{n+1,n+1}^{(n+1)} \end{vmatrix} \\
&= A_{rr}^{(n)}G_n - A_{nr}^{(n)}G_r, \qquad (3.6.15)
\end{aligned}$$

where

$$\begin{aligned}
G_i &= \begin{vmatrix} A_{ir}^{(n+1)} & A_{i,n+1}^{(n+1)} \\ A_{n+1,r}^{(n+1)} & A_{n+1,n+1}^{(n+1)} \end{vmatrix} \\
&= A_{n+1}A_{i,n+1;r,n+1}^{(n+1)}. \qquad (3.6.16)
\end{aligned}$$

Hence,

$$A_nP = A_{n+1}\left[A_{rr}^{(n)}A_{n,n+1;r,n+1}^{(n+1)} - A_{nr}^{(n)}A_{r,n+1;r,n+1}^{(n+1)}\right].$$

But $A^{(n+1)}_{i,n+1;j,n+1} = A^{(n)}_{ij}$. Hence, $A_n P = 0$. The result follows. □

Three particular cases of (B) are required for the proof of the next theorem.

Put $(i, p, q) = (r, r, n)$, $(n-1, r, n)$, (n, r, n) in turn:

$$\begin{vmatrix} A^{(n)}_{rr} & A^{(n)}_{rn} \\ A^{(n+1)}_{n+1,r} & A^{(n+1)}_{n+1,n} \end{vmatrix} - A_n A^{(n+1)}_{r,n+1;rn} = 0, \tag{B$_1$}$$

$$\begin{vmatrix} A^{(n)}_{n-1,r} & A^{(n)}_{n-1,n} \\ A^{(n+1)}_{n+1,r} & A^{(n+1)}_{n+1,n} \end{vmatrix} - A_n A^{(n+1)}_{n-1,n+1;rn} = 0, \tag{B$_2$}$$

$$\begin{vmatrix} A^{(n)}_{nr} & A^{(n)}_{nn} \\ A^{(n+1)}_{n+1,r} & A^{(n+1)}_{n+1,n} \end{vmatrix} - A_n A^{(n+1)}_{n,n+1;rn} = 0. \tag{B$_3$}$$

Theorem 3.7.

$$\begin{vmatrix} A^{(n+1)}_{r,n+1;rn} & A^{(n)}_{rr} & A^{(n)}_{rn} \\ A^{(n+1)}_{n-1,n+1;rn} & A^{(n)}_{n-1,r} & A^{(n)}_{n-1,n} \\ A^{(n+1)}_{n,n+1;rn} & A^{(n)}_{nr} & A^{(n)}_{nn} \end{vmatrix} = 0.$$

PROOF. Denote the determinant by Q. Then,

$$\begin{aligned} Q_{11} &= \begin{vmatrix} A^{(n)}_{n-1,r} & A^{(n)}_{n-1,n} \\ A^{(n)}_{nr} & A^{(n)}_{nn} \end{vmatrix} \\ &= A_n A^{(n)}_{n-1,n;rn} \\ &= A_n A^{(n-1)}_{n-1,r}, \\ Q_{21} &= -A_n A^{(n-1)}_{rr}, \\ Q_{31} &= A_n A^{(n)}_{r,n-1;rn}. \end{aligned} \tag{3.6.17}$$

Hence, expanding Q by the elements in column 1 and applying (B$_1$)–(B$_3$),

$$\begin{aligned} Q &= A_n \big[A^{(n+1)}_{r,n+1;rn} A^{(n-1)}_{n-1,r} - A^{(n+1)}_{n-1,n+1;rn} A^{(n-1)}_{rr} \\ &\quad + A^{(n+1)}_{n,n+1;rn} A^{(n)}_{r,n-1;rn} \big] \end{aligned} \tag{3.6.18}$$

$$\begin{aligned} &= A^{(n-1)}_{n-1,r} \begin{vmatrix} A^{(n)}_{rr} & A^{(n)}_{rn} \\ A^{(n+1)}_{n+1,r} & A^{(n+1)}_{n+1,n} \end{vmatrix} - A^{(n-1)}_{rr} \begin{vmatrix} A^{(n)}_{n-1,r} & A^{(n)}_{n-1,n} \\ A^{(n+1)}_{n+1,r} & A^{(n+1)}_{n+1,n} \end{vmatrix} \\ &\quad + A^{(n)}_{r,n-1;rn} \begin{vmatrix} A^{(n)}_{nr} & A^{(n)}_{nn} \\ A^{(n+1)}_{n+1,r} & A^{(n+1)}_{n+1,n} \end{vmatrix} \\ &= A^{(n+1)}_{n+1,n} \left[A^{(n)}_{nr} A^{(n)}_{r,n-1;rn} - \begin{vmatrix} A^{(n-1)}_{rr} & A^{(n)}_{rr} \\ A^{(n-1)}_{n-1,r} & A^{(n)}_{n-1,r} \end{vmatrix} \right] \\ &\quad - A^{(n+1)}_{n+1,r} \left[A_{n-1} A^{(n)}_{r,n-1;rn} - \begin{vmatrix} A^{(n-1)}_{rr} & A^{(n)}_{rn} \\ A^{(n-1)}_{n-1,r} & A^{(n)}_{n-1,n} \end{vmatrix} \right]. \end{aligned} \tag{3.6.19}$$

The proof is completed by applying (C) and (A_1) with $n \to n-1$. Theorem 3.7 is applied in Section 6.6 on the Matsukidaira–Satsuma equations. □

Theorem 3.8.

$$A_{n+1,r}^{(n+1)} H_n = A_{n+1,n}^{(n+1)} H_r,$$

where

$$H_j = \begin{vmatrix} A_{rr}^{(n-1)} & A_{rj}^{(n+1)} \\ A_{n-1,r}^{(n-1)} & A_{n-1,j}^{(n+1)} \end{vmatrix} - A_{nj}^{(n+1)} A_{r,n-1;rn}^{(n)}.$$

PROOF. Return to (3.6.18), multiply by A_{n+1}/A_n and apply the Jacobi identity:

$$A_{n-1,r}^{(n-1)} \begin{vmatrix} A_{rr}^{(n+1)} & A_{rn}^{(n+1)} \\ A_{n+1,r}^{(n+1)} & A_{n+1,n}^{(n+1)} \end{vmatrix} - A_{rr}^{(n-1)} \begin{vmatrix} A_{n-1,r}^{(n+1)} & A_{n-1,n}^{(n+1)} \\ A_{n+1,r}^{(n+1)} & A_{n+1,n}^{(n+1)} \end{vmatrix}$$

$$+ \ A_{r,n-1;rn}^{(n)} \begin{vmatrix} A_{nr}^{(n+1)} & A_{nn}^{(n+1)} \\ A_{n+1,r}^{(n+1)} & A_{n+1,n}^{(n+1)} \end{vmatrix} = 0,$$

$$A_{n+1,r}^{(n+1)} \big[A_{rr}^{(n-1)} A_{n-1,n}^{(n+1)} - A_{n-1,r}^{(n-1)} A_{rn}^{(n+1)} - A_{nn}^{(n+1)} A_{r,n-1;rn}^{(n)}\big]$$

$$= A_{n+1,n}^{(n+1)} \big[A_{rr}^{(n-1)} A_{n-1,r}^{(n+1)} - A_{rr}^{(n+1)} A_{n-1,r}^{(n-1)} - A_{r,n-1;rn}^{(n)} A_{nr}^{(n+1)}\big],$$

$$A_{n+1,r}^{(n+1)} \left[\begin{vmatrix} A_{rr}^{(n-1)} & A_{rn}^{(n+1)} \\ A_{n-1,r}^{(n-1)} & A_{n-1,n}^{(n+1)} \end{vmatrix} - A_{nn}^{(n+1)} A_{r,n-1;rn}^{(n)}\right]$$

$$= A_{n+1,n}^{(n+1)} \left[\begin{vmatrix} A_{rr}^{(n-1)} & A_{rr}^{(n+1)} \\ A_{n-1,r}^{(n-1)} & A_{n-1,r}^{(n+1)} \end{vmatrix} - A_{nr}^{(n+1)} A_{r,n-1;rn}^{(n)}\right].$$

The theorem follows. □

Exercise. Prove that

$$\begin{vmatrix} A_{i_1j_1}^{(n)} & A_{j_1j_2}^{(n)} & \cdots & A_{i_1j_{r-1}}^{(n)} & A_{i_1,n+1}^{(n+1)} \\ A_{i_2j_1}^{(n)} & A_{i_2j_2}^{(n)} & \cdots & A_{i_2j_{r-1}}^{(n)} & A_{i_2,n+1}^{(n+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{i_rj_1}^{(n)} & A_{i_rj_2}^{(n)} & \cdots & A_{i_rj_{r-1}}^{(n)} & A_{i_r,n+1}^{(n+1)} \end{vmatrix}_r = A_n^{r-1} A_{i_1i_2\ldots i_r;j_1j_2\ldots j_{r-1},n+1}^{(n+1)}.$$

When $r = 2$, this identity degenerates into Variant (A). Generalize Variant (B) in a similar manner.

3.7 Bordered Determinants

3.7.1 Basic Formulas; The Cauchy Expansion

Let

$$\begin{aligned} A_n &= |a_{ij}|_n \\ &= \big|\mathbf{C}_1 \ \mathbf{C}_2 \ \mathbf{C}_3 \cdots \mathbf{C}_n\big|_n \end{aligned}$$

and let B_n denote the determinant of order $(n+1)$ obtained by bordering A_n by the column

$$\mathbf{X} = \left[x_1\ x_2\ x_3 \cdots x_n\right]^T$$

on the right, the row

$$\mathbf{Y} = \left[y_1\ y_2\ y_3 \cdots y_n\right]$$

at the bottom and the element z in position $(n+1, n+1)$. In some detail,

$$B_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & x_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & x_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & x_n \\ y_1 & y_2 & \cdots & y_n & z \end{vmatrix}_{n+1}. \tag{3.7.1}$$

Some authors border on the left and at the top but this method displaces the element a_{ij} to the position $(i+1, j+1)$, which is undesirable for both practical and aesthetic reasons except in a few special cases.

In the theorems which follow, the notation is simplified by discarding the suffix n.

Theorem 3.9.

$$B = zA - \sum_{r=1}^{n} \sum_{s=1}^{n} A_{rs} x_r y_s.$$

PROOF. The coefficient of y_s in B is $(-1)^{n+s+1}F$, where

$$\begin{aligned} F &= \left|\mathbf{C}_1\ \ldots\ \mathbf{C}_{s-1}\ \mathbf{C}_{s+1} \ldots \mathbf{C}_n\ \mathbf{X}\right|_n \\ &= (-1)^{n+s} G, \end{aligned}$$

where

$$G = \left|\mathbf{C}_1 \ldots \mathbf{C}_{s-1}\ \mathbf{X}\ \mathbf{C}_{s+1} \ldots \mathbf{C}_n\right|_n.$$

The coefficient of x_r in G is A_{rs}. Hence, the coefficient of $x_r y_s$ in B is

$$(-1)^{n+s+1+n+s} A_{rs} = -A_{rs}.$$

The only term independent of the x's and y's is zA. The theorem follows. □

Let E_{ij} denote the determinant obtained from A by

a. replacing a_{ij} by z, i, j fixed,
b. replacing a_{rj} by x_r, $1 \le r \le n$, $r \ne i$,
c. replacing a_{is} by y_s, $1 \le s \le n$, $s \ne j$.

Theorem 3.10.

$$B_{ij} = zA_{ij} - \sum_{r=1}^{n} \sum_{s=1}^{n} A_{ir,js} x_r y_s = E_{ij}.$$

PROOF.

$$B_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} & x_1 \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} & x_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} & x_{i-1} \\ a_{i+1,i} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} & x_{i+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} & x_n \\ y_1 & y_2 & \cdots & y_{j-1} & y_{j+1} & \cdots & y_n & z \end{vmatrix}_n .$$

The expansion is obtained by applying arguments to B_{ij} similar to those applied to B in Theorem 3.9. Since the second cofactor is zero when $r = i$ or $s = j$ the double sum contains $(n-1)^2$ nonzero terms, as expected. It remains to prove that $B_{ij} = E_{ij}$.

Transfer the last row of B_{ij} to the ith position, which introduces the sign $(-1)^{n-i}$ and transfer the last column to the jth position, which introduces the sign $(-1)^{n-j}$. The result is E_{ij}, which completes the proof. □

The Cauchy expansion of an arbitrary determinant focuses attention on one arbitrarily chosen element a_{ij} and its cofactor.

Theorem 3.11. *The Cauchy expansion*

$$A = a_{ij}A_{ij} + \sum_{r=1}^{n} \sum_{s=1}^{n} a_{is}a_{rj}A_{ir,sj}.$$

First Proof. The expansion is essentially the same as that given in Theorem 3.10. Transform E_{ij} back to A by replacing z by a_{ij}, x_r by a_{rj} and y_s by a_{is}. The theorem appears after applying the relation

$$A_{ir,js} = -A_{ir,sj}. \tag{3.7.2}$$

Second Proof. It follows from (3.2.3) that

$$\sum_{r=1}^{n} a_{rj}A_{ir,sj} = (1-\delta_{js})A_{is}.$$

Multiply by a_{is} and sum over s:

$$\begin{aligned} \sum_{r=1}^{n} \sum_{s=1}^{n} a_{is}a_{rj}A_{ir,sj} &= \sum_{s=1}^{n} a_{is}A_{is} - \sum_{s=1}^{n} \delta_{js}a_{is}A_{is} \\ &= A - a_{ij}A_{ij}, \end{aligned}$$

which is equivalent to the stated result. □

Theorem 3.12. *If $y_s = 1$, $1 \le s \le n$, and $z = 0$, then*

$$\sum_{j=1}^{n} B_{ij} = 0, \quad 1 \le i \le n.$$

PROOF. It follows from (3.7.2) that

$$\sum_{j=1}^{n}\sum_{s=1}^{n} A_{ir,js} = 0, \quad 1 \le i, r \le n.$$

Expanding B_{ij} by elements from the last column,

$$B_{ij} = -\sum_{r=1}^{n} x_r \sum_{s=1}^{n} A_{ir,js}.$$

Hence

$$\begin{aligned}\sum_{j=1}^{n} B_{ij} &= -\sum_{r=1}^{n} x_r \sum_{j=1}^{n}\sum_{s=1}^{n} A_{ir,js}\\ &= 0.\end{aligned}$$

Bordered determinants appear in other sections including Section 4.10.3 on the Yamazaki–Hori determinant and Section 6.9 on the Benjamin–Ono equation. □

3.7.2 A Determinant with Double Borders

Theorem 3.13.

$$\begin{vmatrix} & & & & u_1 & v_1 \\ & & & & u_2 & v_2 \\ & [a_{ij}]_n & & & \cdots & \cdots \\ & & & & u_n & v_n \\ x_1 & x_2 & \cdots & x_n & \bullet & \bullet \\ y_1 & y_2 & \cdots & y_n & \bullet & \bullet \end{vmatrix}_{n+2} = \sum_{p,q,r,s=1}^{n} u_p v_q x_r y_s A_{pq,rs},$$

where

$$A = |a_{ij}|_n.$$

PROOF. Denote the determinant by B and apply the Jacobi identity to cofactors obtained by deleting one of the last two rows and one of the last two columns

$$\begin{vmatrix} B_{n+1,n+1} & B_{n+1,n+2} \\ B_{n+2,n+1} & B_{n+2,n+2} \end{vmatrix} = \begin{matrix} BB_{n+1,n+2;n+1,n+2} \\ BA. \end{matrix} \tag{3.7.3}$$

Each of the first cofactors is a determinant with single borders

$$B_{n+1,n+1} = \begin{vmatrix} & & & & v_1 \\ & & & & v_2 \\ & [a_{ij}]_n & & & \cdots \\ & & & & v_n \\ y_1 & y_2 & \cdots & y_n & \bullet \end{vmatrix}_{n+1}$$

$$= -\sum_{q=1}^{n}\sum_{s=1}^{n} v_q y_s A_{qs}.$$

Similarly,

$$B_{n+1,n+2} = +\sum_{p=1}^{n}\sum_{s=1}^{n} u_p y_s A_{ps},$$

$$B_{n+2,n+1} = +\sum_{q=1}^{n}\sum_{r=1}^{n} v_q x_r A_{qr},$$

$$B_{n+2,n+2} = -\sum_{p=1}^{n}\sum_{r=1}^{n} u_p x_r A_{pr}.$$

Note the variations in the choice of dummy variables. Hence, (3.7.3) becomes

$$BA = \sum_{p,q,r,s=1}^{n} u_p v_q x_r y_s \begin{vmatrix} A_{pr} & A_{ps} \\ A_{qr} & A_{qs} \end{vmatrix}.$$

The theorem appears after applying the Jacobi identity and dividing by A. □

Exercises

1. Prove the Cauchy expansion formula for A_{ij}, namely

$$A_{ij} = a_{pq}A_{ip,jq} - \sum_{r=1}^{n}\sum_{s=1}^{n} a_{ps}a_{rq}A_{ipr,jqs},$$

where $(p,q) \neq (i,j)$ but are otherwise arbitrary. Those terms in which $r = i$ or p or those in which $s = j$ or q are zero by the definition of higher cofactors.

2. Prove the generalized Cauchy expansion formula, namely

$$A = N_{ij,hk}A_{ij,hk} + \sum_{1\le p\le q\le n}\;\sum_{1\le r\le s\le n} N_{ij,rs}N_{pq,hk}A_{ijpq,rshk},$$

where $N_{ij,hk}$ is a retainer minor and $A_{ij,hk}$ is its complementary cofactor.

4

Particular Determinants

4.1 Alternants

4.1.1 Introduction

Any function of n variables which changes sign when any two of the variables are interchanged is known as an alternating function. It follows that an alternating function vanishes if any two of the variables are equal. Any determinant function which possess these properties is known as an alternant.

The simplest form of alternant is

$$|f_j(x_i)|_n = \begin{vmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \dots & \dots & \dots & \dots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{vmatrix}_n . \tag{4.1.1}$$

The interchange of any two x's is equivalent to the interchange of two rows which gives rise to a change of sign. If any two of the x's are equal, the determinant has two identical rows and therefore vanishes.

The double or two-way alternant is

$$|f(x_i, y_j)|_n = \begin{vmatrix} f(x_1, y_1) & f(x_1, y_2) & \cdots & f(x_1, y_n) \\ f(x_2, y_1) & f(x_2, y_2) & \cdots & f(x_2, y_n) \\ \dots & \dots & \dots & \dots \\ f(x_n, y_1) & f(x_n, y_2) & \cdots & f(x_n, y_n) \end{vmatrix}_n . \tag{4.1.2}$$

If the x's are not distinct, the determinant has two or more identical rows. If the y's are not distinct, the determinant has two or more identical columns. In both cases, the determinant vanishes.

Illustration. The Wronskian $|D_x^{j-1}(f_i)|_n$ is an alternant. The double Wronskian $|D_x^{j-1}D_y^{i-1}(f)|_n$ is a double alternant, $D_x = \partial/\partial x$, etc.

Exercise. Define two third-order alternants ϕ and ψ in column vector notation as follows:

$$\begin{aligned}\phi &= |\mathbf{c}(x_1)\ \mathbf{c}(x_2)\ \mathbf{c}(x_3)|,\\ \psi &= |\mathbf{C}(x_1)\ \mathbf{C}(x_2)\ \mathbf{C}(x_3)|.\end{aligned}$$

Apply l'Hopital's formula to prove that

$$\lim\left(\frac{\phi}{\psi}\right) = \frac{|\mathbf{c}(x)\ \mathbf{c}'(x)\ \mathbf{c}''(x)|}{|\mathbf{C}(x)\ \mathbf{C}'(x)\ \mathbf{C}''(x)|},$$

where the limit is carried out as $x_i \to x$, $1 \le i \le 3$, provided the numerator and denominator are not both zero.

4.1.2 Vandermondians

The determinant

$$\begin{aligned}X_n &= |x_i^{j-1}|_n\\ &= \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1}\\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1}\end{vmatrix}_n\\ &= V(x_1, x_2, \ldots, x_n) \qquad (4.1.3)\end{aligned}$$

is known as the alternant of Vandermonde or simply a Vandermondian.

Theorem.

$$X_n = \prod_{1\le r<s\le n}(x_s - x_r).$$

The expression on the right is known as a difference–product and contains $(n/2) = \frac{1}{2}n(n-1)$ *factors.*

First Proof. The expansion of the determinant consists of the sum of $n!$ terms, each of which is the product of n elements, one from each row and one from each column. Hence, X_n is a polynomial in the x_r of degree

$$0 + 1 + 2 + 3 + \cdots + (n-1) = \tfrac{1}{2}n(n-1).$$

One of the terms in this polynomial is the product of the elements in the leading diagonal, namely

$$+\ x_2 x_3^2 x_4^3 \cdots x_n^{n-1}. \qquad (4.1.4)$$

When any two of the x_r are equal, X_n has two identical rows and therefore vanishes. Hence, very possible difference of the form $(x_s - x_r)$ is a factor of X_n, that is,

$$\begin{aligned} X_n = \quad & K(x_2 - x_1)(x_3 - x_1)(x_4 - x_1)\cdots(x_n - x_1) \\ & \qquad (x_3 - x_2)(x_4 - x_2)\cdots(x_n - x_2) \\ & \qquad\qquad (x_4 - x_3)\cdots(x_n - x_3) \\ & \qquad\qquad\qquad \cdots\cdots \\ & \qquad\qquad\qquad (x_n - x_{n-1}) \\ = K & \prod_{1 \le r < s \le n} (x_s - x_r), \end{aligned}$$

which is the product of K and $\frac{1}{2}n(n-1)$ factors. One of the terms in the expansion of this polynomial is the product of K and the first term in each factor, namely

$$K x_2 x_3^2 x_4^3 \cdots x_n^{n-1}.$$

Comparing this term with (4.1.4), it is seen that $K = 1$ and the theorem is proved.

Second Proof. Perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j - x_n \mathbf{C}_{j-1}$$

in the order $j = n, n-1, n-2, \ldots, 3, 2$. The result is a determinant in which the only nonzero element in the last row is a 1 in position $(n, 1)$. Hence,

$$X_n = (-1)^{n-1} V_{n-1},$$

where V_{n-1} is a determinant of order $(n-1)$. The elements in row s of V_{n-1} have a common factor $(x_s - x_n)$. When all such factors are removed from V_{n-1}, the result is

$$X_n = X_{n-1} \prod_{r=1}^{n-1} (x_n - x_r),$$

which is a reduction formula for X_n. The proof is completed by reducing the value of n by 1 repeatedly and noting that $X_2 = x_2 - x_1$. □

Exercises

1. Let

$$A_n = \left| \binom{j-1}{i-1} (-x_i)^{j-i} \right|_n = 1.$$

Postmultiply the Vandermondian $V_n(\mathbf{x})$ or $V_n(x_1, x_2, \ldots, x_n)$ by A_n, prove the reduction formula

$$V_n(x_1, x_2, \ldots, x_n) = V_{n-1}(x_2 - x_1, x_3 - x_1, \ldots, x_n - x_1) \prod_{p=2}^{n} (x_p - x_1),$$

and hence evaluate $V_n(\mathbf{x})$.

2. Prove that

$$|x_i^{j-1} y_i^{n-j}|_n = \prod_{1 \le r < s \le n} \begin{vmatrix} y_r & x_r \\ y_s & x_s \end{vmatrix}.$$

3. If

$$x_i = \frac{z + c_i}{\rho},$$

prove that

$$|x_i^{j-1}|_n = \rho^{-n(n-1)/2} |c_i^{j-1}|_n,$$

which is independent of z. This relation is applied in Section 6.10.3 on the Einstein and Ernst equations.

4.1.3 Cofactors of the Vandermondian

Theorem 4.1. *The scaled cofactors of the Vandermonian $X_n = |x_{ij}|_n$, where $x_{ij} = x_i^{j-1}$ are given by the quotient formula*

$$X_n^{ij} = \frac{(-1)^{n-j} \sigma_{i,n-j}^{(n)}}{g_{ni}(x_i)},$$

where

$$g_{nr}(x) = \sum_{s=0}^{n-1} (-1)^s \sigma_{rs}^{(n)} x^{n-1-s}.$$

Notes on the symmetric polynomials $\sigma_{rs}^{(n)}$ and the function $g_{nr}(x)$ are given in Appendix A.7.

PROOF. Denote the quotient by F_{ij}. Then,

$$\begin{aligned} \sum_{k=1}^{n} x_{ik} F_{jk} &= \frac{1}{g_{nj}(x_j)} \sum_{k=1}^{n} (-1)^{n-k} \sigma_{j,n-k}^{(n)} x_i^{k-1} \qquad \text{(Put } k = n - s\text{)} \\ &= \frac{1}{g_{nj}(x_j)} \sum_{s=0}^{n-1} (-1)^s \sigma_{js}^{(n)} x_i^{n-s-1} \\ &= \frac{g_{nj}(x_i)}{g_{nj}(x_j)} \\ &= \delta_{ij}. \end{aligned}$$

Hence,

$$\begin{aligned}[x_{ij}]_n[F_{ji}]_n &= \mathbf{I}, \\ [F_{ji}]_n &= [x_{ij}]^{-1} \\ &= [X_n^{ji}]_n.\end{aligned}$$

The theorem follows. □

Theorem 4.2.

$$X_{nj}^{(n)} = (-1)^{n-j} X_{n-1} \sigma_{n-j}^{(n-1)}.$$

Proof. Referring to equations (A.7.1) and (A.7.3) in Appendix A.7,

$$\begin{aligned}X_n &= X_{n-1} \prod_{r=1}^{n-1} (x_n - x_r) \\ &= X_{n-1} f_{n-1}(x_n) \\ &= X_{n-1} g_{nn}(x_n).\end{aligned}$$

From Theorem 4.1,

$$\begin{aligned}X_{nj}^{(n)} &= \frac{(-1)^{n-j} X_n \sigma_{n,n-j}^{(n)}}{g_{nn}(x_n)} \\ &= (-1)^{n-j} X_{n-1} \sigma_{n,n-j}^{(n)}.\end{aligned}$$

The proof is completed using equation (A.7.4) in Appendix A.7. □

4.1.4 A Hybrid Determinant

Let Y_n be a second Vandermondian defined as

$$Y_n = |y_i^{j-1}|_n$$

and let H_{rs} denote the hybrid determinant formed by replacing the rth row of X_n by the sth row of Y_n.

Theorem 4.3.

$$\frac{H_{rs}}{X_n} = \frac{g_{nr}(y_s)}{g_{nr}(x_r)}.$$

Proof.

$$\begin{aligned}\frac{H_{rs}}{X_n} &= \sum_{j=1}^{n} y_s^{j-1} X_n^{rj} \\ &= \frac{1}{g_{nr}(x_r)} \sum_{j=1}^{n} (-1)^{n-j} \sigma_{r,n-j}^{(n)} y_s^{j-1} \qquad \text{(Put } j = n-k\text{)} \\ &= \frac{1}{g_{nr}(x_r)} \sum_{k=0}^{n-1} (-1)^k \sigma_{rk}^{(n)} y_s^{n-1-k}.\end{aligned}$$

This completes the proof of Theorem 4.3 which can be expressed in the form

$$\frac{H_{rs}}{X_n} = \frac{\prod_{i=1}^{n}(y_s - x_i)}{(y_s - x_r)\prod_{\substack{i=1\\i\neq r}}^{n}(x_r - x_i)}. \qquad \square$$

Let

$$\begin{aligned} A_n &= |\sigma_{i,j-1}^{(m)}|_n \\ &= \begin{vmatrix} \sigma_{10}^{(m)} & \sigma_{11}^{(m)} & \cdots & \sigma_{1,n-1}^{(m)} \\ \sigma_{20}^{(m)} & \sigma_{21}^{(m)} & \cdots & \sigma_{2,n-1}^{(m)} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n0}^{(m)} & \sigma_{n1}^{(m)} & \cdots & \sigma_{n,n-1}^{(m)} \end{vmatrix}_n, \quad m \geq n. \end{aligned}$$

Theorem 4.4.

$$A_n = (-1)^{n(n-1)/2} X_n.$$

PROOF.

$$A_n = \left|\mathbf{C}_0\ \mathbf{C}_1\ \mathbf{C}_2 \ldots \mathbf{C}_{n-1},\right|_n$$

where, from the lemma in Appendix A.7,

$$\begin{aligned} \mathbf{C}_j &= \left[\sigma_{1j}^{(m)}\ \sigma_{2j}^{(m)}\ \sigma_{3j}^{(m)}\ \ldots\ \sigma_{nj}^{(m)}\right]^T \\ &= \sum_{p=0}^{j} \sigma_p^{(m)} \left[v_1^{j-p}\ v_2^{j-p}\ v_3^{j-p}\ \ldots\ v_n^{j-p}\right]^T, \quad v_r = -x_r,\ \sigma_0^{(m)} = 1. \end{aligned}$$

Applying the column operations

$$\mathbf{C}'_j = \mathbf{C}_j - \sum_{k=1}^{j} \sigma_k^{(m)} \mathbf{C}_{j-k}$$

in the order $j = 1, 2, 3, \ldots$ so that each new column created by one operation is applied in the next operation, it is found that

$$\mathbf{C}'_j = \left[v_1^j\ v_2^j\ v_3^j\ \ldots\ v_n^j\right]^T, \quad j = 0, 1, 2, \ldots.$$

Hence

$$\begin{aligned} A_n &= |v_i^{j-1}|_n \\ &= (-1)^{n(n-1)/2} |x_i^{j-1}|_n. \end{aligned}$$

Theorem 4.4 follows. $\square$

4.1.5 The Cauchy Double Alternant

The Cauchy double alternant is the determinant

$$A_n = \left| \frac{1}{x_i - y_j} \right|_n,$$

which can be evaluated in terms of the Vandermondians X_n and Y_n as follows.

Perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j - \mathbf{C}_n, \quad 1 \le j \le n-1,$$

and then remove all common factors from the elements of rows and columns. The result is

$$A_n = \frac{\prod_{r=1}^{n-1} (y_r - y_n)}{\prod_{r=1}^{n} (x_r - y_n)} B_n, \tag{4.1.5}$$

where B_n is a determinant in which the last column is

$$\left[1\ 1\ 1 \ldots 1\right]_n^T$$

and all the other columns are identical with the corresponding columns of A_n.

Perform the row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \mathbf{R}_n, \quad 1 \le i \le n-1,$$

on B_n, which then degenerates into a determinant of order $(n-1)$. After removing all common factors from the elements of rows and columns, the result is

$$B_n = \frac{\prod_{r=1}^{n-1} (x_n - x_r)}{\prod_{r=1}^{n-1} (x_n - y_r)} A_{n-1}. \tag{4.1.6}$$

Eliminating B_n from (4.1.5) and (4.1.6) yields a reduction formula for A_n, which, when applied, gives the formula

$$A_n = \frac{(-1)^{n(n-1)/2} X_n Y_n}{\prod_{r,s=1}^{n} (x_r - y_s)}.$$

Exercises

1. Prove the reduction formula

$$A_{ij}^{(n)} = A_{ij}^{(n-1)} \prod_{\substack{r=1 \\ r\neq i}}^{n-1} \left(\frac{x_n - x_r}{x_r - y_n} \right) \prod_{\substack{s=1 \\ s\neq j}}^{n-1} \left(\frac{y_s - y_n}{x_n - y_s} \right).$$

Hence, or otherwise, prove that

$$A_n^{ij} = \frac{1}{x_i - y_j} \frac{f(y_j)g(x_i)}{f'(x_i)g'(y_j)},$$

where

$$f(t) = \prod_{r=1}^{n} (t - x_r),$$

$$g(t) = \prod_{s=1}^{n} (t - y_s).$$

2. Let

$$V_n = \begin{vmatrix} & & & & & f(x_1) \\ & & & & & f(x_2) \\ & & [a_{ij}]_n & & & \vdots \\ & & & & & \vdots \\ & & & & & f(x_n) \\ 1 & 1 & \dots & \dots & 1 & 1 \end{vmatrix}_{n+1},$$

$$W_n = \begin{vmatrix} & & & & & f(x_1) \\ & & & & & f(x_2) \\ & & [a_{ij}]_n & & & \vdots \\ & & & & & \vdots \\ & & & & & f(x_n) \\ -1 & -1 & \dots & \dots & -1 & 1 \end{vmatrix}_{n+1},$$

where

$$a_{ij} = \left(\frac{1 - x_i y_j}{x_i - y_j} \right) f(x_i),$$

$$f(x) = \prod_{i=1}^{n} (x - y_i).$$

Show that

$$V_n = (-1)^{n(n+1)/2} X_n Y_n \prod_{i=1}^{n} (x_i - 1)(y_i + 1),$$

$$W_n = (-1)^{n(n+1)/2} X_n Y_n \prod_{i=1}^{n} (x_i + 1)(y_i - 1).$$

Removing $f(x_1), f(x_2), \ldots, f(x_n)$, from the first n rows in V_n and W_n, and expanding each determinant by the last row and column, deduce that

$$\left| \frac{1 - x_i y_j}{x_i - y_j} \right|_n = \frac{1}{2} \left| \frac{1}{x_i - y_j} \right|_n \left\{ \prod_{i=1}^{n} (x_i + 1)(y_i - 1) + \prod_{i=1}^{n} (x_i - 1)(y_i + 1) \right\}.$$

4.1.6 A Determinant Related to a Vandermondian

Let $P_r(x)$ be a polynomial defined as

$$P_r(x) = \sum_{s=1}^{r} a_{sr} x^{s-1}, \quad r \geq 1.$$

Note that the coefficient is a_{sr}, not the usual a_{rs}.

Let

$$X_n = |x_j^{i-1}|_n.$$

Theorem.

$$|P_i(x_j)|_n = (a_{11}\ a_{22} \cdots a_{nn}) X_n.$$

PROOF. Define an upper triangular determinant U_n as follows:

$$\begin{aligned} U_n &= |a_{ij}|_n, \qquad a_{ij} = 0, \quad i > j, \\ &= a_{11}\ a_{22} \cdots a_{nn}. \end{aligned} \tag{4.1.7}$$

Some of the cofactors of U_i are given by

$$U_{ij}^{(i)} = \begin{cases} 0, & j > i, \\ U_{i-1}, & j = i,\ U_0 = 1. \end{cases}$$

Those cofactors for which $j < i$ are not required in the analysis which follows. Hence, $|U_{ij}^{(i)}|_n$ is also upper triangular and

$$|U_{ij}^{(i)}|_n = \begin{cases} U_{11}^{(1)} U_{22}^{(2)} \cdots U_{nn}^{(n)}, & U_{11}^{(1)} = 1, \\ U_1\ U_2 \cdots U_{n-1}. \end{cases} \tag{4.1.8}$$

Applying the formula for the product of two determinants in Section 1.4,

$$|U_{ij}^{(j)}|_n |P_i(x_j)|_n = |q_{ij}|_n, \tag{4.1.9}$$

where

$$\begin{aligned} q_{ij} &= \sum_{r=1}^{i} U_{ir}^{(i)} P_r(x_j) \\ &= \sum_{r=1}^{i} U_{ir}^{(i)} \sum_{s=1}^{r \to i} a_{sr} x_j^{s-1} \quad (a_{sr} = 0,\ s > r) \\ &= \sum_{s=1}^{i} x_j^{s-1} \sum_{r=1}^{i} a_{sr} U_{ir}^{(i)} \\ &= U_i \sum_{s=1}^{i} x_j^{s-1} \delta_{si} \\ &= U_i x_j^{i-1}. \end{aligned}$$

Hence, referring to (4.1.8),

$$\begin{aligned} |q_{ij}|_n &= \bigl(U_1\ U_2 \cdots U_n\bigr) |x_j^{i-1}| \\ &= U_n |U_{ij}^{(i)}|_n X_n. \end{aligned}$$

The theorem follows from (4.1.7) and (4.1.9). □

4.1.7 A Generalized Vandermondian

Lemma.

$$\left| \sum_{k=1}^{N} y_k x_k^{i+j-2} \right|_n = \sum_{k_1 \ldots k_n = 1}^{N} \left(\prod_{r=1}^{n} y_{k_r} \right) \left(\prod_{s=2}^{n} x_{k_s}^{s-1} \right) \left| x_{k_j}^{i-1} \right|_n.$$

Proof. Denote the determinant on the left by A_n and put

$$a_{ij}^{(k)} = y_k x_k^{i+j-2}$$

in the last identity in Property (g) in Section 2.3.1. Then,

$$A_n = \sum_{k_1 \ldots k_n = 1}^{N} \left| y_{k_j} x_{k_j}^{i+j-2} \right|_n.$$

Now remove the factor $y_{k_j} x_{k_j}^{j-1}$ from column j of the determinant, $1 \le j \le n$. The lemma then appears and is applied in Section 6.10.4 on the Einstein and Ernst equations. □

4.1.8 Simple Vandermondian Identities

Lemmas.

a. $V_n = V_{n-1} \prod_{r=1}^{n-1} (x_n - x_r), \quad n > 1, \quad V(x_1) = 1$

b. $V_n = V(x_2, x_3, \ldots, x_n) \prod_{r=2}^{n} (x_r - x_1)$.

c. $V(x_t, x_{t+1}, \ldots, x_n) = V(x_{t+1}, x_{t+2}, \ldots, x_n) \prod_{r=t+1}^{n} (x_r - x_t)$.

d. $V_{1n}^{(n)} = (-1)^{n+1} V(x_2, x_3, \ldots, x_n) = \dfrac{(-1)^{n+1} V_n}{\prod_{r=2}^{n} (x_r - x_1)}$.

e.
$$\begin{aligned} V_{in}^{(n)} &= (-1)^{n+i} V(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ &= \frac{(-1)^{n+i} V_n}{\prod_{r=1}^{i-1} (x_i - x_r) \prod_{r=i+1}^{n} (x_r - x_i)}, \quad i > 1 \end{aligned}$$

f. If $\{j_1\ j_2 \cdots j_n\}$ is a permutation of $\{1\ 2 \ldots n\}$, then

$$V\big(x_{j_1}, x_{j_2}, \ldots, x_{j_n}\big) = \operatorname{sgn} \left\{ \begin{matrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{matrix} \right\} V(x_1, x_2, \ldots, x_n).$$

The proofs of (a) and (b) follow from the difference–product formula in Section 4.1.2 and are elementary. A proof of (c) can be constructed as follows. In (b), put $n = m-t+1$, then put $x_r = y_{r+t-1}$, $r = 1, 2, 3, \ldots$, and change the dummy variable in the product from r to s using the formula $s = r + t - 1$. The resut is (c) expressed in different symbols. When $t = 1$, (c) reverts to (b). The proofs of (d) and (e) are elementary. The proof of (f) follows from Property (c) in Section 2.3.1 and Appendix A.2 on permutations and their signs.

Let the minors of V_n be denoted by M_{ij}. Then,

$$\begin{aligned} M_i &= M_{in} = V(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \\ M_n &= M_{nn} = V_{n-1}. \end{aligned}$$

Theorems.

a. $\displaystyle\prod_{r=1}^{m} M_r = \frac{V(x_{m+1}, x_{m+2}, \ldots, x_n) V_n^{m-1}}{V(x_1, x_2, \ldots, x_m)}, \quad 1 \le m \le n-1$

b. $\displaystyle\prod_{r=1}^{n} M_r = V_n^{n-2}$

c. $\displaystyle\prod_{r=1}^{m} M_{k_r} = \frac{V(x_{k_{m+1}}, x_{k_{m+2}}, \ldots, x_{k_n}) V_n^{m-1}}{V(x_{k_1}, x_{k_2}, \ldots, x_{k_m})}$

PROOF. Use the method of induction to prove (a), which is clearly valid when $m = 1$. Assume it is valid when $m = s$. Then, from Lemma (e) and referring to Lemma (a) with $n \to s+1$ and Lemma (c) with $m \to s+1$,

$$\prod_{r=1}^{s+1} M_r = \left[\frac{V_n}{\prod_{r=1}^{s}(x_{s+1}-x_r) \prod_{r=s+2}^{n}(x_r - x_{s+1})} \right] \frac{V(x_{s+1}, x_{s+2}, \ldots, x_n) V_n^{s-1}}{V(x_1, x_2, \ldots, x_s)}$$

$$= \frac{V_n^s}{\left[V(x_1, x_2, \ldots, x_s) \prod_{r=1}^{s}(x_{s+1}-x_r)\right]} \left[\frac{V(x_{s+1}, x_{s+2}, \ldots, x_n)}{\prod_{r=s+2}^{n}(x_r - x_{s+1})} \right]$$

$$= \frac{V(x_{s+2}, x_{s+3}, \ldots, x_n) V_n^s}{V(x_1, x_2, \ldots, x_{s+1})}.$$

Hence, (a) is valid when $m = s+1$, which proves (a). To prove (b), put $m = n-1$ in (a) and use $M_n = V_{n-1}$. The details are elementary.

The proof of (c) is obtained by applying the permutation

$$\begin{Bmatrix} 1 & 2 & 3 & \cdots & n \\ k_1 & k_2 & k_3 & \cdots & k_n \end{Bmatrix}$$

to (a). The only complication which arises is the determination of the sign of the expression on the right of (c). It is left as an exercise for the reader to prove that the sign is positive. □

Exercise. Let A_6 denote the determinant of order 6 defined in column vector notation as follows:

$$\mathbf{C}_j = \left[a_j \ a_j x_j \ a_j x_j^2 \ b_j \ b_j y_j \ b_j y_j^2\right]^T, \quad 1 \le j \le 6.$$

Apply the Laplace expansion theorem to prove that

$$A_6 = \sum_{\substack{i<j<k \\ p<q<r}} \sigma a_i a_j a_k b_p b_q b_r V(x_i, x_j, x_k) V(y_p, y_q, y_r),$$

where

$$\sigma = \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ i & j & k & p & q & r \end{Bmatrix}$$

and where the lower set of parameters is a permutation of the upper set. The number of terms in the sum is $\binom{6}{3} = 20$.

Prove also that

$$A_6 = 0 \quad \text{when} \quad a_j = b_j, \quad 1 \le j \le 6.$$

Generalize this result by giving an expansion formula for A_{2n} from the first m rows and the remaining $(2n-m)$ rows using the dummy variables k_r, $1 \le r \le 2n$. The generalized formula and Theorem (c) are applied in Section 6.10.4 on the Einstein and Ernst equations.

4.1.9 Further Vandermondian Identities

The notation

$$\begin{aligned} N_m &= \{1\ 2 \cdots m\}, \\ J_m &= \{j_1\ j_2 \cdots j_m\}, \\ K_m &= \{k_1\ k_2 \cdots k_m\}, \end{aligned}$$

where J_m and K_m are permutations of N_m, is used to simplify the following lemmas.

Lemma 4.5.

$$V(x_1, x_2, \ldots, x_m) = \sum_{J_m}^{N_m} \operatorname{sgn} \left\{ \begin{matrix} N_m \\ J_m \end{matrix} \right\} \sum_{r=1}^{m} x_{j_r}^{r-1}.$$

PROOF. The proof follows from the definition of a determinant in Section 1.2 with $a_{ij} \to x_i^{j-1}$. □

Lemma 4.6.

$$V\big(x_{j_1}, x_{j_2}, \ldots, x_{j_m}\big) = \operatorname{sgn} \left\{ \begin{matrix} N_m \\ J_m \end{matrix} \right\} V(x_1, x_2, \ldots, x_m).$$

This is Lemma (f) in Section 4.1.8 expressed in the present notation with $n \to m$.

Lemma 4.7.

$$\sum_{J_m}^{K_m} F\big(x_{j_1}, x_{j_2}, \ldots, x_{j_m}\big) = \left\{ \begin{matrix} N_m \\ J_m \end{matrix} \right\} \sum_{J_m}^{N_m} F\big(x_{j_1}, x_{j_2}, \ldots, x_{j_m}\big).$$

In this lemma, the permutation symbol is used as a substitution operator. The number of terms on each side is m^2.

Illustration. Put $m = 2$, $F(x_{j_1}, x_{j_2}) = x_{j_1} + x_{j_2}^2$ and denote the left and right sides of the lemma by P and Q respectively. Then,

$$\begin{aligned} P &= x_{k_1} + x_{k_1}^2 + x_{k_2} + x_{k_2}^2 \\ Q &= \left\{ \begin{matrix} 1 & 2 \\ k_1 & k_2 \end{matrix} \right\} (x_1 + x_1^2 + x_2 + x_2^2) \\ &= P. \end{aligned}$$

Theorem.

a. $$\sum_{J_m}^{N_m} \left(\prod_{r=1}^{m} x_{j_r}^{r-1} \right) V\big(x_{j_1}, x_{j_2}, \ldots, x_{j_m}\big) = [V(x_1, x_2, \ldots, x_m)]^2,$$

b. $$\sum_{J_m}^{K_m} \left(\prod_{r=1}^{m} x_{j_r}^{r-1} \right) V\big(x_{j_1}, x_{j_2}, \ldots, x_{j_m}\big) = \big[V\big(x_{k_1}, x_{k_2}, \ldots, x_{k_m}\big)\big]^2.$$

Proof. Denote the left side of (a) by S_m. Then, applying Lemma 4.6,

$$S_m = \sum_{J_m}^{N_m} \left(\prod_{r=1}^{m} x_{j_r}^{r-1} \right) \operatorname{sgn} \left\{ \begin{matrix} N_m \\ J_m \end{matrix} \right\} V(x_1, x_2, \ldots, x_m)$$
$$= V(x_1, x_2, \ldots, x_m) \sum_{J_m}^{N_m} \operatorname{sgn} \left\{ \begin{matrix} N_m \\ J_m \end{matrix} \right\} \prod_{r=1}^{m} x_{j_r}^{r-1}.$$

The proof of (a) follows from Lemma 4.5. The proof of (b) follows by applying the substitution operation $\left\{ \begin{matrix} N_m \\ J_m \end{matrix} \right\}$ to both sides of (a). □

This theorem is applied in Section 6.10.4 on the Einstein and Ernst equations.

4.2 Symmetric Determinants

If $A = |a_{ij}|_n$, where $a_{ji} = a_{ij}$, then A is symmetric about its principal diagonal. By simple reasoning,

$$A_{ji} = A_{ij},$$
$$A_{js,ir} = A_{ir,js},$$

etc. If $a_{n+1-j,n+1-i} = a_{ij}$, then A is symmetric about its secondary diagonal. Only the first type of determinant is normally referred to as symmetric, but the second type can be transformed into the first type by rotation through 90° in either the clockwise or anticlockwise directions. This operation introduces the factor $(-1)^{n(n-1)/2}$, that is, there is a change of sign if $n = 4m + 2$ and $4m + 3$, $m = 0, 1, 2, \ldots$.

Theorem. *If A is symmetric,*

$$\sum_{\text{ep}\{p,q,r\}} A_{pq,rs} = 0,$$

where the symbol $\text{ep}\{p, q, r\}$ *denotes that the sum is carried out over all even permutations of* $\{p, q, r\}$*, including the identity permutation.*

In this simple case the even permutations are also the cyclic permutations [Appendix A.2].

Proof. Denote the sum by S. Then, applying the Jacobi identity (Section 3.6.1),

$$AS = AA_{pq,rs} + AA_{qr,ps} + AA_{rp,qs}$$
$$= \begin{vmatrix} A_{pr} & A_{ps} \\ A_{qr} & A_{qs} \end{vmatrix} + \begin{vmatrix} A_{qp} & A_{qs} \\ A_{rp} & A_{rs} \end{vmatrix} + \begin{vmatrix} A_{rq} & A_{rs} \\ A_{pq} & A_{ps} \end{vmatrix}$$

$$= \begin{vmatrix} A_{pr} & A_{ps} \\ A_{qr} & A_{qs} \end{vmatrix} + \begin{vmatrix} A_{pq} & A_{qs} \\ A_{pr} & A_{rs} \end{vmatrix} + \begin{vmatrix} A_{qr} & A_{rs} \\ A_{pq} & A_{ps} \end{vmatrix}$$
$$= 0.$$

The theorem follows immediately if $A \neq 0$. However, since the identity is purely algebraic, all the terms in the expansion of S as sums of products of elements must cancel out in pairs. The identity must therefore be valid for all values of its elements, including those values for which $A = 0$. The theorem is clearly valid if the sum is carried out over even permutations of any three of the four parameters. □

Notes on skew-symmetric, circulant, centrosymmetric, skew-centrosymmetric, persymmetric (Hankel) determinants, and symmetric Toeplitz determinants are given under separate headings.

4.3 Skew-Symmetric Determinants

4.3.1 Introduction

The determinant $A_n = |a_{ij}|_n$ in which $a_{ji} = -a_{ij}$, which implies $a_{ii} = 0$, is said to be skew-symmetric. In detail,

$$A_n = \begin{vmatrix} \bullet & a_{12} & a_{13} & a_{14} & \cdots \\ -a_{12} & \bullet & a_{23} & a_{24} & \cdots \\ -a_{13} & -a_{23} & \bullet & a_{34} & \cdots \\ -a_{14} & -a_{24} & -a_{34} & \bullet & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n . \tag{4.3.1}$$

Theorem 4.8. *The square of an arbitrary determinant of order n can be expressed as a symmetric determinant of order n if n is odd or a skew-symmetric determinant of order n if n is even.*

PROOF. Let

$$A = |a_{ij}|_n.$$

Reversing the order of the rows,

$$A = (-1)^N |a_{n+1-i,j}|_n, \qquad N = \left[\frac{n}{2}\right]. \tag{4.3.2}$$

Transposing the elements of the original determinant across the secondary diagonal and changing the signs of the elements in the new rows $2, 4, 6, \ldots$,

$$A = (-1)^N |(-1)^{i+1} a_{n+1-j,n+1-i}|_n. \tag{4.3.3}$$

Hence, applying the formula for the product of two determinants in Section 1.4,

$$A^2 = |a_{n+1-i,j}|_n |(-1)^{i+1} a_{n+1-j,n+1-i}|_n$$

$$= |c_{ij}|_n,$$

where

$$\begin{aligned} c_{ij} &= \sum_{r=1}^{n} (-1)^{r+1} a_{n+1-i,r} a_{n+1-j,n+1-r} \qquad \text{(put } r = n+1-s\text{)} \\ &= (-1)^{n+1} \sum_{s=1}^{n} (-1)^{s+1} a_{n+1-j,s} a_{n+1-i,n+1-s} \\ &= (-1)^{n+1} c_{ji}. \end{aligned} \tag{4.3.4}$$

The theorem follows. □

Theorem 4.9. *A skew-symmetric determinant of odd order is identically zero.*

PROOF. Let A^*_{2n-1} denote the determinant obtained from A_{2n-1} by changing the sign of every element. Then, since the number of rows and columns is odd,

$$A^*_{2n-1} = -A_{2n-1}.$$

But,

$$A^*_{2n-1} = A^T_{2n-1} = A_{2n-1}.$$

Hence,

$$A_{2n-1} = 0,$$

which proves the theorem. □

The cofactor $A^{(2n)}_{ii}$ is also skew-symmetric of odd order. Hence,

$$A^{(2n)}_{ii} = 0. \tag{4.3.5}$$

By similar arguments,

$$\begin{aligned} A^{(2n)}_{ji} &= -A^{(2n)}_{ij}, \\ A^{(2n-1)}_{ji} &= A^{(2n-1)}_{ij}. \end{aligned} \tag{4.3.6}$$

It may be verified by elementary methods that

$$A_2 = a^2_{12}, \tag{4.3.7}$$

$$A_4 = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2. \tag{4.3.8}$$

Theorem 4.10. *A_{2n} is the square of a polynomial function of its elements.*

PROOF. Use the method of induction. Applying the Jacobi identity (Section 3.6.1) to the zero determinant A_{2n-1},

$$\begin{vmatrix} A^{(2n-1)}_{ii} & A^{(2n-1)}_{ij} \\ A^{(2n-1)}_{ji} & A^{(2n-1)}_{jj} \end{vmatrix} = 0,$$

$$\left[A_{ij}^{(2n-1)}\right]^2 = A_{ii}^{(2n-1)} A_{jj}^{(2n-1)}. \tag{4.3.9}$$

It follows from the section on bordered determinants (Section 3.7.1) that

$$\begin{vmatrix} & & x_1 \\ & A_{2n-1} & \vdots \\ & & x_{2n-1} \\ \cdots\cdots\cdots & & \\ y_1 \cdots y_{2n-1} & & \bullet \end{vmatrix}_{2n} = -\sum_{i=1}^{2n-1}\sum_{j=1}^{2n-1} A_{ij}^{(2n-1)} x_i y_j. \tag{4.3.10}$$

Put $x_i = a_{i,2n}$ and $y_j = -a_{j,2n}$. Then, the identity becomes

$$\begin{aligned} A_{2n} &= \sum_{i=1}^{2n-1}\sum_{j=1}^{2n-1} A_{ij}^{(2n-1)} a_{i,2n} a_{j,2n} \qquad (4.3.11)\\ &= \sum_{i=1}^{2n-1}\sum_{j=1}^{2n-1} \left[A_{ii}^{(2n-1)} A_{jj}^{(2n-1)}\right]^{1/2} a_{i,2n} a_{j,2n} \\ &= \left[\sum_{i=1}^{2n-1} \left[A_{ii}^{(2n-1)}\right]^{1/2} a_{i,2n}\right] \left[\sum_{j=1}^{2n-1} \left[A_{jj}^{(2n-1)}\right]^{1/2} a_{j,2n}\right] \\ &= \left[\sum_{i=1}^{2n-1} \left[A_{ii}^{(2n-1)}\right]^{1/2} a_{i,2n}\right]^2. \qquad (4.3.12) \end{aligned}$$

However, each $A_{ii}^{(2n-1)}$, $1 \le i \le (2n-1)$, is a skew-symmetric determinant of even order $(2n-2)$. Hence, if each of these determinants is the square of a polynomial function of its elements, then A_{2n} is also the square of a polynomial function of its elements. But, from (4.3.7), it is known that A_2 is the square of a polynomial function of its elements. The theorem follows by induction. □

This proves the theorem, but it is clear that the above analysis does not yield a unique formula for the polynomial since not only is each square root in the series in (4.3.12) ambiguous in sign but each square root in the series for each $A_{ii}^{(2n-1)}$, $1 \le i \le (2n-1)$, is ambiguous in sign.

A unique polynomial for $A_{2n}^{1/2}$, known as a Pfaffian, is defined in a later section. The present section ends with a few theorems and the next section is devoted to the solution of a number of preparatory lemmas.

Theorem 4.11. *If*

$$a_{ji} = -a_{ij},$$

then

a. $|a_{ij} + x|_{2n} = |a_{ij}|_{2n}$,

b. $|a_{ij} + x|_{2n-1} = x \times \begin{pmatrix} \textit{the square of a polyomial function} \\ \textit{of the elements } a_{ij} \end{pmatrix}$

Proof. Let $A_n = |a_{ij}|_n$ and let E_{n+1} and F_{n+1} denote determinants obtained by bordering A_n in different ways:

$$E_{n+1} = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ -x & \bullet & a_{12} & a_{13} & \cdots \\ -x & -a_{12} & \bullet & a_{23} & \cdots \\ -x & -a_{13} & -a_{23} & \bullet & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{n+1}$$

and F_{n+1} is obtained by replacing the first column of E_{n+1} by the column

$$\begin{bmatrix} 0 & -1 & -1 & -1 & \cdots \end{bmatrix}_{n+1}^T.$$

Both A_n and F_{n+1} are skew-symmetric. Then,

$$E_{n+1} = A_n + xF_{n+1}.$$

Return to E_{n+1} and perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j - \mathbf{C}_1, \quad 2 \le j \le n+1,$$

which reduces every element to zero except the first in the first row and increases every other element in columns 2 to $(n+1)$ by x. The result is

$$E_{n+1} = |a_{ij} + x|_n.$$

Hence, applying Theorems 4.9 and 4.10,

$$\begin{aligned} |a_{ij} + x|_{2n} &= A_{2n} + xF_{2n+1} \\ &= A_{2n}, \\ |a_{ij} + x|_{2n-1} &= A_{2n-1} + xF_{2n} \\ &= xF_{2n}. \end{aligned}$$

The theorem follows. □

Corollary. *The determinant*

$$A = |a_{ij}|_{2n}, \quad \textit{where} \quad a_{ij} + a_{ji} = 2x,$$

can be expressed as a skew-symmetric determinant of the same order.

Proof. The proof begins by expressing A in the form

$$A = \begin{vmatrix} x & a_{12} & a_{13} & a_{14} & \cdots \\ 2x - a_{12} & x & a_{23} & a_{24} & \cdots \\ 2x - a_{13} & 2x - a_{23} & x & a_{34} & \cdots \\ 2x - a_{14} & 2x - a_{24} & 2x - a_{34} & x & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{2n}$$

and is completed by subtracting x from each element. □

Let

$$A_n = |a_{ij}|_n, \quad a_{ji} = -a_{ij},$$

and let B_{n+1} denote the skew-symmetric determinant obtained by bordering A_n by the row

$$\begin{bmatrix}-1 & -1 & -1 \cdots -1 & 0\end{bmatrix}_{n+1}$$

below and by the column

$$\begin{bmatrix}1 \; 1 \; 1 \cdots 1 \; 0\end{bmatrix}_{n+1}^T$$

on the right.

Theorem 4.12 (Muir and Metzler**).** *B_{n+1} is expressible as a skew-symmetric determinant of order $(n-1)$.*

PROOF. The row and column operations

$$\mathbf{R}_i' = \mathbf{R}_i + a_{in}\mathbf{R}_{n+1}, \quad 1 \le i \le n-1,$$
$$\mathbf{C}_j' = \mathbf{C}_j + a_{jn}\mathbf{C}_{n+1}, \quad 1 \le j \le n-1,$$

when performed on B_{n+1}, result in the elements a_{ij} and a_{ji} being transformed into a_{ij}^* and a_{ji}^*, where

$$\begin{aligned} a_{ij}^* &= a_{ij} - a_{in} + a_{jn}, \quad 1 \le i \le n-1, \\ a_{ji}^* &= a_{ji} - a_{jn} + a_{in}, \quad 1 \le j \le n-1, \\ &= -a_{ij}^*. \end{aligned}$$

In particular, $a_{in}^* = 0$, so that all the elements except the last in both column n and row n are reduced to zero. Hence, when a Laplace expansion from the last two rows or columns is performed, only one term survives and the formula

$$B_{n+1} = |a_{ij}^*|_{n-1}$$

emerges, which proves the theorem. When n is even, both sides of this formula are identically zero. □

4.3.2 Preparatory Lemmas

Let

$$B_n = |b_{ij}|_n$$

where

$$b_{ij} = \begin{cases} 1, & i < j-1 \\ 0, & i = j-1 \\ -1, & i > j-1. \end{cases}$$

In detail,

$$B_n = \begin{vmatrix} -1 & \bullet & 1 & 1 & \cdots & 1 & 1 \\ -1 & -1 & \bullet & 1 & \cdots & 1 & 1 \\ -1 & -1 & -1 & \bullet & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & -1 & \cdots & -1 & \bullet \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix}_n .$$

Lemma 4.13.

$$B_n = (-1)^n.$$

PROOF. Perform the column operation

$$\mathbf{C}_2' = \mathbf{C}_2 - \mathbf{C}_1$$

and then expand the resulting determinant by elements from the new $\mathbf{C}_2$. The result is

$$B_n = -B_{n-1} = B_{n-2} = \cdots = (-1)^{n-1} B_1.$$

But $B_1 = -1$. The result follows. □

Lemma 4.14.

a. $\displaystyle\sum_{k=1}^{2n} (-1)^{j+k+1} = 0,$

b. $\displaystyle\sum_{k=1}^{i-1} (-1)^{j+k+1} = (-1)^j \delta_{i,\text{even}},$

c. $\displaystyle\sum_{k=i}^{2n} (-1)^{j+k+1} = (-1)^{j+1} \delta_{i,\text{even}},$

where the δ *functions are defined in Appendix A.1. All three identities follow from the elementary identity*

$$\sum_{k=p}^{q} (-1)^k = (-1)^p \delta_{q-p,\text{even}}.$$ □

Define the function E_{ij} *as follows:*

$$E_{ij} = \begin{cases} (-1)^{i+j+1}, & i < j \\ 0, & i = j \\ -(-1)^{i+j+1}, & i > j. \end{cases}$$

Lemma 4.15.

a. $\displaystyle\sum_{k=1}^{2n} E_{jk} = (-1)^{j+1},$

b. $\displaystyle\sum_{k=i}^{2n} E_{jk} = (-1)^{j+1}\delta_{i,\text{odd}}, \quad i \le j$

$\quad = (-1)^{j+1}\delta_{i,\text{even}}, \quad i > j$

c. $\displaystyle\sum_{k=1}^{i-1} E_{jk} = (-1)^{j+1}\delta_{i,\text{even}}, \quad i \le j$

$\quad = (-1)^{j+1}\delta_{i,\text{odd}}, \quad i > j.$

PROOF. Referring to Lemma 4.14(b,c),

$$\begin{aligned}\sum_{k=1}^{2n} E_{jk} &= \sum_{k=1}^{j-1} E_{jk} + E_{jj} + \sum_{k=j+1}^{2n} E_{jk}\\ &= -\sum_{k=1}^{j-1}(-1)^{j+k+1} + 0 + \sum_{k=j+1}^{2n}(-1)^{j+k+1}\\ &= (-1)^{j+1}(\delta_{j,\text{even}} + \delta_{j,\text{odd}})\\ &= (-1)^{j+1},\end{aligned}$$

which proves (a).

If $i \le j$,

$$\begin{aligned}\sum_{k=i}^{2n} E_{jk} &= \left[\sum_{k=1}^{2n} - \sum_{k=1}^{i-1}\right] E_{jk}\\ &= (-1)^{j+1} + \sum_{k=1}^{i-1}(-1)^{j+k+1}\\ &= (-1)^{j+1}(1 - \delta_{j,\text{even}})\\ &= (-1)^{j+1}\delta_{i,\text{odd}}.\end{aligned}$$

If $i > j$,

$$\begin{aligned}\sum_{k=i}^{2n} E_{jk} &= \sum_{k=i}^{2n}(-1)^{j+k+1}\\ &= (-1)^{j+1}\delta_{i,\text{even}},\end{aligned}$$

which proves (b).

$$\sum_{k=1}^{i-1} E_{jk} = \left[\sum_{k=1}^{2n} - \sum_{k=i}^{2n}\right] E_{jk}.$$

Part (c) now follows from (a) and (b). □

Let E_n be a skew-symmetric determinant defined as follows:

$$E_n = |\varepsilon_{ij}|_n,$$

where $\varepsilon_{ij} = 1$, $i < j$, and $\varepsilon_{ji} = -\varepsilon_{ij}$, which implies $\varepsilon_{ii} = 0$.

Lemma 4.16.

$$E_n = \delta_{n,\text{even}}.$$

PROOF. Perform the column operation

$$\mathbf{C}'_n = \mathbf{C}_n + \mathbf{C}_1,$$

expand the result by elements from the new $\mathbf{C}_n$, and apply Lemma 4.13

$$\begin{aligned} E_n &= (-1)^{n-1} B_{n-1} - E_{n-1} \\ &= 1 - E_{n-1} \\ &= 1 - (1 - E_{n-2}) \\ &= E_{n-2} = E_{n-4} = E_{n-6}, \text{ etc.} \end{aligned}$$

Hence, if n is even,

$$E_n = E_2 = 1$$

and if n is odd,

$$E_n = E_1 = 0,$$

which proves the result. □

Lemma 4.17. *The function E_{ij} defined in Lemma 4.15 is the cofactor of the element ε_{ij} in E_{2n}.*

PROOF. Let

$$\lambda_{ij} = \sum_{k=1}^{2n} \varepsilon_{ik} E_{jk}.$$

It is required to prove that $\lambda_{ij} = \delta_{ij}$.

$$\begin{aligned} \lambda_{ij} &= \sum_{k=1}^{i-1} \varepsilon_{ik} E_{jk} + 0 + \sum_{k=i+1}^{2n} \varepsilon_{ik} E_{jk} \\ &= -\sum_{k=1}^{i-1} E_{jk} + \sum_{k=i+1}^{2n} E_{jk} \\ &= \left[\sum_{k=i}^{2n} - \sum_{k=1}^{i-1} \right] E_{jk} - E_{ji}. \end{aligned}$$

If $i < j$,

$$\begin{aligned} \lambda_{ij} &= (-1)^{j+1} \left[\delta_{i,\text{odd}} - \delta_{i,\text{even}} + (-1)^i \right] \\ &= 0. \end{aligned}$$

If $i > j$,

$$\lambda_{ij} = (-1)^{j+1} \left[\delta_{i,\text{even}} - \delta_{i,\text{odd}} - (-1)^i \right]$$

$$
\begin{aligned}
&= 0 \\
\lambda_{ii} &= (-1)^{i+1}\left[\delta_{i,\text{odd}} - \delta_{i,\text{even}}\right] \\
&= 1.
\end{aligned}
$$

This completes the proofs of the preparatory lemmas. The definition of a Pfaffian follows. The above lemmas will be applied to prove the theorem which relates it to a skew-symmetric determinant. □

4.3.3 Pfaffians

The nth-order Pfaffian Pf_n is defined by the following formula, which is similar in nature to the formula which defines the determinant A_n in Section 1.2:

$$
\text{Pf}_n = \sum \text{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \cdots & (2n-1) & 2n \\ i_1 & j_1 & i_2 & j_2 & \cdots & i_n & j_n \end{Bmatrix}_{2n} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}, \tag{4.3.13}
$$

where the sum extends over all possible distinct terms subject to the restriction

$$
1 \le i_s < j_s \le n, \quad 1 \le s \le n.. \tag{4.3.14}
$$

Notes on the permutations associated with Pfaffians are given in Appendix A.2. The number of terms in the sum is

$$
\prod_{s=1}^{n} (2s-1) = \frac{(2n)!}{2^n n!}\,. \tag{4.3.15}
$$

Illustrations

$$
\begin{aligned}
\text{Pf}_1 &= \sum \text{sgn} \begin{Bmatrix} 1 & 2 \\ i & j \end{Bmatrix} a_{ij} \quad \text{(1 term)} \\
&= a_{12}, \\
A_2 &= [\text{Pf}_1]^2 \\
\text{Pf}_2 &= \sum \text{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 \\ i_1 & j_1 & i_2 & j_2 \end{Bmatrix} a_{i_1 j_1} a_{i_2 j_2} \quad \text{(3 terms)}.
\end{aligned} \tag{4.3.16}
$$

Omitting the upper parameters,

$$
\begin{aligned}
\text{Pf}_2 &= \text{sgn}\{1\ 2\ 3\ 4\} a_{12} a_{34} + \text{sgn}\{1\ 3\ 2\ 4\} a_{13} a_{24} + \text{sgn}\{1\ 4\ 2\ 3\} a_{14} a_{23} \\
&= a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23} \\
A_4 &= [\text{Pf}_2]^2.
\end{aligned} \tag{4.3.17}
$$

These results agree with (4.3.7) and (4.3.8).

The coefficient of $a_{r,2n}$, $1 \le r \le (2n-1)$, in Pf_n is found by putting $(i_s, j_s) = (r, 2n)$ for any value of s. Choose $s = 1$. Then, the coefficient is

$$\sum \sigma_r a_{i_2 j_2} a_{i_3 j_3} \cdots a_{i_n j_n},$$

where

$$\begin{aligned}
\sigma_r &= \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & (2n-1) & 2n \\ r & 2n & i_2 & j_2 & \ldots & i_n & j_n \end{Bmatrix}_{2n} \\
&= \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & (2n-1) & 2n \\ r & i_2 & j_2 & i_3 & \ldots & j_n & 2n \end{Bmatrix}_{2n} \\
&= \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & (2n-1) \\ r & i_2 & j_2 & i_3 & \ldots & j_n \end{Bmatrix}_{2n-1} \qquad (4.3.18) \\
&= (-1)^{r+1} \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & (r-1) r (r+1) & \ldots & (2n-1) \\ i_2 & j_2 & i_3 & j_3 & \ldots & r & \ldots & j_n \end{Bmatrix}_{2n-1}, \quad r > 1 \\
&= (-1)^{r+1} \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & (r-1)(r+1) & \ldots & (2n-1) \\ i_2 & j_2 & i_3 & j_3 & \ldots & \ldots & \ldots & j_n \end{Bmatrix}_{2n-2}, \quad r > 1.
\end{aligned}$$

From (4.3.18),

$$\begin{aligned}
\sigma_1 &= \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & (2n-1) \\ 1 & i_2 & j_2 & i_3 & \ldots & j_n \end{Bmatrix}_{2n-1} \\
&= \operatorname{sgn} \begin{Bmatrix} 2 & 3 & 4 & \ldots & (2n-1) \\ i_2 & j_2 & i_3 & \ldots & j_n \end{Bmatrix}_{2n-2}.
\end{aligned}$$

Hence,

$$\mathrm{Pf}_n = \sum_{r=1}^{2n-1} (-1)^{r+1} a_{r,2n} Pf_r^{(n)}, \qquad (4.3.19)$$

where

$$\begin{aligned}
\mathrm{Pf}_r^{(n)} = \sum \operatorname{sgn} &\begin{Bmatrix} 1 & 2 & 3 & 4 & \cdots & (r-1)(r+1) & \cdots & (2n-2) & (2n-1) \\ i_2 & j_2 & i_3 & j_3 & \cdots & \cdots & \cdots & i_n & j_n \end{Bmatrix}_{2n-2} \\
&\times a_{i_2 j_2} a_{i_3 j_3} \cdots a_{i_n j_n}, \qquad 1 < r \le 2n-1, \qquad (4.3.20)
\end{aligned}$$

which is a Pfaffian of order $(n-1)$ in which no element contains the row parameter r or the column parameter $2n$. In particular,

$$\begin{aligned}
\mathrm{Pf}_{2n-1}^{(n)} &= \sum \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \cdots & (2n-3) & (2n-2) \\ i_2 & j_2 & i_3 & j_3 & \cdots & i_n & j_n \end{Bmatrix}_{2n-2} \\
&= \mathrm{Pf}_{n-1}. \qquad (4.3.21)
\end{aligned}$$

Thus, a Pfaffian of order n can be expressed as a linear combination of $(2n-1)$ Pfaffians of order $(n-1)$.

In the particular case in which $a_{ij} = 1$, $i < j$, denote Pf_n by pf_n and denote $\mathrm{Pf}_r^{(n)}$ by $\mathrm{pf}_r^{(n)}$.

Lemma.

$$\mathrm{pf}_n = 1.$$

The proof is by induction. Assume $\mathrm{pf}_m = 1$, $m < n$, which implies $\mathrm{pf}_r^{(n)} = 1$. Then, from (4.3.19),

$$\mathrm{pf}_n = \sum_{r=1}^{2n-1} (-1)^{r+1} = 1.$$

Thus, if every Pfaffian of order $m < n$ is equal to 1, then every Pfaffian of order n is also equal to 1. But from (4.3.16), $\mathrm{pf}_1 = 1$, hence $\mathrm{pf}_2 = 1$, which is confirmed by (4.3.17), $\mathrm{pf}_3 = 1$, and so on.

The following important theorem relates Pfaffians to skew-symmetric determinants.

Theorem.

$$A_{2n} = [\mathrm{Pf}_n]^2.$$

The proof is again by induction. Assume

$$A_{2m} = [\mathrm{Pf}_m]^2, \quad m < n,$$

which implies

$$A_{ii}^{(2n-1)} = \left[\mathrm{Pf}_i^{(n)}\right]^2.$$

Hence, referring to (4.3.9),

$$\begin{aligned} \left[A_{ij}^{(2n-1)}\right]^2 &= A_{ii}^{(2n-1)} A_{jj}^{(2n-1)} \\ &= \left[\mathrm{Pf}_i^{(n)} \mathrm{Pf}_j^{(n)}\right]^2 \end{aligned}$$

$$\frac{A_{ij}^{(2n-1)}}{\mathrm{Pf}_i^{(n)} \mathrm{Pf}_j^{(n)}} = \pm 1 \tag{4.3.22}$$

for all elements a_{ij} for which $a_{ji} = -a_{ij}$. Let $a_{ij} = 1$, $i < j$. Then

$$\begin{aligned} A_{ij}^{(2n-1)} &\to E_{ij}^{(2n-1)} = (-1)^{i+j}, \\ \mathrm{Pf}_i^{(n)} &\to \mathrm{pf}_i^{(n)} = 1. \end{aligned}$$

Hence,

$$\frac{A_{ij}^{(2n-1)}}{\mathrm{Pf}_i^{(n)} \mathrm{Pf}_j^{(n)}} = \frac{E_{ij}^{(2n-1)}}{\mathrm{pf}_i^{(n)} \mathrm{pf}_j^{(n)}} = (-1)^{i+j}, \tag{4.3.23}$$

which is consistent with (4.3.22). Hence,

$$A_{ij}^{(2n-1)} = (-1)^{i+j}\mathrm{Pf}_i^{(n)}\mathrm{Pf}_j^{(n)}. \tag{4.3.24}$$

Returning to (4.3.11) and referring to (4.3.19),

$$\begin{aligned}
A_{2n} &= \left[\sum_{i=1}^{2n-1}(-1)^{i+1}\mathrm{Pf}_i^{(n)}a_{i,2n}\right]\left[\sum_{j=1}^{2n-1}(-1)^{j+1}\mathrm{Pf}_j^{(n)}a_{j,2n}\right]\\
&= \left[\sum_{i=1}^{2n-1}(-1)^{i+1}\mathrm{Pf}_i^{(n)}a_{i,2n}\right]^2\\
&= [\mathrm{Pf}_n]^2,
\end{aligned}$$

which completes the proof of the theorem.

The notation for Pfaffians consists of a triangular array of the elements a_{ij} for which $i < j$:

$$\mathrm{Pf}_n = \left|\begin{matrix} a_{12} & a_{13} & a_{14} & \cdots & a_{1,2n} \\ & a_{23} & a_{24} & \cdots & a_{2,2n} \\ & & a_{34} & \cdots & a_{3,2n} \\ & & & \cdots\cdots\cdots & \\ & & & a_{2n-1,2n} & \end{matrix}\right|_{2n-1}. \tag{4.3.25}$$

Pf_n is a polynomial function of the $n(2n-1)$ elements in the array.

Illustrations

From (4.3.16), (4.3.17), and (4.3.25),

$$\begin{aligned}
\mathrm{Pf}_1 &= |a_{12}| = a_{12},\\
\mathrm{Pf}_2 &= \left|\begin{matrix} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{matrix}\right|\\
&= a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.
\end{aligned}$$

It is left as an exercise for the reader to evaluate Pf_3 directly from the definition (4.3.13) with the aid of the notes given in the section on permutations associated with Pfaffians in Appendix A.2 and to show that

$$\begin{aligned}
\mathrm{Pf}_3 &= \left|\begin{matrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{34} & a_{35} & a_{36} \\ & & & a_{45} & a_{46} \\ & & & & a_{56} \end{matrix}\right|\\
&= a_{16}\left|\begin{matrix} a_{23} & a_{24} & a_{25} \\ & a_{34} & a_{35} \\ & & a_{45} \end{matrix}\right| - a_{26}\left|\begin{matrix} a_{13} & a_{14} & a_{15} \\ & a_{34} & a_{35} \\ & & a_{45} \end{matrix}\right| + a_{36}\left|\begin{matrix} a_{12} & a_{14} & a_{15} \\ & a_{24} & a_{25} \\ & & a_{45} \end{matrix}\right|
\end{aligned}$$

$$
- \begin{array}{c|ccc|} a_{46} & a_{12} & a_{13} & a_{15} \\ & & a_{23} & a_{25} \\ & & & a_{35} \end{array} + \begin{array}{c|ccc|} a_{56} & a_{12} & a_{13} & a_{14} \\ & & a_{23} & a_{24} \\ & & & a_{34} \end{array}
$$

$$
= \sum_{r=1}^{5} (-1)^{r+1} a_{r6} \mathrm{Pf}_r^{(3)}, \tag{4.3.26}
$$

which illustrates (4.3.19). This formula can be regarded as an expansion of Pf_3 by the five elements from the fifth column and their associated second-order Pfaffians. Note that the second of these five Pfaffians, which is multiplied by a_{26}, is *not* obtained from Pf_3 by deleting a particular row and a particular column. It is obtained from Pf_3 by deleting *all* elements whose suffixes include either 2 or 6 whether they be row parameters or column parameters. The other four second-order Pfaffians are obtained in a similar manner.

It follows from the definition of Pf_n that one of the terms in its expansion is

$$
+ a_{12} a_{34} a_{56} \cdots a_{2n-1,2n} \tag{4.3.27}
$$

in which the parameters are in ascending order of magnitude. This term is known as the principal term. Hence, there is no ambiguity in signs in the relations

$$
\begin{aligned}
\mathrm{Pf}_n &= A_{2n}^{1/2} \\
\mathrm{Pf}_i^{(n)} &= \left[A_{ii}^{(2n-1)} \right]^{1/2}.
\end{aligned} \tag{4.3.28}
$$

Skew-symmetric determinants and Pfaffians appear in Section 5.2 on the generalized Cusick identities.

Exercises

1. **Theorem (Muir and Metzler)** *An arbitrary determinant $A_n = |a_{ij}|_n$ can be expressed as a Pfaffian of the same order.*

   Prove this theorem in the particular case in which $n = 3$ as follows: Let

$$
\begin{aligned}
b_{ij} &= \tfrac{1}{2}(a_{ij} + a_{ji}) = b_{ji}, \\
c_{ij} &= \tfrac{1}{2}(a_{ij} - a_{ji}) = -c_{ji}.
\end{aligned}
$$

   Then

$$
\begin{aligned}
b_{ii} &= a_{ii}, \\
c_{ii} &= 0, \\
a_{ij} - b_{ij} &= c_{ij}, \\
a_{ij} + c_{ji} &= b_{ij}.
\end{aligned}
$$

Applying the Laplace expansion formula (Section 3.3) in reverse,

$$A_3^2 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & & & \\ a_{21} & a_{22} & a_{23} & & & \\ a_{31} & a_{32} & a_{33} & & & \\ -b_{31} & -b_{32} & -b_{33} & a_{33} & a_{23} & a_{13} \\ -b_{21} & -b_{22} & -b_{23} & a_{32} & a_{22} & a_{12} \\ -b_{11} & -b_{12} & -b_{13} & a_{31} & a_{21} & a_{11} \end{vmatrix}.$$

Now, perform the column and row operations

$$\begin{aligned} \mathbf{C}'_j &= \mathbf{C}_j + \mathbf{C}_{7-j}, \quad 4 \le j \le 6, \\ \mathbf{R}'_i &= \mathbf{R}_i + \mathbf{R}_{7-i}, \quad 1 \le i \le 3, \end{aligned}$$

and show that the resulting determinant is skew-symmetric. Hence, show that

$$A_3 = \begin{vmatrix} c_{12} & c_{13} & b_{13} & b_{12} & b_{11} \\ & c_{23} & b_{23} & b_{22} & b_{21} \\ & & b_{33} & b_{32} & b_{31} \\ & & & c_{23} & c_{13} \\ & & & & c_{12} \end{vmatrix}.$$

2. **Theorem (Muir and Metzler)** *An arbitrary determinant of order* $2n$ *can be expressed as a Pfaffian of order* n.
Prove this theorem in the particular case in which $n = 2$ as follows: Denote the determinant by A_4, transpose it and interchange first rows 1 and 2 and then rows 3 and 4. Change the signs of the elements in the (new) rows 2 and 4. These operations leave the value of the determinant unaltered. Multiply the initial and final determinants together, prove that the product is skew-symmetric, and, hence, prove that

$$A_4 = \begin{vmatrix} (N_{12,12} + N_{12,34}) & (N_{13,12} + N_{13,34}) & (N_{14,12} + N_{14,34}) \\ & (N_{23,12} + N_{23,34}) & (N_{24,12} + N_{24,34}) \\ & & (N_{34,12} + N_{34,34}) \end{vmatrix}.$$

where $N_{ij,rs}$ is a retainer minor (Section 3.2.1).
3. Expand Pf_3 by the five elements from the first row and their associated second-order Pfaffians.
4. A skew-symmetric determinant A_{2n} is defined as follows:

$$A_{2n} = |a_{ij}|_{2n},$$

where

$$a_{ij} = \frac{x_i - x_j}{x_i + x_j}.$$

Prove that the corresponding Pfaffian is given by the formula

$$\mathrm{Pf}_{2n-1} = \prod_{1 \le i < j \le 2n} a_{ij},$$

that is, the Pfaffian is equal to the product of its elements.

4.4 Circulants

4.4.1 Definition and Notation

A circulant A_n is denoted by the symbol $A(a_1, a_2, a_3, \ldots, a_n)$ and is defined as follows:

$$\begin{aligned} A_n &= A(a_1, a_2, a_3, \ldots, a_n) \\ &= \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{vmatrix}_n . \end{aligned} \tag{4.4.1}$$

Each row is obtained from the previous row by displacing each element, except the last, one position to the right, the last element being displaced to the first position. The name circulant is derived from the circular nature of the displacements.

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} a_{j+1-i}, & j \geq i, \\ a_{n+j+1-i}, & j < i. \end{cases} \tag{4.4.2}$$

4.4.2 Factors

After performing the column operation

$$\mathbf{C}'_1 = \sum_{j=1}^{n} \mathbf{C}_j, \tag{4.4.3}$$

it is easily seen that A_n has the factor $\sum_{r=1}^{n} a_r$ but A_n has other factors. When all the a_r are real, the first factor is real but some of the other factors are complex.

Let ω_r denote the complex number defined as follows and let $\bar{\omega}_r$ denote its conjugate:

$$\begin{aligned} \omega_r &= \exp(2ri\pi/n) \quad 0 \leq r \leq n-1, \\ &= \omega_1^r, \\ \omega_r^n &= 1, \\ \omega_r \bar{\omega}_r &= 1, \\ \omega_0 &= 1. \end{aligned} \tag{4.4.4}$$

ω_r is also a function of n, but the n is suppressed to simplify the notation. The n numbers

$$1, \omega_r, \omega_r^2, \ldots, \omega_r^{n-1} \tag{4.4.5}$$

are the nth roots of unity for any value of r. Two different choices of r give rise to the same set of roots but in a different order. It follows from the third line in (4.4.4) that

$$\sum_{s=0}^{n-1} \omega_r^s = 0, \quad 0 \le r \le n-1. \tag{4.4.6}$$

Theorem.

$$A_n = \prod_{r=0}^{n-1} \sum_{s=1}^{n} \omega_r^{s-1} a_s.$$

PROOF. Let

$$\begin{aligned} z_r &= \sum_{s=1}^{n} \omega_r^{s-1} a_s \\ &= a_1 + \omega_r a_2 + \omega_r^2 a_3 + \cdots + \omega_r^{n-1} a_n, \quad \omega_r^n = 1. \end{aligned} \tag{4.4.7}$$

Then,

$$\left.\begin{aligned} \omega_r z_r &= a_n + \omega_r a_1 + \omega_r^2 a_2 + \cdots + \omega_r^{n-1} a_{n-1} \\ \omega_r^2 z_r &= a_{n-1} + \omega_r a_n + \omega_r^2 a_1 + \cdots + \omega_r^{n-1} a_{n-2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \omega_r^{n-1} z_r &= a_2 + \omega_r a_3 + \omega_r^2 a_4 + \cdots + \omega_r^{n-1} a_1 \end{aligned}\right\}. \tag{4.4.8}$$

Express A_n in column vector notation and perform a column operation:

$$\begin{aligned} A_n &= \left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3 \cdots \mathbf{C}_n\right| \\ &= \left|\mathbf{C}_1'\ \mathbf{C}_2\ \mathbf{C}_3 \cdots \mathbf{C}_n\right|, \end{aligned}$$

where

$$\begin{aligned} \mathbf{C}_1' &= \sum_{j=1}^{n} \omega_r^{j-1} \mathbf{C}_j \\ &= \begin{bmatrix} a_1 \\ a_n \\ a_{n-1} \\ \vdots \\ a_2 \end{bmatrix} + \omega_r \begin{bmatrix} a_2 \\ a_1 \\ a_n \\ \vdots \\ a_3 \end{bmatrix} + \omega_r^2 \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ \vdots \\ a_4 \end{bmatrix} + \cdots + \omega_r^{n-1} \begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ a_1 \end{bmatrix} \\ &= z_r \mathbf{W}_r, \end{aligned}$$

where

$$\mathbf{W}_r = \left[1\ \omega_r\ \omega_r^2 \cdots \omega_r^{n-1}\right]^T. \tag{4.4.9}$$

Hence,

$$A = z_r \left| \mathbf{W}_r \ \mathbf{C}_2 \ \mathbf{C}_3 \cdots \mathbf{C}_n \right|. \tag{4.4.10}$$

It follows that each z_r, $0 \le r \le n-1$, is a factor of A_n. Hence,

$$A_n = K \prod_{r=0}^{n-1} z_r, \tag{4.4.11}$$

but since A_n and the product are homogeneous polynomials of degree n in the a_r, the factor K must be purely numerical. It is clear by comparing the coefficients of a_1^n on each side that $K = 1$. The theorem follows from (4.4.7). □

Illustration. When $n = 3$, $\omega_r = \exp(2ri\pi/3)$, $\omega_r^3 = 1$.

$$\begin{aligned} \omega_0 &= 1, \\ \omega = \omega_1 &= \exp(2i\pi/3), \\ \omega_2 &= \exp(4i\pi/3) = \omega_1^2 = \omega^2 = \bar{\omega}, \\ \omega_2^2 &= \omega_1 = \omega. \end{aligned}$$

Hence,

$$\begin{aligned} A_3 &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{vmatrix} \\ &= (a_1 + a_2 + a_3)(a_1 + \omega_1 a_2 + \omega_1^2 a_3)(a_1 + \omega_2 a_2 + \omega_2^2 a_3) \\ &= (a_1 + a_2 + a_3)(a_1 + \omega a_2 + \omega^2 a_3)(a_1 + \omega^2 a_2 + \omega a_3). \end{aligned} \tag{4.4.12}$$

Exercise. Factorize A_4.

4.4.3 The Generalized Hyperbolic Functions

Define a matrix $\mathbf{W}$ as follows:

$$\begin{aligned} \mathbf{W} &= \left[\omega^{(r-1)(s-1)}\right]_n \quad (\omega = \omega_1) \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2n-2} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \omega^{n-1} & \omega^{2n-2} & \omega^{3n-3} & \cdots & \omega^{(n-1)^2} \end{bmatrix}_n . \end{aligned} \tag{4.4.13}$$

Lemma 4.18.

$$\mathbf{W}^{-1} = \frac{1}{n}\overline{\mathbf{W}}.$$

PROOF.

$$\overline{\mathbf{W}} = \left[\omega^{-(r-1)(s-1)}\right]_n.$$

Hence,

$$\mathbf{W}\overline{\mathbf{W}} = [\alpha_{rs}]_n,$$

where

$$\begin{aligned}\alpha_{rs} &= \sum_{t=1}^{n} \omega^{(r-1)(t-1)-(t-1)(s-1)} \\ &= \sum_{t=1}^{n} \omega^{(t-1)(r-s)}, \\ \alpha_{rr} &= n. \end{aligned} \tag{4.4.14}$$

Put $k = r - s$, $s \neq r$. Then, referring to (4.4.6),

$$\begin{aligned}\alpha_{rs} &= \sum_{t=1}^{n} \omega^{(t-1)k} \quad (\omega^k = \omega_1^k = \omega_k) \\ &= \sum_{t=1}^{n} \omega_k^{t-1} \\ &= 0, \qquad s \neq r. \end{aligned} \tag{4.4.15}$$

Hence,

$$\begin{aligned}[\alpha_{rs}] &= n\mathbf{I}, \\ \mathbf{W}\overline{\mathbf{W}} &= n\mathbf{I}. \end{aligned}$$

The lemma follows. □

The n generalized hyperbolic functions H_r, $1 \leq r \leq n$, of the $(n-1)$ independent variables x_r, $1 \leq r \leq n-1$, are defined by the matrix equation

$$\mathbf{H} = \frac{1}{n}\mathbf{W}\mathbf{E}, \tag{4.4.16}$$

where $\mathbf{H}$ and $\mathbf{E}$ are column vectors defined as follows:

$$\begin{aligned}\mathbf{H} &= \begin{bmatrix} H_1 \; H_2 \; H_3 \ldots H_n \end{bmatrix}^T, \\ \mathbf{E} &= \begin{bmatrix} E_1 \; E_2 \; E_3 \ldots E_n \end{bmatrix}^T, \\ E_r &= \exp\left[\sum_{t=1}^{n-1} \omega^{(r-1)t} x_t\right], \quad 1 \leq r \leq n. \end{aligned} \tag{4.4.17}$$

Lemma 4.19.

$$\prod_{r=1}^{n} E_r = 1.$$

PROOF. Referring to (4.4.15),

$$\prod_{r=1}^{n} E_r = \prod_{r=1}^{n} \exp\left[\sum_{t=1}^{n-1} \omega^{(r-1)t} x_t\right]$$

$$= \exp\left[\sum_{r=1}^{n}\sum_{t=1}^{n-1}\omega^{(r-1)t}x_t\right]$$
$$= \exp\left[\sum_{t=1}^{n-1}x_t\sum_{r=1}^{n}\omega^{(r-1)t}\right]$$
$$= \exp(0).$$

The lemma follows. □

Theorem.

$$A = A(H_1, H_2, H_3, \ldots, H_n) = 1.$$

Proof. The definition (4.4.16) implies that

$$\mathbf{A}(H_1, H_2, H_3, \ldots, H_n) = \begin{bmatrix} H_1 & H_2 & H_3 & \cdots & H_n \\ H_n & H_1 & H_2 & \cdots & H_{n-1} \\ H_{n-1} & H_n & H_1 & \cdots & H_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_2 & H_3 & H_4 & \cdots & H_1 \end{bmatrix}_n$$
$$= \mathbf{W}^{-1}\mathbf{diag}\big(E_1\ E_2\ E_3 \ldots E_n\big)\mathbf{W}. \quad (4.4.18)$$

Taking determinants,

$$A(H_1, H_2, H_3, \ldots, H_n) = \left|\mathbf{W}^{-1}\mathbf{W}\right|\prod_{r=1}^{n}E_r.$$

The theorem follows from Lemma 4.19. □

Illustrations

When $n = 2$, $\omega = \exp(i\pi) = -1$.

$$\mathbf{W} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$
$$\mathbf{W}^{-1} = \tfrac{1}{2}\mathbf{W},$$
$$E_r = \exp[(-1)^{r-1}x_1], \qquad r = 1, 2.$$

Let $x_1 \to x$; then,

$$E_1 = e^x,$$
$$E_2 = e^{-x},$$
$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^x \\ e^{-x} \end{bmatrix},$$
$$H_1 = \operatorname{ch} x,$$
$$H_2 = \operatorname{sh} x,$$

the simple hyperbolic functions;

$$A(H_1, H_2) = \begin{vmatrix} H_1 & H_2 \\ H_2 & H_1 \end{vmatrix} = 1. \tag{4.4.19}$$

When $n = 3$, $\omega_r = \exp(2ri\pi/3)$,

$$\begin{aligned}
\omega_r^3 &= 1, \\
\omega &= \omega_1 = \exp(2i\pi/3), \\
\omega^2 &= \bar{\omega}, \\
\omega\bar{\omega} &= 1. \\
\mathbf{W} &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \\
\mathbf{W}^{-1} &= \tfrac{1}{3}\mathbf{W}, \\
E_r &= \exp\left[\sum_{t=0}^{2} \omega^{(r-1)t} x_t\right] \\
&= \exp\left[\omega^{r-1}x_1 + \omega^{2r-2}x_2\right].
\end{aligned}$$

Let $(x_1, x_2) \to (x, y)$. Then,

$$\begin{aligned}
E_1 &= \exp(x + y), \\
E_2 &= \exp(\omega x + \bar{\omega} y), \\
E_3 &= \exp(\bar{\omega} x + \omega y) = \bar{E}_2.
\end{aligned} \tag{4.4.20}$$

$$\begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \tag{4.4.21}$$

$$\begin{aligned}
H_1 &= \tfrac{1}{3}\left[e^{x+y} + e^{\omega x + \bar{\omega} y} + e^{\bar{\omega} x + \omega y}\right], \\
H_2 &= \tfrac{1}{3}\left[e^{x+y} + \omega e^{\omega x + \bar{\omega} y} + \bar{\omega} e^{\bar{\omega} x + \omega y}\right], \\
H_3 &= \tfrac{1}{3}\left[e^{x+y} + \bar{\omega} e^{\omega x + \bar{\omega} y} + \omega e^{\bar{\omega} x + \omega y}\right].
\end{aligned} \tag{4.4.22}$$

Since the complex terms appear in conjugate pairs, all three functions are real:

$$A(H_1, H_2, H_3) = \begin{vmatrix} H_1 & H_2 & H_3 \\ H_3 & H_1 & H_2 \\ H_2 & H_3 & H_1 \end{vmatrix} = 1. \tag{4.4.23}$$

A bibliography covering the years 1757–1955 on higher-order sine functions, which are closely related to higher-order or generalized hyperbolic functions, is given by Kaufman. Further notes on the subject are given by Schmidt and Pipes, who refer to the generalized hyperbolic functions as cyclical functions and by Izvercianu and Vein who refer to the generalized hyperbolic functions as Appell functions.

Exercises

1. Prove that when $n = 3$ and $(x_1, x_2) \to (x, y)$,

$$\frac{\partial}{\partial x}[H_1, H_2, H_3] = [H_2, H_3, H_1],$$
$$\frac{\partial}{\partial y}[H_1, H_2, H_3] = [H_3, H_1, H_2]$$

and apply these formulas to give an alternative proof of the particular circulant identity

$$A(H_1, H_2, H_3) = 1.$$

If $y = 0$, prove that

$$H_1 = \sum_{r=0}^{\infty} \frac{x^{3r}}{(3r)!},$$
$$H_2 = \sum_{r=0}^{\infty} \frac{x^{3r+2}}{(3r+2)!},$$
$$H_3 = \sum_{r=0}^{\infty} \frac{x^{3r+1}}{(3r+1)!}.$$

2. Apply the partial derivative method to give an alternative proof of the general circulant identity as stated in the theorem.

4.5 Centrosymmetric Determinants

4.5.1 Definition and Factorization

The determinant $A_n = |a_{ij}|_n$, in which

$$a_{n+1-i,n+1-j} = a_{ij} \tag{4.5.1}$$

is said to be centrosymmetric. The elements in row $(n+1-i)$ are identical with those in row i but in reverse order; that is, if

$$\mathbf{R}_i = \left[a_{i1}\ a_{i2} \ldots a_{i,n-1}\ a_{in}\right],$$

then

$$\mathbf{R}_{n+1-i} = \left[a_{in}\ a_{i,n-1} \ldots a_{i2}\ a_{i1}\right].$$

A similar remark applies to columns. A_n is unaltered in form if it is transposed first across one diagonal and then across the other, an operation which is equivalent to rotating A_n in its plane through 180° in either direction. A_n is not necessarily symmetric across either of its diagonals. The most general centrosymmetric determinant of order 5 is of the form

$$A_5 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_2 & c_1 \\ b_5 & b_4 & b_3 & b_2 & b_1 \\ a_5 & a_4 & a_3 & a_2 & a_1 \end{vmatrix}. \tag{4.5.2}$$

Theorem. *Every centrosymmetric determinant can be factorized into two determinants of lower order. A_{2n} has factors each of order n, whereas A_{2n+1} has factors of orders n and $n+1$.*

PROOF. In the row vector

$$\mathbf{R}_i + \mathbf{R}_{n+1-i} = \big[(a_{i1}+a_{in})(a_{i2}+a_{i,n-1})\cdots(a_{i,n-1}+a_{i2})(a_{in}+a_{i1})\big],$$

the $(n+1-j)$th element is identical to the jth element. This suggests performing the row and column operations

$$\mathbf{R}'_i = \mathbf{R}_i + \mathbf{R}_{n+1-i}, \quad 1 \le i \le \left[\tfrac{1}{2}n\right],$$

$$\mathbf{C}'_j = \mathbf{C}_j - \mathbf{C}_{n+1-j}, \quad \left[\tfrac{1}{2}(n+1)\right] + 1 \le j \le n,$$

where $\left[\frac{1}{2}n\right]$ is the integer part of $\frac{1}{2}n$. The result of these operations is that an array of zero elements appears in the top right-hand corner of A_n, which then factorizes by applying a Laplace expansion (Section 3.3). The dimensions of the various arrays which appear can be shown clearly using the notation $\mathbf{M}_{rs}$, etc., for a matrix with r rows and s columns. $\mathbf{0}_{rs}$ is an array of zero elements.

$$\begin{aligned} A_{2n} &= \begin{vmatrix} \mathbf{R}_{nn} & \mathbf{0}_{nn} \\ \mathbf{S}_{nn} & \mathbf{T}_{nn} \end{vmatrix}_{2n} \\ &= |\mathbf{R}_{nn}|\,|\mathbf{T}_{nn}|, \end{aligned} \tag{4.5.3}$$

$$\begin{aligned} A_{2n+1} &= \begin{vmatrix} \mathbf{R}^*_{n+1,n+1} & \mathbf{0}_{n+1,n} \\ \mathbf{S}^*_{n,n+1} & \mathbf{T}^*_{nn} \end{vmatrix}_{2n+1} \\ &= |\mathbf{R}^*_{n+1,n+1}|\,|\mathbf{T}^*_{nn}|. \end{aligned} \tag{4.5.4}$$

□

The method of factorization can be illustrated adequately by factorizing the fifth-order determinant A_5 defined in (4.5.2).

$$A_5 = \begin{vmatrix} a_1+a_5 & a_2+a_4 & 2a_3 & a_4+a_2 & a_5+a_1 \\ b_1+b_5 & b_2+b_4 & 2b_3 & b_4+b_2 & b_5+b_1 \\ c_1 & c_2 & c_3 & c_2 & c_1 \\ b_5 & b_4 & b_3 & b_2 & b_1 \\ a_5 & a_4 & a_3 & a_2 & a_1 \end{vmatrix}$$

$$
= \begin{vmatrix} a_1+a_5 & a_2+a_4 & 2a_3 & \bullet & \bullet \\ b_1+b_5 & b_2+b_4 & 2b_3 & \bullet & \bullet \\ c_1 & c_2 & c_3 & \bullet & \bullet \\ b_5 & b_4 & b_3 & b_2-b_4 & b_1-b_5 \\ a_5 & a_4 & a_3 & a_2-a_4 & a_1-a_5 \end{vmatrix}
$$

$$
= \begin{vmatrix} a_1+a_5 & a_2+a_4 & 2a_3 \\ b_1+b_5 & b_2+b_4 & 2b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} b_2-b_4 & b_1-b_5 \\ a_2-a_4 & a_1-a_5 \end{vmatrix}
$$

$$
= \tfrac{1}{2}|\mathbf{E}|\,|\mathbf{F}|, \tag{4.5.5}
$$

where

$$
\mathbf{E} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} + \begin{bmatrix} a_5 & a_4 & a_3 \\ b_5 & b_4 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix},
$$

$$
\mathbf{F} = \begin{bmatrix} b_2 & b_1 \\ a_2 & a_1 \end{bmatrix} - \begin{bmatrix} b_4 & b_5 \\ a_4 & a_5 \end{bmatrix}. \tag{4.5.6}
$$

Two of these matrices are submatrices of A_5. The other two are submatrices with their rows or columns arranged in reverse order.

Exercise. If a determinant A_n is symmetric about its principal diagonal and persymmetric (Hankel, Section 4.8) about its secondary diagonal, prove analytically that A_n is centrosymmetric.

4.5.2 Symmetric Toeplitz Determinants

The classical Toeplitz determinant A_n is defined as follows:

$$
A_n = |a_{i-j}|_n = \begin{vmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & \cdots & \cdots & \cdots & \cdots & a_0 \end{vmatrix}_n .
$$

The symmetric Toeplitz determinant T_n is defined as follows:

$$
T_n = |t_{|i-j|}|_n = \begin{vmatrix} t_0 & t_1 & t_2 & t_3 & \cdots & t_{n-1} \\ t_1 & t_0 & t_1 & t_2 & \cdots & \cdots \\ t_2 & t_1 & t_0 & t_1 & \cdots & \cdots \\ t_3 & t_2 & t_1 & t_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{n-1} & \cdots & \cdots & \cdots & \cdots & t_0 \end{vmatrix}_n , \tag{4.5.7}
$$

which is centrosymmetric and can therefore be expressed as the product of two determinants of lower order. T_n is also persymmetric about its secondary diagonal.

Let $\mathbf{A}_n$, $\mathbf{B}_n$, and $\mathbf{E}_n$ denote Hankel matrices defined as follows:

$$\begin{aligned}\mathbf{A}_n &= \big[t_{i+j-2}\big]_n,\\ \mathbf{B}_n &= \big[t_{i+j-1}\big]_n,\\ \mathbf{E}_n &= \big[t_{i+j}\big]_n.\end{aligned}\tag{4.5.8}$$

Then, the factors of T_n can be expressed as follows:

$$\begin{aligned}T_{2n-1} &= \tfrac{1}{2}|\mathbf{T}_{n-1} - \mathbf{E}_{n-1}|\,|\mathbf{T}_n + \mathbf{A}_n|,\\ T_{2n} &= |\mathbf{T}_n + \mathbf{B}_n|\,|\mathbf{T}_n - \mathbf{B}_n|.\end{aligned}\tag{4.5.9}$$

Let

$$\begin{aligned}P_n &= \tfrac{1}{2}|\mathbf{T}_n - \mathbf{E}_n| = \tfrac{1}{2}|t_{|i-j|} - t_{i+j}|_n,\\ Q_n &= \tfrac{1}{2}|\mathbf{T}_n + \mathbf{A}_n| = \tfrac{1}{2}|t_{|i-j|} + t_{i+j-2}|_n,\\ R_n &= \tfrac{1}{2}|\mathbf{T}_n + \mathbf{B}_n| = \tfrac{1}{2}|t_{|i-j|} + t_{i+j-1}|_n,\\ S_n &= \tfrac{1}{2}|\mathbf{T}_n - \mathbf{B}_n| = \tfrac{1}{2}|t_{|i-j|} - t_{i+j-1}|_n,\end{aligned}\tag{4.5.10}$$

$$\begin{aligned}U_n &= R_n + S_n,\\ V_n &= R_n - S_n.\end{aligned}\tag{4.5.11}$$

Then,

$$\begin{aligned}T_{2n-1} &= 2P_{n-1}Q_n,\\ T_{2n} &= 4R_nS_n\\ &= U_n^2 - V_n^2.\end{aligned}\tag{4.5.12}$$

Theorem.

a. $T_{2n-1} = U_{n-1}U_n - V_{n-1}V_n$,
b. $T_{2n} = P_nQ_n + P_{n-1}Q_{n+1}$.

PROOF. Applying the Jacobi identity (Section 3.6),

$$\begin{vmatrix} T_{11}^{(n)} & T_{1n}^{(n)} \\ T_{n1}^{(n)} & T_{nn}^{(n)} \end{vmatrix} = T_nT_{1n,1n}^{(n)}.$$

But

$$\begin{aligned}T_{11}^{(n)} &= T_{nn}^{(n)} = T_{n-1},\\ T_{n1}^{(n)} &= T_{1n}^{(n)},\\ T_{1n,1n}^{(n)} &= T_{n-2}.\end{aligned}$$

Hence,

$$T_{n-1}^2 = T_nT_{n-2} + \big(T_{1n}^{(n)}\big)^2.\tag{4.5.13}$$

The element t_{2n-1} does not appear in T_n but appears in the bottom right-hand corner of B_n. Hence,

$$\begin{aligned}\frac{\partial R_n}{\partial t_{2n-1}} &= R_{n-1},\\ \frac{\partial S_n}{\partial t_{2n-1}} &= -S_{n-1}.\end{aligned} \tag{4.5.14}$$

The same element appears in positions $(1, 2n)$ and $(2n, 1)$ in T_{2n}. Hence, referring to the second line of (4.5.12),

$$\begin{aligned}T_{1,2n}^{(2n)} &= \frac{1}{2}\frac{\partial T^{(2n)}}{\partial t_{2n-1}}\\ &= 2\frac{\partial}{\partial t_{2n-1}}(R_n S_n)\\ &= 2(R_{n-1}S_n - R_n S_{n-1}).\end{aligned} \tag{4.5.15}$$

Replacing n by $2n$ in (4.5.13),

$$\begin{aligned}T_{2n-1}^2 &= T_{2n}T_{2n-2} + \left(T_{1,2n}^{(2n)}\right)^2\\ &= 4\left[4R_nS_nR_{n-1}S_{n-1} + (R_{n-1}S_n - R_nS_{n-1})^2\right]\\ &= 4(R_{n-1}S_n + R_nS_{n-1})^2.\end{aligned}$$

The sign of T_{2n-1} is decided by putting $t_0 = 1$ and $t_r = 0$, $r > 0$. In that case, $\mathbf{T}_n = \mathbf{I}_n$, $\mathbf{B}_n = \mathbf{O}_n$, $R_n = S_n = \frac{1}{2}$. Hence, the sign is positive:

$$T_{2n-1} = 2(R_{n-1}S_n + R_nS_{n-1}). \tag{4.5.16}$$

Part (a) of the theorem follows from (4.5.11).

The element t_{2n} appears in the bottom right-hand corner of $\mathbf{E}_n$ but does not appear in either $\mathbf{T}_n$ or $\mathbf{A}_n$. Hence, referring to (4.5.10),

$$\begin{aligned}\frac{\partial P_n}{\partial t_{2n}} &= -P_{n-1},\\ \frac{\partial Q_n}{\partial t_{2n}} &= Q_{n-1}.\end{aligned} \tag{4.5.17}$$

$$\begin{aligned}T_{1,2n+1}^{(2n+1)} &= \frac{1}{2}\frac{\partial T_{2n+1}}{\partial t_{2n}}\\ &= \frac{\partial}{\partial t_{2n}}(P_nQ_{n+1})\\ &= P_nQ_n - P_{n-1}Q_{n+1}.\end{aligned} \tag{4.5.18}$$

Return to (4.5.13), replace n by $2n + 1$, and refer to (4.5.12):

$$\begin{aligned}T_{2n}^2 &= T_{2n+1}T_{2n-1} + \left(T_{1,2n+1}^{(2n+1)}\right)^2\\ &= 4P_nQ_{n+1}P_{n-1}Q_n + (P_nQ_n - P_{n-1}Q_{n+1})^2\\ &= (P_nQ_n + P_{n-1}Q_{n+1})^2.\end{aligned} \tag{4.5.19}$$

When $t_0 = 1$, $t_r = 0$, $r > 0$, $T_{2n} = 1$, $\mathbf{E}_n = \mathbf{O}_n$, $\mathbf{A}_n = \text{diag}[1\ 0\ 0 \ldots 0]$. Hence, $P_n = \frac{1}{2}$, $Q_n = 1$, and the sign of T_{2n} is positive, which proves part (b) of the theorem. □

The above theorem is applied in Section 6.10 on the Einstein and Ernst equations.

Exercise. Prove that

$$T_{12}^{(n)} = T_{n-1,n}^{(n)} = T_{1n;1,n+1}^{(n+1)}.$$

4.5.3 Skew-Centrosymmetric Determinants

The determinant $A_n = |a_{ij}|_n$ is said to be skew-centrosymmetric if

$$a_{n+1-i,n+1-j} = -a_{ij}.$$

In A_{2n+1}, the element at the center, that is, in position $(n+1, n+1)$, is necessarily zero, but in A_{2n}, no element is necessarily zero.

Exercises

1. Prove that A_{2n} can be expressed as the product of two determinants of order n which can be written in the form $(P+Q)(P-Q)$ and hence as the difference between two squares.
2. Prove that A_{2n+1} can be expressed as a determinant containing an $(n+1) \times (n+1)$ block of zero elements and is therefore zero.
3. Prove that if the zero element at the center of A_{2n+1} is replaced by x, then A_{2n+1} can be expressed in the form $x(p+q)(p-q)$.

4.6 Hessenbergians

4.6.1 Definition and Recurrence Relation

The determinant

$$H_n = |a_{ij}|_n,$$

where $a_{ij} = 0$ when $i - j > 1$ or when $j - i > 1$ is known as a Hessenberg determinant or simply a Hessenbergian. If $a_{ij} = 0$ when $i - j > 1$, the Hessenbergian takes the form

$$H_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ & a_{32} & a_{33} & \cdots & \cdots & \cdots \\ & & a_{43} & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots & \cdots \\ & & & & a_{n-1,n-1} & a_{n-1,n} \\ & & & & a_{n,n-1} & a_{nn} \end{vmatrix}_n . \tag{4.6.1}$$

If $a_{ij} = 0$ when $j - i > 1$, the triangular array of zero elements appears in the top right-hand corner. H_n can be expressed neatly in column vector notation.

Let

$$\mathbf{C}_{jr} = \left[a_{1j}\ a_{2j}\ a_{3j} \ldots a_{rj}\ \mathbf{O}_{n-r}\right]_n^T, \tag{4.6.2}$$

where $\mathbf{O}_i$ represents an unbroken sequence of i zero elements. Then

$$H_n = \left|\mathbf{C}_{12}\ \mathbf{C}_{23}\ \mathbf{C}_{34} \ldots \mathbf{C}_{n-1,n}\ \mathbf{C}_{nn}\right|_n. \tag{4.6.3}$$

Hessenbergians satisfy a simple recurrence relation.

Theorem 4.20.

$$H_n = (-1)^{n-1} \sum_{r=0}^{n-1} (-1)^r p_{r+1,n} H_r, \quad H_0 = 1,$$

where

$$p_{ij} = \begin{cases} a_{ij} a_{j,j-1} a_{j-1,j-2} \cdots a_{i+2,i+1} a_{i+1,i}, & j > i \\ a_{ii}, & j = i. \end{cases}$$

PROOF. Expanding H_n by the two nonzero elements in the last row,

$$H_n = a_{nn} H_{n-1} - a_{n,n-1} K_{n-1},$$

where K_{n-1} is a determinant of order $(n-1)$ whose last row also contains two nonzero elements. Expanding K_{n-1} in a similar manner,

$$K_{n-1} = a_{n-1,n} H_{n-2} - a_{n-1,n-2} K_{n-2},$$

where K_{n-2} is a determinant of order $(n-2)$ whose last row also contains two nonzero elements. The theorem appears after these expansions are repeated a sufficient number of times. □

Illustration.

$$\begin{aligned} H_5 = \left|\mathbf{C}_{12}\mathbf{C}_{23}\mathbf{C}_{34}\mathbf{C}_{45}\mathbf{C}_{55}\right| &= a_{55} H_4 - a_{54}\left|\mathbf{C}_{12}\mathbf{C}_{23}\mathbf{C}_{34}\mathbf{C}_{54}\right|, \\ \left|\mathbf{C}_{12}\mathbf{C}_{23}\mathbf{C}_{34}\mathbf{C}_{54}\right| &= a_{45} H_3 - a_{43}\left|\mathbf{C}_{12}\mathbf{C}_{23}\mathbf{C}_{53}\right|, \\ \left|\mathbf{C}_{12}\mathbf{C}_{23}\mathbf{C}_{53}\right| &= a_{35} H_2 - a_{32}\left|\mathbf{C}_{12}\mathbf{C}_{52}\right|, \\ \left|\mathbf{C}_{12}\mathbf{C}_{52}\right| &= a_{25} H_1 - a_{21} a_{15} H_0. \end{aligned}$$

Hence,

$$\begin{aligned} H_5 &= a_{55} H_4 - (a_{45} a_{54}) H_3 + (a_{35} a_{54} a_{43}) H_2 \\ &\quad - (a_{25} a_{54} a_{43} a_{32}) H_1 + (a_{15} a_{54} a_{43} a_{32} a_{21}) H_0 \\ &= p_{55} H_4 - p_{45} H_3 + p_{35} H_2 - p_{25} H_1 + p_{15} H_0. \end{aligned}$$

Muir and Metzler use the term *recurrent* without giving a definition of the term. A *recurrent* is any determinant which satisfies a recurrence relation.

4.6.2 A Reciprocal Power Series

Theorem 4.21. *If*

$$\sum_{r=0}^{\infty}(-1)^r\psi_n t^r = \left[\sum_{r=0}^{\infty}\phi_r t^r\right]^{-1}, \quad \phi_0 = \psi_0 = 1,$$

then

$$\psi_r = \begin{vmatrix} \phi_1 & \phi_0 & & & & \\ \phi_2 & \phi_1 & \phi_0 & & & \\ \phi_3 & \phi_2 & \phi_1 & \phi_0 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ \phi_{n-1} & \phi_{n-2} & \cdots & \cdots & \phi_1 & \phi_0 \\ \phi_n & \phi_{n-1} & \cdots & \cdots & \phi_2 & \phi_1 \end{vmatrix}_n,$$

which is a Hessenbergian.

Proof. The given equation can be expressed in the form

$$(\phi_0 + \phi_1 t + \phi_2 t^2 + \phi_3 t^3 + \cdots)(\psi_0 - \psi_1 t + \psi_2 t^2 - \psi_3 t^3 + \cdots) = 1.$$

Equating coefficients of powers of t,

$$\sum_{i=0}^{n}(-1)^{i+1}\phi_i\psi_{n-i} = 0 \qquad (4.6.4)$$

from which it follows that

$$\phi_n = \sum_{i=1}^{n}(-1)^{i+1}\phi_{n-i}\psi_i. \qquad (4.6.5)$$

In some detail,

$$\begin{aligned} &\phi_0\psi_1 &&= \phi_1 \\ &\phi_1\psi_1 - \phi_0\psi_2 &&= \phi_2 \\ &\phi_2\psi_1 - \phi_1\psi_2 + \phi_0\psi_3 &&= \phi_3 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\phi_{n-1}\psi_1 - \phi_{n-2}\psi_2 + \cdots + (-1)^{n+1}\phi_0\psi_n &&= \phi_n. \end{aligned}$$

These are n equations in the n variables $(-1)^{r-1}\psi_r$, $1 \le r \le n$, in which the determinant of the coefficients is triangular and equal to 1. Hence,

$$(-1)^{n-1}\psi_n = \begin{vmatrix} \phi_0 & & & & & & \phi_1 \\ \phi_1 & \phi_0 & & & & & \phi_2 \\ \phi_2 & \phi_1 & \phi_0 & & & & \phi_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_{n-2} & \phi_{n-3} & \phi_{n-4} & \cdots & \phi_1 & \phi_0 & \phi_{n-1} \\ \phi_{n-1} & \phi_{n-2} & \phi_{n-3} & \cdots & \phi_2 & \phi_1 & \phi_n \end{vmatrix}_n.$$

The proof is completed by transferring the last column to the first position, an operation which introduces the factor $(-1)^{n-1}$. □

In the next theorem, ϕ_m and ψ_m are functions of x.

Theorem 4.22. *If*

$$\phi'_m = (m+a)F\phi_{m-1}, \quad F = F(x),$$

then

$$\psi'_m = (a+2-m)F\psi_{m-1}.$$

PROOF. It follows from (4.6.4) that

$$\psi_n = \sum_{i=1}^{n}(-1)^{i+1}\phi_i\psi_{n-i}. \tag{4.6.6}$$

It may be verified by elementary methods that

$$\begin{aligned}\psi'_1 &= (a+1)F\psi_0,\\ \psi'_2 &= aF\psi_1,\\ \psi'_3 &= (a-1)F\psi_2,\end{aligned}$$

etc., so that the theorem is known to be true for small values of m. Assume it to be true for $1 \le m \le n-1$ and apply the method of induction. Differentiating (4.6.6),

$$\begin{aligned}\psi'_n &= \sum_{i=1}^{n}(-1)^{i+1}(\phi'_i\psi_{n-i} + \phi_i\psi'_{n-i})\\ &= F\sum_{i=1}^{n}(-1)^{i+1}[(i+a)\phi_{i-1}\psi_{n-i} + (a+2-n+i)\phi_i\psi_{n-1-i}]\\ &= F(S_1+S_2+S_3),\end{aligned}$$

where

$$\begin{aligned}S_1 &= \sum_{i=1}^{n}(-1)^{i+1}(i+a)\phi_{i-1}\psi_{n-i},\\ S_2 &= (a+2-n)\sum_{i=1}^{n}(-1)^{i+1}\phi_i\psi_{n-1-i},\\ S_3 &= \sum_{i=1}^{n}(-1)^{i+1}i\phi_i\psi_{n-1-i}.\end{aligned}$$

Since the $i=n$ terms in S_2 and S_3 are zero, the upper limits in these sums can be reduced to $(n-1)$. It follows that

$$S_2 = (a+2-n)\psi_{n-1}.$$

Also, adjusting the dummy variable in S_1 and referring to (4.6.4) with $n \to n-1$,

$$\begin{aligned} S_1 &= \sum_{i=0}^{n-1} (-1)^i (i+1+a)\phi_i \psi_{n-1-i} \\ &= \sum_{i=1}^{n-1} (-1)^i i \phi_i \psi_{n-1-i} + (1+a) \sum_{i=0}^{n-1} (-1)^i \phi_i \psi_{n-1-i} \\ &= -S_3. \end{aligned}$$

Hence, $\psi'_n = (a+2-n)F\psi_{n-1}$, which is equivalent to the stated result. Note that if $\phi'_m = (m-1)\phi_{m-1}$, then $\psi'_m = -(m-1)\psi_{m-1}$. □

4.6.3 A Hessenberg–Appell Characteristic Polynomial

Let

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} a_{j-i+1}, & j \ge i, \\ -j, & j = i-1, \\ 0, & \text{otherwise.} \end{cases}$$

In some detail,

$$A_n = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{n-1} & a_n \\ -1 & a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} \\ & -2 & a_1 & a_2 & \cdots & \cdots & \cdots \\ & & -3 & a_1 & \cdots & \cdots & \cdots \\ & & & & \cdots & \cdots & \cdots \\ & & & & & a_1 & a_2 \\ & & & & & -(n-1) & a_1 \end{vmatrix}_n . \tag{4.6.7}$$

Applying the recurrence relation in Theorem 4.20,

$$A_n = (n-1)! \sum_{r=0}^{n-1} \frac{a_{n-r} A_r}{r!}, \quad n \ge 1, \quad A_0 = 1. \tag{4.6.8}$$

Let $B_n(x)$ denote the characteristic polynomial of the matrix $\mathbf{A}_n$:

$$B_n = \left| \mathbf{A}_n - x\mathbf{I} \right|. \tag{4.6.9}$$

This determinant satisfies the recurrence relation

$$B_n = (n-1)! \sum_{r=0}^{n-1} \frac{b_{n-r} B_r}{r!}, \quad n \ge 1, \quad B_0 = 1, \tag{4.6.10}$$

where

$$b_1 = a_1 - x,$$

$$b_r = a_r, \quad r > 1.$$

$$\begin{aligned} B_n(0) &= A_n, \\ B_{ij}^{(n)}(0) &= A_{ij}^{(n)}. \end{aligned} \tag{4.6.11}$$

Theorem 4.23.

a. $B'_n = -nB_{n-1}$.

b. $\sum_{r=1}^{n} A_{rr}^{(n)} = nA_{n-1}$.

c. $B_n = \sum_{r=0}^{n} \binom{n}{r} A_r(-x)^{n-r}$.

PROOF.

$$\begin{aligned} B_1 &= -x + A_1, \\ B_2 &= x^2 - 2A_1x + A_2, \\ B_3 &= -x^3 + 3A_1x^2 - 3A_2x + A_3, \end{aligned} \tag{4.6.12}$$

etc., which are Appell polynomials (Appendix A.4) so that (a) is valid for small values of n. Assume that

$$B'_r = -rB_{r-1}, \quad 2 \le r \le n-1,$$

and apply the method of induction.

From (4.6.10),

$$\begin{aligned} B_n &= (n-1)! \sum_{r=0}^{n-2} \frac{a_{n-r}B_r}{r!} + (a_1 - x)B_{n-1}, \\ B'_n &= -(n-1)! \sum_{r=1}^{n-2} \frac{a_{n-r}rB_{r-1}}{r!} - (n-1)(a_1 - x)B_{n-2} - B_{n-1} \\ &= -(n-1)! \sum_{r=1}^{n-2} \frac{a_{n-r}B_{r-1}}{(r-1)!} - (n-1)(a_1 - x)B_{n-2} - B_{n-1} \\ &= -(n-1)! \sum_{r=0}^{n-3} \frac{a_{n-1-r}B_r}{r!} - (n-1)(a_1 - x)B_{n-2} - B_{n-1} \\ &= -(n-1)! \sum_{r=0}^{n-2} \frac{b_{n-1-r}B_r}{r!} - B_{n-1} \\ &= -(n-1)B_{n-1} - B_{n-1} \\ &= -nB_{n-1}, \end{aligned}$$

which proves (a).

The proof of (b) follows as a corollary since, differentiating B_n by columns,

$$B'_n = -\sum_{r=1}^{n} B_{rr}^{(n)}.$$

The given result follows from (4.6.11).

To prove (c), differentiate (a) repeatedly, apply the Maclaurin formula, and refer to (4.6.11) again:

$$\begin{aligned} B_n^{(r)} &= \frac{(-1)^r n! B_{n-r}}{(n-r)!}, \\ B_n &= \sum_{r=0}^{n} \frac{B_n^{(r)}(0)}{r!} x^r \\ &= \sum_{r=0}^{n} \binom{n}{r} A_{n-r}(-x)^r. \end{aligned}$$

Put $r = n - s$ and the given formula appears. It follows that B_n is an Appell polynomial for all values of n. □

Exercises

1. Let

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} \psi_{j-i+1}, & j \geq i, \\ j, & j = i - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Prove that if A_n satisfies the Appell equation $A'_n = nA_{n-1}$ for small values of n, then A_n satisfies the Appell equation for all values of n and that the elements must be of the form

$$\begin{aligned} \psi_1 &= x + \alpha_1, \\ \psi_m &= \alpha_m, \qquad m > 1, \end{aligned}$$

where the α's are constants.

2. If

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} \phi_{j-i}, & j \geq i, \\ -j, & j = i - 1, \\ 0, & \text{otherwise,} \end{cases}$$

and where

$$\phi'_m = (m+1)\phi_{m-1}, \quad \phi_0 = \text{constant},$$

prove that

$$A'_n = n(n-1)A_{n-1}.$$

3. Prove that

$$\prod_{r=1}^{n} \begin{vmatrix} 1 & a_r x \\ -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & b_{12}x & b_{13}x^2 & \cdots & \cdots & b_{1,n+1}x^n \\ -1 & 1 & b_{23}x & \cdots & \cdots & b_{2,n+1}x^{n-1} \\ & -1 & 1 & \cdots & \cdots & b_{3,n+1}x^{n-2} \\ & & \cdots & \cdots & \cdots & \cdots \\ & & & & -1 & 1 \end{vmatrix}_{n+1},$$

where

$$b_{ij} = \prod_{r=i}^{j-1} a_r.$$

4. If

$$U_n = \begin{vmatrix} u' & u''/2! & u'''/3! & u^{(4)}/4! & \cdots \\ u & u' & u''/2! & u'''/3! & \cdots \\ & u & u' & u''/2! & \cdots \\ & & u & u' & \cdots \\ & & & \cdots & \cdots \end{vmatrix}_n,$$

prove that

$$U_{n+1} = u'U_n - \frac{uU'_n}{n+1}. \qquad \text{(Burgmeier)}$$

4.7 Wronskians

4.7.1 Introduction

Let $y_r = y_r(x)$, $1 \le r \le n$, denote n functions each with derivatives of orders up to $(n-1)$. These functions are said to be linearly dependent if there exist coefficients λ_r, independent of x and not all zero, such that

$$\sum_{r=1}^{n} \lambda_r y_r = 0 \qquad (4.7.1)$$

for all values of x.

Theorem 4.24. *The necessary condition that the functions y_r be linearly dependent is that*

$$\left| y_j^{(i-1)} \right|_n = 0$$

identically.

PROOF. Equation (4.7.1) together with its first $(n-1)$ derivatives form a set of n homogeneous equations in the n coefficients λ_r. The condition that not all the λ_r be zero is that the determinant of the coefficients of the λ_r be zero, that is,

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = 0$$

for all values of x, which proves the theorem. □

This determinant is known as the Wronskian of the n functions y_r and is denoted by $W(y_1, y_2, \ldots, y_n)$, which can be abbreviated to W_n or W where there is no risk of confusion. After transposition, W_n can be expressed in column vector notation as follows:

$$W_n = W(y_1, y_2, \ldots, y_n) = \left|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}'' \cdots \mathbf{C}^{(n-1)}\right|$$

where

$$\mathbf{C} = \left[y_1\ y_2 \cdots y_n\right]^T. \tag{4.7.2}$$

If $W_n \neq 0$, identically the n functions are linearly independent.

Theorem 4.25. *If* $t = t(x)$,

$$W(ty_1, ty_2, \ldots, ty_n) = t^n W(y_1, y_2, \ldots, y_n).$$

PROOF.

$$\begin{aligned} W(ty_1, ty_2, \ldots, ty_n) &= \left|(t\mathbf{C})\ (t\mathbf{C})'\ (t\mathbf{C})'' \cdots (t\mathbf{C})^{(n-1)}\right| \\ &= \left|\mathbf{K}_1\ \mathbf{K}_2\ \mathbf{K}_3 \cdots \mathbf{K}_n\right|, \end{aligned}$$

where

$$\mathbf{K}_j = (t\mathbf{C})^{(j-1)} = D^{j-1}(t\mathbf{C}), \quad D = \frac{d}{dx}.$$

Recall the Leibnitz formula for the $(j-1)$th derivative of a product and perform the following column operations:

$$\begin{aligned} \mathbf{K}_j' &= \mathbf{K}_j + t\sum_{s=1}^{j-1} \binom{j-1}{s} D^s\left(\frac{1}{t}\right)\mathbf{K}_{j-s}, \quad j = n, n-1, \ldots, 3.2. \\ &= t\sum_{s=0}^{j-1} \binom{j-1}{s} D^s\left(\frac{1}{t}\right)\mathbf{K}_{j-s} \\ &= t\sum_{s=0}^{j-1} \binom{j-1}{s} D^s\left(\frac{1}{t}\right) D^{j-1-s}(t\mathbf{C}) \\ &= tD^{(j-1)}(\mathbf{C}) \\ &= t\mathbf{C}^{(j-1)}. \end{aligned}$$

Hence,

$$W(ty_1, ty_2, \ldots, ty_n) = \left|(t\mathbf{C})\ (t\mathbf{C}')\ (t\mathbf{C}'') \cdots (t\mathbf{C}^{(n-1)})\right|$$
$$= t^n\left|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}'' \cdots \mathbf{C}^{(n-1)}\right|.$$

The theorem follows. □

Exercise. Prove that

$$\frac{d^n x}{dy^n} = \frac{(-1)^{n+1} W\{y'', (y^2)'', (y^3)'' \ldots (y^{n-1})''\}}{1!2!3! \cdots (n-1)!(y')^{n(n+1)/2}},$$

where $y' = dy/dx$, $n \geq 2$. (Mina)

4.7.2 The Derivatives of a Wronskian

The derivative of W_n with respect to x, when evaluated in column vector notation, consists of the sum of n determinants, only one of which has distinct columns and is therefore nonzero. That determinant is the one obtained by differentiating the last column:

$$W_n' = \left|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}'' \cdots \mathbf{C}^{(n-3)}\ \mathbf{C}^{(n-2)}\ \mathbf{C}^{(n)}\right|.$$

Differentiating again,

$$\begin{aligned} W_n'' &= \left|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}'' \cdots \mathbf{C}^{(n-3)}\ \mathbf{C}^{(n-1)}\ \mathbf{C}^{(n)}\right| \\ &\quad + \left|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}'' \cdots \mathbf{C}^{(n-3)}\ \mathbf{C}^{(n-2)}\ \mathbf{C}^{(n+1)}\right|, \end{aligned} \tag{4.7.3}$$

etc. There is no simple formula for $W_n^{(r)}$. In some detail,

$$W_n' = \begin{vmatrix} y_1 & y_1' & \cdots & y_1^{(n-2)} & y_1^{(n)} \\ y_2 & y_2' & \cdots & y_2^{(n-2)} & y_2^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_n & y_n' & \cdots & y_n^{(n-2)} & y_n^{(n)} \end{vmatrix}_n . \tag{4.7.4}$$

The first $(n-1)$ columns of W_n' are identical with the corresponding columns of W_n. Hence, expanding W_n' by elements from its last column,

$$W_n' = \sum_{r=1}^{n} y_r^{(n)} W_{rn}^{(n)}. \tag{4.7.5}$$

Each of the cofactors in the sum is itself a Wronskian of order $(n-1)$:

$$W_{rn}^{(n)} = (-1)^{r+n} W(y_1, y_2, \ldots, y_{r-1}, y_{r+1}, \ldots, y_n). \tag{4.7.6}$$

W_n' is a cofactor of W_{n+1}:

$$W_n' = -W_{n+1,n}^{(n+1)}. \tag{4.7.7}$$

Repeated differentiation of a Wronskian of order n is facilitated by adopting the notation

$$W_{ijk\ldots r} = \left|\mathbf{C}^{(i)}\ \mathbf{C}^{(j)}\ \mathbf{C}^{(k)} \cdots \mathbf{C}^{(r)}\right|$$

$$
\begin{aligned}
&= 0 \quad \text{if the parameters are not distinct}\\
W'_{ijk\ldots r} &= \text{the sum of the determinants obtained by increasing}\\
&\quad\ \text{the parameters one at a time by 1 and discarding}\\
&\quad\ \text{those determinants with two identical parameters.} \qquad (4.7.8)
\end{aligned}
$$

Illustration. Let

$$W = \left|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}''\right| = W_{012}.$$

Then

$$
\begin{aligned}
W' &= W_{013},\\
W'' &= W_{014} + W_{023},\\
W''' &= W_{015} + 2W_{024} + W_{123},\\
W^{(4)} &= W_{016} + 3W_{025} + 2W_{034} + 3W_{124},\\
W^{(5)} &= W_{017} + 4W_{026} + 5W_{035} + 6W_{125} + 5W_{134}, \qquad (4.7.9)
\end{aligned}
$$

etc. Formulas of this type appear in Sections 6.7 and 6.8 on the K dV and KP equations.

4.7.3 The Derivative of a Cofactor

In order to determine formulas for $(W_{ij}^{(n)})'$, it is convenient to change the notation used in the previous section.

Let

$$W = |w_{ij}|_n,$$

where

$$w_{ij} = y_i^{(j-1)} = D^{j-1}(y_i), \quad D = \frac{d}{dx},$$

and where the y_i are arbitrary $(n-1)$ differentiable functions.

Clearly,

$$w'_{ij} = w_{i,j+1}.$$

In column vector notation,

$$W_n = \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_n\right|,$$

where

$$
\begin{aligned}
\mathbf{C}_j &= \left[y_1^{(j-1)}\ y_2^{(j-1)} \cdots y_n^{(j-1)}\right]^T,\\
\mathbf{C}'_j &= \mathbf{C}_{j+1}.
\end{aligned}
$$

Theorem 4.26.

a. $\left(W_{ij}^{(n)}\right)' = -W_{i,j-1}^{(n)} - W_{i,n+1;jn}^{(n+1)}.$

b. $\left(W_{i1}^{(n)}\right)' = -W_{i,n+1;1n}^{(n+1)}.$

c. $\left(W_{in}^{(n)}\right)' = -W_{i,n-1}^{(n)}$.

PROOF. Let $\mathbf{Z}_i$ denote the n-rowed column vector in which the element in row i is 1 and all the other elements are zero.

Then

$$W_{ij}^{(n)} = \left|\mathbf{C}_1 \cdots \mathbf{C}_{j-2}\ \mathbf{C}_{j-1}\ \mathbf{Z}_i\ \mathbf{C}_{j+1} \cdots \mathbf{C}_{n-1}\ \mathbf{C}_n\right|_n, \tag{4.7.10}$$

$$\left(W_{ij}^{(n)}\right)' = \left|\mathbf{C}_1 \cdots \mathbf{C}_{j-2}\ \mathbf{C}_j\ \mathbf{Z}_i\ \mathbf{C}_{j+1} \cdots \mathbf{C}_{n-1}\ \mathbf{C}_n\right|_n + \left|\mathbf{C}_1 \cdots \mathbf{C}_{j-2}\ \mathbf{C}_{j-1}\ \mathbf{Z}_i\ \mathbf{C}_{j+1} \cdots \mathbf{C}_{n-1}\ \mathbf{C}_{n+1}\right|_n. \tag{4.7.11}$$

Formula (a) follows after $\mathbf{C}_j$ and $\mathbf{Z}_i$ in the first determinant are interchanged. Formulas (b) and (c) are special cases of (a) which can be proved by a similar method but may also be obtained from (a) by referring to the definition of first and second cofactors. $W_{i0} = 0$; $W_{rs,tt} = 0$. □

Lemma. *When* $1 \le j, s \le n$,

$$\sum_{r=0}^{n} w_{r,s+1} W_{rj}^{(n)} = \begin{cases} W_n, & s = j-1,\ j \ne 1, \\ -W_{n+1,j}^{(n+1)}, & s = n, \\ 0, & \text{otherwise.} \end{cases}$$

The first and third relations are statements of the sum formula for elements and cofactors (Section 2.3.4):

$$\begin{aligned} \sum_{r=1}^{n} w_{r,n+1} W_{rj}^{(n)} &= \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{j-1}\ \mathbf{C}_{n+1}\ \mathbf{C}_{j+1} \cdots \mathbf{C}_n\right|_n \\ &= (-1)^{n-j}\left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{j-1}\ \mathbf{C}_{j+1} \cdots \mathbf{C}_n\ \mathbf{C}_{n+1}\right|_n. \end{aligned}$$

The second relation follows.

Theorem 4.27.

$$\begin{vmatrix} W_{ij}^{(n)} & W_{in}^{(n)} \\ W_{n+1,j}^{(n+1)} & W_{n+1,n}^{(n+1)} \end{vmatrix} = W_n W_{i,n+1;jn}^{(n+1)}.$$

This identity is a particular case of Jacobi variant (B) (Section 3.6.3) with $(p,q) \to (j,n)$, but the proof which follows is independent of the variant.

PROOF. Applying double-sum relation (B) (Section 3.4),

$$\left(W_n^{ij}\right)' = -\sum_{r=1}^{n}\sum_{s=1}^{n} w'_{rs} W_n^{is} W_n^{rj}.$$

Reverting to simple cofactors and applying the above lemma,

$$\left(\frac{W_{ij}^{(n)}}{W_n}\right)' = -\frac{1}{W_n^2}\sum_r\sum_s w'_{rs} W_{is}^{(n)} W_{rj}^{(n)}$$

$$= -\frac{1}{W_n^2} \sum_{s=j-1,n} W_{is}^{(n)} \sum_r w_{r,s+1} W_{rj}^{(n)},$$

$$W_n \bigl(W_{ij}^{(n)}\bigr)' - W_{ij}^{(n)} W_n' = -W_n W_{i,j-1}^{(n)} + W_{in}^{(n)} W_{n+1,j}^{(n+1)}.$$

Hence, referring to (4.7.7) and Theorem 4.26(a),

$$\begin{aligned} W_{ij}^{(n)} W_{n+1,n}^{(n+1)} - W_{in}^{(n)} W_{n+1,j}^{(n+1)} &= -W_n \bigl[(W_{ij}^{(n)})' + W_{i,j-1}^{(n)}\bigr] \\ &= W_n W_{i,n+1;jn}^{(n+1)}, \end{aligned}$$

which proves Theorem 4.27. □

4.7.4 An Arbitrary Determinant

Since the functions y_i are arbitrary, we may let y_i be a polynomial of degree $(n-1)$. Let

$$y_i = \sum_{r=1}^{n} \frac{a_{ir} x^{r-1}}{(r-1)!}, \tag{4.7.12}$$

where the coefficients a_{ir} are arbitrary. Furthermore, since x is arbitrary, we may let $x = 0$ in algebraic identities. Then,

$$\begin{aligned} w_{ij} &= y_i^{(j-1)}(0) \\ &= a_{ij}. \end{aligned} \tag{4.7.13}$$

Hence, an arbitrary determinant $A_n = |a_{ij}|_n$ can be expressed in the form $(W_n)_{x=0}$ and any algebraic identity which is satisfied by an arbitrary Wronskian is valid for A_n.

4.7.5 Adjunct Functions

Theorem.

$$W(y_1, y_2, \ldots, y_n) W(W^{1n}, W^{2n}, \ldots, W^{nn}) = 1.$$

Proof. Since

$$\bigl|\mathbf{C}\ \mathbf{C}'\ \mathbf{C}'' \cdots \mathbf{C}^{(n-2)}\ \mathbf{C}^{(r)}\bigr| = \begin{cases} 0, & 0 \le r \le n-2 \\ W, & r = n-1, \end{cases}$$

it follows by expanding the determinant by elements from its last column and scaling the cofactors that

$$\sum_{i=1}^{n} y_i^{(r)} W^{in} = \delta_{r,n-1}.$$

Let

$$\varepsilon_{rs} = \sum_{i=1}^{n} y_i^{(r)} (W^{in})^{(s)}. \tag{4.7.14}$$

Then,

$$\varepsilon'_{rs} = \varepsilon_{r+1,s} + \varepsilon_{r,s+1} \tag{4.7.15}$$

and

$$\varepsilon_{r0} = \delta_{r,n-1}. \tag{4.7.16}$$

Differentiating (4.7.16) repeatedly and applying (4.7.15), it is found that

$$\varepsilon_{rs} = \begin{cases} 0, & r+s < n-1 \\ (-1)^s, & r+s = n-1. \end{cases} \tag{4.7.17}$$

Hence,

$$W(y_1, y_2, \ldots, y_n)W(W^{1n}, W^{2n}, \ldots, W^{nn})$$

$$= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}_n \begin{vmatrix} W^{1n} & (W^{1n})' & \cdots & (W^{1n})^{(n-1)} \\ W^{2n} & (W^{2n})' & \cdots & (W^{2n})^{(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ W^{nn} & (W^{nn})' & \cdots & (W^{nn})^{(n-1)} \end{vmatrix}_n$$

$$= \begin{vmatrix} \varepsilon_{00} & \varepsilon_{01} & \varepsilon_{02} & \cdots & \cdots & \varepsilon_{0,n-2} & \varepsilon_{0,n-1} \\ \varepsilon_{10} & \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{1,n-3} & \varepsilon_{1,n-2} & \star \\ \varepsilon_{20} & \varepsilon_{21} & \varepsilon_{22} & \cdots & \varepsilon_{2,n-3} & \star & \star \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \varepsilon_{n-2,0} & \varepsilon_{n-2,1} & \star & \cdots & \star & \star & \star \\ \varepsilon_{n-1,0} & \star & \star & \cdots & \star & \star & \star \end{vmatrix}_n . \tag{4.7.18}$$

From (4.7.17), it follows that those elements which lie above the secondary diagonal are zero: those on the secondary diagonal from bottom left to top right are

$$1, -1, 1, \ldots, (-1)^{n+1}$$

and the elements represented by the symbol $\star$ are irrelevant to the value of the determinant, which is 1 for all values of n. The theorem follows. □

The set of functions $\{W^{1n}, W^{2n}, \ldots, W^{nn}\}$ are said to be adjunct to the set $\{y_1, y_2, \ldots, y_n\}$.

Exercise. Prove that

$$W(y_1, y_2, \ldots, y_n)W(W^{r+1,n}, W^{r+2,n}, \ldots, W^{nn}) = W(y_1, y_2, \ldots, y_r), \quad 1 \le r \le n-1,$$

by raising the order of the second Wronskian from $(n-r)$ to n in a manner similar to that employed in the section of the Jacobi identity.

4.7.6 Two-Way Wronskians

Let

$$W_n = |f^{(i+j-2)}|_n = |D^{i+j-2}f|_n, \quad D = \frac{d}{dx},$$

$$= \begin{vmatrix} f & f' & f'' & \cdots & f^{(n-1)} \\ f' & f'' & f''' & \cdots & \cdots \\ f'' & f''' & f^{(4)} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f^{(n-1)} & \cdots & \cdots & \cdots & f^{(2n-2)} \end{vmatrix}_n . \tag{4.7.19}$$

Then, the rows and columns satisfy the relation

$$\begin{aligned} \mathbf{R}'_i &= \mathbf{R}_{i+1}, \\ \mathbf{C}'_j &= \mathbf{C}_{j+1}, \end{aligned} \tag{4.7.20}$$

which contrasts with the simple Wronskian defined above in which only one of these relations is valid. Determinants of this form are known as two-way or double Wronskians. They are also Hankelians. A more general two-way Wronskian is the determinant

$$W_n = \left| D_x^{i-1} D_y^{j-1}(f) \right|_n \tag{4.7.21}$$

in which

$$\begin{aligned} D_x(\mathbf{R}_i) &= \mathbf{R}_{i+1}, \\ D_y(\mathbf{C}_j) &= \mathbf{C}_{j+1}. \end{aligned} \tag{4.7.22}$$

Two-way Wronskians appear in Section 6.5 on Toda equations.

Exercise. Let A and B denote Wronskians of order n whose columns are defined as follows:

In A,

$$\mathbf{C}_1 = \left[1 \ x \ x^2 \cdots x^{n-1}\right], \quad \mathbf{C}_j = D_x(\mathbf{C}_{j-1}).$$

In B,

$$\mathbf{C}_1 = \left[1 \ y \ y^2 \cdots y^{n-1}\right], \quad \mathbf{C}_j = D_y(\mathbf{C}_{j-1}).$$

Now, let E denote the hybrid determinant of order n whose first r columns are identical with the first r columns of A and whose last s columns are identical with the first s columns of B, where $r + s = n$. Prove that

$$E = \left[0! \ 1! \ 2! \cdots (r-1)!\right]\left[0! \ 1! \ 2! \cdots (s-1)!\right](y-x)^{rs}. \quad \text{(Corduneanu)}$$

4.8 Hankelians 1

4.8.1 Definition and the ϕ_m Notation

A Hankel determinant A_n is defined as

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = f(i+j). \tag{4.8.1}$$

It follows that

$$a_{ji} = a_{ij},$$

so that Hankel determinants are symmetric, but it also follows that

$$a_{i+k,j-k} = a_{ij}, \qquad k = \pm 1, \pm 2, \dots . \tag{4.8.2}$$

In view of this additional property, Hankel determinants are described as persymmetric. They may also be called Hankelians.

A single-suffix notation has an advantage over the usual double-suffix notation in some applications.

Put

$$a_{ij} = \phi_{i+j-2}. \tag{4.8.3}$$

Then,

$$A_n = \begin{vmatrix} \phi_0 & \phi_1 & \phi_2 & \cdots & \phi_{n-1} \\ \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_2 & \phi_3 & \phi_4 & \cdots & \phi_{n+1} \\ \multicolumn{5}{c}{\dotfill} \\ \phi_{n-1} & \phi_n & \phi_{n+1} & \cdots & \phi_{2n-2} \end{vmatrix}_n, \tag{4.8.4}$$

which may be abbreviated to

$$A_n = |\phi_m|_n, \quad 0 \le m \le 2n-2. \tag{4.8.5}$$

In column vector notation,

$$A_n = \left|\mathbf{C}_0\ \mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\right|_n,$$

where

$$\mathbf{C}_j = \left[\phi_j\ \phi_{j+1}\ \phi_{j+2} \cdots \phi_{j+n-1}\right]^T, \quad 0 \le j \le n-1. \tag{4.8.6}$$

The cofactors satisfy $A_{ji} = A_{ij}$, but $A_{ij} \ne F(i+j)$ in general, that is, adj A is symmetric but not Hankelian except possibly in special cases.

The elements ϕ_2 and ϕ_{2n-4} each appear in three positions in A_n. Hence, the cofactor

$$\begin{vmatrix} \phi_2 & \cdots & \phi_{n-1} \\ \vdots & & \vdots \\ \phi_{n-1} & \cdots & \phi_{2n-4} \end{vmatrix} \tag{4.8.7}$$

also appears in three positions in A_n, which yields the identities

$$A^{(n)}_{12;n-1,n} = A^{(n)}_{1n,1n} = A^{(n)}_{n-1,n;12}.$$

Similarly

$$A^{(n)}_{123;n-2,n-1,n} = A^{(n)}_{12n;1,n,n-1} = A^{(n)}_{1n,n-1;12n} = A^{(n)}_{n-2,n-1,n;123}. \tag{4.8.8}$$

Let

$$\begin{aligned} A_n &= |\phi_{i+j-2}|_n = |\phi_m|_n, && 0 \le m \le 2n-2, \\ B_n &= |x^{i+j-2}\phi_{i+j-2}|_n = |x^m\phi_m|_n, && 0 \le m \le 2n-2. \end{aligned} \tag{4.8.9}$$

Lemma.

a. $B_n = x^{n(n-1)}A_n$.
b. $B_{ij}^{(n)} = x^{n(n-1)-(i+j-2)}A_{ij}^{(n)}$.
c. $B_n^{ij} = x^{-(i+j-2)}A_n^{ij}$.

PROOF OF (A). Perform the following operations on B_n: Remove the factor x^{i-1} from row i, $1 \le i \le n$, and the factor x^{j-1} from column j, $1 \le j \le n$. The effect of these operations is to remove the factor x^{i+j-2} from the element in position (i,j).

The result is

$$B_n = x^{2(1+2+3+\cdots+\overline{n-1})}A_n,$$

which yields the stated result. Part (b) is proved in a similar manner, and (c), which contains scaled cofactors, follows by division. □

4.8.2 Hankelians Whose Elements are Differences

The h difference operator Δ_h is defined in Appendix A.8.

Theorem.

$$|\phi_m|_n = |\Delta_h^m\phi_0|_n;$$

that is, a Hankelian remains unaltered in value if each ϕ_m *is replaced by* $\Delta_h^m\phi_0$.

PROOF. *First Proof.* Denote the determinant on the left by A and perform the row operations

$$\mathbf{R}_i' = \sum_{r=0}^{i-1}(-h)^r\binom{i-1}{r}\mathbf{R}_{i-r}, \quad i = n, n-1, n-2, \ldots, 2, \tag{4.8.10}$$

on A. The result is

$$A = \left|\Delta_h^{i-1}\phi_{j-1}\right|_n. \tag{4.8.11}$$

Now, restore symmetry by performing the same operations on the columns, that is,

$$\mathbf{C}_j' = \sum_{r=0}^{j-1}(-h)^r\binom{j-1}{r}\mathbf{C}_{j-r}, \quad j = n, n-1, n-2, \ldots, 2. \tag{4.8.12}$$

The theorem appears. Note that the values of i and j are taken in descending order of magnitude.

The second proof illustrates the equivalence of row and column operations on the one hand and matrix-type products on the other (Section 2.3.2).

Second Proof. Define a triangular matrix $\mathbf{P}(x)$ as follows:

$$\mathbf{P}(x) = \left[\binom{i-1}{j-1} x^{i-j}\right]_n = \begin{bmatrix} 1 & & & \\ x & 1 & & \\ x^2 & 2x & 1 & \\ x^3 & 3x^2 & 3x & 1 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}_n . \tag{4.8.13}$$

Since $|\mathbf{P}(x)| = |\mathbf{P}^T(x)| = 1$ for all values of x.

$$\begin{aligned} A &= |\mathbf{P}(-h)\mathbf{A}\mathbf{P}^T(-h)|_n \\ &= \left|(-h)^{i-j}\binom{i-1}{j-1}\right|_n |\phi_{i+j-2}|_n \left|(-h)^{j-i}\binom{j-1}{i-1}\right|_n \\ &= |\alpha_{ij}|_n \end{aligned} \tag{4.8.14}$$

where, applying the formula for the product of three determinants at the end of Section 3.3.5,

$$\begin{aligned} \alpha_{ij} &= \sum_{r=1}^{i}\sum_{s=1}^{j}(-h)^{i-r}\binom{i-1}{r-1}\phi_{r+s-2}(-h)^{j-s}\binom{j-1}{s-1} \\ &= \sum_{r=0}^{i-1}\binom{i-1}{r}(-h)^{i-1-r}\sum_{s=0}^{j-1}\binom{j-1}{s}(-h)^{j-1-s}\phi_{r+s} \\ &= \sum_{r=0}^{i-1}\binom{i-1}{r}(-h)^{i-1-r}\Delta_h^{j-1}\phi_r \\ &= \Delta_h^{j-1}\sum_{r=0}^{i-1}\binom{i-1}{r}(-h)^{i-1-r}\phi_r \\ &= \Delta_h^{j-1}\Delta_h^{i-1}\phi_0 \\ &= \Delta_n^{i+j-2}\phi_0. \end{aligned} \tag{4.8.15}$$

The theorem follows. Simple differences are obtained by putting $h = 1$. □

Exercise. Prove that

$$\sum_{r=1}^{n}\sum_{s=1}^{n} h^{r+s-2}A_{rs}(x) = A_{11}(x-h).$$

4.8.3 Two Kinds of Homogeneity

The definitions of a function which is homogeneous in its variables and of a function which is homogeneous in the suffixes of its variables are given in Appendix A.9.

Lemma. *The determinant* $A_n = |\phi_m|_n$ *is*

a. *homogeneous of degree* n *in its elements and*
b. *homogeneous of degree* $n(n-1)$ *in the suffixes of its elements.*

Proof. Each of the $n!$ terms in the expansion of A_n is of the form

$$\pm\phi_{1+k_1-2}\phi_{2+k_2-2}\cdots\phi_{n+k_n-2},$$

where $\{k_r\}_1^n$ is a permutation of $\{r\}_1^n$. The number of factors in each term is n, which proves (a). The sum of the suffixes in each term is

$$\begin{aligned}\sum_{r=1}^{n}(r+k_r-2) &= 2\sum_{r=1}^{n} r - 2n\\ &= n(n-1),\end{aligned}$$

which is independent of the choice of $\{k_r\}_1^n$, that is, the sum is the same for each term, which proves (b). □

Exercise. Prove that $A_{ij}^{(n)}$ is homogeneous of degree $(n-1)$ in its elements and homogeneous of degree $(n^2-n+2-i-j)$ in the suffixes of its elements. Prove also that the scaled cofactor A_n^{ij} is homogeneous of degree (-1) in its elements and homogeneous of degree $(2-i-j)$ in the suffixes of its elements.

4.8.4 The Sum Formula

The sum formula for general determinants is given in Section 3.2.4. The sum formula for Hankelians can be expressed in the form

$$\sum_{m=1}^{n}\phi_{m+r-2}A_n^{ms} = \delta_{rs}, \quad 1 \le r,s \le n. \tag{4.8.16}$$

Exercise. Prove that, in addition to the sum formula,

a. $\displaystyle\sum_{m=1}^{n}\phi_{m+n-1}A_{im}^{(n)} = -A_{i,n+1}^{(n+1)}, \quad 1 \le i \le n,$

b. $\displaystyle\sum_{m=1}^{n}\phi_{m+n}A_{im}^{(n)} = A_{1n}^{(n+1)},$

where the cofactors are unscaled. Show also that there exist further sums of a similar nature which can be expressed as cofactors of determinants of orders $(n+2)$ and above.

4.8.5 Turanians

A Hankelian in which $a_{ij} = \phi_{i+j-2+r}$ is called a Turanian by Karlin and Szegö and others.

Let

$$T^{(n,r)} = \begin{cases} |\phi_{m+r}|_n, & 0 \le m \le 2n-2, \\ |\phi_m|_n, & r \le m \le 2n-2+r, \\ \begin{vmatrix} \phi_r & \cdots & \phi_{n-1+r} \\ \cdots\cdots & \cdots & \cdots\cdots \\ \phi_{n-1+r} & \cdots & \phi_{2n-2+r} \end{vmatrix}_n \\ \left|\mathbf{C}_r\ \mathbf{C}_{r+1}\ \mathbf{C}_{r+2} \cdots \mathbf{C}_{n-1+r}\right|. \end{cases} \tag{4.8.17}$$

Theorem 4.28.

$$\begin{vmatrix} T^{(n,r+1)} & T^{(n,r)} \\ T^{(n,r)} & T^{(n,r-1)} \end{vmatrix} = T^{(n+1,r-1)} T^{(n-1,r+1)}.$$

PROOF. Denote the determinant by T. Then, each of the Turanian elements in T is of order n and is a minor of one of the corner elements in $T^{(n+1,r-1)}$. Applying the Jacobi identity (Section 3.6),

$$\begin{aligned} T &= \begin{vmatrix} T_{11}^{(n+1,r-1)} & T_{1,n+1}^{(n+1,r-1)} \\ T_{n+1,1}^{(n+1,r-1)} & T_{n+1,n+1}^{(n+1,r-1)} \end{vmatrix} \\ &= T^{(n+1,r-1)} T_{1,n+1;1,n+1}^{(n+1,r-1)} \\ &= T^{(n+1,r-1)} T^{(n-1,r+1)}, \end{aligned}$$

which proves the theorem. □

Let

$$\begin{aligned} A_n &= T^{(n,0)} = |\phi_{i+j-2}|_n, \\ F_n &= T^{(n,1)} = |\phi_{i+j-1}|_n, \\ G_n &= T^{(n,2)} = |\phi_{i+j}|_n. \end{aligned} \tag{4.8.18}$$

Then, the particular case of the theorem in which $r = 1$ can be expressed in the form

$$A_n G_n - A_{n+1} G_{n-1} = F_n^2. \tag{4.8.19}$$

This identity is applied in Section 4.12.2 on generalized geometric series.

Omit the parameter r in $T^{(n,r)}$ and write T_n.

Theorem 4.29. *For all values of* r,

$$\begin{vmatrix} T_{11}^{(n)} & T_{1,n+1}^{(n+1)} \\ T_{n1}^{(n)} & T_{n,n+1}^{(n+1)} \end{vmatrix} - T_n T_{1n;1,n+1}^{(n+1)} = 0.$$

PROOF. The identity is a particular case of Jacobi variant (A) (Section 3.6.3),

$$\begin{vmatrix} T_{ip}^{(n)} & T_{i,n+1}^{(n+1)} \\ T_{jp}^{(n)} & T_{j,n+1}^{(n+1)} \end{vmatrix} - T_n T_{ij;p,n+1}^{(n+1)} = 0, \tag{4.8.20}$$

where $(i, j, p) = (1, n, 1)$.

Let

$$\begin{aligned} A_n &= T^{(n,r)}, \\ B_n &= T^{(n,r+1)}. \end{aligned}$$

Then Theorem 4.29 is satisfied by both A_n and B_n. □

Theorem 4.30. *For all values of r,*

a. $A_n B_{n+1,n}^{(n+1)} - B_n A_{n+1,n}^{(n+1)} + A_{n+1} B_{n-1} = 0.$
b. $B_{n-1} A_{n+1,n}^{(n+1)} - A_n B_{n,n-1}^{(n)} + A_{n-1} B_n = 0.$

PROOF.

$$\begin{aligned} B_n &= (-1)^n A_{1,n+1}^{(n+1)}, \\ B_{n+1,n}^{(n+1)} &= (-1)^n A_{n1}^{(n+1)}, \\ B_{n-1} &= (-1)^{n-1} A_{1n}^{(n)} \\ &= (-1)^n A_{n,n+1;1,n+1}^{(n+1)}, \\ A_{1n;n,n+1}^{(n+1)} &= A_{1,n-1}^{(n)} \\ &= (-1)^{n-1} B_{n,n-1}^{(n)}. \end{aligned} \tag{4.8.21}$$

Denote the left-hand side of (a) by Y_n. Then, applying the Jacobi identity to A_{n+1},

$$\begin{aligned} (-1)^n Y_n &= \begin{vmatrix} A_{n1}^{(n+1)} & A_{n,n+1}^{(n+1)} \\ A_{n+1,1}^{(n+1)} & A_{n+1,n+1}^{(n+1)} \end{vmatrix} - A_{n+1} A_{n,n+1;1,n+1}^{(n+1)} \\ &= 0, \end{aligned}$$

which proves (a).

The particular case of (4.8.20) in which $(i, j, p) = (n, 1, n)$ and T is replaced by A is

$$\begin{vmatrix} A_{n-1} & A_{n,n+1}^{(n+1)} \\ A_{1n}^{(n)} & A_{1,n+1}^{(n+1)} \end{vmatrix} - A_n A_{n1;n,n+1}^{(n+1)} = 0. \tag{4.8.22}$$

The application of (4.8.21) yields (b). □

This theorem is applied in Section 6.5.1 on Toda equations.

4.8.6 Partial Derivatives with Respect to ϕ_m

In A_n, the elements ϕ_m, ϕ_{2n-2-m}, $0 \le m \le n-2$, each appear in $(m+1)$ positions. The element ϕ_{n-1} appears in n positions, all in the secondary diagonal. Hence, $\partial A_n/\partial\phi_m$ is the sum of a number of cofactors, one for each appearance of ϕ_m. Discarding the suffix n,

$$\frac{\partial A}{\partial \phi_m} = \sum_{p+q=m+2} A_{pq}. \tag{4.8.23}$$

For example, when $n \ge 4$,

$$\begin{aligned}\frac{\partial A}{\partial \phi_3} &= \sum_{p+q=5} A_{pq}\\ &= A_{41} + A_{32} + A_{23} + A_{14}.\end{aligned}$$

By a similar argument,

$$\frac{\partial A_{ij}}{\partial \phi_m} = \sum_{p+q=m+2} A_{ip,jq}, \tag{4.8.24}$$

$$\frac{\partial A_{ir,js}}{\partial \phi_m} = \sum_{p+q=m+2} A_{irp,jsq}. \tag{4.8.25}$$

Partial derivatives of the scaled cofactors A^{ij} and $A^{ir,js}$ can be obtained from (4.8.23)–(4.8.25) with the aid of the Jacobi identity:

$$\frac{\partial A^{ij}}{\partial \phi_m} = -\sum_{p+q=m+2} A^{iq}A^{pj} \tag{4.8.26}$$

$$= \sum_{p+q=m+2} \begin{vmatrix} A^{ij} & A^{iq} \\ A^{pj} & \bullet \end{vmatrix}. \tag{4.8.27}$$

The proof is simple.

Lemma.

$$\frac{\partial A^{ir,js}}{\partial \phi_m} = \sum_{p+q=m+2} \begin{vmatrix} A^{ij} & A^{is} & A^{iq} \\ A^{rj} & A^{rs} & A^{rq} \\ A^{pj} & A^{ps} & \bullet \end{vmatrix}, \tag{4.8.28}$$

which is a development of (4.8.27).

PROOF.

$$\begin{aligned}\frac{\partial A^{ir,js}}{\partial \phi_m} &= \frac{1}{A^2}\left[A\frac{\partial A_{ir,js}}{\partial \phi_m} - A_{ir,js}\frac{\partial A}{\partial \phi_m}\right]\\ &= \frac{1}{A^2}\sum_{p,q}\left[AA_{irp,jsq} - A_{ir,js}A_{pq}\right]\\ &= \sum_{p,q}\left[A^{irp,jsq} - A^{ir,js}A^{pq}\right].\end{aligned} \tag{4.8.29}$$

The lemma follows from the second-order and third-order Jacobi identities. □

4.8.7 Double-Sum Relations

When A_n is a Hankelian, the double-sum relations (A)–(D) in Section 3.4 with $f_r = g_r = \frac{1}{2}$ can be expressed as follows. Discarding the suffix n,

$$\frac{A'}{A} = D(\log A) = \sum_{m=0}^{2n-2} \phi'_m \sum_{p+q=m+2} A^{pq}, \tag{A$_1$}$$

$$(A^{ij})' = -\sum_{m=0}^{2n-2} \phi'_m \sum_{p+q=m+2} A^{ip} A^{jq}, \tag{B$_1$}$$

$$\sum_{m=0}^{2n-2} \phi_m \sum_{p+q=m+2} A^{pq} = n, \tag{C$_1$}$$

$$\sum_{m=0}^{2n-2} \phi_m \sum_{p+q=m+2} A^{ip} A^{jq} = A^{ij}. \tag{D$_1$}$$

Equations (C_1) and (D_1) can be proved by putting $a_{ij} = \phi_{i+j-2}$ in (C) and (D), respectively, and rearranging the double sum, but they can also be proved directly by taking advantage of the first kind of homogeneity of Hankelians and applying the Euler theorem in Appendix A.9.

A_n and $A_{ij}^{(n)}$ are homogeneous polynomial functions of their elements of degrees n and $n-1$, respectively, so that A_n^{ij} is a homogeneous function of degree (-1). Hence, denoting the sums in (C_1) and (D_1) by S_1 and S_2,

$$\begin{aligned}
AS_1 &= \sum_{m=0}^{2n-2} \phi_m \frac{\partial A}{\partial \phi_m} \\
&= nA, \\
S_2 &= -\sum_{m=0}^{2n-2} \phi_m \frac{\partial A^{ij}}{\partial \phi_m} \\
&= A^{ij}.
\end{aligned}$$

which prove (C_1) and (D_1).

Theorem 4.31.

$$\sum_{m=1}^{2n-2} m\phi_m \sum_{p+q=m+2} A^{pq} = n(n-1), \tag{C$_2$}$$

$$\sum_{m=1}^{2n-2} m\phi_m \sum_{p+q=m+1} A^{ip}A^{jq} = (i+j-2)A^{ij}. \tag{D$_2$}$$

These can be proved by putting $a_{ij} = \phi_{i+j-2}$ and $f_r = g_r = r-1$ in (C) and (D), respectively, and rearranging the double sum, but they can also be proved directly by taking advantage of the second kind of homogeneity of Hankelians and applying the modified Euler theorem in Appendix A.9.

PROOF. A_n and A_n^{ij} are homogeneous functions of degree $n(n-1)$ and $(2-i-j)$, respectively, in the suffixes of their elements. Hence, denoting the sums by S_1 and S_2. respectively,

$$\begin{aligned} AS_1 &= \sum_{m=1}^{2n-2} m\phi_m \frac{\partial A}{\partial \phi_m} \\ &= n(n-1)A, \\ S_2 &= -\sum_{m=1}^{2n-2} m\phi_m \frac{\partial A^{ij}}{\partial \phi_m} \\ &= -(2-i-j)A^{ij}. \end{aligned}$$

The theorem follows. □

Theorem 4.32.

$$\sum_{r=1}^{n}\sum_{s=1}^{n}(r+s-2)\phi_{r+s-3}A^{rs} = 0, \tag{E}$$

which can be rearranged in the form

$$\sum_{m=1}^{2n-2} m\phi_{m-1} \sum_{p+q=m+2} A^{pq} = 0 \tag{E$_1$}$$

and

$$\begin{aligned} \sum_{r=1}^{n}\sum_{s=1}^{n}(r+s-2)\phi_{r+s-3}A^{ir}A^{sj} &= iA^{i+1,j} + jA^{i,j+1} \\ &= 0, \qquad (i,j) = (n,n). \end{aligned} \tag{F}$$

which can be rearranged in the form

$$\begin{aligned} \sum_{m=1}^{2n-2} m\phi_{m-1} \sum_{p+q=m+2} A^{ip}A^{jq} &= iA^{i+1,j} + jA^{i,j+1} \\ &= 0, \qquad (i,j) = (n,n). \end{aligned} \tag{F$_1$}$$

Proof of (F). Denote the sum by S and apply the Hankelian relation $\phi_{r+s-3} = a_{r,s-1} = a_{r-1,s}$.

$$S = \sum_{s=1}^{n}(s-1)A^{sj}\sum_{r=1}^{n} a_{r,s-1}A^{ir} + \sum_{r=1}^{n}(r-1)A^{ir}\sum_{s=1}^{n} a_{r-1,s}A^{sj}$$
$$= \sum_{s=1}^{n}(s-1)A^{sj}\delta_{s-1,i} + \sum_{r=1}^{n}(r-1)A^{ir}\delta_{r-1,j}.$$

The proof of (F) follows. Equation (E) is proved in a similar manner. □

Exercises

Prove the following:

1. $\displaystyle\sum_{p+q=m+2} A^{ij,pq} = 0.$

2. $\displaystyle\sum_{m=0}^{2n-2} \phi_m \sum_{q+q=m+2} A_{ip,jq} = (n-1)A_{ij}.$

3. $\displaystyle\sum_{m=1}^{2n-2} m\phi_m \sum_{p+q=m+2} A_{ip,jq} = (n^2-n-i-j+2)A_{ij}.$

4. $\displaystyle\sum_{m=0}^{2n-2} \phi_m \sum_{p+q=m+2} A^{ijp,hkq} = nA^{ij,hk}.$

5. $\displaystyle\sum_{m=1}^{2n-2} m\phi_m \sum_{p+q=m+2} A^{ijp,hkq} = (n^2-n-i-j-h-k-4)A^{ij,hk}.$

6. $\displaystyle\sum_{m=1}^{2n-2} m\phi_{m-1} \sum_{p+q=m+2} A^{ijp,hkq}$
$= iA^{i+1,j;hk} + jA^{i,j+1;hk} + hA^{ij;h+1,k} + kA^{ij;h,k+1}.$

7. $\displaystyle\sum_{m=0}^{2n-2} \sum_{p+q=m+2} \phi_{p+r-1}\phi_{q+r-1}A^{pq} = \phi_{2r}, \quad 0 \le r \le n-1.$

8. $\displaystyle\sum_{m=1}^{2n-2} m \sum_{p+q=m+2} \phi_{p+r-1}\phi_{q+r-1}A^{pq} = 2r\phi_{2r}, \quad 0 \le r \le n-1.$

9. Prove that

$$\sum_{r=1}^{n-1} rA^{r+1,j}\sum_{m=1}^{n} \phi_{m+r-2}A^{im} = iA^{i+1,j}$$

by applying the sum formula for Hankelians and, hence, prove (F_1) directly. Use a similar method to prove (E_1) directly.

4.9 Hankelians 2

4.9.1 The Derivatives of Hankelians with Appell Elements

The Appell polynomial

$$\phi_m = \sum_{r=0}^{m} \binom{m}{r} \alpha_r x^{m-r} \tag{4.9.1}$$

and other functions which satisfy the Appell equation

$$\phi'_m = m\phi_{m-1}, \quad m = 1, 2, 3, \dots, \tag{4.9.2}$$

play an important part in the theory of Hankelians. Extensive notes on these functions are given in Appendix A.4.

Theorem 4.33. *If*

$$A_n = |\phi_m|_n, \quad 0 \le m \le 2n-2,$$

where ϕ_m *satisfies the Appell equation, then*

$$A'_n = \phi'_0 A_{11}^{(n)}.$$

Proof. Split off the $m = 0$ term from the double sum in relation (A_1) in Section 4.8.7:

$$\begin{aligned} \frac{A'}{A} &= \phi'_0 \sum_{p+q=2} A^{pq} + \sum_{m=1}^{2n-2} \phi'_m \sum_{p+q=m+2} A^{pq} \\ &= \phi'_0 A^{11} + \sum_{m=1}^{2n-2} m\phi_{m-1} \sum_{p+q=m+2} A^{pq}. \end{aligned}$$

The theorem follows from (E_1) and remains true if the Appell equation is generalized to

$$\phi'_m = mF\phi_{m-1}, \quad F = F(x). \tag{4.9.3}$$

□

Corollary. *If* ϕ_m *is an Appell polynomial, then* $\phi_0 = \alpha_0 =$ *constant,* $A' = 0$*, and, hence,* A *is independent of* x*, that is,*

$$|\phi_m(x)|_n = |\phi_m(0)|_n = |\alpha_m|_n, \quad 0 \le m \le 2n-2. \tag{4.9.4}$$

This identity is one of a family of identities which appear in Section 5.6.2 on distinct matrices with nondistinct determinants.

If ϕ_m satisfies (4.9.3) and $\phi_0 =$ constant, it does not follows that ϕ_m is an Appell polynomial. For example, if

$$\phi_m = (1 - x^2)^{-m/2} P_m,$$

where P_m is the Legendre polynomial, then ϕ_m satisfies (4.9.3) with

$$F = (1 - x^2)^{-3/2}$$

and $\phi_0 = P_0 = 1$, but ϕ_m is not a polynomial. These relations are applied in Section 4.12.1 to evaluate $|P_m|_n$.

Examples

1. If

$$\phi_m = \frac{1}{m+1} \sum_{r=1}^{k} b_r \{f(x) + c_r\}^{m+1},$$

where $\sum_{r=1}^{k} b_r = 0$, b_r and c_r are independent of x, and k is arbitrary, then

$$\phi'_m = mf'(x)\phi_{m-1},$$
$$\phi_0 = \sum_{r=1}^{k} b_r c_r = \text{constant}.$$

Hence, $A = |\phi_m|_n$ is independent of x.

2. If

$$\phi_m(x, \xi) = \frac{1}{m+1} \left[(\xi + x)^{m+1} - c(\xi - 1)^{m+1} + (c-1)\xi^{m+1} \right],$$

then

$$\frac{\partial \phi_m}{\partial \xi} = m\phi_{m-1},$$
$$\phi_0 = x + c.$$

Hence, A is independent of ξ. This relation is applied in Section 4.11.4 on a nonlinear differential equation.

Exercises

1. Denote the three cube roots of unity by 1, ω, and ω^2, and let$A = |\phi_m|_n$, $0 \le m \le 2n-2$, where

a. $\phi_m = \dfrac{1}{3(m+1)} \left[(x+b+c)^{m+1} + \omega(x+\omega c)^{m+1} + \omega^2(x+\omega^2 c)^{m+1} \right]$,

b. $\phi_m = \dfrac{1}{3(m+1)} \left[(x+b+c)^{m+1} + \omega^2(x+\omega c)^{m+1} + \omega(x+\omega^2 c)^{m+1} \right]$,

c. $\phi_m = \dfrac{1}{3(m+1)(m+2)} \left[(x+c)^{m+2} + \omega^2(x+\omega c)^{m+2} + \omega(x+\omega^2 c)^{m+2} \right]$.

Prove that ϕ_m and hence also A is real in each case, and that in cases (a) and (b), A is independent of x, but in case (c), $A' = cA_{11}$.

2. The Yamazaki–Hori determinant A_n is defined as follows:

$$A_n = |\phi_m|_n, \quad 0 \le m \le 2n-2,$$

where

$$\phi_m = \frac{1}{m+1}\left[p^2(x^2-1)^{m+1} + q^2(y^2-1)^{m+1}\right], \quad p^2+q^2=1.$$

Let

$$B_n = |\psi_m|_n, \quad 0 \le m \le 2n-2,$$

where

$$\psi_m = \frac{\phi_m}{(x^2-y^2)^{m+1}}\,.$$

Prove that

$$\frac{\partial \psi_m}{\partial x} = mF\psi_{m-1},$$

where

$$F = -\frac{2x(y^2-1)}{(x^2-y^2)^2}\,.$$

Hence, prove that

$$\frac{\partial B_n}{\partial x} = FB_{11}^{(n)},$$
$$(x^2-y^2)\frac{\partial A_n}{\partial x} = 2x\left[n^2 A_n - (y^2-1)A_{11}^{(n)}\right].$$

Deduce the corresponding formulas for $\partial B_n/\partial y$ and $\partial A_n/\partial y$ and hence prove that A_n satisfies the equation

$$\left(\frac{x^2-1}{x}\right) z_x + \left(\frac{y^2-1}{y}\right) z_y = 2n^2 z.$$

3. If $A_n = |\phi_m|_n$, $0 \le m \le 2n-2$, where ϕ_m satisfies the Appell equation, prove that

a. $(A_n^{ij})' = -\phi_0' A_n^{i1} A_n^{j1} - (iA_n^{i+1,j} + jA_n^{i,j+1}), \quad (i,j) \ne (n,n),$
b. $(A_n^{nn})' = -\phi_0'(A_n^{1n})^2$.

4. Apply Theorem 4.33 and the Jacobi identity to prove that

$$\left(\frac{A_n}{A_{n-1}}\right)' = \phi_0' \left(\frac{A_{1n}^{(n)}}{A_{n-1}}\right)^2.$$

Hence, prove (3b).

5. If

$$A_n = |\phi_m|_n, \quad 0 \le m \le 2n-2,$$
$$F_n = |\phi_m|_n, \quad 1 \le m \le 2n-1,$$
$$G_n = |\phi_m|_n, \quad 2 \le m \le 2n,$$

where ϕ_m is an Appell polynomial, apply Exercise 3a in which the cofactors are scaled to prove that

$$D(A_{ij}^{(n)}) = -\left(iA_{i+1,j}^{(n)} + jA_{i,j+1}^{(n)}\right)$$

in which the cofactors are unscaled. Hence, prove that

a. $D^r(F_n) = (-1)^{n+r} r! A_{r+1,n+1}^{(n+1)}, \quad 0 \le r \le n;$
b. $D^n(F_n) = n! A_n;$
c. F_n is a polynomial of degree n;
d. $D^r(G_n) = (-1)^r r! \sum_{p+q=r+2} A_{pq}^{(n+1)}, \quad 0 \le r \le 2n;$
e. $D^{2n}(G_n) = (2n)! A_n;$
f. G_n is a polynomial of degree $2n$.

6. Let B_n denote the determinant of order $(n+1)$ obtained by bordering $A_n(0)$ by the row

$$\mathbf{R} = \left[1 \ \ -x \ \ x^2 \ \ -x^3 \cdots (-x)^{n-1} \ \bullet\right]_{n+1}$$

at the bottom and the column $\mathbf{R}^T$ on the right. Prove that

$$B_n = -\sum_{r=0}^{2n-2} (-x)^r \sum_{p+q=r+2} A_{pq}^{(n)}(0).$$

Hence, by applying a formula in the previous exercise and then the Maclaurin expansion formula, prove that

$$B_n = -G_{n-1}.$$

7. Prove that

$$D^r(A_{ij}) = \frac{(-1)^r r!}{(i-1)!(j-1)!} \sum_{s=0}^{r} \frac{(i+r-s-1)!(j+s-1)!}{s!(r-s)!} A_{i+r-s,j+s}.$$

8. Apply the double-sum relation (A_1) in Section 4.8.7 to prove that G_n satisfies the differential equation

$$\sum_{m=0}^{2n-1} \frac{(-1)^m \phi_m D^{m+1}(G_n)}{m!} = 0.$$

4.9.2 The Derivatives of Turanians with Appell and Other Elements

Let

$$T = T^{(n,r)} = \left|\mathbf{C}_r\ \mathbf{C}_{r+1}\ \mathbf{C}_{r+2} \cdots \mathbf{C}_{r+n-1}\right|_n, \tag{4.9.5}$$

where

$$\begin{aligned}\mathbf{C}_j &= \left[\phi_j\ \phi_{j+1}\ \phi_{j+2} \cdots \phi_{j+n-1}\right]^T,\\ \phi'_m &= mF\phi_{m-1}.\end{aligned}$$

Theorem 4.34.

$$T' = rF\left|\mathbf{C}_{r-1}\ \mathbf{C}_{r+1}\ \mathbf{C}_{r+2} \cdots \mathbf{C}_{r+n-1}\right|.$$

PROOF.

$$\mathbf{C}'_j = F\left(j\mathbf{C}_{j-1} + \mathbf{C}^*_j\right),$$

where

$$\mathbf{C}^*_j = \left[0\ \phi_j\ 2\phi_{j+1}\ 3\phi_{j+2} \cdots (n-1)\phi_{j+n-2}\right]^T.$$

Hence,

$$\begin{aligned}T' &= \sum_{j=r}^{r+n-1} \left|\mathbf{C}_r\ \mathbf{C}_{r+1} \cdots \mathbf{C}_{j-1}\ \mathbf{C}'_j \cdots \mathbf{C}_{r+n-1}\right|\\ &= F\sum_{j=r}^{r+n-1} \left|\mathbf{C}_r\ \mathbf{C}_{r+1} \cdots \mathbf{C}_{j-1}(j\mathbf{C}_{j-1} + \mathbf{C}^*_j) \cdots \mathbf{C}_{r+n-1}\right|\\ &= rF\left|\mathbf{C}_{r-1}\ \mathbf{C}_{r+1}\ \mathbf{C}_{r+2} \cdots \mathbf{C}_{r+n-1}\right|\\ &\quad + F\sum_{j=r}^{r+n-1} \left|\mathbf{C}_r\ \mathbf{C}_{r+1} \cdots \mathbf{C}^*_j \cdots \mathbf{C}_{r+n-1}\right|\end{aligned}$$

after discarding determinants with two identical columns. The sum is zero by Theorem 3.1 in Section 3.1 on cyclic dislocations and generalizations. The theorem follows. □

The column parameters in the above definition of T are consecutive. If they are not consecutive, the notation

$$T_{j_1 j_2 \ldots j_n} = \left|\mathbf{C}_{j_1}\ \mathbf{C}_{j_2} \cdots \mathbf{C}_{j_r} \cdots \mathbf{C}_{j_n}\right| \tag{4.9.6}$$

is convenient.

$$T'_{j_1 j_2 \ldots j_n} = F\sum_{r=1}^{n} j_r\left|\mathbf{C}_{j_1}\ \mathbf{C}_{j_2} \cdots \mathbf{C}_{(j_r-1)} \cdots \mathbf{C}_{j_n}\right|. \tag{4.9.7}$$

Higher derivatives may be found by repeated application of this formula, but no simple formula for $D^k(T_{j_1 j_2 \ldots j_n})$ has been found. However, the method can be illustrated adequately by taking the particular case in which $(n,r) = (4,3)$ and ϕ_m is an Appell polynomial so that $F = 1$.

Let

$$T = \left|\mathbf{C}_3\ \mathbf{C}_4\ \mathbf{C}_5\ \mathbf{C}_6\right| = T_{3456}.$$

Then

$$\begin{aligned} D(T) &= 3T_{2456}, \\ D^2(T)/2! &= 3T_{1456} + 6T_{2356}, \\ D^3(T)/3! &= T_{0456} + 8T_{1356} + 10T_{2346}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ D^9(T)/9! &= T_{0126} + 8T_{0135} + 10T_{0234}, \\ D^{10}(T)/10! &= 3T_{0125} + 6T_{0134}, \\ D^{11}(T)/11! &= 3T_{0124}, \\ D^{12}(T)/12! &= T_{0123}, \\ &= |\phi_m|_4, \quad 0 \le m \le 6 \\ &= \text{constant}. \end{aligned} \tag{4.9.8}$$

The array of coefficients is symmetric about the sixth derivative. This result and several others of a similar nature suggest the following conjecture.

Conjecture.

$$\begin{aligned} D^{nr}\{T^{(n,r)}\} &= (nr)!|\phi_m|_n, \quad 0 \le m \le 2n-2 \\ &= \text{constant}. \end{aligned}$$

Assuming this conjecture to be valid, $T^{(n,r)}$ is a polynomial of degree nr and not $n(n+r-1)$ as may be expected by examining the product of the elements in the secondary diagonal. Hence, the loss of degree due to cancellations is $n(n-1)$.

Let

$$T = T^{(n,r)} = \left|\mathbf{C}_r\ \mathbf{C}_{r+1}\ \mathbf{C}_{r+2} \cdots \mathbf{C}_{r+n-1}\right|_n,$$

where

$$\begin{aligned} \mathbf{C}_j &= \left[\psi_{r+j-1}\ \psi_{r+j}\ \psi_{r+j+1} \cdots \psi_{r+j+n-2}\right]_n^T \\ \psi_m &= \frac{f^{(m)}(x)}{m!}, \quad f(x) \text{ arbitrary} \\ \psi'_m &= (m+1)\psi_{m+1}. \end{aligned} \tag{4.9.9}$$

Theorem 4.35.

$$\begin{aligned} T' &= (2n-1+r)\left|\mathbf{C}_r\ \mathbf{C}_{r+1} \cdots \mathbf{C}_{r+n-2}\ \mathbf{C}_{r+n}\right|_n \\ &= -(2n-1+r)T^{(n+1,r)}_{n+1,n}. \end{aligned}$$

PROOF. The sum formula for T can be expressed in the form

$$\sum_{j=1}^{n} \psi_{r+i+j-1} T_{ij}^{(n,r)} = -\delta_{in} T_{n+1,n}^{(n+1,r)}, \tag{4.9.10}$$

$$\mathbf{C}'_j = \left[(r+j)\psi_{r+j}\ (r+j+1)\psi_{r+j+1} \cdots (r+j+n-1)\psi_{r+j+n-1}\right]_n^T. \tag{4.9.11}$$

Let

$$\begin{aligned} \mathbf{C}^*_j &= \mathbf{C}'_j - (r+j)\mathbf{C}_{j+1} \\ &= \left[0\ \psi_{r+j+1}\ 2\psi_{r+j+2} \cdots (n-1)\psi_{r+j+n-1}\right]_n^T. \end{aligned} \tag{4.9.12}$$

Differentiating the columns of T,

$$T' = \sum_{j=1}^{n} U_j,$$

where

$$U_j = \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}'_j\ \mathbf{C}_{j+1} \cdots \mathbf{C}_n\right|_n, \quad 1 \le j \le n.$$

Let

$$\begin{aligned} V_j &= \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}^*_j\ \mathbf{C}_{j+1} \cdots \mathbf{C}_n\right|_n, \quad 1 \le j \le n \\ &= \sum_{i=2}^{n} (i-1)\psi_{r+i+j-1} T_{ij}. \end{aligned} \tag{4.9.13}$$

Then, performing an elementary column operation on U_j,

$$\begin{aligned} U_j &= V_j, \quad 1 \le j \le n-1 \\ U_n &= \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\ \mathbf{C}'_n\right| \\ &= \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\ \mathbf{C}^*_n\right| + (r+n)\left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\ \mathbf{C}_{n+1}\right| \\ &= V_n - (r+n) T_{n+1,n}^{(n+1,r)}. \end{aligned} \tag{4.9.14}$$

Hence,

$$\begin{aligned} T' + (r+n) T_{n+1,n}^{(n+1,r)} &= \sum_{j=1}^{n} V_j \\ &= \sum_{j=1}^{n} (i-1) \sum_{j=1}^{n} \psi_{r+i+j-1} T_{ij} \\ &= -T_{n+1,n}^{(n+1,r)} \sum_{i=2}^{n} (i-1)\delta_{in} \\ &= -(n-1) T_{n+1,n}^{(n+1,r)}. \end{aligned}$$

The theorem follows. □

Theorem 4.36.

$$D(T_{11}^{(n,r)}) = -(2n+r-1)T_{n,n-1}^{(n,r+2)}.$$

PROOF.

$$T_{11}^{(n,r)} = T^{(n-1,r+2)}.$$

The theorem follows by adjusting the parameters in Theorem 4.35. Both these theorems are applied in Section 6.5.3 on the Milne–Thomson equation. □

4.9.3 Determinants with Simple Derivatives of All Orders

Let $\mathbf{Z}_r$ denote the column vector with $(n+1)$ elements defined as

$$\mathbf{Z}_r = \left[0_r\ \phi_0\ \phi_1\ \phi_2 \cdots \phi_{n-r}\right]_{n+1}^T, \quad 1 \le r \le n, \tag{4.9.15}$$

where 0_r denotes an unbroken sequence of r zero elements and ϕ_m is an Appell polynomial.

Let

$$B = \left|\mathbf{Z}_1\ \mathbf{C}_0\ \mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\right|_{n+1}, \tag{4.9.16}$$

where $\mathbf{C}_j$ is defined in (4.9.5). Differentiating B repeatedly, it is found that, apart from a constant factor, only the first column changes:

$$D^r(B) = (-1)^r r! \left|\mathbf{Z}_{r+1}\ \mathbf{C}_0\ \mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\right|_{n+1}, \quad 0 \le r \le n-1.$$

Hence

$$\begin{aligned} D^{n-1}(B) &= (-1)^{n-1}(n-1)!\phi_0 \left|\mathbf{C}_0\ \mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\right|_n \\ &= (-1)^{n-1}(n-1)!\phi_0 |\phi_m|_n, \quad 0 \le m \le 2n-2 \\ &= \text{constant}; \end{aligned}$$

that is, B is a polynomial of degree $(n-1)$ and not (n^2-1), as may be expected by examining the product of the elements in the secondary diagonal of B. Once again, the loss of degree due to cancellations is $n(n-1)$.

Exercise

Let

$$S_m = \sum_{r+s=m} \phi_r \phi_s.$$

This function appears in Exercise 2 at the end of Appendix A.4 on Appell polynomials. Also, let

$$\begin{aligned} \mathbf{C}_j &= \left[S_{j-1}\ S_j\ S_{j+1} \cdots S_{j+n-2}\right]_n^T, \quad 1 \le j \le n, \\ \mathbf{K} &= \left[\bullet\ S_0\ S_1\ S_2 \cdots S_{n-2}\right]_n^T, \\ E &= |S_m|_n, \quad 0 \le m \le 2n-2. \end{aligned}$$

Prove that

$$D^r(E) = (-1)^{r+1} r! \sum_{i=2}^{n} S_{i-2} E_{ir}$$
$$= (-1)^{r+1} r! \big| \mathbf{C}_1 \ \mathbf{C}_2 \cdots \mathbf{C}_{r-1} \ \mathbf{K} \ \mathbf{C}_{r+1} \cdots \mathbf{C}_n \big|_n.$$

4.10 Henkelians 3

4.10.1 The Generalized Hilbert Determinant

The generalized Hilbert determinant K_n is defined as

$$K_n = K_n(h) = |k_{ij}|_n,$$

where

$$k_{ij} = \frac{1}{h+i+j-1}, \quad h \neq 1-i-j, \quad 1 \leq i, j \leq n. \tag{4.10.1}$$

In some detail,

$$K_n = \begin{vmatrix} \frac{1}{h+1} & \frac{1}{h+2} & \cdots & \frac{1}{h+n} \\ \frac{1}{h+2} & \frac{1}{h+3} & \cdots & \frac{1}{h+n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{h+n} & \frac{1}{h+n+1} & \cdots & \frac{1}{h+2n-1} \end{vmatrix}_n . \tag{4.10.2}$$

K_n is of fundamental importance in the evaluation of a number of determinants, not necessarily Hankelians, whose elements are related to k_{ij}. The values of such determinants and their cofactors can, in some cases, be simplified by expressing them in terms of K_n and its cofactors. The given restrictions on h are the only restrictions on h which may therefore be regarded as a continuous variable. All formulas in h given below on the assumption that h is zero, a positive integer, or a permitted negative integer can be modified to include other permitted values by replacing, for example, $(h+n)!$ by $\Gamma(h+n+1)$.

Let $V_{nr} = V_{nr}(h)$ denote a determinantal ratio (not a scaled cofactor) defined as

$$V_{nr} = \frac{1}{K_n} \begin{vmatrix} \frac{1}{h+1} & \frac{1}{h+2} & \cdots & \frac{1}{h+n} \\ \frac{1}{h+2} & \frac{1}{h+3} & \cdots & \frac{1}{h+n+1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{h+n} & \frac{1}{h+n+1} & \cdots & \frac{1}{h+2n-1} \end{vmatrix}_n \text{row } r, \tag{4.10.3}$$

where every element in row r is 1 and all the other elements are identical with the corresponding elements in K_n. The following notes begin with the evaluation of V_{nr} and end with the evaluation of K_n and its scaled cofactor K_n^{rs}.

Identities 1.

$$V_{nr} = \sum_{j=1}^{n} K_n^{rj}, \quad 1 \le r \le n. \tag{4.10.4}$$

$$V_{nr} = \frac{(-1)^{n+r}(h+r+n-1)!}{(h+r-1)!(r-1)!(n-r)!}, \quad 1 \le r \le n. \tag{4.10.5}$$

$$V_{n1} = \frac{(-1)^{n+1}(h+n)!}{h!(n-1)!}. \tag{4.10.6}$$

$$V_{nn} = \frac{(h+2n-1)!}{(h+n-1)!(n-1)!}. \tag{4.10.7}$$

$$K_n^{rs} = \frac{V_{nr}V_{ns}}{h+r+s-1}, \quad 1 \le r,s \le n. \tag{4.10.8}$$

$$K_n^{r1} = \frac{V_{nr}V_{n1}}{h+r}. \tag{4.10.9}$$

$$K_n^{nn} = \frac{K_{n-1}}{K_n} = \frac{V_{nn}^2}{h+2n-1}. \tag{4.10.10}$$

$$K_n^{rs} = \frac{(h+r)(h+s)K_n^{r1}K_n^{s1}}{(h+r+s-1)V_{n1}^2}. \tag{4.10.11}$$

$$K_n = \frac{(n-1)!^2(h+n-1)!^2}{(h+2n-2)!(h+2n-1)!}K_{n-1}. \tag{4.10.12}$$

$$K_n = \frac{[1!\,2!\,3!\cdots(n-1)!]^2 h!(h+1)!\cdots(h+n-1)!}{(h+n)!(h+n+1)!\cdots(h+2n-1)!}. \tag{4.10.13}$$

$$(n-r)V_{nr} + (h+n+r-1)V_{n-1,r} = 0. \tag{4.10.14}$$

$$K_n \prod_{r=1}^{n} V_{nr} = (-1)^{n(n-1)/2}. \tag{4.10.15}$$

PROOF. Equation (4.10.4) is a simple expansion of V_{nr} by elements from row r. The following proof of (4.10.5) is a development of one due to Lane.

Perform the row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \mathbf{R}_r, \quad 1 \le i \le n, \quad i \ne r,$$

on K_n, that is, subtract row r from each of the other rows. The result is

$$K_n = |k'_{ij}|_n,$$

where

$$\begin{aligned} k'_{rj} &= k_{rj}, \\ k'_{ij} &= k_{ij} - k_{rj} \\ &= \left(\frac{r-i}{h+r+j-1}\right) k_{ij}, \quad 1 \le i,j \le n, \quad i \ne r. \end{aligned}$$

After removing the factor $(r-i)$ from each row i, $i \ne r$, and the factor $(h+r+j-1)^{-1}$ from each column j and then canceling K_n the result can be expressed in the form

$$V_{nr} = \prod_{j=1}^{n}(h+r+j-1)\left(\prod_{\substack{i=1\\i\neq r}}^{n}(r-i)\right)^{-1}$$
$$= \frac{(h+r)(h+r-1)\cdots(h+r+n-1)}{[(r-1)(r-2)\cdots 1][(-1)(-2)\cdots(r-n)]},$$

which leads to (4.10.5) and, hence, (4.10.6) and (4.10.7), which are particular cases.

Now, perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j - \mathbf{C}_s, \quad 1 \le j \le n, \quad j \neq s,$$

on V_{nr}. The result is a multiple of a determinant in which the element in position (r,s) is 1 and all the other elements in row r are 0. The other elements in this determinant are given by

$$k''_{ij} = k_{ij} - k_{is}$$
$$= \left(\frac{s-j}{h+i+s-1}\right)k_{ij}, \quad 1 \le i,j \le n, \quad (i,j) \neq (r,s).$$

After removing the factor $(s-j)$ from each column j, $j \neq s$, and the factor $(h+i+s-1)$ from each row i, the cofactor K_{rs} appears and gives the result

$$V_{nr} = K_n^{rs}\prod_{\substack{j=1\\j\neq s}}(s-j)\left(\prod_{\substack{i=1\\i\neq r}}^{n}(h+i+s-1)\right)^{-1},$$

which leads to (4.10.8) and, hence, (4.10.9) and (4.10.10), which are particular cases. Equation (4.10.11) then follows easily. Equation (4.10.12) is a recurrence relation in K_n which follows from (4.10.10) and (4.10.7) and which, when applied repeatedly, yields (4.10.13), an explicit formula for K_n. The proofs of (4.10.14) and (4.10.15) are elementary. □

Exercises

Prove that

1. $K_n(-2n-h) = (-1)^n K_n(h), \quad h = 0,1,2,\ldots.$

2. $\dfrac{\partial}{\partial h}V_{nr} = V_{nr}\displaystyle\sum_{t=0}^{n-1}\frac{1}{h+r+t}.$

3. $\dfrac{\partial}{\partial h}K_n^{rs} = K_n^{rs}\left[\displaystyle\sum_{t=0}^{n-1}\left(\frac{1}{h+r+t}+\frac{1}{h+s+t}\right)-\frac{1}{h+r+s-1}\right].$

4. a. $K_n^{r1}(0) = \dfrac{(-1)^{r+1} n(r+n-1)!}{(r-1)!r!(n-r)!}.$

b. $K_n^{rs}(0) = \dfrac{(-1)^{r+s} rs}{r+s-1} \binom{n-1}{r-1} \binom{n-1}{s-1} \binom{r+n-1}{r} \binom{s+n-1}{s}.$

c. $K_n(0) = \dfrac{[1!2!3! \cdots (n-1)!]^3}{n!(n+1)!(n+2)! \cdots (2n-1)!}.$

5.

$$\begin{aligned} K_n\left(\frac{1}{2}\right) &= 2^n \left| \frac{1}{2i+2j-1} \right|_n \\ &= 2^{2n^2} [1!\,2!\,3! \cdots (n-1)!]^2 \prod_{r=0}^{n-1} \frac{(2r+1)!(r+n)!}{r!(2r+2n+1)!}. \end{aligned}$$

[Apply the Legendre duplication formula in Appendix A.1].

6. By choosing h suitably, evaluate $|1/(2i+2j-3)|_n$.

The next set of identities are of a different nature. The parameter n is omitted from V_{nr}, K_n^{ij}, and so forth.

Identities 2.

$$\sum_j \frac{K^{sj}}{h+r+j-1} = \delta_{rs}, \quad 1 \le r \le n. \tag{4.10.16}$$

$$\sum_j \frac{V_j}{h+r+j-1} = 1, \quad 1 \le r \le n. \tag{4.10.17}$$

$$\sum_j \frac{V_j}{(h+r+j-1)(h+s+j-1)} = \frac{\delta_{rs}}{V_r}, \quad 1 \le r, s \le n. \tag{4.10.18}$$

$$\sum_j \frac{jK^{1j}}{h+r+j-1} = V_1 - h\delta_{r1}, \quad 1 \le r \le n. \tag{4.10.19}$$

$$\sum_j V_j = \sum_i \sum_j K^{ij} = n(n+h). \tag{4.10.20}$$

$$\sum_j jK^{1j} = (n^2 + nh - h)V_1. \tag{4.10.21}$$

PROOF. Equation (4.10.16) is simply the identity

$$\sum_j k_{rj} K^{sj} = \delta_{rs}.$$

To prove (4.10.17), apply (4.10.9) with $r \to j$ and (4.10.4): and (4.10.12),

$$\begin{aligned} V_1 \sum_j \frac{V_j}{h+r+j-1} &= \sum_j \frac{(h+j)K^{j1}}{h+r+j-1} \\ &= \sum_j \left(1 - \frac{r-1}{h+r+j-1}\right) K^{j1} \end{aligned}$$

$$= V_1 - (r-1)\sum_j \frac{K^{j1}}{h+r+j-1}$$
$$= V_1 - (r-1)\delta_{r1}, \quad 1 \le r \le n.$$

The second term is zero. The result follows.

The proof of (4.10.18) when $s \neq r$ follows from the identity

$$\frac{1}{(h+r+j-1)(h+s+j-1)} = \frac{1}{s-r}\left(\frac{1}{h+r+j-1} - \frac{1}{h+s+j-1}\right)$$

and (4.10.15). When $s = r$, the proof follows from (4.10.8) and (4.10.16):

$$V_r \sum_s \frac{V_s}{(h+r+s-1)^2} = \sum_s \frac{K^{rs}}{h+r+s-1} = 1.$$

To prove (4.10.19), apply (4.10.4) and (4.10.16):

$$\sum_j \frac{jK^{1j}}{h+r+j-1} = \sum_j \left(1 - \frac{h+r-1}{h+r+j-1}\right) K^{1j}$$
$$= V_1 - h\delta_{r1} - (r-1)\delta_{r1}, \quad 1 \le r \le n.$$

The third term is zero. The result follows.

Equation (4.10.20) follows from (4.10.4) and the double-sum identity (C) (Section 3.4) with $f_r = r$ and $g_s = s + h - 1$, and (4.10.21) follows from the identity (4.10.9) in the form

$$jK^{1j} = V_1 V_j - hK^{1j}$$

by summing over j and applying (4.10.4) and (4.10.20). □

4.10.2 Three Formulas of the Rodrigues Type

Let

$$R_n(x) = \sum_{j=1}^{n} K^{1j} x^{j-1}$$
$$= \frac{1}{K_n} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{vmatrix}_n.$$

Theorem 4.37.

$$R_n(x) = \frac{(h+n)!}{(n-1)!^2 h! x^{h+1}} D^{n-1}[x^{h+n}(1-x)^{n-1}].$$

PROOF. Referring to (4.10.9), (4.10.5), and (4.10.6),

$$D^{n-1}\left[x^{h+n}(1-x)^{n-1}\right] = \sum_{i=0}^{n-1}(-1)^i \binom{n-1}{i} D^{n-1}(x^{h+n+i})$$

$$= \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \frac{(h+n+i)!}{(h+i+1)!} x^{h+i+1}$$
$$= \sum_{j=1}^{n} (-1)^{j-1} \binom{n-1}{j-1} \frac{(h+n+j-1)!}{(h+j)!} x^{h+j}$$
$$= \frac{(n-1)!^2 h! x^{h+1}}{(h+n)!} \sum_{j=1}^{n} K^{1j} x^{j-1}.$$

The theorem follows. □

Let

$$S_n(x,h) = \sum_{j=1}^{n} K_{nj}^{(n)} (-x)^{j-1}$$
$$= \begin{vmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ k_{n-1,1} & k_{n-1,2} & k_{n-1,3} & \cdots & k_{n-1,n} \\ 1 & -x & x^2 & \cdots & (-x)^{n-1} \end{vmatrix}_n .$$

The column operations

$$\mathbf{C}'_j = \mathbf{C}_j + x\mathbf{C}_{j-1}, \quad 2 \le j \le n,$$

remove the x's from the last row and yield the formula

$$S_n(x,h) = (-1)^{n+1} \left| \frac{x}{h+i+j-1} + \frac{1}{h+i+j} \right|_{n-1} .$$

Let

$$T_n(x,h) = (-1)^{n+1} \left| \frac{1+x}{h+i+j-1} - \frac{1}{h+i+j} \right|_{n-1} .$$

Theorem 4.38.

a. $\dfrac{(h+n-1)!^2}{h!(n-1)!} \dfrac{S_n(x,h)}{S_n(0,h)} = D^{h+n-1}[x^{n-1}(1+x)^{h+n-1}].$

b. $\dfrac{(h+n-1)!}{h!(n-1)!} \dfrac{T_n(x,h)}{T_n(0,h)} = D^{h+n-1}[x^{h+n-1}(1+x)^{n-1}].$

Proof.

$$S_n(0,h) = K_{n1}^{(n)}$$
$$= \frac{K_n(h) V_{nn} V_{n1}}{h+n}$$
$$= (-1)^{n+1} K_n(h) V_{nn} \binom{h+n-1}{h},$$

$$S_n(x,h) = K_n(h)V_{nn}\sum_{j=1}^{n}\frac{V_{nj}(-x)^{j-1}}{h+n+j-1}.$$

Hence,

$$\begin{aligned}
\binom{h+n-1}{h}\frac{S_n(x,h)}{S_n(0,h)} &= (-1)^{n+1}\sum_{j=1}^{n}\frac{V_{nj}(-x)^{j-1}}{h+n+j-1}\\
&= \sum_{j=1}^{n}\frac{1}{(n-j)!(h+j-1)!}\left[\frac{(h+n+j-2)!x^{j-1}}{(j-1)!}\right]\\
&= \frac{1}{(h+n-1)!}\sum_{j=1}^{n}\binom{h+n-1}{h+j-1}D^{h+n-1}(x^{h+n+j-2}),\\
\frac{(h+n-1)!^2}{h!(n-1)!}\frac{S_n(x,h)}{S_n(0,h)} &= D^{h+n-1}\left[x^{n-1}\sum_{j=1}^{n}\binom{h+n-1}{h+j-1}x^{h+j-1}\right]\\
&= D^{h+n-1}\left[x^{n-1}\sum_{r=h}^{h+n-1}\binom{h+n-1}{r}x^{r}\right]\\
&= D^{h+n-1}\left[x^{n-1}(1+x)^{h+n-1} - p_{h+n-2}(x)\right],
\end{aligned}$$

where $p_r(x)$ is a polynomial of degree r. Formula (a) follows. To prove (b), put $x = -1-t$. The details are elementary. □

Further formulas of the Rodrigues type appear in Section 4.11.4.

4.10.3 Bordered Yamazaki–Hori Determinants — 1

Let

$$\begin{aligned}
A &= |a_{ij}|_n = |\theta_m|_n,\\
B &= |b_{ij}|_n = |\phi_m|_n, \quad 0 \le m \le 2n-1,
\end{aligned} \tag{4.10.22}$$

denote two Hankelians, where

$$\begin{aligned}
a_{ij} &= \frac{1}{i+j-1}\left[p^2x^{2(i+j-1)} + q^2y^{2(i+j-1)} - 1\right],\\
\theta_m &= \frac{1}{m+1}\left[p^2x^{2m+2} + q^2y^{2m+2} - 1\right],\\
b_{ij} &= \frac{1}{i+j-1}\left[p^2X^{i+j-1} + q^2Y^{i+j-1}\right],\\
\phi_m &= \frac{1}{m+1}\left[p^2X^{m+1} + q^2Y^{m+1}\right],\\
p^2+q^2 &= 1,\\
X &= x^2-1,\\
Y &= y^2-1.
\end{aligned} \tag{4.10.23}$$

Referring to the section on differences in Appendix A.8,

$$\phi_m = \Delta^m \theta_0$$

so that

$$B = A.$$

The Hankelian B arises in studies by M. Yamazaki and Hori of the Ernst equation of general relativity and A arises in a related paper by Vein.

Define determinants $U(x)$, $V(x)$, and W, each of order $(n+1)$, by bordering A in different ways. Since a_{ij} is a function of x and y, it follows that $U(x)$ and $V(x)$ are also functions of y. The argument x in $U(x)$ and $V(x)$ refers to the variable which appears explicitly in the last row or column.

$$U(x) = \begin{vmatrix} & & & & & x \\ & & & & & x^3/3 \\ & & [a_{ij}]_n & & & x^5/5 \\ & & & & & \cdots \\ & & & & & x^{2n-1}/(2n-1) \\ 1 & 1 & 1 & \cdots & 1 & \bullet \end{vmatrix}_{n+1}$$

$$= -\sum_{r=1}^{n}\sum_{s=1}^{n} \frac{A_{rs}x^{2r-1}}{2r-1}, \tag{4.10.24}$$

$$V(x) = \begin{vmatrix} & & & & & 1 \\ & & & & & 1/3 \\ & & [a_{ij}]_n & & & 1/5 \\ & & & & & \cdots \\ & & & & & 1/(2n-1) \\ x & x^3 & x^5 & \cdots & x^{2n-1} & \bullet \end{vmatrix}_{n+1}$$

$$= -\sum_{r=1}^{n}\sum_{s=1}^{n} \frac{A_{rs}x^{2s-1}}{2r-1}, \tag{4.10.25}$$

$$W = U(1) = V(1). \tag{4.10.26}$$

Theorem 4.39.

$$p^2U^2(x) + q^2U^2(y) = W^2 - AW.$$

PROOF.

$$U^2(x) = \sum_{i,s} \frac{A_{is}x^{2i-1}}{2i-1} \sum_{j,r}^{n} \frac{A_{jr}x^{2j-1}}{2j-1}$$

$$= \sum_{i,j,r,s} \frac{A_{is}A_{jr}x^{2(i+j-1)}}{(2i-1)(2j-1)}.$$

Hence,

$$p^2U^2(x) + q^2U^2(y) - W^2$$

$$\begin{aligned}
&= \sum_{i,j,r,s} \frac{A_{is}A_{jr}}{(2i-1)(2j-1)}[p^2x^{2(i+j-1)} + q^2y^{2(i+j-1)} - 1] \\
&= \sum_{i,j,r,s} \frac{(i+j-1)a_{ij}A_{is}A_{rj}}{(2i-1)(2j-1)} \\
&= \tfrac{1}{2} \sum_{i,j,r,s} \left(\frac{1}{2i-1} + \frac{1}{2j-1} \right) a_{ij}A_{is}A_{rj} \\
&= \sum_{i,j,r,s} \frac{a_{ij}A_{is}A_{rj}}{2i-1} \\
&= \sum_{i,s} \frac{A_{is}}{2i-1} \sum_r \sum_j a_{ij}A_{rj} \\
&= A \sum_{i,s} \frac{A_{is}}{2i-1} \sum_r \delta_{ir} \\
&= -AW
\end{aligned}$$

which proves the theorem. □

Theorem 4.40.

$$p^2V^2(x) + q^2V^2(y) = W^2 - AW.$$

This theorem resembles Theorem 4.39 closely, but the following proof bears little resemblance to the proof of Theorem 4.39. Applying double-sum identity (D) in Section 3.4 with $f_r = r$ and $g_s = s - 1$,

$$\sum_r \sum_s \left[p^2x^{2(r+s-1)} + q^2y^{2(r+s-1)} - 1\right] A^{is}A^{rj} = (i+j-1)A^{ij},$$

$$p^2 \left[\sum_s A^{is}x^{2s-1}\right] \left[\sum_r A^{rj}x^{2r-1}\right] + q^2 \left[\sum_s A^{is}y^{2s-1}\right] \left[\sum_r A^{rj}y^{2r-1}\right] - \left[\sum_s A^{is}\right] \left[\sum_r A^{rj}\right] = (i+j-1)A^{ij}.$$

Put

$$\lambda_i(x) = \sum_j A^{ij}x^{2j-1}.$$

Then,

$$p^2\lambda_i(x)\lambda_j(x) + q^2\lambda_i(y)\lambda_j(y) - \lambda_i(1)\lambda_j(1) = (i+j-1)A^{ij}.$$

Divide by $(2i-1)(2j-1)$, sum over i and j and note that

$$\sum_i \frac{\lambda_i(x)}{2i-1} = -\frac{V(x)}{A}.$$

The result is

$$\frac{1}{A^2}\left[p^2V^2(x)+q^2V^2(y)-W^2\right] = \sum_i\sum_j \frac{i+j-1}{(2i-1)(2j-1)}A^{ij}$$
$$= \tfrac{1}{2}\sum_i\sum_j\left(\frac{1}{2i-1}+\frac{1}{2j-1}\right)A^{ij}$$
$$= -\frac{W}{A}.$$

The theorem follows. The determinant W appears in Section 5.8.6.

Theorem 4.41. *In the particular case in which* $(p,q)=(1,0)$,

$$V(x) = (-1)^{n+1}U(x).$$

PROOF.

$$a_{ij} = \frac{x^{2(i+j-1)}-1}{i+j-1} = a_{ji},$$

which is independent of y. Let

$$Z = \begin{vmatrix} & & & & & 1\\ & & & & & 1\\ & & [c_{ij}]_n & & & 1\\ & & & & & \cdots\\ & & & & & 1\\ x & x^3 & x^5 & \cdots & x^{2n-1} & \bullet \end{vmatrix}_{n+1},$$

where

$$c_{ij} = (i-j)a_{ij}$$
$$= -c_{ji}.$$

The proof proceeds by showing that U and V are each simple multiples of Z. Perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j - x^{2j-1}\mathbf{C}_{n+1}, \quad 1 \le j \le n,$$

on U. This leaves the last column and the last row unaltered, but $[a_{ij}]_n$ is replaced by $[a'_{ij}]_n$, where

$$a'_{ij} = a_{ij} - \frac{x^{2(i+j-1)}}{2i-1}.$$

Now perform the row operations

$$\mathbf{R}'_i = \mathbf{R}_i + \frac{1}{2i-1}\mathbf{R}_{n+1}, \quad 1 \le i \le n.$$

The last column and the last row remain unaltered, but $[a'_{ij}]_n$ is replaced by $[a''_{ij}]_n$, where

$$a''_{ij} = a'_{ij} + \frac{1}{2i-1}$$

$$= \frac{c_{ij}}{2i-1}.$$

After removing the factor $(2i-1)^{-1}$ from row i, $1 \le i \le n$, the result is

$$U = \frac{2^n n!}{(2n)!} \begin{vmatrix} & & & & & x \\ & & & & & x^3 \\ & & [c_{ij}]_n & & & x^5 \\ & & & & & \cdots \\ & & & & & x^{2n-1} \\ 1 & 1 & 1 & \cdots & 1 & \bullet \end{vmatrix}_{n+1}.$$

Transposing,

$$U = \frac{2^n n!}{(2n)!} \begin{vmatrix} & & & & & 1 \\ & & & & & 1 \\ & & [-c_{ij}]_n & & & 1 \\ & & & & & \cdots \\ & & & & & 1 \\ x & x^3 & x^5 & \cdots & x^{2n-1} & \bullet \end{vmatrix}_{n+1}.$$

Now, change the signs of columns 1 to n and row $(n+1)$. This introduces $(n+1)$ negative signs and gives the result

$$U = \frac{(-1)^{n+1} 2^n n!}{(2n)!} Z. \tag{4.10.27}$$

Perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j + \mathbf{C}_{n+1}, \quad 1 \le j \le n,$$

on V. The result is that $[a_{ij}]_n$ is replaced by $[a^*_{ij}]_n$, where

$$a^*_{ij} = a_{ij} + \frac{1}{2i-1}.$$

Perform the row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \frac{x^{2i-1}}{2i-1} \mathbf{R}_{n+1}, \quad 1 \le i \le n,$$

which results in $[a^*_{ij}]_n$ being replaced by $[a^{**}_{ij}]_n$, where

$$\begin{aligned} a^{**}_{ij} &= a^*_{ij} - \frac{x^{2(i+j-1)}}{2i-1} \\ &= \frac{c_{ij}}{2i-1}. \end{aligned}$$

After removing the factor $(2i-1)^{-1}$ from row i, $1 \le i \le n$, the result is

$$V = \frac{2^n n!}{(2n)!} Z. \tag{4.10.28}$$

The theorem follows from (4.10.27) and (4.10.28). □

Let

$$A = |\phi_m|_n, \quad 0 \le m \le 2n-2,$$

where

$$\phi_m = \frac{x^{2m+2}-1}{m+1}.$$

A is identical to $|a_{ij}|_n$, where a_{ij} is defined in Theorem 4.41. Let Y denote the determinant of order $(n+1)$ obtained by bordering A by the row

$$\begin{bmatrix}1 & 1 & 1 \ldots 1 & \bullet\end{bmatrix}_{n+1}$$

below and the column

$$\begin{bmatrix}1 & \frac{1}{3} & \frac{1}{5} \cdots \frac{1}{2n-1} & \bullet\end{bmatrix}^T_{n+1}$$

on the right.

Theorem 4.42.

$$Y = -nK_n\phi_0^{n(n-1)}\sum_{i=1}^{n}\frac{2^{2i-1}(n+i-1)!}{(n-i)!(2i)!}\phi_0^{n-i},$$

where K_n is the simple Hilbert determinant.

PROOF. Perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j - \mathbf{C}_{j-1}$$

in the order $j = n, n-1, n-2, \ldots, 2$. The result is a determinant in which the only nonzero element in the last row is a 1 in position $(n+1, 1)$. Hence,

$$Y = (-1)^n \begin{vmatrix} \Delta\phi_0 & \Delta\phi_1 & \Delta\phi_2 & \cdots & \Delta\phi_{n-2} & 1 \\ \Delta\phi_1 & \Delta\phi_2 & \Delta\phi_3 & \cdots & \Delta\phi_{n-1} & \frac{1}{3} \\ \Delta\phi_2 & \Delta\phi_3 & \Delta\phi_4 & \cdots & \Delta\phi_n & \frac{1}{5} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta\phi_{n-1} & \Delta\phi_n & \Delta\phi_{n+1} & \cdots & \Delta\phi_{2n-3} & \frac{1}{2n-1} \end{vmatrix}_n.$$

Perform the row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \mathbf{R}_{i-1}$$

in the order $i = n, n-1, n-2, \ldots, 2$. The result is

$$Y = (-1)^n \begin{vmatrix} \Delta\phi_0 & \Delta\phi_1 & \Delta\phi_2 & \cdots & \Delta\phi_{n-2} & 1 \\ \Delta^2\phi_0 & \Delta^2\phi_1 & \Delta^2\phi_2 & \cdots & \Delta^2\phi_{n-1} & \Delta\alpha_0 \\ \Delta^2\phi_1 & \Delta^2\phi_2 & \Delta^2\phi_3 & \cdots & \Delta^2\phi_n & \Delta\alpha_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta^2\phi_{n-2} & \Delta^2\phi_{n-1} & \Delta^2\phi_n & \cdots & \Delta^2\phi_{2n-4} & \Delta\alpha_{n-2} \end{vmatrix}_n,$$

where

$$\alpha_m = \frac{1}{2m+1}.$$

Now, perform the row and column operations

$$\mathbf{R}'_i = \sum_{r=0}^{i-2} (-1)^r \binom{i-2}{r} \mathbf{R}_{i-r}, \quad i = n, n-1, n-2, \ldots, 3,$$
$$\mathbf{C}'_j = \sum_{r=0}^{j-1} (-1)^r \binom{j-1}{r} \mathbf{C}_{j-r}, \quad j = n-1, n-2, \ldots, 2.$$

The result is

$$Y = (-1)^n \begin{vmatrix} \Delta\phi_0 & \Delta^2\phi_0 & \Delta^3\phi_0 & \cdots & \Delta^{n-1}\phi_0 & 1 \\ \Delta^2\phi_0 & \Delta^3\phi_0 & \Delta^4\phi_0 & \cdots & \Delta^n\phi_0 & \Delta\alpha_0 \\ \Delta^3\phi_0 & \Delta^4\phi_0 & \Delta^5\phi_0 & \cdots & \Delta^{n+1}\phi_0 & \Delta^2\alpha_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta^n\phi_0 & \Delta^{n+1}\phi_0 & \Delta^{n+2}\phi_0 & \cdots & \Delta^{2n-2}\phi_0 & \Delta^{n-1}\alpha_0 \end{vmatrix}_n,$$

where

$$\Delta^m\phi_0 = \frac{\phi_0^{m+1}}{m+1}.$$

Transfer the last column to the first position, which introduces the sign $(-1)^{n+1}$, and then remove powers of ϕ_0 from all rows and columns except the first column, which becomes

$$\left[1 \ \frac{\Delta\alpha_0}{\phi_0} \ \frac{\Delta^2\alpha_0}{\phi_0^2} \cdots \frac{\Delta^{n-1}\alpha_0}{\phi_0^{n-1}}\right]^T.$$

The other $(n-1)$ columns are identical with the corresponding columns of the Hilbert determinant K_n. Hence, expanding the determinant by elements from the first column,

$$Y = -\phi_0^{n(n-1)} \sum_{i=1}^{n} \left[K_{i1}^{(n)} \Delta^{i-1}\alpha_0\right] \phi_0^{n-i}.$$

The proof is completed with the aid of (4.10.5) and (4.10.8) and the formula for $\Delta^{i-1}\alpha_0$ in Appendix A.8. □

Further notes on the Yamazaki–Hori determinant appear in Section 5.8 on algebraic computing.

4.10.4 A Particular Case of the Yamazaki–Hori Determinant

Let

$$A_n = |\phi_m|_n, \quad 0 \le m \le 2n-2,$$

where

$$\phi_m = \frac{x^{2m+2} - 1}{m+1}. \tag{4.10.29}$$

Theorem.

$$A_n = K_n(x^2-1)^{n^2}, \quad K_n = K_n(0).$$

PROOF.

$$\phi_0 = x^2 - 1.$$

Referring to Example A.3 (with $c = 1$) in the section on differences in Appendix A.8,

$$\Delta^m \phi_0 = \frac{\phi_0^{m+1}}{m+1}.$$

Hence, applying the theorem in Section 4.8.2 on Hankelians whose elements are differences,

$$\begin{aligned} A_n &= |\Delta^m \phi_0|_n \\ &= \left| \frac{\phi_0^{m+1}}{m+1} \right|_n \\ &= \begin{vmatrix} \phi_0 & \frac{1}{2}\phi_0^2 & \frac{1}{3}\phi_0^3 & \cdots & \frac{1}{n}\phi_0^n \\ \frac{1}{2}\phi_0^2 & \frac{1}{3}\phi_0^3 & \frac{1}{4}\phi_0^4 & \cdots & \cdots \\ \frac{1}{3}\phi_0^3 & \frac{1}{4}\phi_0^4 & \frac{1}{5}\phi_0^5 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n}\phi_0^n & \cdots & \cdots & \cdots & \frac{1}{2n-1}\phi_0^{2n-1} \end{vmatrix}_n. \end{aligned}$$

Remove the factor ϕ_0^i from row i, $1 \le i \le n$, and then remove the factor ϕ_0^{j-1} from column j, $2 \le j \le n$. The simple Hilbert determinant K_n appears and the result is

$$\begin{aligned} A_n &= K_n \phi_0^{(1+2+3+\cdots+n)(1+2+3+\cdots+\overline{n-1})} \\ &= K_n \phi_0^{n^2}, \end{aligned}$$

which proves the theorem. □

Exercises

1. Define a triangular matrix $[a_{ij}]$, $1 \le i \le 2n-1$, $1 \le j \le 2n-i$, as follows:

$$\begin{aligned} \text{column } 1 &= \left[1 \; u \; u^2 \cdots u^{2n-2}\right]^T, \\ \text{row } 1 &= \left[1 \; v \; v^2 \cdots v^{2n-2}\right]. \end{aligned}$$

The remaining elements are defined by the rule that the difference between consecutive elements in any one diagonal parallel to the secondary diagonal is constant. For example, one diagonal is

$$\left[u^3 \; \frac{1}{3}(2u^3+v^3) \; \frac{1}{3}(u^3+2v^3) \; v^3\right]$$

in which the column difference is $\frac{1}{3}(v^3 - u^3)$.
Let the determinant of the elements in the first n rows and the first n columns of the matrix be denoted by A_n. Prove that

$$A_n = \frac{K_n n!^3}{(2n)!}(u-v)^{n(n+1)}.$$

2. Define a Hankelian B_n as follows:

$$B_n = \left| \frac{\phi_m}{(m+1)(m+2)} \right|_n, \quad 0 \le m \le 2n-2,$$

where

$$\phi_m = \sum_{r=0}^{m} (m+1-r)u^{m-r}v^r.$$

Prove that

$$B_n = \frac{A_{n+1}}{n!(u-v)^{2n}},$$

where A_n is defined in Exercise 1.

4.11 Hankelians 4

Throughout this section, $K_n = K_n(0)$, the simple Hilbert determinant.

4.11.1 v-Numbers

The integers v_{ni} defined by

$$v_{ni} = V_{ni}(0) = \frac{(-1)^{n+i}(n+i-1)!}{(i-1)!^2(n-i)!} \tag{4.11.1}$$

$$= (-1)^{n+i} i \binom{n-1}{i-1} \binom{n+i-1}{n-1}, \quad 1 \le i \le n, \tag{4.11.2}$$

are of particular interest and will be referred to as v-numbers.

A few values of the v-numbers v_{ni} are given in the following table:

n \ i	1	2	3	4	5
1	1				
2	−2	6			
3	3	−24	30		
4	−4	60	−180	140	
5	5	−120	630	−1120	630

v-Numbers satisfy the identities

$$\sum_{k=1}^{n} \frac{v_{nk}}{i+k-1} = 1, \quad 1 \le i \le n, \tag{4.11.3}$$

$$v_{ni} \sum_{k=1}^{n} \frac{v_{nk}}{(i+k-1)(k+j-1)} = \delta_{ij}, \tag{4.11.4}$$

$$\frac{v_{ni}}{n+i-1} = -\frac{v_{n-1,i}}{n-i}, \tag{4.11.5}$$

$$\sum_{i=1}^{n} v_{ni} = n^2, \tag{4.11.6}$$

and are related to K_n and its scaled cofactors by

$$K_n^{ij} = \frac{v_{ni} v_{nj}}{i+j-1}, \tag{4.11.7}$$

$$K_n \prod_{i=1}^{n} v_{ni} = (-1)^{n(n-1)/2}. \tag{4.11.8}$$

The proofs of these identities are left as exercises for the reader.

4.11.2 Some Determinants with Determinantal Factors

This section is devoted to the factorization of the Hankelian

$$B_n = \det \mathbf{B}_n,$$

where

$$\mathbf{B}_n = [b_{ij}]_n,$$
$$b_{ij} = \frac{x^{2(i+j-1)} - t^2}{i+j-1}, \tag{4.11.9}$$

and to the function

$$G_n = \sum_{j=1}^{n} (x^{2j-1} + t) B_{nj}, \tag{4.11.10}$$

which can be expressed as the determinant $|g_{ij}|_n$ whose first $(n-1)$ rows are identical to the first $(n-1)$ rows of B_n. The elements in the last row are given by

$$g_{nj} = x^{2j-1} + t, \quad 1 \le j \le n.$$

The analysis employs both matrix and determinantal methods.

Define five matrices $\mathbf{K}_n$, $\mathbf{Q}_n$, $\mathbf{S}_n$, $\mathbf{H}_n$, and $\overline{\mathbf{H}}_n$ as follows:

$$\mathbf{K}_n = \left[\frac{1}{i+j-1} \right]_n, \tag{4.11.11}$$

$$\mathbf{Q}_n = \mathbf{Q}_n(x) = \left[\frac{x^{2(i+j-1)}}{i+j-1}\right]_n. \tag{4.11.12}$$

Both $\mathbf{K}_n$ and $\mathbf{Q}_n$ are Hankelians and $\mathbf{Q}_n(1) = \mathbf{K}_n$, the simple Hilbert matrix.

$$\mathbf{S}_n = \mathbf{S}_n(x) = \left[\frac{v_{ni}x^{2j-1}}{i+j-1}\right]_n, \tag{4.11.13}$$

where the v_{ni} are v-numbers.

$$\begin{aligned}\mathbf{H}_n = \mathbf{H}_n(x,t) &= \mathbf{S}_n(x) + t\mathbf{I}_n\\ &= \left[h_{ij}^{(n)}\right]_n,\end{aligned}$$

where

$$\begin{aligned}h_{ij}^{(n)} &= \frac{v_{ni}x^{2j-1}}{i+j-1} + \delta_{ij}t,\\ \overline{\mathbf{H}}_n &= \mathbf{H}_n(x,-t) = \mathbf{S}_n(x) - t\mathbf{I}_n\\ &= \left[\overline{h}_{ij}^{(n)}\right]_n,\end{aligned} \tag{4.11.14}$$

where

$$\begin{aligned}\overline{h}_{ij}^{(n)}(x,t) &= h_{ij}^{(n)}(x,-t),\\ \bar{H}_n(x,-t) &= (-1)^n H_n(-x,t).\end{aligned} \tag{4.11.15}$$

Theorem 4.43.

$$\mathbf{K}_n^{-1}\mathbf{Q}_n = \mathbf{S}_n^2.$$

PROOF. Referring to (4.11.7) and applying the formula for the product of two matrices,

$$\begin{aligned}\mathbf{K}_n^{-1}\mathbf{Q}_n &= \left[\frac{v_{ni}v_{nj}}{i+j-1}\right]_n \left[\frac{x^{2(i+j-1)}}{i+j-1}\right]_n\\ &= \left[\sum_{k=1}^{n} \frac{v_{ni}v_{nk}}{i+k-1}\,\frac{x^{2(k+j-1)}}{k+j-1}\right]_n\\ &= \left[\sum_{k=1}^{n} \left(\frac{v_{ni}x^{2k-1}}{i+k-1}\right)\left(\frac{v_{nk}x^{2j-1}}{k+j-1}\right)\right]_n\\ &= \mathbf{S}_n^2.\end{aligned}$$

□

Theorem 4.44.

$$\mathbf{B}_n = \mathbf{K}_n\mathbf{H}_n\overline{\mathbf{H}}_n,$$

where the symbols can be interpreted as matrices or determinants.

PROOF. Applying Theorem 4.43,

$$\mathbf{B}_n = \mathbf{Q}_n - t^2\mathbf{K}_n$$

$$
\begin{aligned}
&= \mathbf{K}_n\big(\mathbf{K}_n^{-1}\mathbf{Q}_n - t^2\mathbf{I}_n\big) \\
&= \mathbf{K}_n\big(\mathbf{S}_n^2 - t^2\mathbf{I}_n\big) \\
&= \mathbf{K}_n(\mathbf{S}_n + t\mathbf{I}_n)(\mathbf{S}_n - t\mathbf{I}_n) \\
&= \mathbf{K}_n\mathbf{H}_n\overline{\mathbf{H}}_n.
\end{aligned}
$$

□

Corollary.

$$
\begin{aligned}
\mathbf{B}_n^{-1} &= \overline{\mathbf{H}}_n^{-1}\mathbf{H}_n^{-1}\mathbf{K}_n^{-1}, \\
\big[\mathbf{B}_{ji}^{(n)}\big] &= \big[\overline{\mathbf{H}}_{ji}^{(n)}\big]\big[H_{ji}^{(n)}\big]\big[K_{ji}^{(n)}\big].
\end{aligned}
$$

Lemma.

$$
\sum_{i=1}^{n} h_{ij}^{(n)} = x^{2j-1} + t.
$$

The proof applies (4.11.3) and is elementary.

Let E_{n+1} denote the determinant of order $(n+1)$ obtained by bordering H_n as follows:

$$
\begin{aligned}
E_{n+1} &= \begin{vmatrix}
h_{11} & h_{12} & \cdots & h_{1n} & v_{n1}/n \\
h_{21} & h_{22} & \cdots & h_{2n} & v_{n2}/(n+1) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
h_{n1} & h_{n2} & \cdots & h_{nn} & v_{nn}/(2n-1) \\
1 & 1 & \cdots & 1 & \bullet
\end{vmatrix}_{n+1} \\
&= -\sum_{r=1}^{n}\sum_{s=1}^{n} \frac{v_{nr}H_{rs}}{n+r-1}. \qquad (4.11.16)
\end{aligned}
$$

Theorem 4.45.

$$
E_{n+1} = (-1)^n \overline{H}_{n-1}.
$$

The proof consists of a sequence of row and column operations.

PROOF. Perform the column operation

$$
\mathbf{C}_n' = \mathbf{C}_n - x^{2n-1}\mathbf{C}_{n+1} \qquad (4.11.17)
$$

and apply (6b) with $j = n$. The result is

$$
E_{n+1} = \begin{vmatrix}
h_{11} & h_{12} & \cdots & h_{1,n-1} & \bullet & v_{n1}/n \\
h_{21} & h_{22} & \cdots & h_{2,n-1} & \bullet & v_{n2}/(n+1) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
h_{n1} & h_{n2} & \cdots & h_{n,n-1} & t & v_{nn}/(2n-1) \\
1 & 1 & \cdots & 1 & 1 & \bullet
\end{vmatrix}_{n+1}. \qquad (4.11.18)
$$

Remove the element in position (n, n) by performing the row operation

$$
\mathbf{R}_n' = \mathbf{R}_n - t\mathbf{R}_{n+1}. \qquad (4.11.19)
$$

The only element which remains in column n is a 1 in position $(n+1, n)$. Hence,

$$E_{n+1} = -\begin{vmatrix} h_{11} & h_{12} & \cdots & h_{1,n-1} & v_{n1}/n \\ h_{21} & h_{22} & \cdots & h_{2,n-1} & v_{n2}/(n+1) \\ \dots & \dots & \dots & \dots & \dots \\ (h_{n1}-t) & (h_{n2}-t) & \cdots & (h_{n,n-1}-t) & v_{nn}/(2n-1) \end{vmatrix}_n . \tag{4.11.20}$$

It is seen from (4.11.3) (with $i = n$) that the sum of the elements in the last column is unity and it is seen from the lemma that the sum of the elements in column j is x^{2j-1}, $1 \le j \le n-1$. Hence, after performing the row operation

$$\mathbf{R}'_n = \sum_{i=1}^{n} \mathbf{R}_i, \tag{4.11.21}$$

the result is

$$E_{n+1} = \begin{vmatrix} h_{11} & h_{12} & \cdots & h_{1,n-1} & v_{n1}/n \\ h_{21} & h_{22} & \cdots & h_{2,n-1} & v_{n2}/(n+1) \\ \dots & \dots & \dots & \dots & \dots \\ h_{n-1,1} & h_{n-1,2} & \cdots & h_{n-1,n-1} & v_{n,n-1}/(2n-2) \\ x & x^3 & \cdots & x^{2n-3} & 1 \end{vmatrix}_n . \tag{4.11.22}$$

The final set of column operations is

$$\mathbf{C}'_j = \mathbf{C}_j - x^{2j-1}\mathbf{C}_n, \quad 1 \le j \le n-1, \tag{4.11.23}$$

which removes the x's from the last row. The result can then be expressed in the form

$$E_{n+1} = -\left|h_{ij}^{(n)^*}\right|_{n-1}, \tag{4.11.24}$$

where, referring to (4.11.5),

$$\begin{aligned} h_{ij}^{(n)^*} &= h_{ij}^{(n)} - \frac{v_{ni}x^{2j-1}}{n+i-1} \\ &= v_{ni}x^{2j-1}\left(\frac{1}{i+j-1} - \frac{1}{i+n-1}\right) + \delta_{ij}t \\ &= \left(\frac{v_{ni}}{i+n-1}\right)\left(\frac{(n-j)x^{2j-1}}{i+j-1}\right) + \delta_{ij}t \\ &= -\left(\frac{v_{n-1,i}}{n-i}\right)\left(\frac{(n-j)x^{2j-1}}{i+j-1}\right) + \delta_{ij}t \\ &= -\left(\frac{n-j}{n-i}\right)\left(\frac{v_{n-1,i}x^{2j-1}}{i+j-1} - \delta_{ij}t\right) \\ &= -\left(\frac{n-j}{n-i}\right)\bar{h}_{ij}^{(n-1)}, \\ \left|h_{ij}^{(n)^*}\right|_{n-1} &= -\left|-\bar{h}_{ij}^{(n-1)}\right|_{n-1}. \end{aligned} \tag{4.11.25}$$

Theorem 4.45 now follows from (4.11.24). □

Theorem 4.46.

$$G_n = (-1)^{n-1} v_{nn} K_n \overline{H}_n \overline{H}_{n-1},$$

where G_n is defined in (4.11.10).

PROOF. Perform the row operation

$$\mathbf{R}'_i = \sum_{k=1}^{n} \mathbf{R}_k$$

on $\overline{H}_n$ and refer to the lemma. Row i becomes

$$\big[(x+t), (x^3+t), (x^5+t), \ldots, (x^{2n-1}+t)\big].$$

Hence,

$$\overline{H}_n = \sum_{j=1}^{n} (x^{2j-1}+t) \overline{H}_{ij}^{(n)}, \quad 1 \le i \le n. \tag{4.11.26}$$

It follows from the corollary to Theorem 4.44 that

$$B_{ij}^{(n)} = B_{ji}^{(n)} = \sum_{r=1}^{n} \sum_{s=1}^{n} \overline{H}_{jr}^{(n)} H_{rs}^{(n)} K_{si}^{(n)}. \tag{4.11.27}$$

Hence, applying (4.11.7),

$$B_{ij}^{(n)} = K_n v_{ni} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{v_{ns} H_{rs}^{(n)} \overline{H}_{jr}^{(n)}}{i+s-1}. \tag{4.11.28}$$

Put $i = n$, substitute the result into (4.11.10), and apply (4.11.16) and (4.11.24):

$$\begin{aligned} G_n &= K_n v_{nn} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{v_{ns} H_{rs}^{(n)}}{n+s-1} \sum_{j=1}^{n} (x^{2j-1}+t) \overline{H}_{jr}^{(n)} \\ &= K_n v_{nn} \overline{H}_n \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{v_{ns} H_{rs}^{(n)}}{n+s-1} \\ &= -K_n v_{nn} \overline{H}_n E_{n+1}. \end{aligned} \tag{4.11.29}$$

The theorem follows from Theorem 4.45. □

4.11.3 Some Determinants with Binomial and Factorial Elements

Theorem 4.47.

a. $\left| \binom{n+j-2}{n-i} \right|_n = (-1)^{n(n-1)/2},$

b. $\left|\frac{1}{(i+j-2)!}\right| = \frac{(-1)^{n(n-1)/2} 1!\, 2!\, 3! \cdots (n-2)!}{n!\,(n+1)! \cdots (2n-2)!}.$

The second determinant is Hankelian.

PROOF. Denote the first determinant by A_n. Every element in the last row of A_n is equal to 1. Perform the column operations

$$\mathbf{C}'_j = \mathbf{C}_j - \mathbf{C}_{j-1}, \quad j = n, n-1, n-2, \ldots, 2, \tag{4.11.30}$$

which remove all the elements in the last row except the one in position $(n, 1)$. After applying the binomial identity

$$\binom{n}{r} - \binom{n-1}{r} = \binom{n-1}{r-1},$$

the result is

$$A_n = (-1)^{n+1} \left| \binom{n+j-2}{n-i-1} \right|_{n-1}. \tag{4.11.31}$$

Once again, every element in the last row is equal to 1. Repeat the column operations with $j = n-1, n-2, \ldots, 2$ and apply the binomial identity again. The result is

$$A_n = -\left| \binom{n+j-2}{n-i-2} \right|_{n-2}. \tag{4.11.32}$$

Continuing in this way,

$$\begin{aligned} A_n &= +\left| \binom{n+j-2}{n-i-4} \right|_{n-4} \\ &= -\left| \binom{n+j-2}{n-i-6} \right|_{n-6} \\ &= +\left| \binom{n+j-2}{n-i-8} \right|_{n-8} \\ &= \pm \left| \binom{n+j-2}{2-i} \right|_{2} \\ &= \pm 1, \end{aligned} \tag{4.11.33}$$

$$\operatorname{sign}(A_n) = \begin{cases} 1 & \text{when } n = 4m, 4m+1 \\ -1 & \text{when } n = 4m-2, 4m-1, \end{cases} \tag{4.11.34}$$

which proves (a).

Denote the second determinant by B_n. Divide $\mathbf{R}_i$ by $(n-i)!$, $1 \le i \le n-1$, and multiply $\mathbf{C}_j$ by $(n+j-2)!$, $1 \le j \le n$. The result is

$$\frac{(n-1)!\, n!\,(n+1)! \cdots (2n-2)!}{(n-1)!\,(n-2)!\,(n-3)! \cdots 1!} B_n = \left| \frac{(n+j-2)!}{(n-i)!\,(i+j-2)!} \right|_n$$

$$= \left|\binom{n+j-2}{n-i}\right|_n$$
$$= A_n,$$

which proves (b). □

Exercises

Apply similar methods to prove that

1. $\left|\binom{n+j-1}{n-i}\right|_n = (-1)^{n(n-1)/2}$,

2. $\left|\dfrac{1}{(i+j-1)!}\right|_n = \dfrac{(-1)^{n(n-1)/2}K_n}{[1!\,2!\,3!\cdots(n-1)!]^2}$.

Define the number ν_i as follows:

$$(1+z)^{-1/2} = \sum_{i=0}^{\infty} \nu_i z^i. \tag{4.11.35}$$

Then

$$\nu_i = \frac{(-1)^i}{2^{2i}}\binom{2i}{i}. \tag{4.11.36}$$

Let

$$\begin{aligned} A_n &= |\nu_m|_n, \quad 0 \le m \le 2n-2, \\ &= \left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\ \mathbf{C}_n\right|_n \end{aligned} \tag{4.11.37}$$

where

$$\mathbf{C}_j = \left[\nu_{j-1}\ \nu_j \ldots \nu_{n+j-3}\ \nu_{n+j-2}\right]_n^T. \tag{4.11.38}$$

Theorem 4.48.

$$A_n = 2^{-(n-1)(2n-1)}.$$

Proof. Let

$$\lambda_{nr} = \frac{n}{n+r}\binom{n+r}{2r}2^{2r}. \tag{4.11.39}$$

Then, it is shown in Appendix A.10 that

$$\sum_{j=1}^{n} \lambda_{n-1,j-1}\nu_{i+j-2} = \frac{\delta_{in}}{2^{2(n-1)}}, \quad 1 \le i \le n, \tag{4.11.40}$$

$$\begin{aligned} A_n &= 2^{-(2n-3)}\left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\ (\lambda_{n-1,n-1}\mathbf{C}_n)\right|_n, \\ &= 2^{-(2n-3)}\left|\mathbf{C}_1\ \mathbf{C}_2 \cdots \mathbf{C}_{n-1}\ \mathbf{C}'_n\right|_n, \end{aligned} \tag{4.11.41}$$

where

$$\mathbf{C}'_n = \lambda_{n-1,n-1}\mathbf{C}_n + \sum_{j=1}^{n-1} \lambda_{n-1,j-1}\mathbf{C}_j$$
$$= \sum_{j=1}^{n-1} \lambda_{n-1,j-1}\big[\nu_{j-1}\ \nu_j \cdots \nu_{n+j-3}\ \nu_{n+j-2}\big]_n^T$$
$$= 2^{-(4n-5)}\big[0\ 0 \cdots 0\ 1\big]_n^T. \tag{4.11.42}$$

Hence,

$$\begin{aligned} A_n &= 2^{-4(n-1)+1}A_{n-1} \\ A_{n-1} &= 2^{-4(n-2)+1}A_{n-2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ A_2 &= 2^{-4(1)+1}A_1, \qquad (A_1 = \nu_0 = 1). \end{aligned} \tag{4.11.43}$$

The theorem follows by equating the product of the left-hand sides to the product of the right-hand sides. □

It is now required to evaluate the cofactors of A_n.

Theorem 4.49.

a. $A_{nj}^{(n)} = 2^{-(n-1)(2n-3)}\lambda_{n-1,j-1}$,
b. $A_{n1}^{(n)} = 2^{-(n-1)(2n-3)}$,
c. $A_n^{nj} = 2^{2(n-1)}\lambda_{n-1,j-1}$.

Proof. The n equations in (4.11.40) can be expressed in matrix form as follows:

$$\mathbf{A}_n\mathbf{L}_n = \mathbf{C}'_n, \tag{4.11.44}$$

where

$$\mathbf{L}_n = \big[\lambda_{n0}\ \lambda_{n1} \cdots \lambda_{n,n-2}\ \lambda_{n,n-1}\big]_n^T. \tag{4.11.45}$$

Hence,

$$\begin{aligned} \mathbf{L}_n &= \mathbf{A}_n^{-1}\mathbf{C}'_n \\ &= A_n^{-1}\big[A_{ji}^{(n)}\big]_n\mathbf{C}'_n \\ &= 2^{(n-1)(2n-1)-2(n-1)}\big[A_{n1}\ A_{n2} \cdots A_{n,n-1}\ A_{nn}\big]_n^T, \end{aligned} \tag{4.11.46}$$

which yields part (a) of the theorem. Parts (b) and (c) then follow easily. □

Theorem 4.50.

$$A_{ij}^{(n)} = 2^{-n(2n-3)}\left[2^{2i-3}\lambda_{i-1,j-1} + \sum_{r=i+1}^{n-1} \lambda_{r-1,i-1}\lambda_{r-1,j-1}\right], \quad j \le i < n-1.$$

PROOF. Apply the Jacobi identity (Section 3.6.1) to A_r, where $r \geq i+1$:

$$\begin{aligned} \begin{vmatrix} A_{ij}^{(r)} & A_{ir}^{(r)} \\ A_{rj}^{(r)} & A_{rr}^{(r)} \end{vmatrix} &= A_r A_{ir,jr}^{(r)}, \\ &= A_r A_{ij}^{(r-1)}, \\ A_{r-1}A_{ij}^{(r)} - A_r A_{ij}^{(r-1)} &= A_{ir}^{(r)} A_{jr}^{(r)}. \end{aligned} \tag{4.11.47}$$

Scale the cofactors and refer to Theorems 4.48 and 4.49a:

$$\begin{aligned} A_r^{ij} - A_{r-1}^{ij} &= \frac{A_r}{A_{r-1}} \left(A_r^{ri} \ A_r^{rj} \right) \\ &= 2^{-(4r-5)} A_r^{ri} A_r^{rj} \\ &= 2\lambda_{r-1,i-1}\lambda_{r-1,j-1}. \end{aligned} \tag{4.11.48}$$

Hence,

$$\begin{aligned} 2 \sum_{r=i+1}^{n} \lambda_{r-1,i-1}\lambda_{r-1,j-1} &= \sum_{r=i+1}^{n} \left(A_r^{ij} - A_{r-1}^{ij} \right) \\ &= A_n^{ij} - A_i^{ij} \\ &= A_n^{ij} - 2^{2(i-1)}\lambda_{i-1,j-1}, \end{aligned} \tag{4.11.49}$$

which yields a formula for the scaled cofactor A_n^{ij}. The stated formula for the simple cofactor $A_{ij}^{(n)}$ follows from Theorem 4.49a. □

Let

$$E_n = |P_m(0)|_n, \quad 0 \leq m \leq 2n-2, \tag{4.11.50}$$

where $P_m(x)$ is the Legendre polynomial [Appendix A.5]. Then,

$$\begin{aligned} P_{2m+1}(0) &= 0, \\ P_{2m}(0) &= \nu_m. \end{aligned} \tag{4.11.51}$$

Hence,

$$E_n = \begin{vmatrix} \nu_0 & \bullet & \nu_1 & \bullet & \nu_2 & \cdots \\ \bullet & \nu_1 & \bullet & \nu_2 & \bullet & \cdots \\ \nu_1 & \bullet & \nu_2 & \bullet & \nu_3 & \cdots \\ \bullet & \nu_2 & \bullet & \nu_3 & \bullet & \cdots \\ \nu_2 & \bullet & \nu_3 & \bullet & \nu_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n . \tag{4.11.52}$$

Theorem 4.51.

$$E_n = |P_m(0)|_n = (-1)^{n(n-1)/2} 2^{-(n-1)^2}.$$

Proof. By interchanging first rows and then columns in a suitable manner it is easy to show that

$$E_n = \begin{vmatrix} \nu_0 & \nu_1 & \nu_2 & \cdots & & & \\ \nu_1 & \nu_2 & \nu_3 & \cdots & & & \\ \nu_2 & \nu_3 & \nu_4 & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & & & \\ & & & & \nu_1 & \nu_2 & \cdots \\ & & & & \nu_2 & \nu_3 & \cdots \\ & & & & \cdots & \cdots & \cdots \end{vmatrix}_n . \tag{4.11.53}$$

Hence, referring to Theorems 4.11.5 and 4.11.6b,

$$\begin{aligned} E_{2n} &= (-1)^n A_n A_{n+1,1}^{(n+1)} \\ &= (-1)^n 2^{-(2n-1)^2}, \\ E_{2n+1} &= (-1)^n A_{n+1} A_{n+1,1}^{(n+1)} \\ &= (-1)^n 2^{-4n^2}. \end{aligned} \tag{4.11.54}$$

These two results can be combined into one as shown in the theorem which is applied in Section 4.12.1 to evaluate $|P_m(x)|_n$. □

Exercise. If

$$B_n = \left| \binom{2m}{m} \right|_n, \quad 0 \le m \le 2n-2,$$

prove that

$$\begin{aligned} B_n &= 2^{n-1}, \\ B_{ij}^{(n)} &= 2^{2[n(n-1)-(i+j-2)]} A_{ij}^{(n)}, \\ B_{n1}^{(n)} &= 2^{n-1}. \end{aligned}$$

4.11.4 A Nonlinear Differential Equation

Let

$$G_n(x,h,k) = |g_{ij}|_n,$$

where

$$g_{ij} = \begin{cases} \frac{x^{h+i+k-1}}{h+i+k-1}, & j = k \\ \frac{1}{h+i+j-1}, & j \ne k. \end{cases} \tag{4.11.55}$$

Every column in G_n except column k is identical with the corresponding column in the generalized Hilbert determinant $K_n(h)$. Also, let

$$G_n(x,h) = \sum_{k=1}^{n} G_n(x,h,k). \tag{4.11.56}$$

Theorem 4.52.

$$(xG_n')' = K_n(h)x^h P_n^2(x,h),$$

where

$$P_n(x,h) = \frac{D^{h+n}[x^n(1+x)^{h+n-1}]}{(h+n-1)!}.$$

PROOF. Referring to (4.10.8),

$$\begin{aligned} G_n(x) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{K_{ij}^{(n)} x^{h+i+j-1}}{h+i+j-1} \\ &= K_n(h)\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{V_{ni}V_{nj}x^{h+i+j-1}}{(h+i+j-1)^2}. \end{aligned} \tag{4.11.57}$$

Hence,

$$\begin{aligned} (xG_n')' &= K_n(h)x^h \sum_{i=1}^{n}\sum_{j=1}^{n} V_{ni}V_{nj}x^{i+j-2} \\ &= K_n(h)x^h P_n^2(x,h), \end{aligned} \tag{4.11.58}$$

where

$$\begin{aligned} P_n(x,h) &= \sum_{i=1}^{n}(-1)^{n+i}V_{ni}x^{i-1} \\ &= \sum_{i=1}^{n} \frac{(h+n+i-1)!x^{i-1}}{(i-1)!\,(n-i)!\,(h+i-1)!} \\ &= \sum_{i-1}^{n} \frac{D^{h+n}(x^{h+n+i-1})}{(n-i)!\,(h+i-1)!}, \end{aligned} \tag{4.11.59}$$

$$\begin{aligned} (h+n-1)!\,P_n(x,h) &= \sum_{i=1}^{n}\binom{h+n-1}{h+i-1} D^{h+n}(x^{h+n+i-1}) \\ &= D^{h+n}\left[x^n \sum_{i=1}^{n}\binom{h+n-1}{h+i-1}x^{h+i-1}\right] \\ &= D^{h+n}\left[x^n \sum_{r=h}^{h+n-1}\binom{h+n-1}{r}x^r\right] \\ &= D^{h+n}\big[x^n(1+x)^{h+n-1} - p_{h+n-1}(x)\big], \end{aligned} \tag{4.11.60}$$

where $p_r(x)$ is a polynomial of degree r. The theorem follows. □

Let

$$E(x) = |e_{ij}(x)|_{n-1},$$

where

$$e_{ij}(x) = \frac{(1+x)^{i+j+1} - x^{i+j+1}}{i+j+1}. \tag{4.11.61}$$

Theorem 4.53. *The polynomial determinant E satisfies the nonlinear differential equation*

$$\left[\{x(1+x)E\}''\right]^2 = 4n^2(xE)'\{(1+x)E\}'.$$

Proof. Let

$$A(x,\xi) = |\phi_m(x,\xi)|_n, \quad 0 \le m \le 2n-2,$$

where

$$\phi_m(x,\xi) = \frac{1}{m+1}\left[(\xi+x)^{m+1} - c(\xi-1)^{m+1} + (c-1)\xi^{m+1}\right]. \tag{4.11.62}$$

Then,

$$\begin{aligned}\frac{\partial}{\partial\xi}\phi_m(x,\xi) &= m\phi_{m-1}(x,\xi),\\ \phi_0(x,\xi) &= x+c.\end{aligned} \tag{4.11.63}$$

Hence, from Theorem 4.33 in Section 4.9.1, A is independent of ξ. Put $\xi = 0$ and $-x$ in turn and denote the resulting determinants by U and V, respectively. Then,

$$A = U = V, \tag{4.11.64}$$

where

$$\begin{aligned}U(x,c) &= |\phi_m(x,0)|_n\\ &= \left|\frac{x^{m+1}+(-1)^m c}{m+1}\right|_n, \quad 0 \le m \le 2n-2\\ &= \left|\frac{x^{i+j-1}+(-1)^{i+j}c}{i+j-1}\right|_n.\end{aligned} \tag{4.11.65}$$

Put

$$\begin{aligned}\psi_m(x) &= \phi_m(x,-x)\\ &= \frac{(-1)^m}{m+1}[c(1+x)^{m+1} + (1-c)x^{m+1}]\end{aligned} \tag{4.11.66}$$

$$\begin{aligned}V(x,c) &= |\psi_m(x)|_n,\\ &= \left|\frac{c(1+x)^{m+1}+(1-c)x^{m+1}}{m+1}\right|_n\\ &= \left|\frac{c(1+x)^{i+j-1}+(1-c)x^{i+j-1}}{i+j-1}\right|_n.\end{aligned} \tag{4.11.67}$$

Note that $U_{ij} \neq V_{ij}$ in general. Since

$$\begin{aligned}\psi'_m &= -m\psi_{m-1},\\ \psi_0 &= x + c,\end{aligned} \tag{4.11.68}$$

it follows that

$$\begin{aligned}V' &= V_{11}\\ &= \left|\frac{c(1+x)^{i+j+1} + (1-c)x^{i+j+1}}{i+j+1}\right|_{n-1}.\end{aligned} \tag{4.11.69}$$

Expand U and V as a polynomial in c:

$$U(x,c) = V(x,c) = \sum_{r=0}^{n} f_r(x)c^{n-r}. \tag{4.11.70}$$

However, since

$$\psi_m = y_m c + z_m,$$

where z_m is independent of c,

$$y_m = (-1)^m \left[\frac{(1+x)^{m+1} - x^{m+1}}{m+1}\right], \tag{4.11.71}$$

$$\begin{aligned}y'_m &= -my_{m-1},\\ y_0 &= 1,\end{aligned} \tag{4.11.72}$$

it follows from the first line of (4.11.67) that f_0, the coefficient of c^n in V, is given by

$$\begin{aligned}f_0 &= |y_m|_n\\ &= \text{constant}.\end{aligned} \tag{4.11.73}$$

$$c^{n-1}V(x,c^{-1}) = f_0 c^{-1} + f_1 + \sum_{r=1}^{n-1} f_{r+1}c^r,$$

where

$$\begin{aligned}f'_1 &= \left[c^{n-1}D_x V(x,c^{-1})\right]_{c=0}, \quad D_x = \frac{\partial}{\partial x},\\ &= \left[c^{n-1}V_{11}(x,c^{-1})\right]_{c=0}\\ &= \left[c^{n-1}\left|\frac{c^{-1}(1+x)^{i+j+1} + (1-c^{-1})x^{i+j+1}}{i+j+1}\right|_{n-1}\right]_{c=0}\\ &= E.\end{aligned} \tag{4.11.74}$$

Furthermore,

$$D_c\{c^n U(x,c^{-1})\} = D_c\{c^n V(x,c^{-1})\}, \quad D_c = \frac{\partial}{\partial c},$$

$$= D_c \sum_{r=0}^{n} f_r c^r$$
$$= f_1 + \sum_{r=2}^{n} r f_r c^{r-1}. \tag{4.11.75}$$

Hence,

$$\begin{aligned} f_1 &= \left[D_c\{c^n U(x, c^{-1})\}\right]_{c=0} \\ &= D_c \left[c^n \left| \frac{x^{i+j-1} - (-1)^{i+j} c^{-1}}{i+j-1} \right|_n \right]_{c=0} \\ &= \left[D_c \left| \frac{cx^{i+j-1} - (-1)^{i+j}}{i+j-1} \right|_n \right]_{c=0} \\ &= \sum_{k=1}^{n} G_n(x, 0, k) \\ &= G_n(x, 0), \end{aligned} \tag{4.11.76}$$

where $G_n(x, h, k)$ and $G_n(x, h)$ are defined in the first line of (4.11.55) and (4.11.56), respectively.

$$\begin{aligned} E &= G', \\ (xE)' &= (xG')' \\ &= K_n P_n^2, \end{aligned} \tag{4.11.77}$$

where

$$\begin{aligned} K_n &= K_n(0), \\ P_n &= P_n(x, 0) \\ &= \frac{D^n[x^n(1+x)^{n-1}]}{(n-1)!}. \end{aligned} \tag{4.11.78}$$

Let

$$Q_n = \frac{D^n[x^{n-1}(1+x)^n]}{(n-1)!}. \tag{4.11.79}$$

Then,

$$P_n(-1-x) = (-1)^n Q_n.$$

Since

$$E(-1-x) = E(x),$$

it follows that

$$\begin{aligned} \{(1+x)E\}' &= K_n Q_n^2, \\ \{xE\}'\{(1+x)E\}' &= (K_n P_n Q_n)^2. \end{aligned} \tag{4.11.80}$$

The identity

$$xD^n[x^{n-1}(1+x)^n] = nD^{n-1}[x^n(1+x)^{n-1}] \tag{4.11.81}$$

can be proved by showing that both sides are equal to the polynomial

$$n!\sum_{r=1}^{n}\binom{n}{r}\binom{n+r-1}{n}x^r.$$

It follows by differentiating (4.11.79) that

$$(xQ_n)' = nP_n. \tag{4.11.82}$$

Hence,

$$\begin{aligned}\{x(1+x)E\}' &= (1+x)E + x\{(1+x)E\}'\\ &= (1+x)E + K_n xQ_n^2, &(4.11.83)\\ \{x(1+x)E\}'' &= K_nQ_n^2 + K_n(Q_n^2 + 2xQ_nQ_n')\\ &= 2K_nQ_n(xQ_n)'\\ &= 2nK_nP_nQ_n. &(4.11.84)\end{aligned}$$

The theorem follows from (4.11.80). □

A polynomial solution to the differential equation in Theorem 4.47, and therefore the expansion of the determinant E, has been found by Chalkley using a method based on an earlier publication.

Exercises

1. Prove that

$$\left|\frac{(1+x)^{m+1}+c-1}{m+1}\right|_n = U = V, \qquad 0 \le m \le 2n-2.$$

2. Prove that

$$(1+x)D^n[x^n(1+x)^{n-1}] = nD^{n-1}[x^{n-1}(1+x)^n]$$

$$[(1+x)P_n]' = nQ_n.$$

Hence, prove that

$$[X^2(X^2E)'']' = 4n^2X(XE)',$$

where

$$X = \sqrt{x(1+x)}\,.$$

4.12 Hankelians 5

Notes in orthogonal and other polynomials are given in Appendices A.5 and A.6. Hankelians whose elements are polynomials have been evaluated by a variety of methods by Geronimus, Beckenbach et al., Lawden, Burchnall, Seidel, Karlin and Szegö, Das, and others. Burchnall's methods apply the Appell equation but otherwise have little in common with the proof of the first theorem in which $L_m(x)$ is the simple Laguerre polynomial.

4.12.1 Orthogonal Polynomials

Theorem 4.54.

$$\underset{0\le m\le 2n-2}{|L_m(x)|_n} = \frac{(-1)^{n(n-1)/2} 0!\,1!\,2!\cdots(n-2)!}{n!\,(n+1)!\,(n+2)!\cdots(2n-2)!} x^{n(n-1)}, \quad n \ge 2.$$

PROOF. Let

$$\phi_m(x) = x^m L_m\left(\frac{1}{x}\right),$$

then

$$\begin{aligned} \phi_m'(x) &= m\phi_{m-1}(x), \\ \phi_0 &= 1. \end{aligned} \tag{4.12.1}$$

Hence, ϕ_m is an Appell polynomial in which

$$\phi_m(0) = \frac{(-1)^m}{m!}.$$

Applying Theorem 4.33 in Section 4.9.1 on Hankelians with Appell polynomial elements and Theorem 4.47b in Section 4.11.3 on determinants with binomial and factorial elements,

$$\begin{aligned} \left| x^m L_m\left(\frac{1}{x}\right)\right|_n &= |\phi_m(x)|_n, \quad 0 \le m \le 2n-2 \\ &= |\phi_m(0)|_n \\ &= \left|\frac{(-1)^m}{m!}\right|_n \\ &= \left|\frac{1}{m!}\right|_n \\ &= \frac{(-1)^{n(n-1)/2} 0!\,1!\,2!\cdots(n-2)!}{n!\,(n+1)!\,(n+2)!\cdots(2n-2)!}. \end{aligned} \tag{4.12.2}$$

But

$$\left| x^m L_m\left(\frac{1}{x}\right)\right|_n = x^{n(n-1)} \left| L_m\left(\frac{1}{x}\right)\right|_n. \tag{4.12.3}$$

The theorem follows from (4.12.2) and (4.12.3) after replacing x by x^{-1}. □

In the next theorem, $P_m(x)$ is the Legendre polynomial.

Theorem 4.55.

$$|P_m(x)|_n = 2^{-(n-1)^2}(x^2-1)^{n(n-1)/2}.$$
$$0 \le m \le 2n-2$$

First Proof. Let

$$\phi_m(x) = (1-x^2)^{-m/2}P_m(x).$$

Then

$$\phi'_m(x) = mF\phi_{m-1}(x)$$

where

$$\begin{aligned} F &= (1-x^2)^{-3/2} \\ \phi_0 &= P_0(x) = 1. \end{aligned} \tag{4.12.4}$$

Hence, if $A = |\phi_m(x)|_n$, then $A' = 0$ and $A = |\phi_m(0)|_n$.

$$\begin{aligned} |P_m(x)|_n &= \left|(1-x^2)^{m/2}\phi_m(x)\right|_n, \quad 0 \le m \le 2n-2 \\ &= (1-x^2)^{n(n-1)/2}|\phi_m(x)|_n \\ &= (1-x^2)^{n(n-1)/2}|\phi_m(0)|_n \\ &= (1-x^2)^{n(n-1)/2}|P_m(0)|_n. \end{aligned}$$

The formula

$$|P_m(0)|_n = (-1)^{n(n-1)/2}2^{-(n-1)^2}$$

is proved in Theorem 4.50 in Section 4.11.3 on determinants with binomial and factorial elements. The theorem follows. □

Other functions which contain orthogonal polynomials and which satisfy the Appell equation are given by Carlson.

The second proof, which is a modified and detailed version of a proof outlined by Burchnall with an acknowledgement to Chaundy, is preceded by two lemmas.

Lemma 4.56. *The Legendre polynomial $P_n(x)$ is equal to the coefficient of t^n in the polynomial expansion of $[(u+t)(v+t)]^n$, where $u = \frac{1}{2}(x+1)$ and $v = \frac{1}{2}(x-1)$.*

PROOF. Applying the Rodrigues formula for $P_n(x)$ and the Cauchy integral formula for the nth derivative of a function,

$$P_n(x) = \frac{1}{2^n n!}D^n(x^2-1)^n$$

$$\begin{aligned}
&= \frac{1}{2^{n+1}\pi i} \int_C \frac{(\zeta^2-1)^n}{(\zeta-x)^{n+1}}\,d\zeta \qquad (\text{put } \zeta = x+2t) \\
&= \frac{1}{2^{n+1}\pi i} \int_{C'} \frac{[(x+1+2t)(x-1+2t)]^n}{(2t)^{n+1}}\,dt \\
&= \frac{1}{2\pi i} \int_{C'} \frac{g(t)}{t^{n+1}}\,dt \\
&= \frac{g^{(n)}(0)}{n!},
\end{aligned}$$

where

$$g(t) = \left[\left\{\tfrac{1}{2}(x+1)+t\right\}\left\{\tfrac{1}{2}(x-1)+t\right\}\right]^n. \tag{4.12.5}$$

The lemma follows. □

$$\begin{aligned}
[(u+t)(v+t)]^n &= \sum_{r=0}^{n}\sum_{s=0}^{n} \binom{n}{r}\binom{n}{s} u^{n-r}v^{n-s}t^{r+s} \\
&= \sum_{p=0}^{2n} \lambda_{np}t^p
\end{aligned}$$

where

$$\lambda_{np} = \sum_{s=0}^{p} \binom{n}{s}\binom{n}{p-s} u^{n-s}v^{n-p+s}, \quad 0 \le p \le 2n, \tag{4.12.6}$$

which, by symmetry, is unaltered by interchanging u and v.

In particular,

$$\lambda_{00} = 1, \quad \lambda_{n0} = (uv)^n, \quad \lambda_{n,2n} = 1, \quad \lambda_{nn} = P_n(x). \tag{4.12.7}$$

Lemma 4.57.

a. $\lambda_{i,i-r} = (uv)^r\lambda_{i,i+r}$,
b. $\lambda_{i,i-r}\lambda_{j,j+r} = \lambda_{i,i+r}\lambda_{j,j-r}$.

PROOF.

$$\begin{aligned}
\lambda_{n,n+r} &= \sum_{s=0}^{n+r} \binom{n}{s}\binom{n}{n+r-s} u^{n-s}v^{s-r} \\
&= \sum_{s=r}^{n+r} \binom{n}{s}\binom{n}{s-r} u^{n-s}v^{s-r}.
\end{aligned}$$

Changing the sign of r,

$$\lambda_{n,n-r} = \sum_{s=-r}^{n-r} \binom{n}{s}\binom{n}{s+r} u^{n-s}v^{s+r} \qquad (\text{put } s = n-\sigma)$$

$$= \sum_{\sigma=r}^{n+r} \binom{n}{n-\sigma} \binom{n}{n-\sigma+r} u^{\sigma} v^{n-\sigma+r}$$
$$= (uv)^r \sum_{\sigma=r}^{n+r} \binom{n}{\sigma} \binom{n}{\sigma-r} u^{\sigma-r} v^{n-\sigma}. \tag{4.12.8}$$

Part (a) follows after interchanging u and v and replacing n by i. Part (b) then follows easily. □

It follows from Lemma 4.56 that $P_{i+j}(x)$ is equal to the coefficient of t^{i+j} in the expansion of the polynomial

$$\begin{aligned}[(u+t)(v+t)]^{i+j} &= [(u+t)(v+t)]^i[(u+t)(v+t)]^j \\ &= \sum_{r=0}^{2i} \lambda_{ir} t^r \sum_{s=0}^{2j} \lambda_{js} t^s. \end{aligned} \tag{4.12.9}$$

Each sum consists of an odd number of terms, the center terms being $\lambda_{ii}t^i$ and $\lambda_{jj}t^j$ respectively. Hence, referring to Lemma 4.57,

$$\begin{aligned} P_{i+j}(x) &= \sum_{r=1}^{\min(i,j)} \lambda_{i,i-r}\lambda_{j,j+r} + \lambda_{ii}\lambda_{jj} + \sum_{r=1}^{\min(i,j)}{}^{\dagger} \lambda_{i,i+r}\lambda_{j,j-r} \\ &= 2 \sum_{r=0}^{\min(i,j)} \lambda_{i,i+r}\lambda_{j,j-r}, \end{aligned} \tag{4.12.10}$$

where the symbol † denotes that the factor 2 is omitted from the $r = 0$ term. Replacing i by $i-1$ and j by $j-1$,

$$P_{i+j-2}(x) = 2 \sum_{r=0}^{\min(i,j)}{}^{\dagger} \lambda_{i-1,i-1+r}\lambda_{j-1,j-1-r}. \tag{4.12.11}$$

Preparations for the second proof are now complete. Adjusting the dummy variable and applying, in reverse, the formula for the product of two determinants (Section 1.4),

$$\begin{aligned} |P_{i+j-2}|_n &= \left| 2 \sum_{s=1}^{\min(i,j)}{}^{\dagger} \lambda_{i-1,i+s-2}\lambda_{j-1,j-s} \right|_n \\ &= \left|2\lambda^*_{i-1,i+j-2}\right|_n \left|\lambda_{j-1,j-i}\right|_n, \end{aligned} \tag{4.12.12}$$

where the symbol $*$ denotes that the factor 2 is omitted when $j = 1$. Note that $\lambda_{np} = 0$ if $p < 0$ or $p > 2n$. The first determinant is lower triangular and the second is upper triangular so that the value of each determinant is given by the product of the elements in its principal diagonal:

$$|P_{i+j-2}|_n = 2^{n-1} \prod_{i=1}^{n} \lambda_{i-1,2i-2}\lambda_{j-1,0}$$

$$
\begin{aligned}
&= 2^{n-1}\prod_{i=2}^{n}(uv)^{i-1} \\
&= 2^{n-1}(uv)^{1+2+3+\cdots+\overline{n-1}} \\
&= 2^{-(n-1)^2}(x^2-1)^{\frac{1}{2}n(n-1)}.
\end{aligned}
$$

which completes the proof.

Exercises

1. Prove that

$$
\underset{0\le m\le 2n-2}{|H_m(x)|_n} = (-2)^{n(n-1)/2}1!\,2!\,3!\cdots(n-1)!,
$$

where $H_m(x)$ is the Hermite polynomial.

2. If

$$
A_n = \begin{vmatrix} P_{n-1} & P_n \\ P_n & P_{n+1} \end{vmatrix},
$$

prove that

$$
n(n+1)A_n'' = 2(P_n')^2. \qquad \text{(Beckenbach et al.)}
$$

4.12.2 The Generalized Geometric Series and Eulerian Polynomials

Notes on the generalized geometric series $\phi_m(x)$, the closely related function $\psi_m(x)$, the Eulerian polynomial $A_n(x)$, and Lawden's polynomial $S_n(x)$ are given in Appendix A.6.

$$
\begin{aligned}
\psi_m(x) &= \sum_{r=1}^{\infty} r^m x^r, \\
x\psi_m'(x) &= \psi_{m+1}(x), && (4.12.13) \\
S_m(x) &= (1-x)^{m+1}\psi_m, \quad m \ge 0, && (4.12.14) \\
A_m(x) &= S_m(x), \quad m > 0, \\
A_0 &= 1, \quad S_0 = x. && (4.12.15)
\end{aligned}
$$

Theorem (Lawden).

$$
\begin{aligned}
E_n &= |\psi_{i+j-2}|_n = \frac{\lambda_n x^{n(n+1)/2}}{(1-x)^{n^2}}, \\
F_n &= |\psi_{i+j-1}|_n = \frac{\lambda_n n!\, x^{n(n+1)/2}}{(1-x)^{n(n+1)}}, \\
G_n &= |\psi_{i+j}|_n = \frac{\lambda_n (n!)^2 x^{n(n+1)/2}(1-x^{n+1})}{(1-x)^{(n+1)^2}},
\end{aligned}
$$

$$H_n = |S_{i+j-2}|_n = \lambda_n x^{n(n+1)/2},$$
$$J_n = |A_{i+j-2}|_n = \lambda_n x^{n(n-1)/2},$$

where

$$\lambda_n = \left[1!\ 2!\ 3! \cdots (n-1)!\right]^2.$$

The following proofs differ from the originals in some respects.

PROOF. It is proved using a slightly different notation in Theorem 4.28 in Section 4.8.5 on Turanians that

$$E_n G_n - E_{n+1} G_{n-1} = F_n^2,$$

which is equivalent to

$$E_{n-1} G_{n-1} - E_n G_{n-2} = F_{n-1}^2. \tag{4.12.16}$$

Put $x = e^t$ in (4.12.5) so that

$$D_x = e^{-t} D_t$$
$$D_t = x D_x, \quad D_x = \frac{\partial}{\partial x}, \text{ etc.} \tag{4.12.17}$$

Also, put

$$\begin{aligned} \theta_m(t) &= \psi_m(e^t) \\ &= \sum_{r=1}^{\infty} r^m e^{rt} \\ \theta'_m(t) &= \theta_{m+1}(t). \end{aligned} \tag{4.12.18}$$

Define the column vector $\mathbf{C}_j(t)$ as follows:

$$\mathbf{C}_j(t) = \left[\theta_j(t)\ \theta_{j+1}(t)\ \theta_{j+2}(t) \ldots\right]^T$$

so that

$$\mathbf{C}'_j = \mathbf{C}_{j+1}(t). \tag{4.12.19}$$

The number of elements in $\mathbf{C}_j$ is equal to the order of the determinant of which it is a part, that is, n, $n-1$, or $n-2$ in the present context.

Let

$$Q_n(t,\tau) = \left|\mathbf{C}_0(\tau)\ \mathbf{C}_1(t)\ \mathbf{C}_2(t) \cdots \mathbf{C}_{n-1}(t)\right|_n, \tag{4.12.20}$$

where the argument in the first column is τ and the argument in each of the other columns is t. Then,

$$Q_n(t,t) = E_n. \tag{4.12.21}$$

Differentiate Q_n repeatedly with respect to τ, apply (4.12.19), and put $\tau = t$.

$$D_\tau^r\{Q_n(t,t)\} = 0, \quad 1 \le r \le n-1, \tag{4.12.22}$$

$$\begin{aligned} D_\tau^n\{Q_n(t,t)\} &= \big|\mathbf{C}_n(t)\ \mathbf{C}_1(t)\ \mathbf{C}_2(t)\cdots\mathbf{C}_{n-1}(t)\big|_n \\ &= (-1)^{n-1}\big|\mathbf{C}_1(t)\ \mathbf{C}_2(t)\cdots\mathbf{C}_{n-1}(t)\ \mathbf{C}_n(t)\big|_n \\ &= (-1)^{n-1}F_n. \end{aligned} \tag{4.12.23}$$

The cofactors $Q_{i1}^{(n)}$, $1 \le i \le n$, are independent of τ.

$$\begin{aligned} Q_{11}^{(n)}(t) &= E_{11}^{(n)} = G_{n-1}, \\ Q_{n1}^{(n)}(t) &= (-1)^{n+1}\big|\mathbf{C}_1(t)\ \mathbf{C}_2(t)\ \mathbf{C}_3(t)\cdots\mathbf{C}_{n-1}(t)\big|_{n-1} \\ &= (-1)^{n+1}F_{n-1}, \\ Q_{1n}^{(n)}(t,\tau) &= (-1)^{n+1}\big|\mathbf{C}_1(\tau)\ \mathbf{C}_2(t)\ \mathbf{C}_3(t)\cdots\mathbf{C}_{n-1}(t)\big|_{n-1}. \end{aligned} \tag{4.12.24}$$

Hence,

$$\begin{aligned} D_\tau^r\{Q_{1n}^{(n)}(t,t)\} &= 0, \quad 1 \le r \le n-2 \\ D_\tau^{n-1}\{Q_{1n}^{(n)}(t,t)\} &= (-1)^{n+1}\big|\mathbf{C}_n(t)\ \mathbf{C}_2(t)\ \mathbf{C}_3(t)\cdots\mathbf{C}_{n-1}(t)\big|_{n-1} \\ &= -\big|\mathbf{C}_2(t)\ \mathbf{C}_3(t)\cdots\mathbf{C}_{n-1}(t)\ \mathbf{C}_n(t)\big|_{n-1} \\ &= -G_{n-1}, \\ D_\tau^n\{Q_{1n}^{(n)}(t,t)\} &= -D_t(G_{n-1}), \\ Q_{nn}^{(n)}(t,\tau) &= Q_{n-1}(t,\tau), \\ Q_{nn}^{(n)}(t,t) &= E_{n-1}, \end{aligned}$$

$$D_\tau^r\{Q_{nn}^{(n)}(t,t)\} = \begin{cases} 0, & 1 \le r \le n-2 \\ (-1)^n F_{n-1}, & r = n-1 \\ (-1)^n D_t(F_{n-1}), & r = n. \end{cases} \tag{4.12.25}$$

$$Q_{1n,1n}^{(n)}(t) = G_{n-2}. \tag{4.12.26}$$

Applying the Jacobi identity to the cofactors of the corner elements of Q_n,

$$\begin{aligned} \begin{vmatrix} Q_{11}^{(n)}(t) & Q_{1n}^{(n)}(t,\tau) \\ Q_{n1}^{(n)}(t) & Q_{nn}^{(n)}(t,\tau) \end{vmatrix} &= Q_n(t,\tau)Q_{1n,1n}^{(n)}(t), \\ \begin{vmatrix} G_{n-1} & Q_{1n}^{(n)}(t,\tau) \\ (-1)^{n+1}F_{n-1} & Q_{nn}^{(n)}(t,\tau) \end{vmatrix} &= Q_n(t,\tau)G_{n-2}. \end{aligned} \tag{4.12.27}$$

The first column of the determinant is independent of τ, hence, differentiating n times with respect to τ and putting $\tau = t$,

$$\begin{aligned} \begin{vmatrix} G_{n-1} & D_t(G_{n-1}) \\ (-1)^{n+1}F_{n-1} & (-1)^n D_t(F_{n-1}) \end{vmatrix} &= (-1)^{n+1}F_n G_{n-2}, \\ G_{n-1}D_t(F_{n-1}) - F_{n-1}D_t(G_{n-1}) &= -F_n G_{n-2}, \\ D_t\left[\frac{G_{n-1}}{F_{n-1}}\right] &= \frac{F_n G_{n-2}}{F_{n-1}^2}. \end{aligned}$$

Reverting to x and referring to (4.12.17),

$$xD_x\left[\frac{G_{n-1}}{F_{n-1}}\right] = \frac{F_n G_{n-2}}{F_{n-1}^2}, \tag{4.12.28}$$

where the elements in the determinants are now $\psi_m(x)$, $m = 0, 1, 2, \ldots$.

The difference formula

$$\Delta^m \psi_0 = x\psi_m, \quad m = 1, 2, 3, \ldots, \tag{4.12.29}$$

is proved in Appendix A.8. Hence, applying the theorem in Section 4.8.2 on Hankelians whose elements are differences,

$$\begin{aligned} E_n &= |\psi_m|_n, \quad 0 \le m \le 2n-2 \\ &= |\Delta^m \psi_0|_n \\ &= \begin{vmatrix} \psi_0 & x\psi_1 & x\psi_2 & \cdots \\ x\psi_1 & x\psi_2 & x\psi_3 & \cdots \\ x\psi_2 & x\psi_3 & x\psi_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n . \end{aligned} \tag{4.12.30}$$

Every element except the one in position $(1, 1)$ contains the factor x. Hence, removing these factors and applying the relation

$$\begin{aligned} \psi_0/x &= \psi_0 + 1, \\ E_n &= x^n \begin{vmatrix} \psi_0 + 1 & \psi_1 & \psi_2 & \cdots \\ \psi_1 & \psi_2 & \psi_3 & \cdots \\ \psi_2 & \psi_3 & \psi_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n \\ &= x^n\left(E_n + E_{11}^{(n)}\right). \end{aligned} \tag{4.12.31}$$

Hence

$$E_{11}^{(n)} = G_{n-1} = \left(\frac{1 - x^n}{x^n}\right) E_n. \tag{4.12.32}$$

Put

$$\begin{aligned} u_n &= \frac{G_n}{F_n}, \\ v_n &= \frac{E_{n-1}}{E_n}. \end{aligned} \tag{4.12.33}$$

The theorem is proved by deducing and solving a differential–difference equation satisfied by u_n:

$$\frac{v_n}{v_{n+1}} = \frac{E_{n-1}E_{n+1}}{E_n^2}.$$

From (4.12.32),

$$\frac{G_{n-1}}{G_n} = \frac{x(1 - x^n)v_{n+1}}{1 - x^{n+1}}. \tag{4.12.34}$$

From (4.12.28) and (4.12.33),

$$xu'_{n-1} = \left(\frac{G_{n-1}}{F_{n-1}}\right)^2 \left(\frac{F_n}{G_n}\right) \left(\frac{G_{n-2}}{G_{n-1}}\right) \left(\frac{G_n}{G_{n-1}}\right),$$

$$\frac{u_n u'_{n-1}}{u_{n-1}^2} = \frac{(1-x^{n-1})(1-x^{n+1})}{x(1-x^n)^2} \left[\frac{v_n}{v_{n+1}}\right]. \qquad (4.12.35)$$

From (4.12.16),

$$\begin{aligned}\left(\frac{F_{n-1}}{G_{n-1}}\right)^2 &= \frac{E_{n-1}}{G_{n-1}} - \left(\frac{E_n}{G_{n-1}}\right)\left(\frac{G_{n-2}}{G_{n-1}}\right)\\ &= \left(\frac{E_{n-1}}{E_n}\right)\left(\frac{E_n}{G_{n-1}}\right)\left[1 - \left(\frac{E_n}{E_{n-1}}\right)\left(\frac{G_{n-2}}{G_{n-1}}\right)\right]\\ \frac{1}{u_{n-1}^2} &= v_n \left(\frac{x^n}{1-x^n}\right)\left[1 - \frac{x(1-x^{n-1})}{1-x^n}\right]\\ &= \frac{x^n(1-x)}{(1-x^n)^2} v_n.\end{aligned}$$

Replacing n by $n+1$,

$$\frac{1}{u_n^2} = \frac{x^{n+1}(1-x)}{(1-x^{n+1})^2} v_{n+1}. \qquad (4.12.36)$$

Hence,

$$\left(\frac{u_n}{u_{n-1}}\right)^2 = \frac{1}{x}\left(\frac{1-x^{n+1}}{1-x^n}\right)^2 \left[\frac{v_n}{v_{n+1}}\right]. \qquad (4.12.37)$$

Eliminating v_n/v_{n+1} from (4.12.35) yields the differential–difference equation

$$u_n = \left(\frac{1-x^{n+1}}{1-x^{n-1}}\right) u'_{n-1}. \qquad (4.12.38)$$

Evaluating u_n as defined by (4.12.33) for small values of n, it is found that

$$u_1 = \frac{1!(1-x^2)}{(1-x)^2}, \quad u_2 = \frac{2!(1-x^3)}{(1-x)^3}, \quad u_3 = \frac{3!(1-x^4)}{(1-x)^4}. \qquad (4.12.39)$$

The solution which satifies (4.12.38) and (4.12.39) is

$$u_n = \frac{G_n}{F_n} = \frac{n!(1-x^{n+1})}{(1-x)^{n+1}}. \qquad (4.12.40)$$

From (4.12.36),

$$v_n = \frac{E_{n-1}}{E_n} = \frac{(1-x)^{2n-1}}{(n-1)!^2 x^n},$$

which yields the difference equation

$$E_n = \frac{(n-1)!^2 x^n}{(1-x)^{2n-1}} E_{n-1}. \qquad (4.12.41)$$

Evaluating E_n for small values of n, it is found that

$$E_1 = \frac{x}{1-x}, \quad E_2 = \frac{1!^2 x^3}{(1-x)^4}, \quad E_3 = \frac{[1!\,2!]^2 x^6}{(1-x)^9}. \tag{4.12.42}$$

The solution which satisfies (4.12.41) and (4.12.42) is as given in the theorem. It is now a simple exercise to evaluate F_n and G_n. G_n is found in terms of E_{n+1} by replacing n by $n+1$ in (4.12.32) and then F_n is given in terms of G_n by (4.12.40). The results are as given in the theorem. The proof of the formula for H_n follows from (4.12.14).

$$\begin{aligned} H_n &= |S_m|_n \\ &= |(1-x)^{m+1}\psi_m|_n \\ &= (1-x)^{n^2}|\psi_m|_n \\ &= (1-x)^{n^2}E_n. \end{aligned} \tag{4.12.43}$$

The given formula follows. The formula for J_n is proved as follows:

$$J_n = |A_m|_n.$$

Since

$$A_0 = 1 = (1-x)(\psi_0 + 1), \tag{4.12.44}$$

it follows by applying the second line of (4.12.31) that

$$\begin{aligned} J_n &= \begin{vmatrix} (1-x)(\psi_0+1) & (1-x)^2\psi_1 & (1-x)^3\psi_2 & \cdots \\ (1-x)^2\psi_1 & (1-x)^3\psi_2 & (1-x)^4\psi_3 & \cdots \\ (1-x)^3\psi_2 & (1-x)^4\psi_3 & (1-x)^5\psi_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n \\ &= (1-x)^{n^2}\begin{vmatrix} \psi_0+1 & \psi_1 & \psi_2 & \cdots \\ \psi_1 & \psi_2 & \psi_3 & \cdots \\ \psi_2 & \psi_3 & \psi_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n \\ &= (1-x)^{n^2}x^{-n}E_n \\ &= x^{-n}H_n, \end{aligned} \tag{4.12.45}$$

which yields the given formula and completes the proofs of all five parts of Lawden's theorem. □

4.12.3 A Further Generalization of the Geometric Series

Let A_n denote the Hankel–Wronskian defined as

$$A_n = |D^{i+j-2}f|_n, \quad D = \frac{d}{dt}, \quad A_0 = 1, \tag{4.12.46}$$

where f is an arbitrary function of t. Then, it is proved that Section 6.5.2 on Toda equations that

$$D^2(\log A_n) = \frac{A_{n+1}A_{n-1}}{A_n^2}\,. \tag{4.12.47}$$

Put

$$g_n = D^2(\log A_n). \tag{4.12.48}$$

Theorem 4.58. *g_n satisfies the differential–difference equation*

$$g_n = ng_1 + \sum_{r=1}^{n-1}(n-r)D^2(\log g_r).$$

PROOF. From (4.12.47),

$$\frac{A_{r+1}A_{r-1}}{A_r^2} = g_r,$$

$$\prod_{r=1}^{s}\frac{A_{r+1}}{A_r}\prod_{r=1}^{s}\frac{A_{r-1}}{A_r} = \prod_{r=1}^{s} g_r,$$

which simplifies to

$$\frac{A_{s+1}}{A_s} = A_1\prod_{r=1}^{s} g_r. \tag{4.12.49}$$

Hence,

$$\prod_{s=1}^{n-1}\frac{A_{s+1}}{A_s} = A_1^{n-1}\prod_{s=1}^{n-1}\prod_{r=1}^{s} g_r,$$

$$A_n = A_1^n\prod_{r=1}^{n-1} g_r^{n-r}$$

$$= A_1^n\prod_{r=1}^{n-1} g_{n-r}^r, \tag{4.12.50}$$

$$\log A_n = n\log A_1 + \sum_{r=1}^{n-1}(n-r)\log g_r. \tag{4.12.51}$$

The theorem appears after differentiating twice with respect to t and referring to (4.12.48). □

In certain cases, the differential–difference equation can be solved and A_n evaluated from (4.12.50). For example, let

$$f = \left(\frac{e^t}{1-e^t}\right)^p$$

$$= \sum_{r=0}^{\infty} \frac{(p)_r e^{(r+p)t}}{r!},$$

where

$$(p)_r = p(p+1)(p+2)\cdots(p+r-1) \tag{4.12.52}$$

and denote the corresponding determinant by $E_n^{(p)}$:

$$E_n^{(p)} = \left|\psi_m^{(p)}\right|_n, \quad 0 \le m \le 2n-2,$$

where

$$\begin{aligned}\psi_m^{(p)} &= D^m f \\ &= \sum_{r=0}^{\infty} \frac{(p)_r (r+p)^m e^{(r+p)t}}{r!}.\end{aligned} \tag{4.12.53}$$

Theorem 4.59.

$$E_n^{(p)} = \frac{e^{n(2p+n-1)t/2}}{(1-e^t)^{n(p+n-1)}} \prod_{r=1}^{n-1} r!(p)_r.$$

PROOF. Put

$$g_r = \frac{\alpha_r e^t}{(1-e^t)^2}, \quad \alpha_r \text{ constant},$$

and note that, from (4.12.48),

$$\begin{aligned}g_1 &= D^2(\log f) \\ &= \frac{pe^t}{(1-e^t)^2},\end{aligned}$$

so that $\alpha_1 = p$ and

$$\begin{aligned}\log g_r &= \log \alpha_r + t - 2\log(1-e^t), \\ D^2(\log g_r) &= \frac{2e^t}{(1-e^t)^2}.\end{aligned} \tag{4.12.54}$$

Substituting these functions into the differential–difference equation, it is found that

$$\begin{aligned}\alpha_n &= n\alpha_1 + 2\sum_{r=1}^{n-1}(n-r) \\ &= n(p+n-1).\end{aligned} \tag{4.12.55}$$

Hence,

$$\begin{aligned}g_n &= \frac{n(p+n-1)e^t}{(1-e^t)^2}, \\ g_{n-r} &= \frac{(n-r)(p+n-r-1)e^t}{(1-e^t)^2}.\end{aligned} \tag{4.12.56}$$

Substituting this formula into (4.12.50) with $A_n \to E_n^{(p)}$ and $E_n^{(1)} = f$,

$$E_n^{(p)} = \left(\frac{e^t}{1-e^t}\right)^p \prod_{r=1}^{n-1} \left[(n-r)(p+n-r-1)\frac{e^t}{(1-e^t)^2}\right]^r, \qquad (4.12.57)$$

which yields the stated formula. □

Note that the substitution $x = e^t$ yields

$$\begin{aligned} \psi_m^{(1)} &= \psi_m, \\ E_n^{(1)} &= E_n, \end{aligned}$$

so that $\psi_m^{(p)}$ may be regarded as a further generalization of the geometric series ψ_m and $E_n^{(p)}$ is a generalization of Lawden's determinant E_n.

Exercise. If

$$f = \begin{cases} \sec^p x \\ \operatorname{cosec}^p x \end{cases},$$

prove that

$$A_n = \begin{cases} \sec^{n(p+n-1)} x \\ \operatorname{cosec}^{n(p+n-1)} \end{cases} \prod_{r=1}^{n-1} r!\,(p)_r.$$

4.13 Hankelians 6

4.13.1 Two Matrix Identities and Their Corollaries

Define three matrices $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{N}$ of order n as follows:

$$\begin{aligned} \mathbf{M} &= [\alpha_{ij}]_n && \text{(symmetric)}, \\ \mathbf{K} &= [2^{i+j-1} k_{i+j-2}]_n && \text{(Hankel)}, \\ \mathbf{N} &= [\beta_{ij}]_n && \text{(lower triangular)}, \end{aligned} \qquad (4.13.1)$$

where

$$\alpha_{ij} = \begin{cases} (-1)^{j-1} u_{i-j} + u_{i+j-2}, & j \le i \\ (-1)^{i-1} u_{j-i} + u_{i+j+2}, & j \ge i; \end{cases} \qquad (4.13.2)$$

$$u_r = \sum_{j=1}^{N} a_j f_r(x_j), \quad a_j \text{ arbitrary}; \qquad (4.13.3)$$

$$f_r(x) = \tfrac{1}{2}\left\{(x+\sqrt{1+x^2})^r + (x-\sqrt{1+x^2})^r\right\}; \qquad (4.13.4)$$

$$k_r = \sum_{j=1}^{N} a_j x_j^r; \qquad (4.13.5)$$

$$\beta_{ij} = 0, \qquad j > i \text{ or } i+j \text{ odd},$$

$$\begin{aligned}
\beta_{2i,2i} &= \lambda_{ii}, \qquad i \ge 1, \\
\beta_{2i+1,2j+1} &= \lambda_{ij}, \qquad 0 \le j \le i, \\
\beta_{2i+2,2j} &= \lambda_{i+1,j} - \lambda_{ij}, \qquad 1 \le j \le i+1, \qquad (4.13.6) \\
\lambda_{ij} &= \frac{i}{i+j}\binom{i+j}{2j}; \\
\lambda_{ii} &= \tfrac{1}{2}, \quad i > 0; \quad \lambda_{i0} = 1, \ i \ge 0. \qquad (4.13.7)
\end{aligned}$$

The functions λ_{ij} and $f_r(x)$ appear in Appendix A.10.

Theorem 4.60.

$$\mathbf{M} = \mathbf{N}\mathbf{K}\mathbf{N}^T.$$

Proof. Let

$$\mathbf{G} = [\gamma_{ij}]_n = \mathbf{N}\mathbf{K}\mathbf{N}^T. \qquad (4.13.8)$$

Then

$$\begin{aligned}
\mathbf{G}^T &= \mathbf{N}\mathbf{K}^T\mathbf{N}^T \\
&= \mathbf{N}\mathbf{K}\mathbf{N}^T \\
&= \mathbf{G}.
\end{aligned}$$

Hence, $\mathbf{G}$ is symmetric, and since $\mathbf{M}$ is also symmetric, it is sufficient to prove that $\alpha_{ij} = \gamma_{ij}$ for $j \le i$. There are four cases to consider:

i. i, j both odd,
ii. i odd, j even,
iii. i even, j odd,
iv. i, j both even.

To prove case (i), put $i = 2p+1$ and $j = 2q+1$ and refer to Appendix A.10, where the definition of $g_r(x)$ is given in (A.10.7), the relationships between $f_r(x)$ and $g_r(x)$ are given in Lemmas (a) and (b) and identities among the $g_r(x)$ are given in Theorem 4.61.

$$\begin{aligned}
\alpha_{2p+1,2q+1} &= u_{2q+2p} + u_{2q-2p} \\
&= \sum_{j=1}^{N} a_j \left\{ f_{2q+2p}(x_j) + f_{2q-2p}(x_j) \right\} \\
&= \sum_{j=1}^{N} a_j \left\{ g_{q+p}(x_j) + g_{q-p}(x_j) \right\} \\
&= 2\sum_{j=1}^{N} a_j g_p(x_j) g_q(x_j). \qquad (4.13.9)
\end{aligned}$$

It follows from (4.13.8) and the formula for the product of three matrices (the exercise at the end of Section 3.3.5) with appropriate adjustments to the upper limits that

$$\gamma_{ij} = \sum_{r=1}^{i} \sum_{s=1}^{j} \beta_{ir} 2^{r+s-1} k_{r+s-2} \beta_{js}.$$

Hence,

$$\gamma_{2p+1,2q+1} = 2 \sum_{r=1}^{2p+1} \sum_{s=1}^{2q+1} \beta_{2p+1,r} 2^{r+s-2} k_{r+s-2} \beta_{2q+1,s}. \tag{4.13.10}$$

From the first line of (4.13.6), the summand is zero when r and s are even. Hence, replace r by $2r+1$, replace s by $2s+1$ and refer to (4.13.5) and (4.13.6),

$$\begin{aligned}
\gamma_{2p+1,2q+1} &= 2 \sum_{r=0}^{p} \sum_{s=0}^{q} \beta_{2p+1,2r+1} \beta_{2q+1,2s+1} 2^{2r+2s} k_{2r+2s} \\
&= 2 \sum_{r=0}^{p} \sum_{s=0}^{q} \lambda_{pr} \lambda_{qs} \sum_{j=1}^{N} a_j (2x_j)^{2r+2s} \\
&= 2 \sum_{j=1}^{N} a_j \sum_{r=0}^{p} \lambda_{pr} (2x_j)^{2r} \sum_{s=0}^{q} \lambda_{qs} (2x_j)^{2s} \\
&= 2 \sum_{j=1}^{N} a_j g_p(x_j) g_q(x_j) \\
&= \alpha_{2p+1,2q+1},
\end{aligned} \tag{4.13.11}$$

which completes the proof of case (i). Cases (ii)–(iv) are proved in a similar manner. □

Corollary.

$$\begin{aligned}
|\alpha_{ij}|_n = |\mathbf{M}|_n &= |\mathbf{N}|_n^2 |\mathbf{K}|_n \\
&= |\beta_{ij}|_n^2 |2^{i+j-1} k_{i+j-2}|_n \\
&= \left(\prod_{i=1}^{n} \beta_{ii} \right)^2 2^n |2^{i+j-2} k_{i+j-2}|_n.
\end{aligned} \tag{4.13.12}$$

But, $\beta_{11} = 1$ *and* $\beta_{ii} = \frac{1}{2}$, $2 \le i \le n$. *Hence, referring to Property (e) in Section 2.3.1,*

$$|\alpha_{ij}|_n = 2^{n^2-2n+2} |k_{i+j-2}|_n. \tag{4.13.13}$$

Thus, **M** *can be expressed as a Hankelian.*

Define three other matrices $\mathbf{M}'$, $\mathbf{K}'$, and $\mathbf{N}'$ of order n as follows:

$$\begin{aligned}
\mathbf{M}' &= [\alpha'_{ij}]_n && \text{(symmetric)},\\
\mathbf{K}' &= [2^{i+j-1}(k_{i+j}+k_{i+j-2})]_n && \text{(Hankel)}, \qquad (4.13.14)\\
\mathbf{N}' &= [\beta'_{ij}]_n && \text{(lower triangular)},
\end{aligned}$$

where k_r is defined in (4.13.5);

$$\alpha'_{ij} = \begin{cases} (-1)^{j-1}u_{i-j} + u_{i+j}, & j \le i \\ (-1)^{i-1}u_{j-i} + u_{i+j}, & j \ge i, \end{cases} \qquad (4.13.15)$$

$$\begin{aligned}
\beta'_{ij} &= 0, \qquad j > i \text{ or } i+j \text{ odd},\\
\beta'_{2i,2j} &= \tfrac{1}{2}\mu_{ij}, \qquad 1 \le j \le i,\\
\beta'_{2i+1,2j+1} &= \lambda_{ij} + \tfrac{1}{2}\mu_{ij}, \qquad 0 \le j \le i. \qquad (4.13.16)
\end{aligned}$$

The functions λ_{ij} and $\frac{1}{2}\mu_{ij}$ appear in Appendix A.10. $\mu_{ij} = (2j/i)\lambda_{ij}$.

Theorem 4.61.

$$\mathbf{M} = \mathbf{N}'\mathbf{K}(\mathbf{N}')^T.$$

The details of the proof are similar to those of Theorem 4.60.
Let

$$\mathbf{N}'\mathbf{K}'(\mathbf{N}')^T = [\gamma'_{ij}]_n$$

and consider the four cases separately. It is found with the aid of Theorem A.8(e) in Appendix A.10 that

$$\begin{aligned}
\gamma'_{2p+1,2q+1} &= \sum_{j=1}^{N} a_{ij}\{g_{q-p}(x_j) + g_{q+p+1}(x_j)\}\\
&= \alpha'_{2p+1,2q+1} \qquad (4.13.17)
\end{aligned}$$

and further that $\gamma'_{ij} = \alpha'_{ij}$ for all values of i and j.

Corollary.

$$\begin{aligned}
|\alpha'_{ij}|_n = |\mathbf{M}'|_n &= |\mathbf{N}'|_n^2|\mathbf{K}'|_n\\
&= |\beta'_{ij}|_n^2 2^n \left|2^{i+j-2}(k_{i+j}+k_{i+j-2})\right|_n\\
&= 2^{n^2}|k_{i+j}+k_{i+j-2}|_n \qquad (4.13.18)
\end{aligned}$$

since $\beta'_{ii} = 1$ for all values of i. Thus, $\mathbf{M}'$ can also be expressed as a Hankelian.

4.13.2 The Factors of a Particular Symmetric Toeplitz Determinant

The determinants

$$P_n = \tfrac{1}{2}|p_{ij}|_n,$$

$$Q_n = \tfrac{1}{2}|q_{ij}|_n, \tag{4.13.19}$$

where

$$\begin{aligned} p_{ij} &= t_{|i-j|} - t_{i+j}, \\ q_{ij} &= t_{|i-j|} + t_{i+j-2}, \end{aligned} \tag{4.13.20}$$

appear in Section 4.5.2 as factors of a symmetric Toeplitz determinant.

Put

$$t_r = \omega^r u_r, \quad (\omega^2 = -1).$$

Then,

$$\begin{aligned} p_{ij} &= \omega^{i+j-2}\alpha'_{ij}, \\ q_{ij} &= \omega^{i+j-2}\alpha_{ij}, \end{aligned} \tag{4.13.21}$$

where α'_{ij} and α_{ij} are defined in (4.13.15) and (4.13.2), respectively. Hence, referring to the corollaries in Theorems 4.60 and 4.61,

$$\begin{aligned} P_n &= \tfrac{1}{2}\left|\omega^{i+j-2}\alpha'_{ij}\right|_n \\ &= \tfrac{1}{2}\omega^{n(n-1)}|\alpha'_{ij}|_n \\ &= (-1)^{n(n-1)/2}2^{n^2-1}|k_{i+j} + k_{i+j-2}|_n. \end{aligned} \tag{4.13.22}$$

$$\begin{aligned} Q_n &= \tfrac{1}{2}|\omega^{i+j-2}\alpha_{ij}|_n \\ &= (-1)^{n(n-1)/2}2^{(n-1)^2}|k_{i+j-2}|_n. \end{aligned} \tag{4.13.23}$$

Since P_n and Q_n each have a factor $\omega^{n(n-1)}$ and $n(n-1)$ is even for all values of n, these formulas remain valid when ω is replaced by $(-\omega)$ and are applied in Section 6.10.5 on the Einstein and Ernst equations.

4.14 Casoratians — A Brief Note

The Casoratian $K_n(x)$, which arises in the theory of difference equations, is defined as follows:

$$\begin{aligned} K_n(x) &= |f_i(x+j-1)|_n \\ &= \begin{vmatrix} f_1(x) & f_1(x+1) & \cdots & f_1(x+n-1) \\ f_2(x) & f_2(x+1) & \cdots & f_2(x+n-1) \\ \cdots & \cdots & \cdots & \cdots \\ f_n(x) & f_n(x+1) & \cdots & f_n(x+n-1) \end{vmatrix}_n. \end{aligned}$$

The role played by Casoratians in the theory of difference equations is similar to the role played by Wronskians in the theory of differential equations. Examples of their applications are given by Milne-Thomson, Brand, and Browne and Nillsen. Some applications of Casoratians in mathematical physics are given by Hirota, Kajiwara et al., Liu, Ohta et al., and Yuasa.

5

Further Determinant Theory

5.1 Determinants Which Represent Particular Polynomials

5.1.1 Appell Polynomial

Notes on Appell polynomials are given in Appendix A.4.

Let

$$\psi_n(x) = (-1)^n \sum_{r=0}^{n} \binom{n}{r} \alpha_{n-r}(-x)^r. \tag{5.1.1}$$

Theorem.

$$\textbf{a.}\ \psi_n(x) = \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \alpha_n \\ 1 & x & x^2 & x^3 & \cdots & \cdots & \binom{n}{0} x^n \\ & 1 & 2x & 3x^2 & \cdots & \cdots & \binom{n}{1} x^{n-1} \\ & & 1 & 3x & \cdots & \cdots & \binom{n}{2} x^{n-2} \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1 & nx \end{vmatrix}_{n+1},$$

$$\textbf{b.}\ \psi_n(x) = \frac{1}{n!}\begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \alpha_n \\ n & x & & & & & \\ & n-1 & 2x & & & & \\ & & n-2 & 3x & & & \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1 & nx \end{vmatrix}_{n+1}.$$

Both determinants are Hessenbergians (Section 4.6).

Proof of (a). Denote the determinant by H_{n+1}, expand it by the two elements in the last row, and repeat this operation on the determinants of lower order which appear. The result is

$$H_{n+1}(x) = \sum_{r=1}^{n} \binom{n}{r} H_{n+1-r}(-x)^r + (-1)^n \alpha_n.$$

The H_{n+1} term can be absorbed into the sum, giving

$$(-1)^n \alpha_n = \sum_{r=0}^{n} \binom{n}{r} H_{n+1-r}(-x)^r.$$

This is an Appell polynomial whose inverse relation is

$$H_{n+1}(x) = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \alpha_{n-r} x^r,$$

which is equivalent to the stated result.

Proof of (b). Denote the determinant by H^*_{n+1} and note that some of its elements are functions of n, so that the minor obtained by removing its last row and column is *not* equal to H^*_n and hence there is no obvious recurrence relation linking H^*_{n+1}, H^*_n, H^*_{n-1}, etc.

The determinant H^*_{n+1} can be obtained by transforming H_{n+1} by a series of row operations which reduce some of its elements to zero. Multiply $\mathbf{R}_i$ by $(n+2-i)$, $2 \le i \le n+1$, and compensate for the unwanted factor $n!$ by dividing the determinant by that factor. Now perform the row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \left(\frac{i-1}{n+1-i} \right) x \mathbf{R}_{i+1}$$

first with $2 \le i \le n$, which introduces $(n-1)$ zero elements into $\mathbf{C}_{n+1}$, then with $2 \le i \le n-1$, which introduces $(n-2)$ zero elements into $\mathbf{C}_n$, then with $2 \le i \le n-2$, etc., and, finally, with $i = 2$. The determinant H^*_{n+1} appears. □

5.1.2 *The Generalized Geometric Series and Eulerian Polynomials*

Notes on the generalized geometric series $\psi_n(x)$ and the Eulerian polynomials $A_n(x)$ are given in Appendix A.6.

$$A_n(x) = (1-x)^{n+1}\psi_n(x). \tag{5.1.2}$$

Theorem (Lawden).

$$\frac{A_n}{n!\,x} = \begin{vmatrix} 1 & 1-x & & & & \\ 1/2! & 1 & 1-x & & & \\ 1/3! & 1/2! & 1 & 1-x & & \\ \cdots & \cdots & \cdots & \cdots & & \\ 1/(n-1)! & 1/(n-2)! & \cdots & & 1 & 1-x \\ 1/n! & 1/(n-1)! & \cdots & & 1/2! & 1 \end{vmatrix}_n .$$

The determinant is a Hessenbergian.

PROOF. It is proved in the section on differences (Appendix A.8) that

$$\Delta^m \psi_0 = \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \psi_s = x\psi_m. \tag{5.1.3}$$

Put

$$\psi_s = (-1)^s s!\,\phi_s. \tag{5.1.4}$$

Then,

$$\sum_{s=0}^{m-1} \frac{\phi_s}{(m-s)!} + (1-x)\phi_m = 0, \quad m = 1, 2, 3, \ldots . \tag{5.1.5}$$

In some detail,

$$\begin{aligned} &\phi_0 + (1-x)\phi_1 = 0,\\ &\phi_0/2! + \phi_1 + (1-x)\phi_2 = 0,\\ &\phi_0/3! + \phi_1/2 + \phi_2 + (1-x)\phi_3 = 0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\phi_0/n! + \phi_1/(n-1)! + \phi_2/(n-2)! + \cdots + \phi_{n-1} + (1-x)\phi_n = 0. \end{aligned} \tag{5.1.6}$$

When these n equations in the $(n+1)$ variables ϕ_r, $0 \le r \le n$, are augmented by the relation

$$(1-x)\phi_0 = x, \tag{5.1.7}$$

the determinant of the coefficients is triangular so that its value is $(1-x)^{n+1}$. Solving the $(n+1)$ equations by Cramer's formula (Section 2.3.5),

$$\phi_n = \frac{1}{(1-x)^{n+1}} \begin{vmatrix} 1-x & & & & & & x \\ 1 & 1-x & & & & & 0 \\ 1/2! & 1 & 1-x & & & & 0 \\ 1/3! & 1/2! & 1 & 1-x & & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1/(n-1)! & 1/(n-2)! & \cdots & \cdots & 1 & 1-x & 0 \\ 1/n! & 1/(n-1)! & \cdots & \cdots & 1/2! & 1 & 0 \end{vmatrix}_{n+1} \quad (5.1.8)$$

After expanding the determinant by the single nonzero element in the last column, the theorem follows from (5.1.2) and (5.1.4). □

Exercises

Prove that

1. $$\sum_{r=0}^{n} \alpha_r x^{n-r} y^r = \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \alpha_n \\ -y & x & & & & & \\ & -y & x & & & & \\ & & -y & x & & & \\ & & & \cdots & \cdots & \cdots & \\ & & & & & -y & x \end{vmatrix}_{n+1}.$$

2. $$(x+y)^n = \begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^{n-1} & x^n \\ -1 & y & xy & x^2y & \cdots & x^{n-2}y & x^{n-1}y \\ & -1 & y & xy & \cdots & x^{n-3}y & x^{n-2}y \\ & & -1 & y & \cdots & x^{n-4}y & x^{n-3}y \\ & & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & -1 & y \end{vmatrix}_{n+1}.$$

3. $$(-b)^n {}_2F_0\left(\frac{x}{a}, -n; -\frac{b}{a}\right) = \begin{vmatrix} -c_1 & b & & & & \\ a & -c_2 & b & & & \\ & 2a & -c_3 & b & & \\ & & 3a & -c_4 & & \\ & & & \cdots & \cdots & \cdots \\ & & & & -c_{n-1} & b \\ & & & & (n-1)a & -c_n \end{vmatrix}_n,$$

where

$$c_r = (r-1)a + b + x. \qquad \text{(Frost and Sackfield)}$$

and ${}_2F_0$ is the generalized hypergeometric function.

5.1.3 Orthogonal Polynomials

Determinants which represent orthogonal polynomials (Appendix A.5) have been constructed using various methods by Pandres, Rösler, Yahya, Stein et al., Schleusner, and Singhal, Frost and Sackfield and others. The following method applies the Rodrigues formulas for the polynomials.

Let

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = \binom{j-1}{i-1} u^{(j-i)} - \binom{j-1}{i-2} v^{(j-i+1)}, \quad u^{(r)} = D^r(u), \text{ etc.,}$$
$$u = \frac{vy'}{y} = v(\log y)'. \tag{5.1.9}$$

In some detail,

$$A_n = \begin{vmatrix} u & u' & u'' & u''' & \cdots & u^{(n-2)} & u^{(n-1)} \\ -v & u-v' & 2u'-v'' & 3u''-v''' & \cdots & \cdots & \cdots \\ & -v & u-2v' & 3u'-3v'' & \cdots & \cdots & \cdots \\ & & -v & u-3v' & \cdots & \cdots & \cdots \\ & & & -v & \cdots & \cdots & \cdots \\ & & & & \cdots & \cdots & \cdots \\ & & & & & -v & u-(n-1)v' \end{vmatrix}_n . \tag{5.1.10}$$

Theorem.

a. $A^{(n+1)}_{n+1,n} = -A'_n$,

b. $A_n = \dfrac{v^n D^n(y)}{y}$.

PROOF. Express A_n in column vector notation:

$$A_n = \left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3 \cdots \mathbf{C}_n\right|_n,$$

where

$$\mathbf{C}_j = \left[a_{1j}\ a_{2j}\ a_{3j} \cdots a_{j+1,j}\ O_{n-j-1}\right]^T_n \tag{5.1.11}$$

where O_r represents an unbroken sequence of r zero elements.

Let $\mathbf{C}^*_j$ denote the column vector obtained by dislocating the elements of $\mathbf{C}_j$ one position downward, leaving the uppermost position occupied by a zero element:

$$\mathbf{C}^*_j = \left[O\ a_{1j}\ a_{2j} \cdots a_{jj}\ a_{j+1,j}\ O_{n-j-2}\right]^T_n. \tag{5.1.12}$$

Then,

$$\mathbf{C}'_j + \mathbf{C}^*_j = \left[a'_{1j}\ (a'_{2j} + a_{1j})\ (a'_{3j} + a_{2j}) \cdots (a'_{j+1,j} + a_{jj})\ a_{j+1,j}\ O_{n-j-2}\right]^T_n.$$

But

$$\begin{aligned} a'_{ij} + a_{i-1,j} &= \left[\binom{j-1}{i-1} + \binom{j-1}{i-2} \right] u^{(j-i+1)} \\ &\quad - \left[\binom{j-1}{i-2} + \binom{j-1}{i-3} \right] v^{(j-i+2)} \\ &= \binom{j}{i-1} u^{(j-i+1)} - \binom{j}{i-2} v^{(j-i+2)} \\ &= a_{i,j+1}. \end{aligned} \tag{5.1.13}$$

Hence,

$$\mathbf{C}'_j + \mathbf{C}^*_j = \mathbf{C}_{j+1}, \tag{5.1.14}$$

$$A'_n = \sum_{j=1}^{n} \left| \mathbf{C}_1 \ \mathbf{C}_2 \cdots \mathbf{C}'_j \ \mathbf{C}_{j+1} \cdots \mathbf{C}_{n-1} \ \mathbf{C}_n \right|_n,$$

$$A^{(n+1)}_{n+1,n} = -\left| \mathbf{C}_1 \ \mathbf{C}_2 \cdots \mathbf{C}_j \ \mathbf{C}_{j+1} \cdots \mathbf{C}_{n-1} \ \mathbf{C}_{n+1} \right|_n. \tag{5.1.15}$$

Hence,

$$\begin{aligned} A'_n + A^{(n+1)}_{n+1,n} &= \sum_{j=1}^{n} \left| \mathbf{C}_1 \ \mathbf{C}_2 \cdots (\mathbf{C}'_j - \mathbf{C}_{j+1}) \ \mathbf{C}_{j+1} \cdots \mathbf{C}_n \right|_n \\ &= -\sum_{j=1}^{n} \left| \mathbf{C}_1 \ \mathbf{C}_2 \cdots \mathbf{C}^*_j \cdots \mathbf{C}_n \right| \\ &= 0 \end{aligned}$$

by Theorem 3.1 on cyclic dislocations and generalizations in Section 3.1, which proves (a).

Expanding A_{n+1} by the two elements in its last row,

$$\begin{aligned} A_{n+1} &= (u - nv')A_n - vA^{(n+1)}_{n+1,n} \\ &= (u - nv')A_n + vA'_n \\ &= v \left[A'_n + \left(\frac{u}{v} - \frac{nv'}{v} \right) A_n \right], \\ \frac{yA_{n+1}}{v^{n+1}} &= \frac{y}{v^n} \left[A'_n + \left(\frac{y'}{y} - \frac{nv'}{v} \right) A_n \right] \\ &= \frac{yA'_n}{v^n} + \left(\frac{y}{v^n} \right)' A_n \\ &= D \left(\frac{yA_n}{v^n} \right) \\ &= D^2 \left(\frac{yA_{n-1}}{v^{n-1}} \right) \\ &= D^r \left(\frac{yA_{n-r+1}}{v^{n-r+1}} \right), \quad 0 \le r \le n \end{aligned}$$

$$= D^n\left(\frac{yA_1}{v}\right), \quad \left(A_1 = u = \frac{vy'}{y}\right)$$
$$= D^{n+1}(y).$$

Hence,

$$A_{n+1} = \frac{v^{n+1}D^{n+1}(y)}{y},$$

which is equivalent to (b).

The Rodrigues formula for the generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ is

$$L_n^{(\alpha)}(x) = \frac{x^n D^n(e^{-x}x^{n+\alpha})}{n!\, e^{-x}x^{n+\alpha}}\,. \tag{5.1.16}$$

Hence, choosing

$$v = x,$$
$$y = e^{-x}x^{n+\alpha},$$

so that

$$u = x - n - \alpha, \tag{5.1.17}$$

formula (b) becomes

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\times \tag{5.1.18}$$
$$\begin{vmatrix} n+\alpha-x & 1 & & & & \\ -x & n+\alpha-x-1 & 2 & & & \\ & -x & n+\alpha-x-2 & 3 & & \\ & & \cdots & \cdots & \cdots & \cdots \\ & & & & 2+\alpha-x & n-1 \\ & & & & -x & 1+\alpha-x \end{vmatrix}_{\text{n}}.$$

□

Exercises

Prove that

1. $$L_n^{(\alpha)}(x) = \frac{1}{n!}\begin{vmatrix} n+\alpha-x & n+\alpha & n+\alpha & n+\alpha & \cdots \\ 1 & n+\alpha-x & n+\alpha & n+\alpha & \cdots \\ & 2 & n+\alpha-x & n+\alpha & \cdots \\ & & 3 & n+\alpha-x & \cdots \\ & & \cdots & \cdots & \cdots \end{vmatrix}_n$$

(Pandres).

2. $$H_n(x) = \begin{vmatrix} 2x & 2 & & & & & \\ 1 & 2x & 4 & & & & \\ & 1 & 2x & 6 & & & \\ & & 1 & 2x & 8 & & \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & 2x & 2n-2 \\ & & & & & 1 & 2x \end{vmatrix}_n.$$

3. $$P_n(x) = \frac{1}{n!}\begin{vmatrix} x & 1 & & & & & \\ 1 & 3x & 2 & & & & \\ & 2 & 5x & 3 & & & \\ & & 3 & 7x & 4 & & \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & (2n-3)x & n-1 \\ & & & & & n-1 & (2n-1)x \end{vmatrix}_n$$

(Muir and Metzler).

4. $$P_n(x) = \frac{1}{2^n n!} \begin{vmatrix} 2nx & 2n & & & & \\ 1-x^2 & (2n-2)x & 4n-2 & & & \\ & 1-x^2 & (2n-4)x & 6n-6 & & \\ & & 1-x^2 & (2n-6)x & & \\ & & & \cdots & \cdots & \cdots \\ & & & & 4x & n^2+n-2 \\ & & & & 1-x^2 & 2x \end{vmatrix}_n.$$

5. Let

$$A_n = |a_{ij}|_n = \left|\mathbf{C}_1\ \mathbf{C}_2\ \mathbf{C}_3 \cdots \mathbf{C}_n\right|,$$

where

$$\begin{aligned} a_{ij} &= u^{(j-1)} \\ &= \begin{cases} \binom{j-1}{i-2} v^{(j-i+1)}, & 2 \le i \le j+1, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

and let

$$\mathbf{C}_j^* = \left[O_2\ a_{2j}\ a_{3j} \cdots a_{n-1,j}\right]^T.$$

Prove that

$$\mathbf{C}_j' + \mathbf{C}_j^* = \mathbf{C}_{j+1},$$
$$A_n' + A_{n+1,n}^{(n+1)} = 0$$

and hence prove that

$$A_n = (-1)^{n+1} v^n D^{n-1}\left(\frac{u}{v}\right).$$

6. Prove that the determinant A_n in (5.1.10) satisfies the relation

$$A_{n+1} = vA'_n + (u - nv')A_n.$$

Put $v = 1$ to get

$$A_{n+1} = A'_n + A_1 A_n$$

where

$$A_n = \begin{vmatrix} u & u' & u'' & u''' & \cdots \\ -1 & u & 2u' & 3u'' & \cdots \\ & -1 & u & 3u' & \cdots \\ & & -1 & u & \cdots \\ & & & \cdots & \cdots \end{vmatrix}_n.$$

These functions appear in a paper by Yebbou on the calculation of determining factors in the theory of differential equations. Yebbou uses the notation $\alpha^{[n]}$ in place of A_n.

5.2 The Generalized Cusick Identities

The principal Cusick identity in its generalized form relates a particular skew-symmetric determinant (Section 4.3) to two Hankelians (Section 4.8).

5.2.1 Three Determinants

Let ϕ_r and ψ_r, $r \geq 1$, be two sets of arbitrary functions and define three power series as follows:

$$\begin{aligned} \Phi_i &= \sum_{r=i}^{\infty} \phi_r t^{r-i}, \quad i \geq 1; \\ \Psi_i &= \sum_{r=i}^{\infty} \psi_r t^{r-i}, \quad i \geq 1; \\ G_i &= \Phi_i \Psi_i. \end{aligned} \tag{5.2.1}$$

Let

$$G_i = \sum_{j=i+1}^{\infty} a_{ij} t^{j-i-1}, \quad i \geq 1. \tag{5.2.2}$$

Then, equating coefficients of t^{j-i-1},

$$a_{ij} = \sum_{s=1}^{j-i} \phi_{s+i-1} \psi_{j-s}, \quad i < j. \tag{5.2.3}$$

In particular,

$$a_{i,2n} = \sum_{s=1}^{2n-i} \phi_{s+i-1}\psi_{2n-s}, \quad 1 \le i \le 2n-1. \tag{5.2.4}$$

Let A_{2n} denote the skew-symmetric determinant of order $2n$ defined as

$$A_{2n} = |a_{ij}|_{2n}, \tag{5.2.5}$$

where a_{ij} is defined by (5.2.3) for $1 \le i \le j \le 2n$ and $a_{ji} = -a_{ij}$, which implies $a_{ii} = 0$.

Let H_n and K_n denote Hankelians of order n defined as

$$H_n = \begin{cases} |h_{ij}|_n, & h_{ij} = \phi_{i+j-1} \\ |\phi_m|_n, & 1 \le m \le 2n-1; \end{cases} \tag{5.2.6}$$

$$K_n = \begin{cases} |k_{ij}|_n, & k_{ij} = \psi_{i+j-1} \\ |\psi_m|_n, & 1 \le m \le 2n-1. \end{cases} \tag{5.2.7}$$

All the elements ϕ_r and ψ_r which appear in H_n and K_n, respectively, also appear in $a_{1,2n}$ and therefore also in A_{2n}. The principal identity is given by the following theorem.

Theorem 5.1.

$$A_{2n} = H_n^2 K_n^2.$$

However, since

$$A_{2n} = \mathrm{Pf}_n^2,$$

where Pf_n *is a Pfaffian (Section 4.3.3), the theorem can be expressed in the form*

$$\mathrm{Pf}_n = H_n K_n. \tag{5.2.8}$$

Since Pfaffians are uniquely defined, there is no ambiguity in sign in this relation.

The proof uses the method of induction. It may be verified from (4.3.25) and (5.2.3) that

$$\mathrm{Pf}_1 = a_{12} = \phi_1\psi_1 = H_1K_1,$$

$$\begin{aligned} \mathrm{Pf}_2 &= \left| \begin{matrix} \phi_1\psi_1 & \phi_1\psi_2 + \phi_2\psi_1 & \phi_1\psi_3 + \phi_2\psi_2 + \phi_3\psi_1 \\ & \phi_2\psi_2 & \phi_2\psi_3 + \phi_3\psi_2 \\ & & \phi_3\psi_3 \end{matrix} \right| \\ &= \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_3 \end{vmatrix} \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_2 & \psi_3 \end{vmatrix} \\ &= H_2K_2 \end{aligned} \tag{5.2.9}$$

so that the theorem is known to be true when $n = 1$ and 2.

Assume that

$$\mathrm{Pf}_m = H_m K_m, \quad m < n. \tag{5.2.10}$$

The method by which the theorem is proved for all values of n is outlined as follows.

Pf_n is expressible in terms of Pfaffians of lower order by the formula

$$\mathrm{Pf}_n = \sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} a_{i,2n}, \tag{5.2.11}$$

where, in this context, $a_{i,2n}$ is defined as a sum in (5.2.4) so that Pf_n is expressible as a double sum. The introduction of a variable x enables the inductive assumption (5.2.10) to be expressed as the equality of two polynomials in x. By equating coefficients of one particular power of x, an identity is found which expresses $\mathrm{Pf}_i^{(n)}$ as the sum of products of cofactors of H_n and K_n (Lemma 5.5). Hence, Pf_n is expressible as a triple sum containing the cofactors of H_n and K_n. Finally, with the aid of an identity in Appendix A.3, it is shown that the triple sum simplifies to the product $H_n K_n$.

The following Pfaffian identities will also be applied.

$$\mathrm{Pf}_i^{(n)} = \left(A_{ii}^{(2n-1)}\right)^{1/2}, \tag{5.2.12}$$

$$(-1)^{i+j} \mathrm{Pf}_i^{(n)} \mathrm{Pf}_j^{(n)} = A_{ij}^{(2n-1)}, \tag{5.2.13}$$

$$\mathrm{Pf}_{2n-1}^{(n)} = \mathrm{Pf}_{n-1}. \tag{5.2.14}$$

The proof proceeds with a series of lemmas.

5.2.2 Four Lemmas

Let a_{ij}^* be the function obtained from a_{ij} by replacing each ϕ_r by $(\phi_r - x\phi_{r+1})$ and by replacing each ψ_r by $(\psi_r - x\psi_{r+1})$.

Lemma 5.2.

$$a_{ij}^* = a_{ij} - (a_{i,j+1} + a_{i+1,j})x + a_{i+1,j+1}x^2.$$

PROOF.

$$\begin{aligned} a_{ij}^* &= \sum_{s=1}^{j-i} (\phi_{s+i-1} - x\phi_{s+i})(\psi_{j-s} - x\psi_{j-s+1}) \\ &= a_{ij} - (s_1 + s_2)x + s_3 x^2, \end{aligned}$$

where

$$\begin{aligned} s_1 &= \sum_{s=1}^{j-i} \phi_{s+i-1}\psi_{j-s+1} \\ &= a_{i,j+1} - \phi_i \psi_j, \end{aligned}$$

$$s_2 = a_{i+1,j} + \phi_i \psi_j,$$
$$s_3 = a_{i+1,j+1}.$$

The lemma follows. □

Let

$$\begin{aligned} A^*_{2n} &= |a^*_{ij}|_{2n}, \\ \mathrm{Pf}^*_n &= \left(A^*_{2n}\right)^{1/2}. \end{aligned} \tag{5.2.15}$$

Lemma 5.3.

$$\sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} x^{2n-i-1} = \mathrm{Pf}^*_{n-1}.$$

PROOF. Denote the sum by F_n. Then, referring to (5.2.13) and Section 3.7 on bordered determinants,

$$\begin{aligned} F_n^2 &= \sum_{i=1}^{2n-1} \sum_{j=1}^{2n-1} (-1)^{i+j} \mathrm{Pf}_i^{(n)} \mathrm{Pf}_j^{(n)} x^{4n-i-j-2} \\ &= \sum_{i=1}^{2n-1} \sum_{j=1}^{2n-1} A_{ij}^{(2n-1)} x^{4n-i-j-2} \\ &= -\left| \begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1,2n-1} & x^{2n-2} \\ a_{21} & a_{22} & \cdots & a_{2,2n-1} & x^{2n-3} \\ \multicolumn{5}{c}{\dotfill} \\ a_{2n-1,1} & a_{2n-1,2} & \cdots & a_{2n-1,2n-1} & 1 \\ x^{2n-2} & x^{2n-3} & \cdots & 1 & \bullet \end{array} \right|_{2n}. \end{aligned} \tag{5.2.16}$$

(It is not necessary to put $a_{ii} = 0$, etc., in order to prove the lemma.)

Eliminate the x's from the last column and row by means of the row and column operations

$$\begin{aligned} \mathbf{R}'_i &= \mathbf{R}_i - x\mathbf{R}_{i+1}, \quad 1 \le i \le 2n-2, \\ \mathbf{C}'_j &= \mathbf{C}_j - x\mathbf{C}_{j+1}, \quad 1 \le j \le 2n-2. \end{aligned} \tag{5.2.17}$$

The result is

$$\begin{aligned} F_n^2 &= -\left| \begin{array}{ccccc} a^*_{11} & a^*_{12} & \cdots & a^*_{1,2n-1} & \bullet \\ a^*_{21} & a^*_{22} & \cdots & a^*_{2,2n-1} & \bullet \\ \multicolumn{5}{c}{\dotfill} \\ a^*_{2n-1,1} & a^*_{2n-1,2} & \cdots & a^*_{2n-1,2n-1} & 1 \\ \bullet & \bullet & \cdots & 1 & \bullet \end{array} \right|_{2n} \\ &= +|a^*_{ij}|_{2n-2} \\ &= A^*_{2n-2}. \end{aligned}$$

The lemma follows by taking the square root of each side. □

Let H^*_{n-1} and K^*_{n-1} denote the determinants obtained from H_{n-1} and K_{n-1}, respectively, by again replacing each ϕ_r by $(\phi_r - x\phi_{r+1})$ and by replacing each ψ_r by $(\psi_r - x\psi_{r+1})$. In the notation of the second and fourth lines of (5.2.6),

$$\begin{aligned} H^*_{n-1} &= \left|\phi_m - x\phi_{m+1}\right|_n, \quad 1 \le m \le 2n-3, \\ K^*_{n-1} &= \left|\psi_m - x\psi_{m+1}\right|_n, \quad 1 \le m \le 2n-3. \end{aligned} \tag{5.2.18}$$

Lemma 5.4.

a. $\displaystyle\sum_{i=1}^{n} H_{in}^{(n)} x^{n-i} = H^*_{n-1}$,

b. $\displaystyle\sum_{i=1}^{n} K_{in}^{(n)} x^{n-i} = K^*_{n-1}$.

Proof of (a).

$$\sum_{i=1}^{n} H_{in}^{(n)} x^{n-i} = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_{n-1} & x^{n-1} \\ \phi_2 & \phi_3 & \cdots & \phi_n & x^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_{n-1} & \phi_n & \cdots & \phi_{2n-3} & x \\ \phi_n & \phi_{n+1} & \cdots & \phi_{2n-2} & 1 \end{vmatrix}_n .$$

The result follows by eliminating the x's from the last column by means of the row operations:

$$\mathbf{R}'_i = \mathbf{R}_i - x\mathbf{R}_{i+1}, \quad 1 \le i \le n-1.$$

Part (b) is proved in a similar manner. □

Lemma 5.5.

$$(-1)^{i+1}\mathrm{Pf}_i^{(n)} = \sum_{j=1}^{n} H_{jn}^{(n)} K_{i-j+1,n}^{(n)}, \quad 1 \le i \le 2n-1.$$

Since $K_{mn}^{(n)} = 0$ when $m < 1$ and when $m > n$, the true upper limit in the sum is i, but it is convenient to retain n in order to simplify the analysis involved in its application.

Proof. It follows from the inductive assumption (5.2.10) that

$$\mathrm{Pf}^*_{n-1} = H^*_{n-1}K^*_{n-1}. \tag{5.2.19}$$

Hence, applying Lemmas 5.3 and 5.4,

$$\begin{aligned} \sum_{i=1}^{2n-1} (-1)^{i+1}\mathrm{Pf}_i^{(n)} x^{2n-i-1} &= \left[\sum_{i=1}^{n} H_{in}^{(n)} x^{n-i}\right]\left[\sum_{s=1}^{n} K_{sn}^{(n)} x^{n-s}\right] \\ &= \left[\sum_{j=1}^{n}\sum_{s=1}^{n} H_{jn}^{(n)} K_{sn}^{(n)} x^{2n-j-s}\right] \quad \left[\begin{matrix} s = i-j+1 \\ s \to i \end{matrix}\right] \end{aligned}$$

$$= \sum_{j=1}^{n} \sum_{i=j}^{n+j-1} H_{jn}^{(n)} K_{i-j+1,n}^{(n)} x^{2n-i-1}$$

$$= \sum_{i=1}^{2n-1} x^{2n-i-1} \sum_{j=1}^{n} H_{jn}^{(n)} K_{i-j+1,n}^{(n)}. \tag{5.2.20}$$

Note that the changes in the limits of the i-sum have introduced only zero terms. The lemma follows by equating coefficients of x^{2n-i-1}. □

5.2.3 *Proof of the Principal Theorem*

A double-sum identity containing the symbols c_{ij}, f_i, and g_i is given in Appendix A.3. It follows from Lemma 5.5 that the conditions defining the validity of the double-sum identity are satisfied if

$$\begin{aligned} f_i &= (-1)^{i+1} \mathrm{Pf}_i^{(n)}, \\ c_{ij} &= H_{in}^{(n)} K_{jn}^{(n)}, \\ g_i &= a_{i,2n}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} a_{i,2n} &= \sum_{i=1}^{n} \sum_{j=1}^{n} H_{in}^{(n)} K_{jn}^{(n)} a_{i+j-1,2n} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} H_{in}^{(n)} K_{jn}^{(n)} \sum_{s=1}^{2n-i-j+1} \phi_{s+i+j-2} \psi_{2n-s}. \end{aligned}$$

From (5.2.11), the sum on the left is equal to Pf_n. Also, since the interval $(1, 2n-i-j+1)$ can be split into the intervals $(1, n+1-j)$ and $(n+2-j, 2n-i-j+1)$, it follows from the note in Appendix A.3 on a triple sum that

$$\mathrm{Pf}_n = \sum_{j=1}^{n} K_{jn}^{(n)} X_j + \sum_{i=1}^{n-1} H_{in}^{(n)} Y_i,$$

where

$$\begin{aligned} X_j &= \sum_{i=1}^{n} H_{in}^{(n)} \sum_{s=1}^{n+1-j} \phi_{s+i+j-2} \psi_{2n-s} \\ &= \sum_{s=1}^{n+1-j} \psi_{2n-s} \sum_{i=1}^{n} \phi_{s+i+j-2} H_{in}^{(n)} \\ &= \sum_{s=1}^{n+1-j} \psi_{2n-s} \sum_{i=1}^{n} h_{i,s+j-1} H_{in}^{(n)} \end{aligned}$$

$$= H_n \sum_{s=1}^{n+1-j} \psi_{2n-s}\delta_{s,n-j+1}$$

$$= H_n \psi_{n+j-1}, \quad 1 \le j \le n; \tag{5.2.21}$$

$$Y_i = \sum_{j=1}^{n} K_{jn}^{(n)} \sum_{s=n+2-j}^{2n-i-j+1} \phi_{t+i+2}\psi_{2n-s}, \qquad \begin{bmatrix} s = t-j \\ s \to t \end{bmatrix}$$

$$= \sum_{j=1}^{n} K_{jn}^{(n)} \sum_{t=n+2}^{2n-i+1} \phi_{t+i-2}\psi_{2n+j-t}$$

$$= \sum_{t=n+2}^{2n-i+1} \phi_{t+i-2} \sum_{j=1}^{n} \psi_{2n+j-t} K_{jn}^{(n)}$$

$$= \sum_{t=n+2}^{2n-i+1} \phi_{t+i-2} \sum_{j=1}^{n} k_{j+n+1-t,n} K_{jn}^{(n)}$$

$$= K_n \sum_{t=n+2}^{2n-i+1} \phi_{t+i-2}\delta_{t,n+1}$$

$$= 0, \quad 1 \le i \le n-1, \tag{5.2.22}$$

since $t > n+1$. Hence,

$$\begin{aligned} \mathrm{Pf}_n &= H_n \sum_{j=1}^{n} K_{jn}^{(n)} \psi_{n+j-1} \\ &= H_n \sum_{j=1}^{n} k_{jn} K_{jn}^{(n)} \\ &= H_n K_n, \end{aligned}$$

which completes the proof of Theorem 5.1.

5.2.4 Three Further Theorems

The principal theorem, when expressed in the form

$$\sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} a_{i,2n} = H_n K_n, \tag{5.2.23}$$

yields two corollaries by partial differentiation. Since the only elements in Pf_n which contain ϕ_{2n-1} and ψ_{2n-1} are $a_{i,2n}$, $1 \le i \le 2n-1$, and $\mathrm{Pf}_i^{(n)}$ is independent of $a_{i,2n}$, it follows that $\mathrm{Pf}_i^{(n)}$ is independent of ϕ_{2n-1} and ψ_{2n-1}. Moreover, these two functions occur only once in H_n and K_n, respectively, both in position (n, n).

From (5.2.4),

$$\frac{\partial a_{i,2n}}{\partial \phi_{2n-1}} = \psi_i.$$

Also,

$$\frac{\partial H_n}{\partial \phi_{2n-1}} = H_{n-1}.$$

Hence,

$$\sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} \psi_i = H_{n-1} K_n. \tag{5.2.24}$$

Similarly,

$$\sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} \phi_i = H_n K_{n-1}. \tag{5.2.25}$$

The following three theorems express modified forms of $|a_{ij}|_n$ in terms of the Hankelians.

Let $B_n(\phi)$ denote the determinant which is obtained from $|a_{ij}|_n$ by replacing the last row by the row

$$\left[\phi_1 \ \phi_2 \ \phi_3 \ldots \phi_n\right].$$

Theorem 5.6.

a. $B_{2n-1}(\phi) = H_{n-1} H_n K_{n-1}^2$,
b. $B_{2n-1}(\psi) = H_{n-1}^2 K_{n-1} K_n$,
c. $B_{2n}(\phi) = -H_n^2 K_{n-1} K_n$,
d. $B_{2n}(\psi) = -H_{n-1} H_n K_n^2$.

PROOF. Expanding $B_{2n-1}(\phi)$ by elements from the last row and their cofactors and referring to (5.2.13), (5.2.14), and (5.2.25),

$$\begin{aligned} B_{2n-1}(\phi) &= \sum_{j=1}^{2n-1} \phi_j A_{2n-1,j}^{(2n-1)} \\ &= \mathrm{Pf}_{2n-1}^{(n)} \sum_{i=1}^{2n-1} (-1)^{i+1} \mathrm{Pf}_i^{(n)} \phi_i \\ &= \mathrm{Pf}_{n-1} H_n K_{n-1}. \end{aligned} \tag{5.2.26}$$

Part (a) now follows from Theorem 5.1 and (b) is proved in a similar manner.

Expanding $B_{2n}(\phi)$ with the aid of Theorem 3.9 on bordered determinants (Section 3.7) and referring to (5.2.11) and (5.2.25),

$$B_{2n}(\phi) = -\sum_{i=1}^{2n-1} \sum_{j=1}^{2n-1} a_{i,2n} \phi_j A_{ij}^{(2n-1)}$$

$$= -\left[\sum_{i=1}^{2n-1}(-1)^{i+1}\mathrm{Pf}_i^{(n)}a_{i,2n}\right]\left[\sum_{j=1}^{2n-1}(-1)^{j+1}\mathrm{Pf}_j^{(n)}\phi_j\right]$$
$$= -\mathrm{Pf}_n H_n K_{n-1}. \tag{5.2.27}$$

Part (c) now follows from Theorem 5.1 and (d) is proved in a similar manner. □

Let $\mathbf{R}(\phi)$ denote the row vector defined as

$$\mathbf{R}(\phi) = \left[\phi_1\ \phi_2\ \phi_3 \cdots \phi_{2n-1}\ \bullet\right]$$

and let $B_{2n}(\phi,\psi)$ denote the determinant of order $2n$ which is obtained from $|a_{ij}|_{2n}$ by replacing the last row by $-\mathbf{R}(\phi)$ and replacing the last column by $\mathbf{R}^T(\psi)$.

Theorem 5.7.

$$B_{2n}(\phi,\psi) = H_{n-1}H_nK_{n-1}K_n.$$

Proof.

$$B_{2n}(\phi,\psi) = \sum_{i=1}^{2n-1}\sum_{j=1}^{2n-1}\psi_i\phi_jA_{ij}^{(2n-1)}.$$

The theorem now follows (5.2.13), (5.2.24), and (5.2.25). □

Theorem 5.8.

$$B_{2n}(\phi,\psi) = A_{2n-1,2n}^{(2n)}.$$

Proof. Applying the Jacobi identity (Section 3.6),

$$\begin{vmatrix} A_{2n-1,2n-1}^{(2n)} & A_{2n-1,2n}^{(2n)} \\ A_{2n,2n-1}^{(2n)} & A_{2n,2n}^{(2n)} \end{vmatrix} = A_{2n}A_{2n-1,2n;2n-1,2n}^{(2n)}. \tag{5.2.28}$$

But, $A_{ii}^{(2n)}$, $i = 2n-1,\ 2n$, are skew-symmetric of odd order and are therefore zero. The other two first cofactors are equal in magnitude but opposite in sign. Hence,

$$\left(A_{2n-1,2n}^{(2n)}\right)^2 = A_{2n}A_{2n-2},$$
$$A_{2n-1,2n}^{(2n)} = \mathrm{Pf}_n\mathrm{Pf}_{n-1}. \tag{5.2.29}$$

Theorem 5.8 now follows from Theorems 5.1 and 5.7. □

If $\psi_r = \phi_r$, then $K_n = H_n$ and Theorems 5.1, 5.6a and c, and 5.7 degenerate into identities published in a different notation by Cusick, namely,

$$A_{2n} = H_n^4,$$
$$B_{2n-1}(\phi) = H_{n-1}^3H_n,$$

$$\begin{aligned} B_{2n}(\phi) &= -H_{n-1}H_n^3, \\ B_{2n}(\phi,\phi) &= H_{n-1}^2 H_n^2. \end{aligned} \tag{5.2.30}$$

These identities arose by a by-product in a study of Littlewood's Diophantime approximation problem.

The negative sign in the third identity, which is not required in Cusick's notation, arises from the difference between the methods by which $B_n(\phi)$ and Cusick's determinant T_n are defined. Note that $B_{2n}(\phi,\phi)$ is skew-symmetric of even order and is therefore expected to be a perfect square.

Exercises

1. Prove that

$$\begin{aligned} A_{1,2n}^{(2n)} &= -H_n H_{1n}^{(n)} K_n K_{1n}^{(n)}, \\ A_{1,2n-1}^{(2n-1)} &= H_{n-1} H_{1n}^{(n)} K_{n-1} K_{1n}^{(n)}. \end{aligned}$$

2. Let $V_n(\phi)$ be the determinant obtained from $A_{1,2n}^{(2n)}$ by replacing the last row by $\mathbf{R}_{2n}(\phi)$ and let $W_n(\phi)$ be the determinant obtained from $A_{1,2n-1}^{(2n-1)}$ by replacing the last row by $\mathbf{R}_{2n-1}(\phi)$. Prove that

$$\begin{aligned} V_n(\phi) &= -H_n H_{1n}^{(n)} K_{n-1} K_{1n}^{(n)}, \\ W_n(\phi) &= -H_{n-1} H_{1n}^{(n)} K_{n-1} K_{1,n-1}^{(n-1)}. \end{aligned}$$

3. Prove that

$$A_{i,2n}^{(2n)} = (-1)^{i+1}\mathrm{Pf}_n \mathrm{Pf}_i^{(n-1)}.$$

5.3 The Matsuno Identities

Some of the identities in this section appear in Appendix II in a book on the bilinear transformation method by Y. Matsuno, but the proofs have been modified.

5.3.1 A General Identity

Let

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} u_{ij}, & j \neq i \\ x - \sum\limits_{\substack{r=1 \\ r\neq i}}^{n} u_{ir}, & j = i, \end{cases} \tag{5.3.1}$$

and

$$u_{ij} = \frac{1}{x_i - x_j} = -u_{ji}, \tag{5.3.2}$$

where the x_i are distinct but otherwise arbitrary.

Illustration.

$$A_3 = \begin{vmatrix} x - u_{12} - u_{13} & u_{12} & u_{13} \\ u_{21} & x - u_{21} - u_{23} & u_{23} \\ u_{31} & u_{32} & x - u_{31} - u_{32} \end{vmatrix}.$$

Theorem.

$$A_n = x^n.$$

[This theorem appears in a section of Matsuno's book in which the x_i are the zeros of classical polynomials but, as stated above, it is valid for all x_i, provided only that they are distinct.]

PROOF. The sum of the elements in each row is x. Hence, after performing the column operations

$$\begin{aligned} \mathbf{C}'_n &= \sum_{j=1}^{n} \mathbf{C}_j \\ &= x\big[1\ 1\ 1 \cdots 1\big]^T, \end{aligned}$$

it is seen that A_n is equal to x times a determinant in which every element in the last column is 1. Now, perform the row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \mathbf{R}_n, \quad 1 \le i \le n-1,$$

which remove every element in the last column except the element 1 in position (n, n). The result is

$$A_n = xB_{n-1},$$

where

$$\begin{aligned} B_{n-1} &= |b_{ij}|_{n-1}, \\ b_{ij} &= \begin{cases} u_{ij} - u_{nj} = \frac{u_{ij}u_{nj}}{u_{ni}}, & j \ne i \\ x - \sum\limits_{\substack{r=1 \\ r \ne i}}^{n-1} u_{ir}, & j = i. \end{cases} \end{aligned}$$

It is now found that, after row i has been multiplied by the factor u_{ni}, $1 \le i \le n-1$, the same factor can be canceled from column i, $1 \le i \le n-1$, to give the result

$$B_{n-1} = A_{n-1}.$$

Hence,

$$A_n = xA_{n-1}.$$

But $A_2 = x^2$. The theorem follows. □

5.3.2 Particular Identities

It is shown in the previous section that $A_n = x^n$ provided only that the x_i are distinct. It will now be shown that the diagonal elements of A_n can be modified in such a way that $A_n = x^n$ as before, but only if the x_i are the zeros of certain orthogonal polynomials. These identities supplement those given by Matsuno.

It is well known that the zeros of the Laguerre polynomial $L_n(x)$, the Hermite polynomial $H_n(x)$, and the Legendre polynomial $P_n(x)$ are distinct. Let $p_n(x)$ represent any one of these polynomials and let its zeros be denoted by x_i, $1 \le i \le n$. Then,

$$p_n(x) = k\prod_{r=1}^{n}(x - x_r), \tag{5.3.3}$$

where k is a constant. Hence,

$$\begin{aligned} \log p_n(x) &= \log k + \sum_{r=1}^{n}\log(x - x_r), \\ \frac{p_n'(x)}{p_n(x)} &= \sum_{r=1}^{n}\frac{1}{x - x_r}. \end{aligned} \tag{5.3.4}$$

It follows that

$$\sum_{\substack{r=1 \\ r\neq i}}^{n}\frac{1}{x - x_r} = \frac{(x - x_i)p_n'(x) - p_n(x)}{(x - x_i)p_n(x)}. \tag{5.3.5}$$

Hence, applying the l'Hopital limit theorem twice,

$$\begin{aligned} \sum_{\substack{r=1 \\ r\neq i}}^{n}\frac{1}{x_i - x_r} &= \lim_{x\to x_i}\left[\frac{(x - x_i)p_n'(x) - p_n(x)}{(x - x_i)p_n(x)}\right] \\ &= \lim_{x\to x_i}\left[\frac{(x - x_i)p_n'''(x) + p_n''(x)}{(x - x_i)p_n''(x) + 2p_n'(x)}\right] \\ &= \frac{p_n''(x_i)}{2p_n'(x_i)}. \end{aligned} \tag{5.3.6}$$

The sum on the left appears in the diagonal elements of A_n. Now redefine A_n as follows:

$$A_n = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} u_{ij}, & j \neq i \\ x - \frac{p_n''(x_i)}{2p_n'(x_i)}, & j = i. \end{cases} \tag{5.3.7}$$

This A_n clearly has the same value as the original A_n since the left-hand side of (5.3.6) has been replaced by the right-hand side, its algebraic equivalent.

The right-hand side of (5.3.6) will now be evaluated for each of the three particular polynomials mentioned above with the aid of their differential equations (Appendix A.5).

Laguerre Polynomials.

$$\begin{aligned} xL_n''(x) + (1-x)L_n'(x) + nL_n(x) &= 0, \\ L_n(x_i) = 0, \quad 1 \leq i \leq n, \\ \frac{L_n''(x_i)}{2L_n'(x_i)} = \frac{x_i - 1}{x_i}\,. \end{aligned} \tag{5.3.8}$$

Hence, if

$$a_{ij} = \begin{cases} u_{ij}, & j \neq i \\ x - \frac{x_i - 1}{2x_i}, & j = i, \end{cases}$$

then

$$A_n = |a_{ij}|_n = x^n. \tag{5.3.9}$$

Hermite Polynomials.

$$\begin{aligned} H_n''(x) - 2xH_n'(x) + 2nH_n(x) &= 0, \\ H_n(x_i) = 0, \qquad 1 \leq i \leq n, \\ \frac{H_n''(x_i)}{2H_n'(x_i)} = x_i. \end{aligned} \tag{5.3.10}$$

Hence if,

$$a_{ij} = \begin{cases} u_{ij}, & j \neq i \\ x - x_i, & j = i, \end{cases}$$

then

$$A_n = |a_{ij}|_n = x^n. \tag{5.3.11}$$

Legendre Polynomials.

$$\begin{aligned} (1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) &= 0, \\ P_n(x_i) = 0, \quad 1 \leq i \leq n, \\ \frac{P_n''(x_i)}{2P_n'(x_i)} = \frac{x_i}{1 - x_i^2}\,. \end{aligned} \tag{5.3.12}$$

Hence, if

$$a_{ij} = \begin{cases} u_{ij}, & j \neq i \\ x - \frac{x_i}{1-x_i^2}, & j = i, \end{cases}$$

then

$$A_n = |a_{ij}|_n = x^n. \tag{5.3.13}$$

Exercises

1. Let A_n denote the determinant defined in (5.3.9) and let

$$B_n = |b_{ij}|_n,$$

where

$$b_{ij} = \begin{cases} \frac{2}{x_i - x_j}, & j \neq i \\ x + \frac{1}{x_i}, & j = i, \end{cases}$$

where, as for $A_n(x)$, the x_i denote the zeros of the Laguerre polynomial. Prove that

$$B_n(x-1) = 2^n A_n\left(\frac{x}{2}\right)$$

and, hence, prove that

$$B_n(x) = (x+1)^n.$$

2. Let

$$A_n^{(p)} = |a_{ij}^{(p)}|_n,$$

where

$$a_{ij}^{(p)} = \begin{cases} u_{ij}^p, & j \neq i \\ x - \sum\limits_{\substack{r=1 \\ r \neq i}}^{n} u_{ir}^p, & j = i, \end{cases}$$

$$u_{ij} = \frac{1}{x_i - x_j} = -u_{ji}$$

and the x_i are the zeros of the Hermite polynomial $H_n(x)$. Prove that

$$A_n^{(2)} = \prod_{r=1}^{n} [x - (r-1)],$$

$$A_n^{(4)} = \prod_{r=1}^{n} \left[x - \frac{1}{6}(r^2 - 1)\right].$$

5.4 The Cofactors of the Matsuno Determinant

5.4.1 Introduction

Let

$$E_n = |e_{ij}|_n,$$

where

$$e_{ij} = \begin{cases} \frac{1}{c_i - c_j}, & j \neq i \\ x_i, & j = i, \end{cases} \tag{5.4.1}$$

and where the c's are distinct but otherwise arbitrary and the x's are arbitrary. In some detail,

$$E_n = \begin{vmatrix} x_1 & \frac{1}{c_1 - c_2} & \frac{1}{c_1 - c_3} & \cdots & \frac{1}{c_1 - c_n} \\ \frac{1}{c_2 - c_1} & x_2 & \frac{1}{c_2 - c_3} & \cdots & \cdots \\ \frac{1}{c_3 - c_1} & \frac{1}{c_3 - c_2} & x_3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{c_n - c_1} & \cdots & \cdots & \cdots & x_n \end{vmatrix}_n . \tag{5.4.2}$$

This determinant is known here as the Matsuno determinant in recognition of Matsuno's solutions of the Kadomtsev–Petviashvili (KP) and Benjamin–Ono (BO) equations (Sections 6.8 and 6.9), where it appears in modified forms. It is shown below that the first and higher scaled cofactors of E satisfy a remarkably rich set of algebraic multiple-sum identities which can be applied to simplify the analysis in both of Matsuno's papers.

It is convenient to introduce the symbol † into a double sum to denote that those terms in which the summation variables are equal are omitted from the sum. Thus,

$$\sum_r \sum_s {}^{\dagger} u_{rs} = \sum_r \sum_s u_{rs} - \sum_r u_{rr}. \tag{5.4.3}$$

It follows from the partial derivative formulae in the first line of (3.2.4), (3.6.7), (3.2.16), and (3.2.17) that

$$\begin{aligned} \frac{\partial E_{pq}}{\partial x_i} &= E_{ip,iq}, \\ \frac{\partial E_{pr,qs}}{\partial x_i} &= E_{ipr,iqs} \\ \frac{\partial E^{pq}}{\partial x_i} &= -E^{pi} E^{iq}, \\ \left(E^{ii} + \frac{\partial}{\partial x_i} \right) E^{pq} &= E^{ip,iq}, \\ \left(E^{ii} + \frac{\partial}{\partial x_i} \right) E^{pr,qs} &= E^{ipr,iqs}, \end{aligned}$$

$$\left(E^{ii} + \frac{\partial}{\partial x_i}\right) E^{pru,qsv} = e^{ipru,iqsv}, \tag{5.4.4}$$

etc.

5.4.2 First Cofactors

When $f_r + g_r = 0$, the double-sum identities (C) and (D) in Section 3.4 become

$$\sum_{r=1}^{n} \sum_{s=1}^{n}{}^{\dagger} (f_r + g_s) a_{rs} A^{rs} = 0, \tag{C†}$$

$$\sum_{r=1}^{n} \sum_{s=1}^{n}{}^{\dagger} (f_r + g_s) a_{rs} A^{is} A^{rj} = (f_i + g_j) A^{ij}. \tag{D†}$$

Applying (C†) to E with $f_r = -g_r = c_r^m$,

$$\sum_{r=1}^{n} \sum_{s=1}^{n}{}^{\dagger} \left(\frac{c_r^m - c_s^m}{c_r - c_s}\right) E^{rs} = 0. \tag{5.4.5}$$

Putting $m = 1, 2, 3$ yields the following particular cases:

$m = 1$: $\quad \sum_r \sum_s{}^{\dagger} E^{rs} = 0,$

which is equivalent to

$$\sum_r \sum_s E^{rs} = \sum_r E^{rr}; \tag{5.4.6}$$

$m = 2$: $\quad \sum_r \sum_s{}^{\dagger} (c_r + c_s) E^{rs} = 0,$

which is equivalent to

$$\sum_r \sum_s (c_r + c_s) E^{rs} = 2 \sum_r c_r E^{rr}; \tag{5.4.7}$$

$m = 3$: $\quad \sum_r \sum_s{}^{\dagger} (c_r^2 + c_r c_s + c_s^2) E^{rs} = 0,$

which is equivalent to

$$\sum_r \sum_s (c_r^2 + c_r c_s + c_s^2) E^{rs} = 3 \sum_r c_r^2 E^{rr}. \tag{5.4.8}$$

Applying (D†) to E, again with $f_r = -g_r = c_r^m$,

$$\sum_r \sum_s{}^{\dagger} \left(\frac{c_r^m - c_s^m}{c_r - c_s}\right) E^{is} E^{rj} = (c_i^m - c_j^m) E^{ij}. \tag{5.4.9}$$

Putting $m = 1, 2$ yields the following particular cases:

$m = 1$: $$\sum_r \sum_s {}^{\dagger} E^{is} E^{rj} = (c_i - c_j) E^{ij},$$

which is equivalent to

$$\sum_r \sum_s E^{is} E^{rj} - \sum_r E^{ir} E^{rj} = (c_i - c_j) E^{ij}; \tag{5.4.10}$$

$m = 2$: $$\sum_r \sum_s {}^{\dagger} (c_r + c_s) E^{is} E^{rj} = (c_i^2 - c_j^2) E^{ij},$$

which is equivalent to

$$\sum_r \sum_s (c_r + c_s) E^{is} E^{rj} - 2 \sum_r c_r E^{ir} E^{rj} = (c_i^2 - c_j^2) E^{ij}, \tag{5.4.11}$$

etc. Note that the right-hand side of (5.4.9) is zero when $j = i$ for all values of m. In particular, (5.4.10) becomes

$$\sum_r \sum_s E^{is} E^{ri} = \sum_r E^{ir} E^{ri} \tag{5.4.12}$$

and the equation in item $m = 2$ becomes

$$\sum_r \sum_s (c_r + c_s) E^{is} E^{ri} = 2 \sum_r c_r E^{ir} E^{ri}. \tag{5.4.13}$$

5.4.3 First and Second Cofactors

The following five identities relate the first and second cofactors of E: They all remain valid when the parameters are lowered.

$$\sum_{r,s} {}^{\dagger} E^{ir,js} = -(c_i - c_j) E^{ij}, \tag{5.4.14}$$

$$\sum_{r,s} {}^{\dagger} (c_r + c_s) E^{ir,js} = -(c_i^2 - c_j^2) E^{ij}, \tag{5.4.15}$$

$$\sum_{r,s} (c_r - c_s) E^{rs} = \sum_{r,s} E^{rs,rs}, \tag{5.4.16}$$

$$2 \sum_{r,s} {}^{\dagger} c_r E^{rs} = -2 \sum_{r,s} {}^{\dagger} c_s E^{rs} = \sum_{r,s} E^{rs,rs}, \tag{5.4.17}$$

$$\sum_{r<s} (c_s E^{rs} + c_r E^{sr} + E^{rs,rs}) = 0. \tag{5.4.18}$$

To prove (5.4.14), apply the Jacobi identity to $E^{ir,js}$ and refer to (5.4.6) and the equation in item $m = 1$.

$$\begin{aligned} \sum_{r,s} {}^{\dagger} E^{ir,js} &= \sum_{r,s} {}^{\dagger} \begin{vmatrix} E^{ij} & E^{is} \\ E^{rj} & E^{rs} \end{vmatrix} \\ &= E^{ij} \sum_{r,s} {}^{\dagger} E^{rs} - \sum_{r,s} {}^{\dagger} E^{is} E^{rj} \end{aligned}$$

$$= -(c_i - c_j)E^{ij}.$$

Equation (5.4.15) can be proved in a similar manner by appling (5.4.7) and the equation in item $m = 2$. The proof of (5.4.16) is a little more difficult. Modify (5.4.12) by making the following changes in the parameters. First $i \to k$, then $(r, s) \to (i, j)$, and, finally, $k \to r$. The result is

$$\sum_{i,j}{}^{\dagger} E^{rj} E^{ir} = \sum_i E^{ri} E^{ir}. \tag{5.4.19}$$

Now sum (5.4.10) over i, j and refer to (5.4.19) and (5.4.6):

$$\begin{aligned}
\sum_{i,j} (c_i - c_j) E^{ij} &= \sum_{i,j,r,s} E^{is} E^{rj} - \sum_r \left[\sum_{i,j} E^{ir} E^{rj} \right] \\
&= \left[\sum_{i,s} E^{is} \right] \left[\sum_{r,j} E^{rj} \right] - \sum_r \sum_i E^{ri} E^{ir} \\
&= \sum_i E^{ii} \sum_r E^{rr} - \sum_{i,r} E^{ri} E^{ir} \\
&= \sum_{i,r} \begin{vmatrix} E^{ii} & E^{ir} \\ E^{ri} & E^{rr} \end{vmatrix} \\
&= \sum_{i,r} E^{ir,ir},
\end{aligned}$$

which is equivalent to (5.4.16). The symbol † can be attached to the sum on the left without affecting its value. Hence, this identity together with (5.4.7) yields (5.4.17), which can then be expressed in the symmetric form (5.4.18) in which $r < s$.

5.4.4 Third and Fourth Cofactors

The following identities contain third and fourth cofactors of E:

$$\sum_{r,s} (c_r - c_s) E^{rt,st} = \sum_{r,s} E^{rst,rst}, \tag{5.4.20}$$

$$\sum_{r,s} (c_r - c_s) E^{rtu,stu} = \sum_{r,s} E^{rstu,rstu}, \tag{5.4.21}$$

$$\sum_{r,s} (c_r^2 - c_s^2) E^{rs} = 2 \sum_{r,s} c_r E^{rs,rs}, \tag{5.4.22}$$

$$\sum_{r,s} (c_r - c_s)^2 E^{rs} = \sum_{r,s,t} E^{rst,rst}, \tag{5.4.23}$$

$$\sum_{r,s} (c_r - c_s)^2 E^{ru,su} = \sum_{r,s,t} E^{rstu,rstu}, \tag{5.4.24}$$

$$\sum_{r,s}{}^{\dagger} c_r c_s E^{rs} = -\sum_{r,s}{}^{\dagger}(c_r^2 + c_s^2)E^{rs}$$

$$= -\tfrac{1}{3}\sum_{r,s,t} E^{rst,rst}, \tag{5.4.25}$$

$$\sum_{r,s} c_r c_s E^{rs} = \sum_r c_r^2 E^{rr} - \frac{1}{3}\sum_{r,s,t} E^{rst,rst}, \tag{5.4.26}$$

$$\sum_{r,s}(c_r^2 + c_s^2)E^{rs} = 2\sum_r c_r^2 E^{rr} + \frac{1}{3}\sum_{r,s,t} E^{rst,rst}, \tag{5.4.27}$$

$$\sum_{r,s}{}^{\dagger} c_r^2 E^{rs} = \frac{1}{6}\sum_{r,s,t} E^{rst,rst} + \sum_{r,s} c_r E^{rs,rs}, \tag{5.4.28}$$

$$\sum_{r,s}{}^{\dagger} c_s^2 E^{rs} = \frac{1}{6}\sum_{r,s,t} E^{rst,rst} - \sum_{r,s} c_r E^{rs,rs}. \tag{5.4.29}$$

To prove (5.4.20), apply the second equation of (5.4.4) and (5.4.16).

$$E_{pr,ps} = \frac{\partial E_{rs}}{\partial x_p}.$$

Multiply by $(c_r - c_s)$ and sum over r and s:

$$\sum_{r,s}(c_r - c_s)E_{pr,ps} = \frac{\partial}{\partial x_p}\sum_{r,s}(c_r - c_s)E_{rs}$$

$$= \frac{\partial}{\partial x_p}\sum_{r,s} E_{rs,rs}$$

$$= \sum_{r,s} E_{prs,prs},$$

which is equivalent to (5.4.20). The application of the fifth equation in (5.4.4) with the modification $(i, p, r, q, s) \to (u, r, t, s, t)$ to (5.4.20) yields (5.4.21).

To prove (5.4.22), sum (5.4.11) over i and j, change the dummy variables as indicated

$$\sum_{i,j}(c_i^2 - c_j^2)E^{ij} = F - G$$

where, referring to (5.4.6) and (5.4.7),

$$F = \left[\sum_{i,s} E^{is}\right]\left[\sum_{r,j} c_r E^{rj}\right] + \left[\sum_{r,j} E^{rj}\right]\left[\sum_{i,s} c_s E^{is}\right]$$

$$= \sum_i E^{ii}\left[\sum_{\substack{r,j \\ (j\to s)}} c_r E^{rj} + \sum_{\substack{i,s \\ (i\to r)}} c_s E^{is}\right]$$

$$= \sum_i E^{ii} \sum_{r,s} (c_r + c_s) E^{rs}$$

$$= 2 \sum_i E^{ii} \sum_r c_r E^{rr}, \tag{5.4.30}$$

$$G = 2 \sum_{i,j,r} c_r E^{ir} E^{rj}. \tag{5.4.31}$$

Modify (5.4.10) with $j = i$ by making the changes $i \leftrightarrow r$ and $s \to j$. This gives

$$G = 2 \sum_r c_r \sum_i E^{ir} E^{ri}. \tag{5.4.32}$$

Hence,

$$\sum_{i,j} (c_i^2 - c_j^2) E^{ij} = 2 \sum_{i,r} c_r \begin{vmatrix} E^{ii} & E^{ir} \\ E^{ri} & E^{rr} \end{vmatrix}$$

$$= 2 \sum_{i,r} E^{ir,ir},$$

which is equivalent to (5.4.22).

To prove (5.4.23) multiply (5.4.10) by $(c_i - c_j)$, sum over i and j, change the dummy variables as indicated, and refer to (5.4.6):

$$\sum_{i,j} (c_i - c_j)^2 E^{ij} = H - J, \tag{5.4.33}$$

where

$$H = \sum_{i,j} (c_i - c_j) \sum_{r,s} E^{is} E^{rj}$$

$$= \left[\sum_{r,j} E^{rj} \right] \left[\sum_{\substack{i,s \\ (s \to j)}} c_i E^{is} \right] - \left[\sum_{i,s} E^{is} \right] \left[\sum_{\substack{r,j \\ (r \to i)}} c_j E^{rj} \right]$$

$$= \sum_r E^{rr} \sum_{i,j} (c_i - c_j) E^{ij}, \tag{5.4.34}$$

$$J = \sum_{i,j} (c_i - c_j) \sum_r E^{ir} E^{rj}. \tag{5.4.35}$$

Hence, referring to (5.4.20) with suitable changes in the dummy variables,

$$\sum_{i,j} (c_i - c_j)^2 E^{ij} = \sum_{i,j,r} (c_i - c_j) \begin{vmatrix} E^{ij} & E^{ir} \\ E^{rj} & E^{rr} \end{vmatrix}$$

$$= \sum_{i,j,r} (c_i - c_j) E^{ir,jr}$$

$$= \sum_{i,j,r} E^{ijr,ijr},$$

which is equivalent to (5.4.23). The application of a suitably modified the fourth line of (5.4.4) to (5.4.23) yields (5.4.24). Identities (5.4.27)–(5.4.29) follow from (5.4.8), (5.4.22), (5.4.24), and the identities

$$\begin{aligned}
3c_r c_s &= (c_r^2 + c_r c_s + c_s^2) - (c_r - c_s)^2, \\
6c_r^2 &= 2(c_r^2 + c_r c_s + c_s^2) + (c_r - c_s)^2 + 3(c_r^2 - c_s^2), \\
6c_s^2 &= 2(c_r^2 + c_r c_s + c_s^2) + (c_r - c_s)^2 - 3(c_r^2 - c_s^2).
\end{aligned}$$

5.4.5 Three Further Identities

The identities

$$\sum_{r,s}(c_r^2 + c_s^2)(c_r - c_s)E^{rs} = 2\sum_{r,s} c_r^2 E^{rs,rs} + \frac{1}{3}\sum_{r,s,u,v} E^{rsuv,rsuv}, \tag{5.4.36}$$

$$\sum_{r,s}(c_r^2 - c_s^2)(c_r + c_s)E^{rs} = 2\sum_{r,s} c_r(c_r + c_s)E^{rs,rs} - \frac{1}{6}\sum_{r,s,u,v} E^{rsuv,rsuv}, \tag{5.4.37}$$

$$\sum_{r,s} c_r c_s(c_r - c_s)E^{rs} = \sum_{r,s} c_r c_s E^{rs,rs} - \frac{1}{4}\sum_{r,s,u,v} E^{rsuv,rsuv} \tag{5.4.38}$$

are more difficult to prove than those in earlier sections. The last one has an application in Section 6.8 on the KP equation, but its proof is linked to those of the other two.

Denote the left sides of the three identities by P, Q, and R, respectively. To prove (5.4.36), multiply the second equation in (5.4.10) by $(c_i^2 + c_j^2)$, sum over i and j and refer to (5.4.4), (5.4.6), and (5.4.27):

$$\begin{aligned}
P &= \sum_{i,j,r,s} (c_i^2 + c_j^2)E^{is}E^{rj} + \sum_{i,j,r}(c_i^2 + c_j^2)E^{ir}E^{rj} \\
&= \left[\sum_{j,r} E^{rj}\right]\left[\sum_{\substack{i,s \\ (s\to j)}} c_i^2 E^{is}\right] + \left[\sum_{i,s} E^{is}\right]\left[\sum_{\substack{j,r \\ (r\to i)}} c_j^2 E^{rj}\right] \\
&\quad + \sum_{i,j,r}(c_i^2 + c_j^2)\frac{\partial E^{ij}}{\partial x_r}
\end{aligned}$$

$$= \sum_r E^{rr} \sum_{i,j} (c_i^2 + c_j^2) E^{ij} + \sum_r \frac{\partial}{\partial x_r} \sum_{i,j} (c_i^2 + c_j^2) E^{ij}$$
$$= \sum_r \left(E^{rr} + \frac{\partial}{\partial x_r} \right) \sum_{i,j} (c_i^2 + c_j^2) E^{ij}$$
$$= \sum_v \left(E^{vv} + \frac{\partial}{\partial x_v} \right) \left[2 \sum_r c_r^2 E^{rr} + \frac{1}{3} \sum_{r,s,t} E^{rst,rst} \right]$$
$$= 2 \sum_{r,v} c_r^2 E^{rv,rv} + \frac{1}{3} \sum_{r,s,t,v} E^{rstv,rstv},$$

which is equivalent to (5.4.36).

Since

$$(c_r^2 - c_s^2)(c_r + c_s) - 2c_r c_s (c_r - c_s) = (c_r^2 + c_s^2)(c_r - c_s),$$

it follows immedately that

$$Q - 2R = P. \tag{5.4.39}$$

A second relation between Q and R is found as follows. Let

$$U = \sum_r c_r E^{rr},$$
$$V = \frac{1}{2} \sum_{r,s} E^{rs,rs}. \tag{5.4.40}$$

It follows from (5.4.17) that

$$V = \sum_{r,s} c_r E^{rs} - \sum_r c_r E^{rr}$$
$$= \sum_r c_r E^{rr} - \sum_{r,s} c_s E^{rs}.$$

Hence

$$\sum_{r,s} c_r E^{rs} = U + V,$$
$$\sum_{r,s} c_s E^{rs} = U - V. \tag{5.4.41}$$

To obtain a formula for R, multiply (5.4.10) by $c_i c_j$, sum over i and j, and apply the third equation of (5.4.4):

$$R = \sum_{i,j,r,s} c_i c_j E^{is} E^{rj} - \sum_{i,j,r} c_i c_j E^{ir} E^{rj}$$
$$= \left[\sum_{i,s} c_i E^{is} \right] \left[\sum_{j,r} c_j E^{rj} \right] + \sum_r \frac{\partial}{\partial x_r} \sum_{i,j} c_i c_j E^{ij}$$

$$= U^2 - V^2 + \sum_r \frac{\partial S}{\partial x_r}, \tag{5.4.42}$$

where

$$S = \sum_{i,j} c_i c_j E^{ij}. \tag{5.4.43}$$

This function is identical to the left-hand side of (5.4.26). Let

$$T = \sum_{i,j,r,s} (c_i + c_j)(c_r + c_s) E^{is} E^{rj}. \tag{5.4.44}$$

Then, applying (5.4.6),

$$\begin{aligned} T &= \sum_{i,s} c_i E^{is} \sum_{j,r} c_r E^{rj} + \sum_{j,r} E^{rj} \sum_{i,s} c_i c_s E^{is} \\ &\quad + \sum_{i,s} E^{is} \sum_{j,r} c_j c_r E^{rj} + \sum_{j,r} c_j E^{rj} \sum_{i,s} c_s E^{is} \\ &= (U+V)^2 + 2S \sum_{r,s} E^{rs} + (U-V)^2 \\ &= 2(U^2 + V^2) + 2S \sum_r E^{rr}. \end{aligned} \tag{5.4.45}$$

Eliminating V from (5.4.42),

$$T + 2R = 4U^2 + 2\sum_r \left(E^{rr} + \frac{\partial}{\partial x_r} \right) S. \tag{5.4.46}$$

To obtain a formula for Q, multiply (5.4.11) by $(c_i + c_j)$, sum over i and j, and apply (5.4.13) with the modifications $(i, j) \leftrightarrow (r, s)$ on the left and $(i, r) \to (r, s)$ on the right:

$$\begin{aligned} Q &= \sum_{i,j,r,s} (c_i + c_j)(c_r + c_s) E^{is} E^{rj} - 2 \sum_{i,j,r} c_r (c_i + c_j) E^{ir} E^{rj} \\ &= T - 2 \sum_r c_r \sum_{i,j} (c_i + c_j) E^{ir} E^{rj} \\ &= T - 4 \sum_{r,s} c_r c_s E^{rs} E^{sr}. \end{aligned} \tag{5.4.47}$$

Eliminating T from (5.4.46) and applying (5.4.26) and the fourth and sixth lines of (5.4.4),

$$\begin{aligned} Q + 2R &= 4 \sum_{r,s} c_r c_s (E^{rr} E^{ss} - E^{sr} E^{rs}) \\ &\quad + 2 \sum_r \left(E^{rr} + \frac{\partial}{\partial x_r} \right) \left[\sum_s c_s^2 E^{ss} - \frac{1}{3} \sum_{s,t,u} E^{stu,stu} \right] \end{aligned}$$

$$= 4\sum_{r,s} c_r c_s E^{rs,rs} + 2\sum_{r,s} c_s^2 E^{rs,rs}$$
$$-\frac{2}{3}\sum_{r,s,t,u} E^{rstu,rstu}. \tag{5.4.48}$$

This is the second relation between Q and R, the first being (5.4.39). Identities (5.4.37), (5.4.38), and (5.3) follow by solving these two equations for Q and R, where P is given by (5.1).

Exercise. Prove that

$$\sum_{r,s}(c_r - c_s)\phi_n(c_r, c_s)E^{rs} = \sum_{r,s}\phi_n(c_r, c_s)E^{rs,rs}, \quad n = 1, 2,$$

where

$$\phi_1(c_r, c_s) = c_r + c_s,$$
$$\phi_2(c_r, c_s) = 3c_r^2 + 4c_r c_s + 3c_s^2.$$

Can this result be generalized?

5.5 Determinants Associated with a Continued Fraction

5.5.1 *Continuants and the Recurrence Relation*

Define a continued fraction f_n as follows:

$$f_n = \frac{1}{1+}\,\frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_{n-1}}{a_{n-1}+}\,\frac{b_n}{a_n}, \quad n = 1, 2, 3, \ldots . \tag{5.5.1}$$

f_n is obtained from f_{n-1} by adding b_n/a_n to a_{n-1}.

Examples.

$$f_1 = \frac{1}{1 + \frac{b_1}{a_1}}$$
$$= \frac{a_1}{a_1 + b_1},$$
$$f_2 = \frac{1}{1 + \frac{b_1}{a_1 + \frac{b_2}{a_2}}}$$
$$= \frac{a_1 a_2 + b_2}{a_1 a_2 + b_2 + a_2 b_1},$$
$$f_3 = \frac{1}{1 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3}}}}$$

$$= \frac{a_1a_2a_3 + a_1b_3 + a_3b_2}{a_1a_2a_3 + a_1b_3 + a_3b_2 + a_2a_3b_1 + b_1b_3}.$$

Each of these fractions can be expressed in the form H_{11}/H, where H is a tridiagonal determinant:

$$f_1 = \frac{|a_1|}{\begin{vmatrix} 1 & b_1 \\ -1 & a_1 \end{vmatrix}},$$

$$f_2 = \frac{\begin{vmatrix} a_1 & b_2 \\ -1 & a_2 \end{vmatrix}}{\begin{vmatrix} 1 & b_1 & \\ -1 & a_1 & b_2 \\ & -1 & a_2 \end{vmatrix}},$$

$$f_3 = \frac{\begin{vmatrix} a_1 & b_2 & \\ -1 & a_2 & b_3 \\ & -1 & a_3 \end{vmatrix}}{\begin{vmatrix} 1 & b_1 & & \\ -1 & a_1 & b_2 & \\ & -1 & a_2 & b_3 \\ & & -1 & a_3 \end{vmatrix}}.$$

Theorem 5.9.

$$f_n = \frac{H_{11}^{(n+1)}}{H_{n+1}},$$

where

$$H_{n+1} = \begin{vmatrix} 1 & b_1 & & & & & \\ -1 & a_1 & b_2 & & & & \\ & -1 & a_2 & b_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & a_{n-2} & b_{n-1} & \\ & & & & -1 & a_{n-1} & b_n \\ & & & & & -1 & a_n \end{vmatrix}_{n+1}. \tag{5.5.2}$$

PROOF. Use the method of induction. Assume that

$$f_{n-1} = \frac{H_{11}^{(n)}}{H_n},$$

which is known to be true for small values of n. Hence, adding b_n/a_n to a_{n-1},

$$f_n = \frac{K_{11}^{(n)}}{K_n}, \tag{5.5.3}$$

where

$$K_n = \begin{vmatrix} 1 & b_1 & & & & & \\ -1 & a_1 & b_2 & & & & \\ & -1 & a_2 & b_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & a_{n-3} & b_{n-2} & \\ & & & & -1 & a_{n-2} & b_{n-1} \\ & & & & & -1 & a_{n-1} + (b_n/a_n) \end{vmatrix}_n . \tag{5.5.4}$$

Return to H_{n+1}, remove the factor a_n from the last column, and then perform the column operation

$$\mathbf{C}'_n = \mathbf{C}_n + \mathbf{C}_{n+1}.$$

The result is a determinant of order $(n+1)$ in which the only element in the last row is 1 in the right-hand corner.

It then follows that

$$H_{n+1} = a_n K_n.$$

Similarly,

$$H_{11}^{(n+1)} = a_n K_{11}^{(n-1)}.$$

The theorem follows from (5.5.3). □

Tridiagonal determinants of the form H_n are called continuants. They are also simple Hessenbergians which satisfy the three-term recurrence relation. Expanding H_{n+1} by the two elements in the last row, it is found that

$$H_{n+1} = a_n H_n + b_n H_{n-1}.$$

Similarly,

$$H_{11}^{(n+1)} = a_n H_{11}^{(n)} + b_n H_{11}^{(n)}. \tag{5.5.5}$$

The theorem can therefore be reformulated as follows:

$$f_n = \frac{Q_n}{P_n}, \tag{5.5.6}$$

where P_n and Q_n each satisfy the recurrence relation

$$R_n = a_n R_{n-1} + b_n R_{n-2} \tag{5.5.7}$$

with the initial values $P_0 = 1$, $P_1 = a_1 + b_1$, $Q_0 = 1$, and $Q_1 = a_1$.

5.5.2 Polynomials and Power Series

In the continued fraction f_n defined in (5.5.1) in the previous section, replace a_r by 1 and replace b_r by $a_r x$. Then,

$$f_n = \frac{1}{1+}\,\frac{a_1x}{1+}\,\frac{a_2x}{1+}\cdots\frac{a_{n-1}x}{1+}\,\frac{a_nx}{1}$$

$$= \frac{Q_n}{P_n}, \tag{5.5.8}$$

where P_n and Q_n each satisfy the recurrence relation

$$R_n = R_{n-1} + a_n x R_{n-2} \tag{5.5.9}$$

with $P_0 = 1$, $P_1 = 1 + a_1 x$, $Q_0 = 1$, and $Q_1 = 1$. It follows that

$$\begin{aligned}
P_2 &= 1 + (a_1 + a_2)x, \\
Q_2 &= 1 + a_2 x, \\
P_3 &= 1 + (a_1 + a_2 + a_3)x + a_1 a_3 x^2, \\
Q_3 &= 1 + (a_2 + a_3)x, \\
P_4 &= 1 + (a_1 + a_2 + a_3 + a_4)x + (a_1 a_3 + a_1 a_4 + a_2 a_4)x^2, \\
Q_4 &= 1 + (a_2 + a_3 + a_4)x + a_2 a_4 x^2.
\end{aligned} \tag{5.5.10}$$

It also follows from the previous section that $P_n = H_{n+1}$, etc., where

$$H_{n+1} = \begin{vmatrix} 1 & a_1 x & & & & & \\ -1 & 1 & a_2 x & & & & \\ & -1 & 1 & a_3 x & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 1 & a_{n-2}x & \\ & & & & -1 & 1 & a_{n-1}x \\ & & & & & -1 & 1 \end{vmatrix}_{n+1}. \tag{5.5.11}$$

The alternative formula

$$H_{n+1} = \begin{vmatrix} 1 & x & & & & & \\ -a_1 & 1 & x & & & & \\ & -a_2 & 1 & x & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -a_{n-3} & 1 & x & \\ & & & & -a_{n-2} & 1 & x \\ & & & & & -a_{n-1} & 1 \end{vmatrix}_{n+1} \tag{5.5.12}$$

can be proved by showing that the second determinant satisfies the same recurrence relation as the first determinant and has the same initial values. Also,

$$Q_n = H_{11}^{(n+1)}. \tag{5.5.13}$$

Using elementary methods, it is found that

$$\begin{aligned}
f_1 &= 1 - a_1 x + a_1^2 x^2 + \cdots, \\
f_2 &= 1 - a_1 x + a_1(a_1 + a_2)x^2 - a_1(a_1^2 + 2a_1 a_2 + a_2^2)x^3 + \cdots, \\
f_3 &= 1 - a_1 x + a_1(a_1 + a_2)x^2 - a_1(a_1^2 + 2a_1 a_2 + a_2^2 + a_2 a_3)x^3 + \cdots \\
&\quad + a_1(a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + 2a_2^3 + 2a_2^2 a_3 \\
&\qquad + a_2 a_3^2 + 2a_1 a_2 a_3)x^4 + \cdots,
\end{aligned} \tag{5.5.14}$$

etc. These formulas lead to the following theorem.

Theorem 5.10.

$$f_n - f_{n-1} = (-1)^n (a_1 a_2 a_3 \cdots a_n) x^n + O(x^{n+1}),$$

that is, the coefficients of x^r, $1 \le r \le n-1$, *in the series expansion of* f_n *are identical to those in the expansion of* f_{n-1}.

PROOF. Applying the recurrence relation (5.5.9),

$$\begin{aligned}
P_{n-1}Q_n - P_nQ_{n-1} &= P_{n-1}(Q_{n-1} + a_n x Q_{n-2}) - (P_{n-1} + a_n x P_{n-2})Q_{n-1} \\
&= -a_n x (P_{n-2}Q_{n-1} - P_{n-1}Q_{n-2}) \\
&= a_{n-1}a_n x^2 (P_{n-3}Q_{n-2} - P_{n-2}Q_{n-3}) \\
&\;\;\vdots \\
&= (-1)^n (a_3 a_4 \cdots a_n) x^{n-2} (P_1Q_2 - P_2Q_1) \\
&= (-1)^n (a_1 a_2 \cdots a_n) x^n \qquad (5.5.15)\\
f_n - f_{n-1} &= \frac{Q_n}{P_n} - \frac{Q_{n-1}}{P_{n-1}} \\
&= \frac{P_{n-1}Q_n - P_nQ_{n-1}}{P_nP_{n-1}} \\
&= \frac{(-1)^n (a_1 a_2 \cdots a_n) x^n}{P_nP_{n-1}}\,. \qquad (5.5.16)
\end{aligned}$$

The theorem follows since $P_n(x)$ is a polynomial with $P_n(0) = 1$. □

Let

$$f_n(x) = \sum_{r=0}^{\infty} c_r x^r. \qquad (5.5.17)$$

From the third equation in (5.5.14),

$$\begin{aligned}
c_0 &= 1, \\
c_1 &= -a_1, \\
c_2 &= a_1(a_1 + a_2), \\
c_3 &= -a_1(a_1^2 + 2a_1a_2 + a_2^2 + a_2a_3), \\
c_4 &= a_1(a_1^2a_2 + 2a_1a_2^2 + a_2^3 + 2a_2^2a_3 + a_1^2a_3 + 2a_1a_2a_3 + a_2a_3^2 \\
&\qquad + a_1^2a_4 + a_1a_2a_4 + a_2a_3a_4), \qquad (5.5.18)
\end{aligned}$$

etc. Solving these equations for the a_r,

$$\begin{aligned}
a_1 &= -|c_1|, \\
a_2 &= \frac{\begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix}}{|c_1|},
\end{aligned}$$

$$a_3 = \frac{|c_0| \begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix}}{|c_1| \begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix}}, \tag{5.5.19}$$

etc. Determinantal formulas for a_{2n-1}, a_{2n}, and two other functions will be given shortly.

Let

$$\begin{aligned} A_n &= |c_{i+j-2}|_n, \\ B_n &= |c_{i+j-1}|_n, \end{aligned} \tag{5.5.20}$$

with $A_0 = B_0 = 1$. Identities among these determinants and their cofactors appear in Hankelians 1.

It follows from the recurrence relation (5.5.9) and the initial values of P_n and Q_n that P_{2n-1}, P_{2n}, Q_{2n+1}, and Q_{2n} are polynomials of degree n. In all four polynomials, the constant term is 1. Hence, we may write

$$\begin{aligned} P_{2n-1} &= \sum_{r=0}^{n} p_{2n-1,r} x^r, \\ Q_{2n+1} &= \sum_{r=0}^{n} q_{2n+1,r} x^r, \\ P_{2n} &= \sum_{r=0}^{n} p_{2n,r} x^r, \\ Q_{2n} &= \sum_{r=0}^{n} q_{2n,r} x^r, \end{aligned} \tag{5.5.21}$$

where both p_{mr} and q_{mr} satisfy the recurrence relation

$$u_{mr} = u_{m-1,r} + a_m u_{m-2,r-1}$$

and where

$$\begin{aligned} p_{m0} = q_{m0} &= 1, \qquad \text{all } m, \\ p_{2n-1,r} = p_{2n,r} &= 0, \qquad r < 0 \text{ or } r > n. \end{aligned} \tag{5.5.22}$$

Theorem 5.11.

a. $p_{2n-1,r} = \dfrac{A^{(n+1)}_{n+1,n+1-r}}{A_n}, \quad 0 \le r \le n,$

b. $p_{2n,r} = \dfrac{B^{(n+1)}_{n+1,n+1-r}}{B_n}, \quad 0 \le r \le n,$

c. $a_{2n+1} = -\dfrac{A_n B_{n+1}}{A_{n+1} B_n},$

d. $a_{2n} = -\dfrac{A_{n+1} B_{n-1}}{A_n B_n}.$

PROOF. Let

$$f_{2n-1}P_{2n-1} - Q_{2n-1} = \sum_{r=0}^{\infty} h_{nr}x^r, \tag{5.5.23}$$

where f_n is defined by the infinite series (5.5.17). Then, from (5.5.8),

$$h_{nr} = 0, \quad \text{all } n \text{ and } r,$$

where

$$h_{nr} = \begin{cases} \sum_{t=0}^{r} c_{r-t}p_{2n-1,t} - q_{2n-1,r}, & 0 \le r \le n-1 \\ \sum_{t=0}^{r} c_{r-t}p_{2n-1,t}, & r \ge n. \end{cases} \tag{5.5.24}$$

The upper limit n in the second sum arises from (5.5.22).

The n equations

$$h_{nr} = 0, \qquad n \le r \le 2n-1,$$

yield

$$\sum_{t=1}^{n} c_{r-t}p_{2n-1,t} + c_r = 0. \tag{5.5.25}$$

Solving these equations by Cramer's formula yields part (a) of the theorem.

Part (b) is proved in a similar manner. Let

$$f_{2n}P_{2n} - Q_{2n} = \sum_{r=0}^{\infty} k_{nr}x^r. \tag{5.5.26}$$

Then,

$$k_{nr} = 0, \qquad \text{all } n \text{ and } r,$$

where

$$k_{rn} = \begin{cases} \sum_{t=0}^{r} c_{r-t}p_{2n,t} - q_{2n,r}, & 0 \le r \le n \\ \sum_{t=0}^{n} c_{r-t}p_{2n,t}, & r \ge n+1. \end{cases} \tag{5.5.27}$$

The n equations

$$k_{nr} = 0, \quad n+1 \le r \le 2n,$$

yield

$$\sum_{t=1}^{n} c_{r-t}p_{2n,t} + c_r = 0. \tag{5.5.28}$$

Solving these equations by Cramer's formula yields part (b) of the theorem.

The equation

$$h_{n,2n+1} = 0$$

yields

$$\sum_{t=0}^{n+1} c_{2n+1-t} p_{2n+1,t} = 0. \tag{5.5.29}$$

Applying the recurrence relation (5.5.22) and then parts (a) and (b) of the theorem,

$$\sum_{t=0}^{n} c_{2n+1-t} p_{2n,t} + a_{2n+1} \sum_{t=1}^{n+1} c_{2n+1-t} p_{2n-1,t-1} = 0,$$

$$\frac{1}{B_n} \sum_{t=0}^{n} c_{2n+1-t} B_{n+1,n+1-t}^{(n+1)} + \frac{a_{2n+1}}{A_n} \sum_{t=1}^{n+1} c_{2n+1-t} A_{n+1,n+2-t}^{(n+1)} = 0,$$

$$\frac{B_{n+1}}{B_n} + a_{2n+1} \frac{A_{n+1}}{A_n} = 0,$$

which proves part (c).

Part (d) is proved in a similar manner. The equation

$$k_{n,2n} = 0$$

yields

$$\sum_{t=0}^{n} c_{2n-t} p_{2n,t} = 0. \tag{5.5.30}$$

Applying the recurrence relation (5.5.22) and then parts (a) and (b) of the theorem,

$$\sum_{t=0}^{n} c_{2n-t} p_{2n-1,t} + a_{2n} \sum_{t=1}^{n} c_{2n-t} p_{2n-2,t-1} = 0,$$

$$\frac{1}{A_n} \sum_{t=0}^{n} c_{2n-t} A_{n+1,n+1-t}^{(n+1)} + \frac{a_{2n}}{B_{n-1}} \sum_{t=1}^{n} c_{2n-t} B_{n,n+1-t}^{(n)} = 0,$$

$$\frac{A_{n+1}}{A_n} + a_{2n} \frac{B_n}{B_{n-1}} = 0,$$

which proves part (d). □

Exercise. Prove that

$$P_6 = 1 + x \sum_{r=1}^{6} a_r + x^2 \sum_{r=1}^{4} a_r \sum_{s=r+2}^{6} a_s + x^3 \sum_{r=1}^{2} a_r \sum_{s=r+2}^{4} a_s \sum_{t=s+2}^{6} a_t$$

and find the corresponding formula for Q_7.

5.5.3 *Further Determinantal Formulas*

Theorem 5.12.

a. $$P_{2n-1} = \frac{1}{A_n}\begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \\ x^n & x^{n-1} & x^{n-2} & \cdots & 1 \end{vmatrix}_{n+1},$$

b. $$P_{2n} = \frac{1}{B_n}\begin{vmatrix} c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n} \\ x^n & x^{n-1} & x^{n-2} & \cdots & 1 \end{vmatrix}_{n+1}.$$

PROOF. Referring to the first line of (5.5.21) and to Theorem 5.11a,

$$\begin{aligned} P_{2n-1} &= \frac{1}{A_n}\sum_{r=0}^{n} A^{(n+1)}_{n+1,n+1-r}x^r \\ &= \frac{1}{A_n}\sum_{j=1}^{n+1} A^{(n+1)}_{n+1,j}x^{n+1-j}. \end{aligned}$$

Part (a) follows and part (b) is proved in a similar manner with the aid of the third line in (5.5.21) and Theorem 5.11b. □

Lemmas.

a. $$\sum_{r=0}^{n-1} u_r \sum_{t=0}^{r} c_{r-t}v_{n+1-t} = \sum_{j=1}^{n} v_{j+1}\sum_{r=0}^{j-1} c_r u_{n+r-j},$$

b. $$\sum_{r=0}^{n} u_r \sum_{t=0}^{r} c_{r-t}v_{n+1-t} = \sum_{j=0}^{n} v_{j+1}\sum_{r=0}^{j} c_r u_{n+r-j}.$$

These two lemmas differ only in some of their limits and could be regarded as two particular cases of one lemma whose proof is elementary and consists of showing that both double sums represent the sum of the same triangular array of terms.

Let

$$\psi_m = \sum_{r=0}^{m} c_r x^r. \qquad (5.5.31)$$

Theorem 5.13.

a. $$Q_{2n-1} = \frac{1}{A_n}\begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \\ \psi_0 x^n & \psi_1 x^{n-1} & \psi_2 x^{n-2} & \cdots & \psi_n \end{vmatrix}_{n+1},$$

b. $Q_{2n} = \dfrac{1}{B_n}\begin{vmatrix} c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n} \\ \psi_0 x^n & \psi_1 x^{n-1} & \psi_2 x^{n-2} & \cdots & \psi_n \end{vmatrix}_{n+1}.$

PROOF. From the second equation in (5.5.24) in the previous section and referring to Theorem 5.11a,

$$\begin{aligned} q_{2n-1,r} &= \sum_{t=0}^{r} c_{r-t} p_{2n-1,t}, \quad 0 \le r \le n-1 \\ &= \frac{1}{A_n} \sum_{t=0}^{r} c_{r-t} A_{n+1,n+1-t}^{(n+1)}. \end{aligned}$$

Hence, from the second equation in (5.5.21) with $n \to n-1$ and applying Lemma (a) with $u_r \to x^r$ and $v_s \to A_{n+1,s}^{(n+1)}$,

$$\begin{aligned} A_n Q_{2n-1} &= \sum_{r=0}^{n-1} x^r \sum_{t=0}^{r} c_{r-t} A_{n+1,n+1-t}^{(n+1)} \\ &= \sum_{j=1}^{n} A_{n+1,j+1}^{(n+1)} \sum_{r=0}^{j-1} c_r x^{n+r-j} \\ &= \sum_{j=1}^{n} x^{n-j} A_{n+1,j+1}^{(n+1)} \sum_{r=0}^{j-1} c_r x^r \\ &= \sum_{j=1}^{n} \psi_{j-1} x^{n-j} A_{n+1,j+1}^{(n+1)}. \end{aligned}$$

This sum represents a determinant of order $(n+1)$ whose first n rows are identical with the first n rows of the determinant in part (a) of the theorem and whose last row is

$$\left[0 \;\; \psi_0 x^{n-1} \;\; \psi_1 x^{n-2} \;\; \psi_2 x^{n-3} \cdots \psi_{n-1}\right]_{n+1}.$$

The proof of part (a) is completed by performing the row operation

$$\mathbf{R}'_{n+1} = \mathbf{R}_{n+1} + x^n \mathbf{R}_1.$$

The proof of part (b) of the theorem applies Lemma (b) and gives the required result directly, that is, without the necessity of performing a row operation. From (5.5.27) in the previous section and referring to Theorem 5.11b,

$$q_{2n,r} = \sum_{t=0}^{r} c_{r-t} p_{2n,t}, \quad 0 \le r \le n$$

$$= \frac{1}{B_n} \sum_{t=0}^{r} c_{r-t} B_{n+1,n+1-t}^{(n+1)}.$$

Hence, from the fourth equation in (5.5.11) and applying Lemma (b) and (5.5.31),

$$\begin{aligned} B_n Q_{2n} &= \sum_{r=0}^{n} x^r \sum_{t=0}^{r} c_{r-t} B_{n+1,n+1-t}^{(n+1)} \\ &= \sum_{j=0}^{n} B_{n+1,j+1}^{(n+1)} \sum_{r=0}^{j} c_r x^{n+r-j} \\ &= \sum_{j=0}^{n} \psi_j x^{n-j} B_{n+1,j+1}^{(n+1)}. \end{aligned}$$

This sum is an expansion of the determinant in part (b) of the theorem. This completes the proofs of both parts of the theorem. □

Exercise. Show that the equations

$$\begin{aligned} h_{n,2n+j} &= 0, \quad j \geq 2, \\ k_{n,2n+j} &= 0, \quad j \geq 1, \end{aligned}$$

lead respectively to

$$S_{n+2} = 0, \quad \text{all } n, \tag{X}$$

$$T_{n+1} = 0, \quad \text{all } n, \tag{Y}$$

where S_{n+2} denotes the determinant obtained from A_{n+2} by replacing its last row by the row

$$\left[c_{n+j-1}\ c_{n+j}\ c_{n+j+1} \cdots c_{2n+j}\right]_{n+2}$$

and T_{n+1} denotes the determinant obtained from B_{n+1} by replacing its last row by the row

$$\left[c_{n+j}\ c_{n+j+1}\ c_{n+j+2} \cdots c_{2n+j}\right]_{n+1}.$$

Regarding (X) and (Y) as conditions, what is their significance?

5.6 Distinct Matrices with Nondistinct Determinants

5.6.1 Introduction

Two matrices $[a_{ij}]_m$ and $[b_{ij}]_n$ are equal if and only if $m = n$ and $a_{ij} = b_{ij}$, $1 \leq i,j \leq n$. No such restriction applies to determinants. Consider determinants with constant elements. It is a trivial exercise to find two determinants $A = |a_{ij}|_n$ and $B = |b_{ij}|_n$ such that $a_{ij} \neq b_{ij}$ for any pair (i, j) and the elements a_{ij} are not merely a rearrangement of the elements b_{ij}, but $A = B$. It is an equally trivial exercise to find two determinants of different orders which have the same value. If the elements are polynomials, then the determinants are also polynomials and the exercises are more difficult.

It is the purpose of this section to show that there exist families of distinct matrices whose determinants are not distinct for the reason that they represent identical polynomials, apart from a possible change in sign. Such determinants may be described as equivalent.

5.6.2 Determinants with Binomial Elements

Let $\phi_m(x)$ denote an Appell polynomial (Appendix A.4):

$$\phi_m(x) = \sum_{r=0}^{m} \binom{m}{r} \alpha_{m-r} x^r. \tag{5.6.1}$$

The inverse relation is

$$\alpha_m = \sum_{r=0}^{m} \binom{m}{r} \phi_{m-r}(x)(-x)^r. \tag{5.6.2}$$

Define infinite matrices $\mathbf{P}(x)$, $\mathbf{P}^T(x)$, $\mathbf{A}$, and $\Phi(x)$ as follows:

$$\mathbf{P}(x) = \left[\overleftrightarrow{\binom{i-1}{j-1}} x^{i-j} \right], \qquad i, j \geq 1, \tag{5.6.3}$$

where the symbol $\longleftrightarrow$ denotes that the order of the columns is to be reversed. $\mathbf{P}^T$ denotes the transpose of $\mathbf{P}$. Both $\mathbf{A}$ and Φ are defined in Hankelian notation (Section 4.8):

$$\begin{aligned} \mathbf{A} &= [\alpha_m], \qquad m \geq 0, \\ \Phi(x) &= [\phi_m(x)], \quad m \geq 0. \end{aligned} \tag{5.6.4}$$

Now define block matrices $\mathbf{M}$ and $\mathbf{M}^*$ as follows:

$$\mathbf{M} = \begin{bmatrix} \mathbf{O} & \mathbf{P}^T(x) \\ \mathbf{P}(x) & \Phi(x) \end{bmatrix}, \tag{5.6.5}$$

$$\mathbf{M}^* = \begin{bmatrix} \mathbf{O} & \mathbf{P}^T(-x) \\ \mathbf{P}(-x) & \mathbf{A} \end{bmatrix}. \tag{5.6.6}$$

These matrices are shown in some detail below. They are triangular, symmetric, and infinite in all four directions. Denote the diagonals containing the unit elements in both matrices by diag(1).

It is now required to define a number of determinants of submatrices of either $\mathbf{M}$ or $\mathbf{M}^*$. Many statements are abbreviated by omitting references to submatrices and referring directly to subdeterminants.

Define a Turanian T_{nr} (Section 4.9.2) as follows:

$$T_{nr} = \begin{vmatrix} \phi_{r-2n+2} & \cdots & \phi_{r-n+2} \\ \vdots & & \vdots \\ \phi_{r-n+1} & \cdots & \phi_r \end{vmatrix}_n, \quad r \geq 2n-2, \tag{5.6.7}$$

which is a subdeterminant of $\mathbf{M}$.

$$\begin{bmatrix}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & & & & & & & & & & 1 & \cdots \\
\cdots & & & & & & & & & 1 & 4x & \cdots \\
\cdots & & & & & & & & 1 & 3x & 6x^2 & \cdots \\
\cdots & & & & & & & 1 & 2x & 3x^2 & 4x^3 & \cdots \\
\cdots & & & & & & 1 & x & x^2 & x^3 & x^4 & \cdots \\
\cdots & & & & & 1 & \phi_0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 & \cdots \\
\cdots & & & & 1 & x & \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \cdots \\
\cdots & & & 1 & 2x & x^2 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \cdots \\
\cdots & & 1 & 3x & 3x^2 & x^3 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_7 & \cdots \\
\cdots & 1 & 4x & 6x^2 & 4x^3 & x^4 & \phi_4 & \phi_5 & \phi_6 & \phi_7 & \phi_8 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}$$

The infinite matrix $\mathbf{M}$

$$\begin{bmatrix}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & & & & & & & & & & 1 & \cdots \\
\cdots & & & & & & & & & 1 & -4x & \cdots \\
\cdots & & & & & & & & 1 & -3x & 6x^2 & \cdots \\
\cdots & & & & & & & 1 & -2x & 3x^2 & -4x^3 & \cdots \\
\cdots & & & & & & 1 & -x & x^2 & -x^3 & x^4 & \cdots \\
\cdots & & & & & 1 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots \\
\cdots & & & & 1 & -x & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \cdots \\
\cdots & & & 1 & -2x & x^2 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \cdots \\
\cdots & & 1 & -3x & 3x^2 & -x^3 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \cdots \\
\cdots & 1 & -4x & 6x^2 & -4x^3 & x^4 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}$$

The infinite matrix $\mathbf{M}^*$

The element α_r occurs $(r+1)$ times in $\mathbf{M}^*$. Consider all the subdeterminants of $\mathbf{M}^*$ which contain the element α_r in the bottom right-hand corner and whose order n is sufficiently large for them to contain the element α_0 but sufficiently small for them not to have either unit or zero elements along their secondary diagonals. Denote these determinants by B_s^{nr}, $s = 1, 2, 3, \ldots$. Some of them are symmetric and unique whereas others occur in pairs, one of which is the transpose of the other. They are coaxial in the sense that all their secondary diagonals lie along the same diagonal parallel to diag(1) in $\mathbf{M}^*$.

Theorem 5.14. *The determinants B_s^{nr}, where n and r are fixed, $s = 1, 2, 3, \ldots$, represent identical polynomials of degree $(r+2-n)(2n-2-r)$.*

Denote their common polynomial by B_{nr}.

Theorem 5.15.

$$T_{r+2-n,r} = (-1)^k B_{nr}, \quad r \geq 2n-2, \quad n = 1, 2, 3, \ldots$$

where

$$k = n + r + \left[\tfrac{1}{2}(r+2)\right].$$

Both of these theorems have been proved by Fiedler using the theory of S-matrices but in order to relate the present notes to Fiedler's, it is necessary to change the sign of x.

When $r = 2n-2$, Theorem 5.15 becomes the symmetric identity

$$T_{n,2n-2} = B_{n,2n-2},$$

that is

$$\begin{vmatrix} \phi_0 & \cdots & \phi_{n-1} \\ \vdots & & \vdots \\ \phi_{n-1} & \cdots & \phi_{2n-2} \end{vmatrix}_n = \begin{vmatrix} \alpha_0 & \cdots & \alpha_{n-1} \\ \vdots & & \vdots \\ \alpha_{n-1} & \cdots & \alpha_{2n-2} \end{vmatrix}_n \quad \text{(degree 0)}$$

$$|\phi_m|_n = |\alpha_m|_n, \quad 0 \leq m \leq 2n-2,$$

which is proved by an independent method in Section 4.9 on Hankelians 2.

Theorem 5.16. *To each identity, except one, described in Theorems 5.14 and 5.15 there corresponds a dual identity obtained by reversing the role of* $\mathbf{M}$ *and* $\mathbf{M}^*$, *that is, by interchanging $\phi_m(x)$ and α_m and changing the sign of each x where it occurs explicitly. The exceptional identity is the symmetric one described above which is its own dual.*

The following particular identities illustrate all three theorems. Where $n = 1$, the determinants on the left are of unit order and contain a single element. Each identity is accompanied by its dual.

$(n, r) = (1, 1)$:

$$|\phi_1| = \begin{vmatrix} 1 & -x \\ \alpha_0 & \alpha_1 \end{vmatrix},$$

$$|\alpha_1| = \begin{vmatrix} 1 & x \\ \phi_0 & \phi_1 \end{vmatrix}; \tag{5.6.8}$$

$(n, r) = (3, 2)$:

$$|\phi_2| = -\begin{vmatrix} & 1 & -2x \\ 1 & -x & x^2 \\ \alpha_0 & \alpha_1 & \alpha_2 \end{vmatrix} = -\begin{vmatrix} & 1 & -x \\ 1 & \alpha_0 & \alpha_1 \\ -x & \alpha_1 & \alpha_2 \end{vmatrix} \quad \text{(symmetric)},$$

$$|\alpha_2| = -\begin{vmatrix} & 1 & 2x \\ 1 & x & x^2 \\ \phi_0 & \phi_1 & \phi_2 \end{vmatrix} = -\begin{vmatrix} & 1 & x \\ 1 & \phi_0 & \phi_1 \\ x & \phi_1 & \phi_2 \end{vmatrix} \text{ (symmetric);} \tag{5.6.9}$$

$(n, r) = (4, 3)$:

$$|\phi_3| = -\begin{vmatrix} & & 1 & -2x \\ & 1 & -x & x^2 \\ 1 & \alpha_0 & \alpha_1 & \alpha_2 \\ -x & \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix} = -\begin{vmatrix} & & 1 & -3x \\ & 1 & -2x & 3x^2 \\ 1 & -x & x^2 & -x^3 \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix},$$

$$|\alpha_3| = -\begin{vmatrix} & & 1 & 2x \\ & 1 & x & x^2 \\ 1 & \phi_0 & \phi_1 & \phi_2 \\ x & \phi_1 & \phi_2 & \phi_3 \end{vmatrix} = -\begin{vmatrix} & & 1 & 3x \\ & 1 & 2x & 3x^2 \\ 1 & x & x^2 & x^3 \\ \phi_0 & \phi_1 & \phi_2 & \phi_3 \end{vmatrix}; \tag{5.6.10}$$

$(n, r) = (3, 3)$:

$$\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_3 \end{vmatrix} = \begin{vmatrix} 1 & -x & x^2 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix},$$

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ \phi_0 & \phi_1 & \phi_2 \\ \phi_1 & \phi_2 & \phi_3 \end{vmatrix}; \tag{5.6.11}$$

$(n, r) = (4, 4)$:

$$\begin{vmatrix} \phi_2 & \phi_3 \\ \phi_3 & \phi_4 \end{vmatrix} = -\begin{vmatrix} & 1 & -2x & 3x^2 \\ 1 & -x & x^2 & -x^3 \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix} = \begin{vmatrix} & 1 & -x & x^2 \\ 1 & \alpha_0 & \alpha_1 & \alpha_2 \\ -x & \alpha_1 & \alpha_2 & \alpha_3 \\ x^2 & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix},$$

$$\begin{vmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{vmatrix} = -\begin{vmatrix} & 1 & 2x & 3x^2 \\ 1 & x & x^2 & x^3 \\ \phi_0 & \phi_1 & \phi_2 & \phi_3 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{vmatrix} = \begin{vmatrix} & 1 & x & x^2 \\ 1 & \phi_0 & \phi_1 & \phi_2 \\ x & \phi_1 & \phi_2 & \phi_3 \\ x^2 & \phi_2 & \phi_3 & \phi_4 \end{vmatrix} \tag{5.6.12}$$

The coaxial nature of the determinants B_s^{nr} is illustrated for the case $(n, r) = (6, 6)$ as follows:

$$\begin{vmatrix} \phi_4 & \phi_5 \\ \phi_5 & \phi_6 \end{vmatrix} = \left\{ \begin{array}{l} \text{each of the three determinants of order 6} \\ \text{enclosed within overlapping dotted frames} \\ \text{in the following display:} \end{array} \right.$$

$$\begin{array}{cccccccc}
 & & & & & 1 & -4x & 10x^2 \\
 & & & & 1 & -3x & 6x^2 & -10x^2 \\
 & & & 1 & -2x & 3x^2 & -4x^3 & 5x^4 \\
 & & 1 & -x & x^2 & -x^3 & x^4 & -x^5 \\
 & 1 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\
1 & -x & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
-2x & x^2 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \\
3x^2 & -x^3 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & &
\end{array} \tag{5.6.13}$$

These determinants are B_s^{66}, $s = 1, 2, 3$, as indicated at the corners of the frames. B_1^{66} is symmetric and is a bordered Hankelian. The dual identities are found in the manner described in Theorem 5.16.

All the determinants described above are extracted from consecutive rows and columns of $\mathbf{M}$ or $\mathbf{M}^*$. A few illustrations are sufficient to demonstrate the existence of identities of a similar nature in which the determinants are extracted from nonconsecutive rows and columns of $\mathbf{M}$ or $\mathbf{M}^*$.

In the first two examples, either the rows or the columns are nonconsecutive:

$$\begin{vmatrix} \phi_0 & \phi_2 \\ \phi_1 & \phi_3 \end{vmatrix} = -\begin{vmatrix} & 1 & -2x \\ \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}, \tag{5.6.14}$$

$$\begin{vmatrix} \phi_1 & \phi_3 \\ \phi_2 & \phi_4 \end{vmatrix} = \begin{vmatrix} & & 1 & -2x \\ 1 & \alpha_0 & \alpha_1 & \alpha_2 \\ -x & \alpha_1 & \alpha_2 & \alpha_3 \\ x^2 & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix} = \begin{vmatrix} & & 1 & -3x \\ 1 & -x & x^2 & -x^3 \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix}. \tag{5.6.15}$$

In the next example, both the rows and columns are nonconsecutive:

$$\begin{vmatrix} \phi_0 & \phi_2 \\ \phi_2 & \phi_4 \end{vmatrix} = -\begin{vmatrix} & & 1 & -2x \\ & \alpha_0 & \alpha_1 & \alpha_2 \\ 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ -2x & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix}. \tag{5.6.16}$$

The general form of these identities is not known and hence no theorem is known which includes them all.

In view of the wealth of interrelations between the matrices $\mathbf{M}$ and $\mathbf{M}^*$, each can be described as the dual of the other.

Exercise. Verify these identities and their duals by elementary methods. The above identities can be generalized by introducing a second variable y. A few examples are sufficient to demonstrate their form.

$$\phi_1(x+y) = \begin{vmatrix} 1 & -x \\ \phi_0(y) & \phi_1(y) \end{vmatrix} = \begin{vmatrix} 1 & -y \\ \phi_0(x) & \phi_1(x) \end{vmatrix}, \tag{5.6.17}$$

$$\phi_1(y) = \begin{vmatrix} 1 & x \\ \phi_0(x+y) & \phi_1(x+y) \end{vmatrix}, \tag{5.6.18}$$

$$\begin{vmatrix} \phi_1(x+y) & \phi_2(x+y) \\ \phi_2(x+y) & \phi_3(x+y) \end{vmatrix} = \begin{vmatrix} 1 & -x & x^2 \\ \phi_0(y) & \phi_1(y) & \phi_2(y) \\ \phi_1(y) & \phi_2(y) & \phi_3(y) \end{vmatrix},$$

$$= \begin{vmatrix} 1 & -y & y^2 \\ \phi_0(x) & \phi_1(x) & \phi_2(x) \\ \phi_1(x) & \phi_2(x) & \phi_3(x) \end{vmatrix} \qquad (5.6.19)$$

$$\begin{vmatrix} \phi_2(x+y) & \phi_3(x+y) \\ \phi_3(x+y) & \phi_4(x+y) \end{vmatrix} = \begin{vmatrix} & 1 & -x & x^2 \\ 1 & \phi_0(y) & \phi_1(y) & \phi_2(y) \\ -x & \phi_1(y) & \phi_2(y) & \phi_3(y) \\ x^2 & \phi_2(y) & \phi_3(y) & \phi_4(y) \end{vmatrix}$$

$$= \begin{vmatrix} & 1 & -2x & 3x^2 \\ 1 & -x & x^2 & -x^3 \\ \phi_0(y) & \phi_1(y) & \phi_2(y) & \phi_3(y) \\ \phi_1(y) & \phi_2(y) & \phi_3(y) & \phi_4(y) \end{vmatrix}. \qquad (5.6.20)$$

Do these identities possess duals?

5.6.3 Determinants with Stirling Elements

Matrices $\mathbf{s}_n(x)$ and $\mathbf{S}_n(x)$ whose elements contain Stirling numbers of the first and second kinds, s_{ij} and S_{ij}, respectively, are defined in Appendix A.1.

Let the matrix obtained by rotating $\mathbf{S}_n(x)$ through 90° in the anticlockwise direction be denotes by $\overset{\curvearrowleft}{\mathbf{S}_n}(x)$. For example,

$$\overset{\curvearrowleft}{\mathbf{S}_5}(x) = \begin{bmatrix} & & & & 1 \\ & & & 1 & 10x \\ & & 1 & 6x & 25x^2 \\ & 1 & 3x & 7x^2 & 15x^3 \\ 1 & x & x^2 & x^3 & x^4 \end{bmatrix}.$$

Define another nth-order triangular matrix $\mathbf{B}_n(x)$ as follows:

$$\mathbf{B}_n(x) = \overleftrightarrow{[b_{ij}x^{i-j}]}, \quad n \ge 2, \quad 1 \le i, j \le n,$$

where

$$b_{ij} = \frac{1}{(j-1)!} \sum_{r=0}^{j-1} (-1)^r \binom{j-1}{r} (n-r-1)^{i-1}, \quad i \ge j. \qquad (5.6.21)$$

These numbers are integers and satisfy the recurrence relation

$$b_{ij} = b_{i-1,j-1} + (n-j)b_{i-j,j},$$

where

$$b_{11} = 1. \qquad (5.6.22)$$

Once again the symbol $\longleftrightarrow$ denotes that the columns are arranged in reverse order.

Illustrations

$$\mathbf{B}_2(x) = \begin{bmatrix} & 1 \\ 1 & x \end{bmatrix},$$

$$\mathbf{B}_3(x) = \begin{bmatrix} & & 1 \\ & 1 & 2x \\ 1 & 3x & 4x^2 \end{bmatrix},$$

$$\mathbf{B}_4(x) = \begin{bmatrix} & & & 1 \\ & & 1 & 3x \\ & 1 & 5x & 9x^2 \\ 1 & 6x & 19x^2 & 27x^3 \end{bmatrix},$$

$$\mathbf{B}_5(x) = \begin{bmatrix} & & & & 1 \\ & & & 1 & 4x \\ & & 1 & 7x & 16x^2 \\ & 1 & 9x & 37x^2 & 64x^3 \\ 1 & 10x & 55x^2 & 175x^3 & 256x^4 \end{bmatrix}.$$

Since b_{ij} is a function of n, $\mathbf{B}_n$ is not a submatrix of $\mathbf{B}_{n+1}$. Finally, define a block matrix $\mathbf{N}_{2n}$ of order $2n$ as follows:

$$\mathbf{N}_{2n} = \begin{bmatrix} \mathbf{O} & \overset{\curvearrowleft}{\mathbf{S}_n(x)} \\ \mathbf{B}_n(x) & \mathbf{A}_n \end{bmatrix}, \tag{5.6.23}$$

where $\mathbf{A}_n = [\alpha_m]_n$, as before.

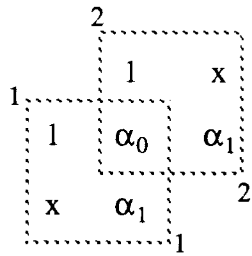
Illustrations

$$\mathbf{N}_4 = \begin{bmatrix} & & & 1 \\ & & 1 & x \\ & 1 & \alpha_0 & \alpha_1 \\ 1 & x & \alpha_1 & \alpha_2 \end{bmatrix},$$

$$\mathbf{N}_6 = \begin{bmatrix} & & & & & 1 \\ & & & & 1 & 3x \\ & & & 1 & x & x^2 \\ & & 1 & \alpha_0 & \alpha_1 & \alpha_2 \\ & 1 & 2x & \alpha_1 & \alpha_2 & \alpha_3 \\ 1 & 3x & 4x^2 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix}$$

$\mathbf{N}_{2n}$ is symmetric only when $n = 2$.

A subset of $\mathbf{N}_4$ is:

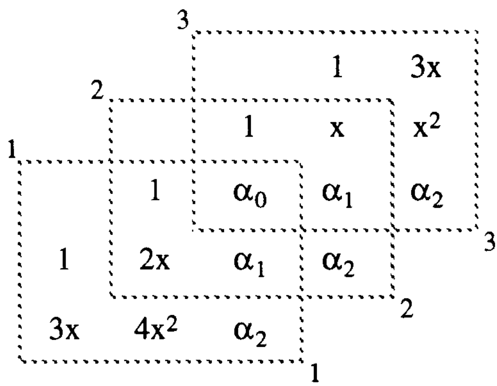


Each of the two overlapping coaxial second-order matrices indicated by frames has a determinant equal to

$$-(\alpha_0 x - \alpha_1) = \sum_{r=1}^{2} s_{2r}\alpha_{r-1}x^{2-r}. \tag{5.6.24}$$

In this case, the equality is a trivial one, as one matrix is merely the transpose of the other.

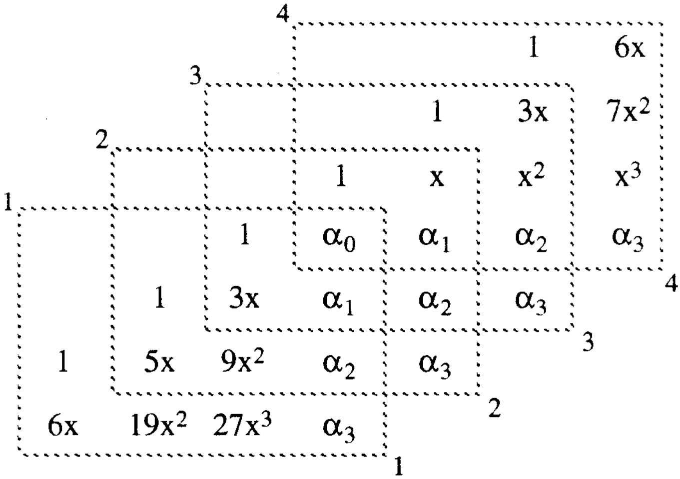
A subset of $\mathbf{N}_6$ is



Each of the three distinct overlapping coaxial third-order matrices indicated by frames has a determinant equal to

$$-(2\alpha_0 x^2 - 3\alpha_1 x + \alpha_2) = \sum_{r=1}^{3} s_{3r}\alpha_{r-1}x^{3-r}. \tag{5.6.25}$$

A subset of $\mathbf{N}_8$ is



Each of the four distinct overlapping coaxial fourth-order matrices indicated by frames has a determinant equal to

$$-(6\alpha_0 x^3 - 11\alpha_1 x^2 + 6\alpha_2 x - \alpha_3) = \sum_{r=1}^{4} s_{4r}\alpha_{r-1}x^{4-r}. \tag{5.6.26}$$

It does not appear to be possible to construct dual families of determinantal identities by interchanging the roles of s_{ij} and S_{ij}, but there exists the following simple identity in which the elements of the determinant contain Stirling numbers of the first kind and the sum contains Stirling numbers of the second kind:

$$\begin{vmatrix} \alpha_0 & 1 & & & \\ \alpha_1 & s_{21}x & 1 & & \\ \alpha_2 & s_{31}x^2 & s_{32}x & 1 & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-2} & s_{n-1,1}x^{n-2} & \cdots & \cdots & 1 \\ \alpha_{n-1} & s_{n1}x^{n-1} & \cdots & \cdots & s_{n,n-1}x \end{vmatrix}_n = (-1)^{n-1}\sum_{r=1}^{n} S_{nr}\alpha_{r-1}x^{n-r}. \tag{5.6.27}$$

The determinant is a Hessenbergian (Section 4.6) and is obtained from $s_n(x)$ by removing the last column, which contains a single nonzero element, and adding a column of α's on the left. The proof is left as an exercise for the reader.

5.7 The One-Variable Hirota Operator

5.7.1 Definition and Taylor Relations

Several nonlinear equations of mathematical physics, including the Korteweg–de Vries, Kadomtsev–Petviashvili, Boussinesq, and Toda equations, can be expressed neatly in terms of multivariable Hirota operators. The ability of an equation to be expressible in Hirota form is an important factor in the investigation of its integrability.

The one-variable Hirota operator, denoted here by H^n, is defined as follows: If $f = f(x)$ and $g = g(x)$, then

$$\begin{aligned} H^n(f,g) &= \left[\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x)g(x')\right]_{x'=x} \\ &= \sum_{r=0}^{n} (-1)^r \binom{n}{r} D^{n-r}(f)D^r(g), \quad D = \frac{d}{dx}. \end{aligned} \tag{5.7.1}$$

The factor $(-1)^r$ distinguishes this sum from the Leibnitz formula for $D^n(fg)$. The notation H_x, H_{xx}, etc., is convenient in some applications.

Examples.

$$\begin{aligned} H_x(f,g) = H^1(f,g) &= f_x g - f g_x \\ &= -H_x(g,f), \\ H_{xx}(f,g) = H^2(f,g) &= f_{xx}g - 2f_x g_x + f g_{xx} \\ &= H_{xx}(f,g). \end{aligned}$$

Lemma.

$$e^{zH}(f,g) = f(x+z)g(x-z).$$

PROOF. Using the notation $r = i\ (\to j)$ defined in Appendix A.1,

$$\begin{aligned} e^{zH}(f,g) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} H^n(f,g) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{r=0}^{n(\to\infty)} (-1)^r \binom{n}{r} D^{n-r}(f)D^r(g) \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r D^r(g)}{r!} \sum_{n=0(\to r)}^{\infty} \frac{z^n D^{n-r}(f)}{(n-r)!} \quad (\text{put } s = n-r) \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r z^r D^r(g)}{r!} \sum_{s=0}^{\infty} \frac{z^s D^s(f)}{s!}. \end{aligned}$$

These sums are Taylor expansions of $g(x-z)$ and $f(x+z)$, respectively, which proves the lemma. □

Applying Taylor's theorem again,

$$\tfrac{1}{2}\{\phi(x+z)-\phi(x-z)\} = \sum_{n=0}^{\infty} \frac{z^{2n+1}D^{2n+1}(\phi)}{(2n+1)!},$$
$$\tfrac{1}{2}\{\psi(x+z)+\psi(x-z)\} = \sum_{n=0}^{\infty} \frac{z^{2n}D^{2n}(\psi)}{(2n)!}. \tag{5.7.2}$$

5.7.2 *A Determinantal Identity*

Define functions ϕ, ψ, u_n and a Hessenbergian E_n as follows:

$$\begin{aligned}
\phi &= \log(fg),\\
\psi &= \log(f/g) && (5.7.3)\\
u_{2n} &= D^{2n}(\phi),\\
u_{2n+1} &= D^{2n+1}(\psi), && (5.7.4)\\
E_n &= |e_{ij}|_n,
\end{aligned}$$

where

$$e_{ij} = \begin{cases} \binom{j-1}{i-1} u_{j-i+1}, & j \ge i,\\ -1, & j = i-1,\\ 0, & \text{otherwise.} \end{cases} \tag{5.7.5}$$

It follows from (5.7.3) that

$$\begin{aligned}
f &= e^{(\phi+\psi)/2},\\
g &= e^{(\phi-\psi)/2},\\
fg &= e^{\phi}. && (5.7.6)
\end{aligned}$$

Theorem.

$$\frac{H^n(f,g)}{fg} = E_n$$
$$= \begin{vmatrix} u_1 & u_2 & u_3 & u_4 & \cdots & u_{n-1} & u_n \\ -1 & u_1 & 2u_2 & 3u_3 & \cdots & \cdots & \binom{n-1}{n-2} u_{n-1} \\ & -1 & u_1 & 3u_2 & \cdots & \cdots & \binom{n-1}{n-3} u_{n-2} \\ & & -1 & u_1 & \cdots & \cdots & \cdots \\ & & & & \cdots & \cdots & \cdots \\ & & & & & -1 & u_1 \end{vmatrix}_n .$$

This identity was conjectured by one of the authors and proved by Caudrey in 1984. The correspondence was private. Two proofs are given below. The first is essentially Caudrey's but with additional detail.

PROOF. *First proof* (Caudrey). The Hessenbergian satisfies the recurrence relation (Section 4.6)

$$E_{n+1} = \sum_{r=0}^{n} \binom{n}{r} u_{r+1} E_{n-r}. \tag{5.7.7}$$

Let

$$F_n = \frac{H^n(f,g)}{fg}, \quad f = f(x),\ g = g(x),\ F_0 = 1. \tag{5.7.8}$$

The theorem will be proved by showing that F_n satisfies the same recurrence relation as E_n and has the same initial values.

Let

$$K = \begin{cases} \frac{e^{zH}(f,g)}{fg} \\ \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{H^n(f,g)}{fg} \\ \sum_{n=0}^{\infty} \frac{z^n F_n}{n!}. \end{cases} \tag{5.7.9}$$

Then,

$$\frac{\partial K}{\partial z} = \sum_{n=1}^{\infty} \frac{z^{n-1} F_n}{(n-1)!} \tag{5.7.10}$$

$$= \sum_{n=0}^{\infty} \frac{z^n F_{n+1}}{n!}. \tag{5.7.11}$$

From the lemma and (5.7.6),

$$K = \frac{f(x+z)g(x-z)}{f(x)g(x)} = \exp\left[\tfrac{1}{2}\{\phi(x+z) + \phi(x-z) + \psi(x+z) - \psi(x-z) - 2\phi(x)\}\right]. \tag{5.7.12}$$

Differentiate with respect to z, refer to (5.7.11), note that

$$D_z(\phi(x-z)) = -D_x(\phi(x-z))$$

etc., and apply the Taylor relations (5.7.2) from the previous section. The result is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{z^n F_{n+1}}{n!} &= D\left[\tfrac{1}{2}\{\phi(x+z) - \phi(x-z) + \psi(x+z) + \psi(x-z)\}\right] K \\
&= \left[\sum_{n=0}^{\infty} \frac{z^{2n+1} D^{2n+2}(\phi)}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{z^{2n} D^{2n+1}(\psi)}{(2n)!}\right] K \\
&= \left[\sum_{n=0}^{\infty} \frac{z^{2n+1} u_{2n+2}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{z^{2n} u_{2n+1}}{(2n)!}\right] K
\end{aligned}$$

$$= \sum_{m=0}^{\infty} \frac{z^m u_{m+1}}{m!} \sum_{r=0}^{\infty} \frac{z^r F_r}{r!}. \tag{5.7.13}$$

Equating coefficients of z^n,

$$\frac{F_{n+1}}{n!} = \sum_{r=0}^{n} \frac{u_{r+1} F_{n-r}}{r!\,(n-r)!},$$

$$F_{n+1} = \sum_{r=0}^{n} \binom{n}{r} u_{r+1} F_{n-r}. \tag{5.7.14}$$

This recurrence relation in F_n is identical in form to the recurrence relation in E_n given in (5.7.7). Furthermore,

$$E_1 = F_1 = u_1,$$
$$E_2 = F_2 = u_1^2 + u_2.$$

Hence,

$$E_n = F_n$$

which proves the theorem.

Second proof. Express the lemma in the form

$$\sum_{i=0}^{\infty} \frac{z^i}{i!} H^i(f, g) = f(x+z)g(x-z). \tag{5.7.15}$$

Hence,

$$H^i(f, g) = \left[D_z^i\{f(x+z)g(x-z)\}\right]_{z=0}. \tag{5.7.16}$$

Put

$$f(x) = e^{F(x)},$$
$$g(x) = e^{G(x)},$$
$$w = F(x+z) + G(x-z).$$

Then,

$$\begin{aligned} H^i(e^F, e^G) &= \left[D_z^i(e^w)\right]_{z=0} \\ &= \left[D_z^{i-1}(e^w w_z)\right]_{z=0} \\ &= \sum_{j=0}^{i-1} \binom{i-1}{j} \left[D_z^{i-j}(w) D_z^j(e^w)\right]_{z=0} \\ &= \sum_{j=0}^{i-1} \binom{i-1}{j} \psi_{i-j} H^j(e^F, e^G), \quad i \ge 1, \end{aligned} \tag{5.7.17}$$

where

$$\psi_r = \left[D_z^r(w)\right]_{z=0}$$

$$= D^r\{F(x) + (-1)^r G(x)\}, \quad D = \frac{d}{dx}. \tag{5.7.18}$$

Hence,

$$\begin{aligned}\psi_{2r} &= D^{2r}\log(fg)\\ &= D^{2r}(\phi)\\ &= u_{2r}.\end{aligned}$$

Similarly,

$$\psi_{2r+1} = u_{2r+1}.$$

Hence, $\psi_r = u_r$ for all values of r.

In (5.7.17), put

$$H_i = H^i(e^F, e^G),$$

so that

$$H_0 = e^{F+G}$$

and put

$$\begin{aligned}a_{ij} &= \binom{i-1}{j}\psi_{i-j}, \quad j < i,\\ a_{ii} &= -1.\end{aligned}$$

Then,

$$a_{i0} = \psi_i = u_i$$

and (5.7.17) becomes

$$\sum_{j=0}^{i} a_{ij}H_j = 0, \quad i \geq 1,$$

which can be expressed in the form

$$\begin{aligned}\sum_{j=1}^{i} a_{ij}H_j &= -a_{i0}H_0\\ &= -e^{F+G}u_i, \quad i \geq 1.\end{aligned} \tag{5.7.19}$$

This triangular system of equations in the H_j is similar in form to the triangular system in Section 2.3.5 on Cramer's formula. The solution of that system is given in terms of a Hessenbergian. Hence, the solution of (5.7.19) is also expressible in terms of a Hessenbergian,

$$H_j = e^{F+G}\begin{vmatrix} u_1 & -1 & & & \\ u_2 & u_1 & -1 & & \\ u_3 & 2u_2 & u_1 & -1 & \\ u_4 & 3u_3 & 3u_2 & u_1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n,$$

which, after transposition, is equivalent to the stated result. □

Exercises

1. Prove that

$$\sum_{k=1}^{i} b_{ik} u_k = H_i,$$

where

$$b_{ik} = \binom{i-1}{k-1} H_{i-k}$$

and hence express u_k as a Hessenbergian whose elements are the H_i.

2. Prove that

$$H(A^{is}, A^{rj}) = \sum_{p=1}^{n} \sum_{q=1}^{n} a'_{pq} \begin{vmatrix} A^{iq} & A^{ir,sq} \\ A^{pj} & A^{pr,sj} \end{vmatrix}.$$

5.8 Some Applications of Algebraic Computing

5.8.1 *Introduction*

In the early days of electronic digital computing, it was possible to perform, in a reasonably short time, long and complicated calculations with real numbers such as the evaluation of π to 1000 decimal places or the evaluation of a determinant of order 100 with real numerical elements, but no system was able to operate with complex numbers or to solve even the simplest of algebraic problems such as the factorization of a polynomial or the evaluation of a determinant of low order with symbolic elements.

The first software systems designed to automate symbolic or algebraic calculations began to appear in the 1950s, but for many years, the only people who were able to profit from them were those who had easy access to large, fast computers. The situation began to improve in the 1970s and by the early 1990s, small, fast personal computers loaded with sophisticated software systems had sprouted like mushrooms from thousands of desktops and it became possible for most professional mathematicians, scientists, and engineers to carry out algebraic calculations which were hitherto regarded as too complicated even to attempt.

One of the branches of mathematics which can profit from the use of computers is the investigation into the algebraic and differential properties of determinants, for the work involved in manipulating determinants of orders greater than 5 is usually too complicated to tackle unaided. Remember that the expansion of a determinant of order n whose elements are monomials consists of the sum of $n!$ terms each with n factors and that many formulas in determinant theory contain products and quotients involving several determinants of order n or some function of n.

Computers are invaluable in the initial stages of an investigation. They can be used to study the behavior of determinants as their orders increase and to assist in the search for patterns. Once a pattern has been observed, it may be possible to formulate a conjecture which, when proved analytically, becomes a theorem. In some cases, it may be necessary to evaluate determinants of order 10 or more before the nature of the conjecture becomes clear or before a previously formulated conjecture is realized to be false.

In Section 5.6 on distinct matrices with nondistinct determinants, there are two theorems which were originally published as conjectures but which have since been proved by Fiedler. However, that section also contains a set of simple isolated identities which still await unification and generalization. The nature of these identities is comparatively simple and it should not be difficult to make progress in this field with the aid of a computer.

The following pages contain several other conjectures which await proof or refutation by analytic methods and further sets of simple isolated identities which await unification and generalization. Here again the use of a computer should lead to further progress.

5.8.2 Hankel Determinants with Hessenberg Elements

Define a Hessenberg determinant H_n (Section 4.6) as follows:

$$H_n = \begin{vmatrix} h_1 & h_2 & h_3 & h_4 & \cdots & h_{n-1} & h_n \\ 1 & h_1 & h_2 & h_3 & \cdots & \cdots & \cdots \\ & 1 & h_1 & h_2 & \cdots & \cdots & \cdots \\ & & 1 & h_1 & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1 & h_1 \end{vmatrix}_n,$$

$$H_0 = 1. \tag{5.8.1}$$

Conjecture 1.

$$\begin{vmatrix} H_{n+r} & H_{n+r+1} & \cdots & H_{2n+r-1} \\ H_{n+r-1} & H_{n+r} & \cdots & H_{2n+r-2} \\ \cdots & \cdots & \cdots & \cdots \\ H_{r+1} & H_{r+2} & \cdots & H_{n+r} \end{vmatrix}_n = \begin{vmatrix} h_n & h_{n+1} & \cdots & h_{2n+r-1} \\ h_{n-1} & h_n & \cdots & h_{2n+r-2} \\ \cdots & \cdots & \cdots & \cdots \\ h_{1-r} & h_{2-r} & \cdots & h_n \end{vmatrix}_{n+r}.$$

$h_0 = 1$, $h_m = 0$, $m < 0$.

Both determinants are of Hankel form (Section 4.8) but have been rotated through 90° from their normal orientations. Restoration of normal orientations introduces negative signs to determinants of orders $4m$ and $4m+1$, $m \ge 1$. When $r = 0$, the identity is unaltered by interchanging H_s and h_s, $s = 1, 2, 3 \ldots$. The two determinants merely change sides. The identities in which $r = \pm 1$ form a dual pair in the sense that one can be transformed into the other by interchanging H_s and h_s, $s = 0, 1, 2, \ldots$.

Examples.

$(n, r) = (2, 0)$:

$$\begin{vmatrix} H_2 & H_3 \\ H_1 & H_2 \end{vmatrix} = \begin{vmatrix} h_2 & h_3 \\ h_1 & h_2 \end{vmatrix};$$

$(n, r) = (3, 0)$:

$$\begin{vmatrix} H_3 & H_4 & H_5 \\ H_2 & H_3 & H_4 \\ H_1 & H_2 & H_3 \end{vmatrix} = \begin{vmatrix} h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \end{vmatrix};$$

$(n, r) = (2, 1)$:

$$\begin{vmatrix} H_3 & H_4 \\ H_2 & H_3 \end{vmatrix} = \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 \end{vmatrix};$$

$(n, r) = (3, -1)$:

$$\begin{vmatrix} H_2 & H_3 & H_4 \\ H_1 & H_2 & H_3 \\ 1 & H_1 & H_2 \end{vmatrix} = \begin{vmatrix} h_3 & h_4 \\ h_2 & h_3 \end{vmatrix}.$$

Conjecture 2.

$$\begin{vmatrix} H_n & H_{n+1} \\ 1 & H_1 \end{vmatrix} = \begin{vmatrix} h_2 & h_3 & h_4 & h_5 & \cdots & h_n & h_{n+1} \\ 1 & h_1 & h_2 & h_3 & \cdots & h_{n-2} & h_{n-1} \\ & 1 & h_1 & h_2 & \cdots & h_{n-3} & h_{n-2} \\ & & 1 & h_1 & \cdots & h_{n-4} & h_{n-3} \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1 & h_1 \end{vmatrix}_n.$$

Note that, in the determinant on the right, there is a break in the sequence of suffixes from the first row to the second.

The following set of identities suggest the existence of a more general relation involving determinants in which the sequence of suffixes from one row to the next or from one column to the next is broken.

$$\begin{vmatrix} H_1 & H_3 \\ 1 & H_2 \end{vmatrix} = \begin{vmatrix} h_1 & h_3 \\ 1 & h_2 \end{vmatrix},$$

$$\begin{vmatrix} H_2 & H_4 \\ H_1 & H_3 \end{vmatrix} = \begin{vmatrix} h_1 & h_3 & h_4 \\ 1 & h_2 & h_3 \\ & h_1 & h_2 \end{vmatrix},$$

$$\begin{vmatrix} H_3 & H_5 \\ H_2 & H_4 \end{vmatrix} = \begin{vmatrix} h_1 & h_3 & h_4 & h_5 \\ 1 & h_2 & h_3 & h_4 \\ & h_1 & h_2 & h_3 \\ & 1 & h_1 & h_2 \end{vmatrix},$$

$$\begin{vmatrix} H_2 & H_4 & H_5 \\ H_1 & H_3 & H_4 \\ 1 & H_2 & H_3 \end{vmatrix} = \begin{vmatrix} h_2 & h_4 & h_5 \\ h_1 & h_3 & h_4 \\ 1 & h_2 & h_3 \end{vmatrix},$$

$$\begin{vmatrix} H_3 & H_5 \\ H_1 & H_3 \end{vmatrix} = \begin{vmatrix} h_1 & h_3 & h_4 & h_5 \\ 1 & h_2 & h_3 & h_4 \\ & h_1 & h_2 & h_3 \\ & & 1 & h_1 \end{vmatrix}. \tag{5.8.2}$$

5.8.3 Hankel Determinants with Hankel Elements

Let

$$A_n = |\phi_{r+m}|_n, \quad 0 \le m \le 2n-2, \tag{5.8.3}$$

which is an Hankelian (or a Turanian).

Let

$$\begin{aligned} B_r &= A_2 \\ &= \begin{vmatrix} \phi_r & \phi_{r+1} \\ \phi_{r+1} & \phi_{r+2} \end{vmatrix}. \end{aligned} \tag{5.8.4}$$

Then B_r, B_{r+1}, and B_{r+2} are each Hankelians of order 2 and are each minors of A_3:

$$\begin{aligned} B_r &= A_{33}^{(3)}, \\ B_{r+1} &= A_{31}^{(3)} = A_{13}^{(3)}, \\ B_{r+2} &= A_{11}^{(3)}. \end{aligned} \tag{5.8.5}$$

Hence, applying the Jacobi identity (Section 3.6),

$$\begin{aligned} \begin{vmatrix} B_{r+2} & B_{r+1} \\ B_{r+1} & B_r \end{vmatrix} &= \begin{vmatrix} A_{11}^{(3)} & A_{13}^{(3)} \\ A_{31}^{(3)} & A_{33}^{(3)} \end{vmatrix} \\ &= A_3 A_{13,13}^{(3)} \\ &= \phi_2 A_3. \end{aligned} \tag{5.8.6}$$

Now redefine B_r. Let $B_r = A_3$. Then, $B_r, B_{r+1}, \ldots, B_{r+4}$ are each second minors of A_5:

$$\begin{aligned} B_r &= A_{45,45}^{(5)}, \\ B_{r+1} &= -A_{15,45}^{(5)} = -A_{45,15}^{(5)}, \\ B_{r+2} &= A_{12,45}^{(5)} = A_{15,15}^{(5)} = A_{45,12}^{(5)}, \\ B_{r+3} &= -A_{12,15}^{(5)} = -A_{15,12}^{(5)}, \\ B_{r+4} &= A_{12,12}^{(5)}. \end{aligned} \tag{5.8.7}$$

Hence,

$$\begin{vmatrix} B_{r+4} & B_{r+3} & B_{r+2} \\ B_{r+3} & B_{r+2} & B_{r+1} \\ B_{r+2} & B_{r+1} & B_r \end{vmatrix} = \begin{vmatrix} A^{(5)}_{12,12} & A^{(5)}_{12,15} & A^{(5)}_{12,45} \\ A^{(5)}_{15,12} & A^{(5)}_{15,15} & A^{(5)}_{15,45} \\ A^{(5)}_{45,12} & A^{(5)}_{45,15} & A^{(5)}_{45,45} \end{vmatrix}. \tag{5.8.8}$$

Denote the determinant on the right by V_3. Then, V_3 is not a standard third-order Jacobi determinant which is of the form

$$|A^{(n)}_{pq}|_3 \quad \text{or} \quad |A^{(n)}_{gp,hq}|_3, \quad p = i, j, k, \quad q = r, s, t.$$

However, V_3 can be regarded as a generalized Jacobi determinant in which the elements have vector parameters:

$$V_3 = |A^{(5)}_{\mathbf{uv}}|_3, \tag{5.8.9}$$

where $\mathbf{u}$ and $\mathbf{v} = [1, 2]$, $[1, 5]$, and $[4, 5]$, and $A^{(5)}_{\mathbf{uv}}$ is interpreted as a second cofactor of A_5. It may be verified that

$$V_3 = A^{(5)}_{125;125} A^{(5)}_{145;145} A_5 + \phi_4 (A^{(5)}_{15})^2 \tag{5.8.10}$$

and that if

$$V_3 = |A^{(4)}_{\mathbf{uv}}|_3, \tag{5.8.11}$$

where $\mathbf{u}$ and $\mathbf{v} = [1, 2]$, $[1, 4]$, and $[3, 4]$, then

$$V_3 = A^{(4)}_{124;124} A^{(4)}_{134;134} A_4 + (A^{(4)}_{14})^2. \tag{5.8.12}$$

These results suggest the following conjecture:

Conjecture. If

$$V_3 = |A^{(n)}_{\mathbf{uv}}|_3,$$

where $\mathbf{u}$ and $\mathbf{v} = [1, 2]$, $[1, n]$, and $[n-1, n]$, then

$$V_3 = A^{(n)}_{12n;12n} A^{(n)}_{1,n-1,n;1,n-1,n} A_n + A^{(n)}_{12,n-1,n;12,n-1,n} (A^{(n)}_{1n})^2.$$

Exercise. If

$$V_3 = |A^{(4)}_{\mathbf{uv}}|,$$

where

$$\mathbf{u} = [1, 2], [1, 3], \text{ and } [2, 4],$$
$$\mathbf{v} = [1, 2], [1, 3], \text{ and } [2, 3].$$

prove that

$$V_3 = -\phi_5 \phi_6 A_4.$$

5.8.4 Hankel Determinants with Symmetric Toeplitz Elements

The symmetric Toeplitz determinant T_n (Section 4.5.2) is defined as follows:

$$T_n = |t_{|i-j|}|_n,$$

with

$$T_0 = 1. \tag{5.8.13}$$

For example,

$$\begin{aligned} T_1 &= t_0, \\ T_2 &= t_0^2 - t_1^2, \\ T_3 &= t_0^3 - 2t_0t_1^2 - t_0t_2^2 + 2t_1^2t_2, \end{aligned} \tag{5.8.14}$$

etc. In each of the following three identities, the determinant on the left is a Hankelian with symmetric Toeplitz elements, but when the rows or columns are interchanged they can also be regarded as second-order subdeterminants of $|T_{|i-j|}|_n$, which is a symmetric Toeplitz determinant with symmetric Toeplitz elements. The determinants on the right are subdeterminants of T_n with a common principal diagonal.

$$\begin{aligned} \begin{vmatrix} T_0 & T_1 \\ T_1 & T_2 \end{vmatrix} &= -|t_1|^2, \\ \begin{vmatrix} T_1 & T_2 \\ T_2 & T_3 \end{vmatrix} &= -\begin{vmatrix} t_1 & t_0 \\ t_2 & t_1 \end{vmatrix}^2, \\ \begin{vmatrix} T_2 & T_3 \\ T_3 & T_4 \end{vmatrix} &= -\begin{vmatrix} t_1 & t_0 & t_1 \\ t_2 & t_1 & t_0 \\ t_3 & t_2 & t_1 \end{vmatrix}^2. \end{aligned} \tag{5.8.15}$$

Conjecture.

$$\begin{vmatrix} T_{n-1} & T_n \\ T_n & T_{n+1} \end{vmatrix} = -\begin{vmatrix} t_1 & t_0 & t_1 & t_2 & \cdots & t_{n-2} \\ t_2 & t_1 & t_0 & t_1 & \cdots & t_{n-3} \\ t_3 & t_2 & t_1 & t_0 & \cdots & t_{n-4} \\ t_4 & t_3 & t_2 & t_1 & \cdots & t_{n-5} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 \end{vmatrix}_n^2.$$

Other relations of a similar nature include the following:

$$\begin{vmatrix} T_0 & T_1 \\ T_2 & T_3 \end{vmatrix} = \begin{vmatrix} t_0 & t_1 & t_2 \\ t_1 & t_0 & t_1 \\ t_2 & t_1 & \end{vmatrix},$$

$$\begin{vmatrix} T_1 & T_2 & T_3 \\ T_2 & T_3 & T_4 \\ T_3 & T_4 & T_5 \end{vmatrix} \quad \text{has a factor} \quad \begin{vmatrix} t_0 & t_1 & t_2 \\ t_1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \end{vmatrix}. \tag{5.8.16}$$

5.8.5 Hessenberg Determinants with Prime Elements

Let the sequence of prime numbers be denoted by $\{p_n\}$ and define a Hessenberg determinant H_n (Section 4.6) as follows:

$$H_n = \begin{vmatrix} p_1 & p_2 & p_3 & p_4 & \cdots \\ 1 & p_1 & p_2 & p_3 & \cdots \\ & 1 & p_1 & p_2 & \cdots \\ & & 1 & p_1 & \cdots \\ & & & \cdots & \cdots \end{vmatrix}_n .$$

This determinant satisfies the recurrence relation

$$H_n = \sum_{r=0}^{n-1} (-1)^r p_{r+1} H_{n-1-r}, \quad H_0 = 1.$$

A short list of primes and their associated Hessenberg numbers is given in the following table:

n	1	2	3	4	5	6	7	8	9	10
p_n	2	3	5	7	11	13	17	19	23	29
H_n	2	1	1	2	3	7	10	13	21	26

n	11	12	13	14	15	16	17	18	19	20
p_n	31	37	41	43	47	53	59	61	67	71
H_n	33	53	80	127	193	254	355	527	764	1149

Conjecture. The sequence $\{H_n\}$ is monotonic from H_3 onward.

This conjecture was contributed by one of the authors to an article entitled "Numbers Count" in the journal *Personal Computer World* and was published in June 1991. Several readers checked its validity on computers, but none of them found it to be false. The article is a regular one for computer buffs and is conducted by Mike Mudge, a former colleague of the author.

Exercise. Prove or refute the conjecture analytically.

5.8.6 Bordered Yamazaki–Hori Determinants — 2

A bordered determinant W of order $(n+1)$ is defined in Section 4.10.3 and is evaluated in Theorem 4.42 in the same section. Let that determinant be denoted here by W_{n+1} and verify the formula

$$W_{n+1} = -\frac{K_n}{4}(x^2-1)^{n(n-1)}\{(x+1)^n - (x-1)^n\}^2$$

for several values of n. K_n is the simple Hilbert determinant.

Replace the last column of W_{n+1} by the column

$$\left[1\ 3\ 5 \cdots (2n-1)\ \bullet\right]^T$$

and denote the result by Z_{n+1}. Verify the formula

$$Z_{n+1} = -n^2 K_n (x^2 - 1)^{n^2-2}(x^2 - n^2)$$

for several values of n.

Both formulas have been proved analytically, but the details are complicated and it has been decided to omit them.

Exercise. Show that

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & x \\ a_{21} & a_{22} & \cdots & a_{2n} & x^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & x^n \\ 1 & \frac{x}{3} & \cdots & \frac{x^{n-1}}{2n-1} & -\frac{1}{2} \end{vmatrix} = -\tfrac{1}{2} K_n F\left(n, -n; \tfrac{1}{2}; -x\right),$$

where

$$a_{ij} = \frac{(1+x)^{i+j-1} - x^{i+j-1}}{i+j-1}$$

and where $F(a, b; c; x)$ is the hypergeometric function.

5.8.7 Determinantal Identities Related to Matrix Identities

If $\mathbf{M}_r$, $1 \le r \le s$, denote matrices of order n and

$$\sum_{r=1}^{s} \mathbf{M}_r = \mathbf{0}, \quad s > 2,$$

then, in general,

$$\sum_{r=1}^{s} |\mathbf{M}_r| \ne 0, \quad s > 2,$$

that is, the corresponding determinantal identity is *not* valid. However, there are nontrivial exceptions to this rule.

Let $\mathbf{P}$ and $\mathbf{Q}$ denote arbitrary matrices of order n. Then

1. **a.** $(\mathbf{PQ} + \mathbf{QP}) + (\mathbf{PQ} - \mathbf{QP}) - 2\mathbf{PQ} = \mathbf{0}$, all n,
b. $|\mathbf{PQ} + \mathbf{QP}| + |\mathbf{PQ} - \mathbf{QP}| - |2\mathbf{PQ}| = 0$, $n = 2$.
2. **a.** $(\mathbf{P} - \mathbf{Q})(\mathbf{P} + \mathbf{Q}) - (\mathbf{P}^2 - \mathbf{Q}^2) - (\mathbf{PQ} - \mathbf{QP}) = \mathbf{0}$, all n,
b. $|(\mathbf{P} - \mathbf{Q})(\mathbf{P} + \mathbf{Q})| - |\mathbf{P}^2 - \mathbf{Q}^2| - |\mathbf{PQ} - \mathbf{QP}| = 0$, $n = 2$.
3. **a.** $(\mathbf{P} - \mathbf{Q})(\mathbf{P} + \mathbf{Q}) - (\mathbf{P}^2 - \mathbf{Q}^2) + (\mathbf{PQ} + \mathbf{QP}) - 2\mathbf{PQ} = \mathbf{0}$, all n,
b. $|(\mathbf{P} - \mathbf{Q})(\mathbf{P} + \mathbf{Q})| - |\mathbf{P}^2 - \mathbf{Q}^2| + |\mathbf{PQ} + \mathbf{QP}| - |2\mathbf{PQ}| = 0$, $n = 2$.

The matrix identities 1(a), 2(a), and 3(a) are obvious. The corresponding determinantal identities 1(b), 2(b), and 3(b) are not obvious and no neat proofs have been found, but they can be verified manually or on a computer. Identity 3(b) can be obtained from 1(b) and 2(b) by eliminating $|\mathbf{PQ} - \mathbf{QP}|$.

It follows that there exist at least two solutions of the equation

$$|\mathbf{X}+\mathbf{Y}| = |\mathbf{X}| + |\mathbf{Y}|, \quad n=2,$$

namely

$$\begin{aligned} \mathbf{X} &= \mathbf{PQ}+\mathbf{QP} \quad \textit{or} \quad \mathbf{P}^2-\mathbf{Q}^2, \\ \mathbf{Y} &= \mathbf{PQ}-\mathbf{QP}. \end{aligned}$$

Furthermore, the equation

$$|\mathbf{X}-\mathbf{Y}+\mathbf{Z}| = |\mathbf{X}| - |\mathbf{Y}| + |\mathbf{Z}|, \quad n=2,$$

is satisfied by

$$\begin{aligned} \mathbf{X} &= \mathbf{P}^2-\mathbf{Q}^2, \\ \mathbf{Y} &= \mathbf{PQ}+\mathbf{QP}, \\ \mathbf{Z} &= 2\mathbf{PQ}. \end{aligned}$$

Are there any other determinantal identities of a similar nature?

6

Applications of Determinants in Mathematical Physics

6.1 Introduction

This chapter is devoted to verifications of the determinantal solutions of several equations which arise in three branches of mathematical physics, namely lattice, relativity, and soliton theories. All but one are nonlinear.

Lattice theory can be defined as the study of elements in a two- or three-dimensional array under the influence of neighboring elements. For example, it may be required to determine the electromagnetic state of one loop in an electrical network under the influence of the electromagnetic field generated by neighboring loops or to study the behavior of one atom in a crystal under the influence of neighboring atoms.

Einstein's theory of general relativity has withstood the test of time and is now called classical gravity. The equations which appear in this chapter arise in that branch of the theory which deals with stationary axisymmetric gravitational fields.

A soliton is a solitary wave and soliton theory can be regarded as a branch of nonlinear wave theory.

The term *determinantal solution* needs clarification since it can be argued that any function can be expressed as a determinant and, hence, any solvable equation has a solution which can be expressed as a determinant. The term *determinantal solution* shall mean a solution containing a determinant which has not been evaluated in simple form and may possibly be the simplest form of the function it represents. A number of determinants have been evaluated in a simple form in earlier chapters and elsewhere, but they are exceptional. In general, determinants cannot be evaluated in simple form. The definition of a determinant as a sum of products of elements is not, in general, a simple form as it is not, in general, amenable to many of the processes of analysis, especially repeated differentiation.

There may exist a section of the mathematical community which believes that if an equation possesses a determinantal solution, then the determinant must emerge from a matrix like an act of birth, for it cannot materialize in any other way! This belief has not, so far, been justified. In some cases, the determinants do indeed emerge from sets of equations and hence, by implication, from matrices, but in other cases, they arise as nonlinear algebraic and differential forms with no mother matrix in sight. However, we do not exclude the possibility that new methods of solution can be devised in which every determinant emerges from a matrix.

Where the integer n appears in the equation, as in the Dale and Toda equations, n or some function of n appears in the solution as the order of the determinant. Where n does not appear in the equation, it appears in the solution as the arbitrary order of a determinant.

The equations in this chapter were originally solved by a variety of methods including the application of the Gelfand–Levitan–Marchenko (GLM) integral equation of inverse scattering theory, namely

$$K(x,y,t) + R(x+y,t) + \int_x^\infty K(x,z,t)R(y+z,t)\,dz = 0$$

in which the kernel $R(u,t)$ is given and $K(x,y,t)$ is the function to be determined. However, in this chapter, all solutions are verified by the purely determinantal techniques established in earlier chapters.

6.2 Brief Historical Notes

In order to demonstrate the extent to which determinants have entered the field of differential and other equations we now give brief historical notes on the origins and solutions of these equations. The detailed solutions follow in later sections.

6.2.1 The Dale Equation

The Dale equation is

$$(y'')^2 = y'\left(\frac{y}{x}\right)'\left(\frac{y+4n^2}{1+x}\right)',$$

where n is a positive integer. This equation arises in the theory of stationary axisymmetric gravitational fields and is the only nonlinear ordinary equation to appear in this chapter. It was solved in 1978. Two related equations, which appear in Section 4.11.4, were solved in 1980. Cosgrove has published an equation which can be transformed into the Dale equation.

6.2.2 The Kay–Moses Equation

The one-dimensional Schrödinger equation, which arises in quantum theory, is

$$\left[D^2 + \varepsilon^2 - V(x)\right]y = 0, \quad D = \frac{d}{dx},$$

and is the only linear ordinary equation to appear in this chapter.

The solution for arbitrary $V(x)$ is not known, but in a paper published in 1956 on the reflectionless transmission of plane waves through dielectrics, Kay and Moses solved it in the particular case in which

$$V(x) = -2D^2(\log A),$$

where A is a certain determinant of arbitrary order whose elements are functions of x. The equation which Kay and Moses solved is therefore

$$\left[D^2 + \varepsilon^2 + 2D^2(\log A)\right]y = 0.$$

6.2.3 The Toda Equations

The differential–difference equations

$$D(R_n) = \exp(-R_{n-1}) - \exp(-R_{n+1}),$$
$$D^2(R_n) = 2\exp(-R_n) - \exp(-R_{n-1}) - \exp(-R_{n+1}), \quad D = \frac{d}{dx},$$

arise in nonlinear lattice theory. The first appeared in 1975 in a paper by Kac and van Moerbeke and can be regarded as a discrete analog of the KdV equation (Ablowitz and Segur, 1981). The second is the simplest of a series of equations introduced by Toda in 1967 and can be regarded as a second-order development of the first. For convenience, these equations are referred to as first-order and second-order Toda equations, respectively.

The substitutions

$$R_n = -\log y_n,$$
$$y_n = D(\log u_n)$$

transform the first-order equation into

$$D(\log y_n) = y_{n+1} - y_{n-1} \tag{6.2.1}$$

and then into

$$D(u_n) = \frac{u_n u_{n+1}}{u_{n-1}}. \tag{6.2.2}$$

The same substitutions transform the second-order equation first into

$$D^2(\log y_n) = y_{n+1} - 2y_n + y_{n-1}$$

and then into

$$D^2(\log u_n) = \frac{u_{n+1}u_{n-1}}{u_n^2}\,. \tag{6.2.3}$$

Other equations which are similar in nature to the transformed second-order Toda equations are

$$\begin{aligned} D_x D_y(\log u_n) &= \frac{u_{n+1}u_{n-1}}{u_n^2},\\ (D_x^2 + D_y^2)\log u_n &= \frac{u_{n+1}u_{n-1}}{u_n^2},\\ \frac{1}{\rho}D_\rho\big[\rho D_\rho(\log u_n)\big] &= \frac{u_{n+1}u_{n-1}}{u_n^2}\,. \end{aligned} \tag{6.2.4}$$

All these equations are solved in Section 6.5.

Note that (6.2.1) can be expressed in the form

$$D(y_n) = y_n(y_{n+1} - y_{n-1}), \tag{6.2.1a}$$

which appeared in 1974 in a paper by Zacharov, Musher, and Rubenchick on Langmuir waves in a plasma and was solved in 1987 by S. Yamazaki in terms of determinants P_{2n-1} and P_{2n} of order n. Yamazaki's analysis involves a continued fraction. The transformed equation (6.2.2) is solved below without introducing a continued fraction but with the aid of the Jacobi identity and one of its variants (Section 3.6).

The equation

$$D_x D_y(R_n) = \exp(R_{n+1} - R_n) - \exp(R_n - R_{n-1}) \tag{6.2.5}$$

appears in a 1991 paper by Kajiwara and Satsuma on the q-difference version of the second-order Toda equation.

The substitution

$$R_n = \log\left(\frac{u_{n+1}}{u_n}\right)$$

reduces it to the first line of (6.2.4).

In the chapter on reciprocal differences in his book *Calculus of Finite Differences*, Milne-Thomson defines an operator r_n by the relations

$$\begin{aligned} r_0 f(x) &= f(x),\\ r_1 f(x) &= \frac{1}{f'(x)},\\ \big[r_{n+1} - r_{n-1} - (n+1)r_1 r_n\big] f(x) &= 0. \end{aligned}$$

Put

$$r_n f = y_n.$$

Then,

$$y_{n+1} - y_{n-1} - (n+1)r_1(y_n) = 0,$$

that is,

$$y_n'(y_{n+1} - y_{n-1}) = n + 1.$$

This equation will be referred to as the Milne-Thomson equation. Its origin is distinct from that of the Toda equations, but it is of a similar nature and clearly belongs to this section.

6.2.4 The Matsukidaira–Satsuma Equations

The following pairs of coupled differential–difference equations appeared in a paper on nonlinear lattice theory published by Matsukidaira and Satsuma in 1990.

The first pair is

$$\begin{aligned} q_r' &= q_r(u_{r+1} - u_r), \\ \frac{u_r'}{u_r - u_{r-1}} &= \frac{q_r'}{q_r - q_{r-1}}. \end{aligned}$$

These equations contain two dependent variables q and u, and two independent variables, x which is continuous and r which is discrete. The solution is expressed in terms of a Hankel–Wronskian of arbitrary order n whose elements are functions of x and r.

The second pair is

$$\begin{aligned} (q_{rs})_y &= q_{rs}(u_{r+1,s} - u_{rs}), \\ \frac{(u_{rs})_x}{u_{rs} - u_{r,s-1}} &= \frac{q_{rs}(v_{r+1,s} - v_{rs})}{q_{rs} - q_{r,s-1}}. \end{aligned}$$

These equations contain three dependent variables, q, u, and v, and four independent variables, x and y which are continuous and r and s which are discrete. The solution is expressed in terms of a two-way Wronskian of arbitrary order n whose elements are functions of x, y, r, and s.

In contrast with Toda equations, the discrete variables do not appear in the solutions as orders of determinants.

6.2.5 The Korteweg–de Vries Equation

The Korteweg–de Vries (KdV) equation, namely

$$u_t + 6uu_x + u_{xxx} = 0,$$

where the suffixes denote partial derivatives, is nonlinear and first arose in 1895 in a study of waves in shallow water. However, in the 1960s, interest in the equation was stimulated by the discovery that it also arose in studies of magnetohydrodynamic waves in a warm plasma, ion acoustic waves, and acoustic waves in an anharmonic lattice. Of all physically significant nonlinear partial differential equations with known analytic solutions, the KdV equation is one of the simplest. The KdV equation can be regarded as a particular case of the Kadomtsev–Petviashvili (KP) equation but it is of such fundamental importance that it has been given detailed individual attention in this chapter.

A method for solving the KdV equation based on the GLM integral equation was described by Gardner, Greene, Kruskal, and Miura (GGKM) in 1967. The solution is expressed in the form

$$u = 2D_x\{K(x,x,t)\}, \quad D_x = \frac{\partial}{\partial x}\,.$$

However, GGKM did not give an explicit solution of the integral equation and the first explicit solution of the KdV equation was given by Hirota in 1971 in terms of a determinant with well-defined elements but of arbitrary order. He used an independent method which can be described as heuristic, that is, obtained by trial and error. In another pioneering paper published the same year, Zakharov solved the KdV equation using the GGKM method. Wadati and Toda also applied the GGKM method and, in 1972, published a solution which agrees with Hirota's.

In 1979, Satsuma showed that the solution of the KdV equation can be expressed in terms of a Wronskian, again with well-defined elements but of arbitrary order. In 1982, Pöppe transformed the KdV equation into an integral equation and solved it by the Fredholm determinant method. Finally, in 1983, Freeman and Nimmo solved the KdV equation directly in Wronskian form.

6.2.6 *The Kadomtsev–Petviashvili Equation*

The Kadomtsev–Petviashvili (KP) equation, namely

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0,$$

arises in a study published in 1970 of the stability of solitary waves in weakly dispersive media. It can be regarded as a two-dimensional generalization of the KdV equation to which it reverts if u is independent of y.

The non-Wronskian solution of the KP equation was obtained from inverse scattering theory (Lamb, 1980) and verified in 1989 by Matsuno using a method based on the manipulation of bordered determinants. In 1983, Freeman and Nimmo solved the KP equation directly in Wronskian form, and in 1988, Hirota, Ohta, and Satsuma found a solution containing a two-way (right and left) Wronskian. Again, all determinants have well-defined elements but are of arbitrary order. Shortly after the Matsuno paper appeared, A. Nakamura solved the KP equation by means of four linear operators and a determinant of arbitrary order whose elements are defined as integrals.

The verifications given in Sections 6.7 and 6.8 of the non-Wronskian solutions of both the KdV and KP equations apply purely determinantal methods and are essentially those published by Vein and Dale in 1987.

6.2.7 The Benjamin–Ono Equation

The Benjamin–Ono (BO) equation is a nonlinear integro-differential equation which arises in the theory of internal waves in a stratified fluid of great depth and in the propagation of nonlinear Rossby waves in a rotating fluid. It can be expressed in the form

$$u_t + 4uu_x + H\{u_{xx}\} = 0,$$

where $H\{f(x)\}$ denotes the Hilbert transform of $f(x)$ defined as

$$H\{f(x)\} = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(y)}{y-x}\, dy$$

and where P denotes the principal value.

In a paper published in 1988, Matsuno introduced a complex substitution into the BO equation which transformed it into a more manageable form, namely

$$2A_x A_x^* = A^*(A_{xx} + \omega A_t) + A(A_{xx} + \omega A_t)^* \quad (\omega^2 = -1),$$

where A^* is the complex conjugate of A, and found a solution in which A is a determinant of arbitrary order whose diagonal elements are linear in x and t and whose nondiagonal elements contain a sequence of distinct arbitrary constants.

6.2.8 The Einstein and Ernst Equations

In the particular case in which a relativistic gravitational field is axially symmetric, the Einstein equations can be expressed in the form

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \mathbf{P}}{\partial \rho}\mathbf{P}^{-1}\right) + \frac{\partial}{\partial z}\left(\rho \frac{\partial \mathbf{P}}{\partial z}\mathbf{P}^{-1}\right) = 0,$$

where the matrix $\mathbf{P}$ is defined as

$$\mathbf{P} = \frac{1}{\phi}\begin{bmatrix} 1 & \psi \\ \psi & \phi^2 + \psi^2 \end{bmatrix}. \tag{6.2.6}$$

ϕ is the gravitational potential and is real and ψ is either real, in which case it is the twist potential, or it is purely imaginary, in which case it has no physical significance. (ρ, z) are cylindrical polar coordinates, the angular coordinate being absent as the system is axially symmetric.

Since $\det \mathbf{P} = 1$,

$$\begin{aligned}
\mathbf{P}^{-1} &= \frac{1}{\phi}\begin{bmatrix} \phi^2+\psi^2 & -\psi \\ -\psi & 1 \end{bmatrix}, \\
\frac{\partial \mathbf{P}}{\partial \rho} &= \frac{1}{\phi^2}\begin{bmatrix} -\phi_\rho & \phi\psi_\rho - \psi\phi_\rho \\ \phi\psi_\rho - \psi\phi_\rho & \phi^2\phi_\rho + 2\phi\psi\psi_\rho - \psi^2\phi_\rho \end{bmatrix}, \\
\frac{\partial \mathbf{P}}{\partial \rho}\mathbf{P}^{-1} &= \frac{\mathbf{M}}{\phi^2}, \\
\frac{\partial \mathbf{P}}{\partial z}\mathbf{P}^{-1} &= \frac{\mathbf{N}}{\phi^2},
\end{aligned}$$

where

$$\mathbf{M} = \begin{bmatrix} -(\phi\phi_\rho + \psi\psi_\rho) & \psi_\rho \\ (\phi^2-\psi^2)\psi_\rho - 2\phi\psi\phi_\rho & \phi\phi_\rho + \psi\psi_\rho \end{bmatrix}$$

and $\mathbf{N}$ is the matrix obtained from $\mathbf{M}$ by replacing ϕ_ρ by ϕ_z and ψ_ρ by ψ_z.

The equation above (6.2.6) can now be expressed in the form

$$\frac{\mathbf{M}}{\rho} - \frac{2}{\phi}(\phi_\rho\mathbf{M} + \phi_z\mathbf{N}) + (\mathbf{M}_\rho + \mathbf{N}_z) = 0 \tag{6.2.7}$$

where

$$\phi_\rho\mathbf{M} + \phi_z\mathbf{N} = \begin{bmatrix} -\left\{ \begin{matrix} \phi(\phi_\rho^2+\phi_z^2) \\ +\psi(\phi_\rho\psi_\rho + \phi_z\psi_z) \end{matrix} \right\} & \{\phi_\rho\psi_\rho + \phi_z\psi_z\} \\ \left\{ \begin{matrix} (\phi^2-\psi^2)(\phi_\rho\psi_\rho + \phi_z\psi_z) \\ -2\phi\psi(\phi_\rho^2+\phi_z^2) \end{matrix} \right\} & \left\{ \begin{matrix} \phi(\phi_\rho^2+\phi_z^2) \\ +\psi(\phi_\rho\psi_\rho + \phi_z\psi_z) \end{matrix} \right\} \end{bmatrix},$$

$$\begin{aligned}
&\mathbf{M}_\rho + \mathbf{N}_z \\
&= \begin{bmatrix} -\left\{ \begin{matrix} \phi(\phi_{\rho\rho}+\phi_{zz}) + \psi(\psi_{\rho\rho}+\psi_{zz}) \\ +\phi_\rho^2+\phi_z^2+\psi_\rho^2+\psi_z^2 \end{matrix} \right\} & \{\psi_{\rho\rho}+\psi_{zz}\} \\ \left\{ \begin{matrix} (\phi^2-\psi^2)(\psi_{\rho\rho}+\psi_{zz}) - 2\phi\psi(\phi_{\rho\rho}+\phi_{zz}) \\ -2\psi(\phi_\rho^2+\phi_z^2+\psi_\rho^2+\psi_z^2) \end{matrix} \right\} & \left\{ \begin{matrix} \phi(\phi_{\rho\rho}+\phi_{zz}) + \psi(\psi_{\rho\rho}+\psi_{zz}) \\ +\phi_\rho^2+\phi_z^2+\psi_\rho^2+\psi_z^2 \end{matrix} \right\} \end{bmatrix}
\end{aligned}$$

The Einstein equations can now be expressed in the form

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = 0,$$

where

$$\begin{aligned}
f_{12} &= \frac{1}{\phi}\left[\phi\left(\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz}\right) - 2(\phi_\rho\psi_\rho + \phi_z\psi_z)\right] = 0, \\
f_{11} &= -\psi f_{12} - \left[\phi\left(\phi_{\rho\rho} + \frac{1}{\rho}\phi_\rho + \phi_{zz}\right) - \phi_\rho^2 - \phi_z^2 + \psi_\rho^2 + \psi_z^2\right] = 0, \\
f_{21} &= (\phi^2-\psi^2)f_{12} - 2\psi\left[\phi\left(\phi_{\rho\rho} + \frac{1}{\rho}\phi_\rho + \phi_{zz}\right) - \phi_\rho^2 - \phi_z^2 + \psi_\rho^2 + \psi_z^2\right] = 0, \\
f_{22} &= -f_{11} = 0,
\end{aligned}$$

which yields only two independent scalar equations, namely

$$\phi\left(\phi_{\rho\rho} + \frac{1}{\rho}\phi_\rho + \phi_{zz}\right) - \phi_\rho^2 - \phi_z^2 + \psi_\rho^2 + \psi_z^2 = 0, \tag{6.2.8}$$

$$\phi\left(\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz}\right) - 2(\phi_\rho\psi_\rho + \phi_z\psi_z) = 0. \tag{6.2.9}$$

The second equation can be rearranged into the form

$$\frac{\partial}{\partial\rho}\left(\frac{\rho\psi_\rho}{\phi^2}\right) + \frac{\partial}{\partial z}\left(\frac{\rho\psi_z}{\phi^2}\right) = 0.$$

Historically, the scalar equations (6.2.8) and (6.2.9) were formulated before the matrix equation (6.2.1), but the modern approach to relativity is to formulate the matrix equation first and to derive the scalar equations from them.

Equations (6.2.8) and (6.2.9) can be contracted into the form

$$\phi\nabla^2\phi - (\nabla\phi)^2 + (\nabla\psi)^2 = 0, \tag{6.2.10}$$

$$\phi\nabla^2\psi - 2\nabla\phi\cdot\nabla\psi = 0, \tag{6.2.11}$$

which can be contracted further into the equations

$$\tfrac{1}{2}(\zeta_+ + \zeta_-)\nabla^2\zeta_\pm = (\nabla\zeta_\pm)^2, \tag{6.2.12}$$

where

$$\begin{aligned} \zeta_+ &= \phi + \omega\psi, \\ \zeta_- &= \phi - \omega\psi \qquad (\omega^2 = -1). \end{aligned} \tag{6.2.13}$$

The notation

$$\begin{aligned} \zeta &= \phi + \omega\psi, \\ \zeta^* &= \phi - \omega\psi, \end{aligned} \tag{6.2.14}$$

where ζ^* is the complex conjugate of ζ, can be used only when ϕ and ψ are real. In that case, the two equations (6.2.12) reduce to the single equation

$$\tfrac{1}{2}(\zeta + \zeta^*)\nabla^2\zeta = (\nabla\zeta)^2. \tag{6.2.15}$$

In 1983, Y. Nakamura conjectured the existence two related infinite sets of solutions of (6.2.8) and (6.2.9). He denoted them by

$$\begin{aligned} &\phi'_n,\ \psi'_n, \quad n \geq 1, \\ &\phi_n,\ \psi_n, \quad n \geq 2, \end{aligned} \tag{6.2.16}$$

and deduced the first few members of each set with the aid of the pair of coupled difference–differential equations given in Appendix A.11 and the Bäcklund transformations β and γ given in Appendix A.12. The general Nakamura solutions were given by Vein in 1985 in terms of cofactors associated with a determinant of arbitrary order whose elements satisfy the difference–differential equations. These solutions are reproduced with minor modifications in Section 6.10.2. In 1986, Kyriakopoulos approached the same problem from another direction and obtained the same determinant in a different form.

The Nakamura–Vein solutions are of great interest mathematically but are not physically significant since, as can be seen from (6.10.21) and (6.10.22), ϕ_n and ψ_n can be complex functions when the elements of B_n are complex. Even when the elements are real, ψ_n and ψ'_n are purely imaginary when n is odd. The Nakamura–Vein solutions are referred to as intermediate solutions.

The Neugebauer family of solutions published in 1980 contains as a particular case the Kerr–Tomimatsu–Sato class of solutions which represent the gravitational field generated by a spinning mass. The Ernst complex potential ξ in this case is given by the formula

$$\xi = F/G \tag{6.2.17}$$

where F and G are determinants of order $2n$ whose column vectors are defined as follows:

In F,

$$\mathbf{C}_j = \left[\tau_j \;\; c_j\tau_j \;\; c_j^2\tau_j \cdots c_j^{n-2}\tau_j \;\; 1 \;\; c_j \;\; c_j^2 \ldots c_j^n\right]_{2n}^T, \tag{6.2.18}$$

and in G,

$$\mathbf{C}_j = \left[\tau_j \;\; c_j\tau_j \;\; c_j^2\tau_j \cdots c_j^{n-1}\tau_j \;\; 1 \;\; c_j \;\; c_j^2 \ldots c_j^{n-1}\right]_{2n}^T, \tag{6.2.19}$$

where

$$\tau_j = e^{\omega\theta_j}\left[\rho^2 + (z + c_j)^2\right]^{\frac{1}{2}} \quad (\omega^2 = -1) \tag{6.2.20}$$

and $1 \le j \le 2n$. The c_j and θ_j are arbitrary real constants which can be specialized to give particular solutions such as the Yamazaki–Hori solutions and the Kerr–Tomimatsu–Sato solutions.

In 1993, Sasa and Satsuma used the Nakamura–Vein solutions as a starting point to obtain physically significant solutions. Their analysis included a study of Vein's quasicomplex symmetric Toeplitz determinant A_n and a related determinant E_n. They showed that A_n and E_n satisfy two equations containing Hirota operators. They then applied these equations to obtain a solution of the Einstein equations and verified with the aid of a computer that their solution is identical with the Neugebauer solution for small values of n. The equations satisfied by A_n and E_n are given as exercises at the end of Section 6.10.2 on the intermediate solutions.

A wholly analytic method of obtaining the Neugebauer solutions is given in Sections 6.10.4 and 6.10.5. It applies determinantal identities and other relations which appear in this chapter and elsewhere to modify the Nakamura–Vein solutions by means of algebraic Bäcklund transformations.

The substitution

$$\zeta = \frac{1-\xi}{1+\xi} \tag{6.2.21}$$

transforms equation (6.2.15) into the Ernst equation, namely

$$(\xi\xi^* - 1)\nabla^2\xi = 2\xi^*(\nabla\xi \cdot \nabla\xi) \tag{6.2.22}$$

which appeared in 1968.

In 1977, M. Yamazaki conjectured and, in 1978, Hori proved that a solution of the Ernst equation is given by

$$\xi_n = \frac{pxu_n - \omega qyv_n}{w_n} \quad (\omega^2 = -1), \tag{6.2.23}$$

where x and y are prolate spheroidal coordinates and u_n, v_n, and w_n are determinants of arbitrary order n in which the elements in the first columns of u_n and v_n are polynomials with complicated coefficients. In 1983, Vein showed that the Yamazaki–Hori solutions can be expressed in the form

$$\xi_n = \frac{pU_{n+1} - \omega qV_{n+1}}{W_{n+1}} \tag{6.2.24}$$

where U_{n+1}, V_{n+1}, and W_{n+1} are bordered determinants of order $n+1$ with comparatively simple elements. These determinants are defined in detail in Section 4.10.3.

Hori's proof of (6.2.23) is long and involved, but no neat proof has yet been found. The solution of (6.2.24) is stated in Section 6.10.6, but since it was obtained directly from (6.2.23) no neat proof is available.

6.2.9 The Relativistic Toda Equation

The relativistic Toda equation, namely

$$\begin{aligned}\ddot{R}_n = {} & \left(1 + \frac{\dot{R}_{n-1}}{c}\right)\left(1 + \frac{\dot{R}_n}{c}\right)\frac{\exp(R_{n-1} - R_n)}{1 + (1/c^2)\exp(R_{n-1} - R_n)} \\ & - \left(1 - \frac{\dot{R}_n}{c}\right)\left(1 + \frac{\dot{R}_{n+1}}{c}\right)\frac{\exp(R_n - R_{n+1})}{1 + (1/c^2)\exp(R_n - R_{n+1})},\end{aligned} \tag{6.2.25}$$

where $\dot{R}_n = dR_n/dt$, etc., was introduced by Rujisenaars in 1990. In the limit as $c \to \infty$, (6.2.25) degenerates into the equation

$$\ddot{R}_n = \exp(R_{n-1} - R_n) - \exp(R_n - R_{n+1}). \tag{6.2.26}$$

The substitution

$$R_n = \log\left\{\frac{U_{n-1}}{U_n}\right\} \tag{6.2.27}$$

reduces (6.2.26) to (6.2.3).

Equation (6.2.25) was solved by Ohta, Kajiwara, Matsukidaira, and Satsuma in 1993. A brief note on the solutions is given in Section 6.11.

6.3 The Dale Equation

Theorem. *The Dale equation, namely*

$$(y'')^2 = y'\left(\frac{y}{x}\right)'\left(\frac{y+4n^2}{1+x}\right)',$$

where n is a positive integer, is satisfied by the function

$$y = 4(c-1)xA_n^{11},$$

where A_n^{11} is a scaled cofactor of the Hankelian $A_n = |a_{ij}|_n$ in which

$$a_{ij} = \frac{x^{i+j-1} + (-1)^{i+j}c}{i+j-1}$$

and c is an arbitrary constant. The solution is clearly defined when $n \geq 2$ but can be made valid when $n = 1$ by adopting the convention $A_{11} = 1$ so that $A^{11} = (x+c)^{-1}$.

PROOF. Using Hankelian notation (Section 4.8),

$$A = |\phi_m|_n, \quad 0 \leq m \leq 2n-2,$$

where

$$\phi_m = \frac{x^{m+1} + (-1)^m c}{m+1}. \tag{6.3.1}$$

Let

$$P = |\psi_m|_n,$$

where

$$\psi_m = (1+x)^{-m-1}\phi_m.$$

Then,

$$\psi'_m = mF\psi_{m-1}$$

(the Appell equation), where

$$F = (1+x)^{-2}. \tag{6.3.2}$$

Hence, by Theorem 4.33 in Section 4.9.1 on Hankelians with Appell elements,

$$\begin{aligned} P' &= \psi'_0 P_{11} \\ &= \frac{(1-c)P_{11}}{(1+x)^2}. \end{aligned} \tag{6.3.3}$$

Note that the theorem cannot be applied to A directly since ϕ_m does not satisfy the Appell equation for any $F(x)$.

Using the identity

$$|t^{i+j-2}a_{ij}|_n = t^{n(n-1)}|a_{ij}|_n,$$

it is found that

$$\begin{aligned} P &= (1+x)^{-n^2}A, \\ P_{11} &= (1+x)^{-n^2+1}A_{11}. \end{aligned} \tag{6.3.4}$$

Hence,

$$(1+x)A' = n^2A - (c-1)A_{11}. \tag{6.3.5}$$

Let

$$\alpha_i = \sum_r x^{r-1}A^{ri}, \tag{6.3.6}$$

$$\beta_i = \sum_r (-1)^r A^{ri}, \tag{6.3.7}$$

$$\begin{aligned} \lambda &= \sum_r (-1)^r \alpha_r \\ &= \sum_r \sum_s (-1)^r x^{s-1}A^{rs} \\ &= \sum_s x^{s-1}\beta_s, \end{aligned} \tag{6.3.8}$$

where r and $s = 1, 2, 3, \ldots, n$ in all sums.

Applying double-sum identity (D) in Section 3.4 with $f_r = r$ and $g_s = s-1$, then (B),

$$\begin{aligned} (i+j-1)A^{ij} &= \sum_r \sum_s [x^{r+s-1} + (-1)^{r+s}c]A^{ri}A^{sj} \\ &= x\alpha_i\alpha_j + c\beta_i\beta_j \end{aligned} \tag{6.3.9}$$

$$\begin{aligned} (A^{ij})' &= -\sum_r \sum_s x^{i+j-2}A^{is}A^{rj} \\ &= -\alpha_i\alpha_j. \end{aligned} \tag{6.3.10}$$

Hence,

$$\begin{aligned} x(A^{ij})' + (i+j-1)A^{ij} &= c\beta_i\beta_j, \\ (x^{i+j-1}A^{ij})' &= c(x^{i-1}\beta_i)(x^{j-1}\beta_j). \end{aligned}$$

In particular,

$$\begin{aligned} (A^{11})' &= -\alpha_1^2, \\ (xA^{11})' &= c\beta_1^2. \end{aligned} \tag{6.3.11}$$

Applying double-sum identities (C) and (A),

$$\begin{aligned}\sum_{r=1}^{n}\sum_{s=1}^{n}[x^{r+s-1}+(-1)^{r+s}c]A^{rs} &= \sum_{r=1}^{n}(2r-1)\\ &= n^2 \end{aligned} \tag{6.3.12}$$

$$\begin{aligned}\frac{xA'}{A} &= \sum_{r=1}^{n}\sum_{s=1}^{n} x^{r+s-1}A^{rs}\\ &= n^2 - c\sum_{r=1}^{n}\sum_{s=1}^{n}(-1)^{r+s}A^{rs}. \end{aligned} \tag{6.3.13}$$

Differentiating and using (6.3.10),

$$\begin{aligned}\left(\frac{xA'}{A}\right)' &= c\sum_{r}^{n}\sum_{s}^{n}(-1)^{r+s}\alpha_r\alpha_s\\ &= c\lambda^2. \end{aligned} \tag{6.3.14}$$

It follows from (6.3.5) that

$$\begin{aligned}\frac{xA'}{A} &= \left[1-\frac{1}{1+x}\right][n^2-(c-1)A^{11}]\\ &= n^2 - \left[\frac{(c-1)xA^{11}+n^2}{1+x}\right]. \end{aligned} \tag{6.3.15}$$

Hence, eliminating xA'/A and using (6.3.14),

$$\left[\frac{(c-1)xA^{11}+n^2}{1+x}\right]' = -c\lambda^2. \tag{6.3.16}$$

Differentiating (6.3.7) and using (6.3.10) and the first equation in (6.3.8),

$$\beta_i' = \lambda\alpha_i. \tag{6.3.17}$$

Differentiating the second equation in (6.3.11) and using (6.3.17),

$$(xA^{11})'' = 2c\lambda\alpha_1\beta_1. \tag{6.3.18}$$

All preparations for proving the theorem are now complete.

Put

$$y = 4(c-1)xA^{11}.$$

Referring to the second equation in (6.3.11),

$$\begin{aligned}y' &= 4(c-1)(xA^{11})'\\ &= 4c(c-1)\beta_1^2. \end{aligned} \tag{6.3.19}$$

Referring to the first equation in (6.3.11),

$$\left(\frac{y}{x}\right)' = 4(c-1)(A^{11})'$$

$$= -4c(c-1)\alpha_1^2. \tag{6.3.20}$$

Referring to (6.3.16),

$$\begin{aligned}\left(\frac{y+4n^2}{1+x}\right)' &= 4\left[\frac{(c-1)xA^{11}+n^2}{1+x}\right]' \\ &= -4c\lambda^2. \end{aligned}\tag{6.3.21}$$

Differentiating (6.3.19) and using (6.3.17),

$$y'' = 8c(c-1)\lambda\alpha_1\beta_1. \tag{6.3.22}$$

The theorem follows from (6.3.19) and (6.3.22). □

6.4 The Kay–Moses Equation

Theorem. *The Kay–Moses equation, namely*

$$\left[D^2 + \varepsilon^2 + 2D^2(\log A)\right]y = 0 \tag{6.4.1}$$

is satisfied by the equation

$$y = e^{-\omega\varepsilon x}\left[1 - \sum_{i,j=1}^{n} \frac{e^{(c_i+c_j)\omega\varepsilon x}A^{ij}}{c_j - 1}\right], \quad \omega^2 = -1,$$

where

$$\begin{aligned} A &= |a_{rs}|_n, \\ a_{rs} &= \delta_{rs}b_r + \frac{e^{(c_r+c_s)\omega\varepsilon x}}{c_r + c_s}. \end{aligned}$$

The b_r, $r \geq 1$, are arbitrary constants and the c_r, $r \geq 1$, are constants such that $c_j \neq 1$, $1 \leq j \leq n$ and $c_r + c_s \neq 0$, $1 \leq r, s \leq n$, but are otherwise arbitrary.

The analysis which follows differs from the original both in the form of the solution and the method by which it is obtained.

PROOF. Let $A = |a_{rs}(u)|_n$ denote the symmetric determinant in which

$$\begin{aligned} a_{rs} &= \delta_{rs}b_r + \frac{e^{(c_r+c_s)u}}{c_r + c_s} = a_{sr}, \\ a'_{rs} &= e^{(c_r+c_s)u}. \end{aligned}\tag{6.4.2}$$

Then the double-sum relations (A)–(D) in Section 3.4 with $f_r = g_r = c_r$ become

$$(\log A)' = \sum_{r,s} e^{(c_r+c_s)u}A^{rs}, \tag{6.4.3}$$

$$(A^{ij})' = -\sum_r e^{c_r u} A^{rj} \sum_s e^{c_s u} A^{is}, \tag{6.4.4}$$

$$2\sum_r b_r c_r A^{rr} + \sum_{r,s} e^{(c_r+c_s)u} A^{rs} = 2\sum_r c_r, \tag{6.4.5}$$

$$2\sum_r b_r c_r A^{ir} A^{rj} + \sum_r e^{c_r u} A^{rj} \sum_s e^{c_s u} A^{is} = (c_i + c_j) A^{ij}. \tag{6.4.6}$$

Put

$$\phi_i = \sum_s e^{c_s u} A^{is}. \tag{6.4.7}$$

Then (6.4.4) and (6.4.6) become

$$(A^{ij})' = -\phi_i \phi_j, \tag{6.4.8}$$

$$2\sum_r b_r c_r A^{ir} A^{rj} + \phi_i \phi_j = (c_i + c_j) A^{ij}. \tag{6.4.9}$$

Eliminating the $\phi_i \phi_j$ terms,

$$(A^{ij})' + (c_i + c_j) A^{ij} = 2\sum_r b_r c_r A^{ir} A^{rj},$$

$$\left[e^{(c_i+c_j)u} A^{ij}\right]' = 2e^{(c_i+c_j)u} \sum_r b_r c_r A^{ir} A^{rj}. \tag{6.4.10}$$

Differentiating (6.4.3),

$$\begin{aligned}(\log A)'' &= \sum_{i,j} \left[e^{(c_i+c_j)u} A^{ij}\right]' \\ &= 2\sum_r b_r c_r \sum_i e^{c_i u} A^{ir} \sum_j e^{c_j u} A^{rj} \\ &= 2\sum_r b_r c_r \phi_r^2. \end{aligned} \tag{6.4.11}$$

Replacing s by r in (6.4.7),

$$\begin{aligned} e^{c_i u} \phi_i &= \sum_r e^{(c_i+c_r)u} A^{ir}, \\ \left(e^{c_j u} \phi_i\right)' &= 2\sum_r b_r c_r \left(e^{c_i u} A^{ir}\right) \sum_j e^{c_j u} A^{rj} \\ &= 2\sum_r b_r c_r \phi_r e^{c_i u} A^{ir}, \\ \phi_i' + c_i \phi_i &= 2\sum_r b_r c_r \phi_r A^{ir}. \end{aligned}$$

Interchange i and r, multiply by $b_r c_r A^{rj}$, sum over r, and refer to (6.4.9):

$$\sum_r b_r c_r A^{rj} (\phi_r' + c_r \phi_r) = 2\sum_i b_i c_i \phi_i \sum_r b_r c_r A^{ir} A^{rj}$$

$$
\begin{aligned}
&= \sum_i b_i c_i \phi_i [(c_i + c_j) A^{ij} - \phi_i \phi_j] \\
&= \sum_r b_r c_r \phi_r [(c_r + c_j) A^{rj} - \phi_r \phi_j],
\end{aligned}
$$

$$\sum_r b_r c_r A^{rj} \phi_r' = \sum_r b_r c_r \phi_r [c_j A^{rj} - \phi_r \phi_j], \tag{6.4.12}$$

$$\sum_r b_r c_r A^{rj} (\phi_r' - \phi_r) = \sum_r b_r c_r \phi_r (c_j - 1) A^{rj} - \sum_r b_r c_r \phi_r^2 \phi_j.$$

Multiply by $e^{c_j u}/(c_j - 1)$, sum over j, and refer to (6.4.7):

$$
\begin{aligned}
\sum_{j,r} \frac{b_r c_r A^{rj} e^{c_j u} (\phi_r' - \phi_r)}{c_j - 1} &= \sum_r b_r c_r \phi_r^2 - \sum_r b_r c_r \phi_r^2 \sum_j \frac{e^{c_j u} \phi_j}{c_j - 1} \\
&= F \sum_r b_r c_r \phi_r^2 \\
&= \tfrac{1}{2} F (\log A)'',
\end{aligned} \tag{6.4.13}
$$

where

$$
\begin{aligned}
F &= 1 - \sum_j \frac{e^{c_j u} \phi_j}{c_j - 1} \\
&= 1 - \sum_{i,j} \frac{e^{(c_i + c_j) u} A^{ij}}{c_j - 1}.
\end{aligned} \tag{6.4.14}
$$

Differentiate and refer to (6.4.9):

$$
\begin{aligned}
F' &= -2 \sum_r b_r c_r \sum_j \frac{e^{c_j u} A^{rj}}{c_j - 1} \sum_i e^{c_i u} A^{ir} \\
&= -2 \sum_r b_r c_r \sum_j \frac{\phi_r e^{c_j u} A^{rj}}{c_j - 1}.
\end{aligned} \tag{6.4.15}
$$

Differentiate again and refer to (6.4.8):

$$
\begin{aligned}
F'' &= 2 \sum_r b_r c_r \sum_j \frac{e^{c_j u}}{c_j - 1} \left[\phi_r^2 \phi_j - c_j \phi_r A^{rj} - \phi_r' A^{rj} \right] \\
&= P - Q - R,
\end{aligned} \tag{6.4.16}
$$

where

$$
\begin{aligned}
P &= 2 \sum_j \frac{e^{c_j u} \phi_j}{c_j - 1} \sum_r b_r c_r \phi_r^2 \\
&= (1 - F)(\log A)''
\end{aligned} \tag{6.4.17}
$$

$$Q = 2 \sum_{j,r} \frac{b_r c_r c_j \phi_r e^{c_j u} A^{rj}}{c_j - 1}$$

$$
\begin{aligned}
&= 2\sum_r b_r c_r \phi_r \sum_j e^{c_j u} A^{rj} + 2\sum_r b_r c_r \sum_j \frac{\phi_r e^{c_j u} A^{rj}}{c_j - 1}\\
&= 2\sum_r b_r c_r \phi_r^2 - F'\\
&= (\log A)'' - F', \qquad (6.4.18)\\
R &= 2\sum_j \frac{e^{c_j u}}{c_j - 1} \sum_r b_r c_r \phi_r' A^{rj}\\
&= 2\sum_j \frac{e^{c_j u}}{c_j - 1} \sum_r b_r c_r \phi_r [c_j A^{rj} - \phi_r \phi_j]\\
&= Q - P. \qquad (6.4.19)
\end{aligned}
$$

Hence, eliminating P, Q, and R from (6.4.16)–(6.4.19),

$$\frac{d^2F}{du^2} - 2\frac{dF}{du} + 2F(\log A)'' = 0. \qquad (6.4.20)$$

Put

$$F = e^u y. \qquad (6.4.21)$$

Then, (6.4.20) is transformed into

$$\frac{d^2y}{du^2} - y + 2y\frac{d^2}{du^2}(\log A) = 0. \qquad (6.4.22)$$

Finally, put $u = \omega\varepsilon x$, $(\omega^2 = -1)$. Then, (6.4.22) is transformed into

$$\frac{d^2y}{dx^2} + \varepsilon^2 y + 2y\frac{d^2}{dx^2}(\log A) = 0,$$

which is identical with (6.4.1), the Kay–Moses equation. This completes the proof of the theorem. □

6.5 The Toda Equations

6.5.1 The First-Order Toda Equation

Define two Hankel determinants (Section 4.8) A_n and B_n as follows:

$$
\begin{aligned}
A_n &= |\phi_m|_n, \quad 0 \le m \le 2n-2,\\
B_n &= |\phi_m|_n, \quad 1 \le m \le 2n-1,\\
A_0 &= B_0 = 1. \qquad (6.5.1)
\end{aligned}
$$

The algebraic identities

$$A_n B_{n+1,n}^{(n+1)} - B_n A_{n+1,n}^{(n+1)} + A_{n+1} B_{n-1} = 0, \qquad (6.5.2)$$

$$B_{n-1} A_{n+1,n}^{(n+1)} - A_n B_{n,n-1}^{(n)} + A_{n-1} B_n = 0 \qquad (6.5.3)$$

are proved in Theorem 4.30 in Section 4.8.5 on Turanians.

Let the elements in both A_n and B_n be defined as

$$\phi_m(x) = f^{(m)}(x), \quad f(x) \text{ arbitrary,}$$

so that

$$\phi'_m = \phi_{m+1} \tag{6.5.4}$$

and both A_n and B_n are Wronskians (Section 4.7) whose derivatives are given by

$$\begin{aligned} A'_n &= -A^{(n+1)}_{n+1,n}, \\ B'_n &= -B^{(n+1)}_{n+1,n}. \end{aligned} \tag{6.5.5}$$

Theorem 6.1. *The equation*

$$u'_n = \frac{u_n u_{n+1}}{u_{n-1}}$$

is satisfied by the function defined separately for odd and even values of n as follows:

$$\begin{aligned} u_{2n-1} &= \frac{A_n}{B_{n-1}}, \\ u_{2n} &= \frac{B_n}{A_n}. \end{aligned}$$

PROOF.

$$\begin{aligned} B^2_{n-1} u'_{2n-1} &= B_{n-1} A'_n - A_n B'_{n-1} \\ &= -B_{n-1} A^{(n+1)}_{n+1,n} + A_n B^{(n)}_{n,n-1} \\ B^2_{n-1} \left(\frac{u_{2n-1} u_{2n}}{u_{2n-2}} \right) &= A_{n-1} B_n. \end{aligned}$$

Hence, referring to (6.5.3),

$$\begin{aligned} B^2_{n-1} \left[\frac{u_{2n-1} u_{2n}}{u_{2n-2}} - u'_{2n-1} \right] &= A_{n-1} B_n + B_{n-1} A^{(n+1)}_{n+1,n} - A_n B^{(n)}_{n,n-1} \\ &= 0, \end{aligned}$$

which proves the theorem when n is odd.

$$\begin{aligned} A^2_n u'_{2n} &= A_n B'_n - B_n A'_n \\ &= -A_n B^{(n+1)}_{n+1,n} + B_n A^{(n+1)}_{n,n+1}, \\ A^2_n \left(\frac{u_{2n} u_{2n+1}}{u_{2n-1}} \right) &= A_{n+1} B_{n-1}. \end{aligned}$$

Hence, referring to (6.5.2),

$$A^2_n \left[\frac{u_{2n} u_{2n+1}}{u_{2n-1}} - u'_{2n} \right] = A_{n+1} B_{n-1} + A_n B^{(n+1)}_{n+1,n} - B_n A^{(n+1)}_{n,n+1}$$

$$= 0,$$

which proves the theorem when n is even. □

Theorem 6.2. *The function*

$$y_n = D(\log u_n), \quad D = \frac{d}{dx},$$

is given separately for odd and even values of n as follows:

$$y_{2n-1} = \frac{A_{n-1}B_n}{A_n B_{n-1}},$$
$$y_{2n} = \frac{A_{n+1}B_{n-1}}{A_n B_n}.$$

PROOF.

$$\begin{aligned} y_{2n-1} &= D\log\left(\frac{A_n}{B_{n-1}}\right) \\ &= \frac{1}{A_n B_{n-1}}\left(B_{n-1}A'_n - A_n B'_{n-1}\right) \\ &= \frac{1}{A_n B_{n-1}}\left[-B_{n-1}A^{(n+1)}_{n+1,n} + A_n B^{(n)}_{n,n-1}\right]. \end{aligned}$$

The first part of the theorem follows from (6.5.3).

$$\begin{aligned} y_{2n} &= D\log\left(\frac{B_n}{A_n}\right) \\ &= \frac{1}{A_n B_n}\left(A_n B'_n - B_n A'_n\right) \\ &= \frac{1}{A_n B_n}\left[-A_n B^{(n+1)}_{n+1,n} + B_n A^{(n+1)}_{n+1,n}\right]. \end{aligned}$$

The second part of the theorem follows from (6.5.2). □

6.5.2 The Second-Order Toda Equations

Theorem 6.3. *The equation*

$$D_x D_y(\log u_n) = \frac{u_{n+1}u_{n-1}}{u_n^2}, \quad D_x = \frac{\partial}{\partial x}, \text{ etc.}$$

is satisfied by the two-way Wronskian

$$u_n = A_n = \left|D_x^{i-1}D_y^{j-1}(f)\right|_n,$$

where the function $f = f(x, y)$ is arbitrary.

PROOF. The equation can be expressed in the form

$$\begin{vmatrix} D_x D_y(A_n) & D_x(A_n) \\ D_y(A_n) & A_n \end{vmatrix} = A_{n+1}A_{n-1}. \tag{6.5.6}$$

The derivative of A_n with respect to x, as obtained by differentiating the rows, consists of the sum of n determinants, only one of which is nonzero. That determinant is a cofactor of A_{n+1}:

$$D_x(A_n) = -A^{(n+1)}_{n,n+1}.$$

Differentiating the columns with respect to y and then the rows with respect to x,

$$D_y(A_n) = -A^{(n+1)}_{n+1,n},$$
$$D_xD_y(A_n) = A^{(n+1)}_{nn}. \tag{6.5.7}$$

Denote the determinant in (6.5.6) by E. Then, applying the Jacobi identity (Section 3.6) to A_{n+1},

$$\begin{aligned} E &= \begin{vmatrix} A^{(n+1)}_{nn} & -A^{(n+1)}_{n,n+1} \\ -A^{(n+1)}_{n+1,n} & A^{(n+1)}_{n+1,n+1} \end{vmatrix} \\ &= A_{n+1}A^{(n+1)}_{n,n+1;n,n+1} \end{aligned}$$

which simplifies to the right side of (6.5.6).

It follows as a corollary that the equation

$$D^2(\log u_n) = \frac{u_{n+1}u_{n-1}}{u_n^2}, \quad D = \frac{d}{dx},$$

is satisfied by the Hankel–Wronskian

$$u_n = A_n = |D^{i+j-2}(f)|_n,$$

where the function $f = f(x)$ is arbitrary. □

Theorem 6.4. *The equation*

$$\frac{1}{\rho}D_\rho\big[\rho D_\rho(\log u_n)\big] = \frac{u_{n+1}u_{n-1}}{u_n^2}, \quad D_\rho = \frac{d}{d\rho},$$

is satisfied by the function

$$u_n = A_n = e^{-n(n-1)x}B_n, \tag{6.5.8}$$

where

$$B_n = \big|(\rho D_\rho)^{i+j-2}f(\rho)\big|_n, \quad f(\rho) \textit{ arbitrary.}$$

PROOF. Put $\rho = e^x$. Then, $\rho D_\rho = D_x$ and the equation becomes

$$D_x^2(\log A_n) = \frac{\rho^2 A_{n+1}A_{n-1}}{A_n^2}. \tag{6.5.9}$$

Applying (6.5.8) to transform this equation from A_n to B_n,

$$\begin{aligned} D_x^2(\log B_n) &= D_x^2(\log A_n) \\ &= \frac{\rho^2 B_{n+1}B_{n-1}}{B_n^2}e^{-[(n+1)n+(n-1)(n-2)-2n(n-1)]x} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\rho^2 B_{n+1} B_{n-1} e^{-2x}}{B_n^2} \\
&= \frac{B_{n+1} B_{n-1}}{B_n^2}\,.
\end{aligned}
$$

This equation is identical in form to the equation in the corollary to Theorem 6.3. Hence,

$$
B_n = \left| D_x^{i+j-2} g(x) \right|_n, \quad g(x) \text{ arbitrary},
$$

which is equivalent to the stated result. □

Theorem 6.5. *The equation*

$$
(D_x^2 + D_y^2) \log u_n = \frac{u_{n+1} u_{n-1}}{u_n^2}
$$

is satisfied by the function

$$
u_n = A_n = \left| D_z^{i-1} D_{\bar{z}}^{j-1}(f) \right|_n,
$$

where $z = \frac{1}{2}(x + iy)$, $\bar{z}$ *is the complex conjugate of* z *and the function* $f = f(z, \bar{z})$ *is arbitrary.*

PROOF.

$$
\begin{aligned}
D_x^2(\log A_n) &= \tfrac{1}{4}\left(D_z^2 + 2D_z D_{\bar{z}} + D_{\bar{z}}^2\right) \log A_n, \\
D_y^2(\log A_n) &= -\tfrac{1}{4}\left(D_z^2 - 2D_z D_{\bar{z}} + D_{\bar{z}}^2\right) \log A_n.
\end{aligned}
$$

Hence, the equation is transformed into

$$
D_z D_{\bar{z}}(\log A_n) = \frac{A_{n+1} A_{n-1}}{A_n^2},
$$

which is identical in form to the equation in Theorem 6.3. The present theorem follows. □

6.5.3 The Milne-Thomson Equation

Theorem 6.6. *The equation*

$$
y_n'(y_{n+1} - y_{n-1}) = n + 1
$$

is satisfied by the function defined separately for odd and even values of n *as follows:*

$$
\begin{aligned}
y_{2n-1} &= \frac{B_{11}^{(n)}}{B_n} = B_n^{11}, \\
y_{2n} &= \frac{A_{n+1}}{A_{11}^{(n+1)}} = \frac{1}{A_{n+1}^{11}},
\end{aligned}
$$

where A_n *and* B_n *are Hankelians defined as*

$$\begin{aligned} A_n &= |\phi_m|_n, \qquad 0 \le m \le 2n-2, \\ B_n &= |\phi_m|_n, \qquad 1 \le m \le 2n-1, \\ \phi'_m &= (m+1)\phi_{m+1}. \end{aligned}$$

PROOF.

$$\begin{aligned} B_{1n}^{(n)} &= (-1)^{n+1} A_{11}^{(n)}, \\ A_{1,n+1}^{(n+1)} &= (-1)^n B_n. \end{aligned} \tag{6.5.10}$$

It follows from Theorems 4.35 and 4.36 in Section 4.9.2 on derivatives of Turanians that

$$\begin{aligned} D(A_n) &= -(2n-1)A_{n+1,n}^{(n+1)}, \\ D(B_n) &= -2nB_{n,n+1}^{(n+1)}, \\ D(A_{11}^{(n)}) &= -(2n-1)A_{1,n+1;1n}^{(n+1)}, \\ D(B_{11}^{(n)}) &= -2nB_{1,n+1;1,n}^{(n+1)}. \end{aligned} \tag{6.5.11}$$

The algebraic identity in Theorem 4.29 in Section 4.8.5 on Turanians is satisfied by both A_n and B_n.

$$\begin{aligned} B_n^2 y'_{2n-1} &= B_n D(B_{11}^{(n)}) - B_{11}^{(n)} D(B_n) \\ &= 2n\bigl[B_{11}^{(n)} B_{n,n+1}^{(n+1)} - B_n B_{1,n+1;1n}^{(n+1)}\bigr] \\ &= 2nB_{1n}^{(n)} B_{1,n+1}^{(n+1)} \\ &= -2nA_{11}^{(n)} A_{11}^{(n+1)}. \end{aligned}$$

Applying the Jacobi identity,

$$\begin{aligned} A_{11}^{(n)} A_{11}^{(n+1)}(y_{2n} - y_{2n-2}) &= A_{n+1}A_{11}^{(n)} - A_n A_{11}^{(n+1)} \\ &= A_{n+1}A_{1,n+1;1,n+1}^{(n+1)} - A_{n+1,n+1}^{n+1} A_{11}^{(n+1)} \\ &= -\bigl[A_{1,n+1}^{(n+1)}\bigr]^2 \\ &= -B_n^2. \end{aligned}$$

Hence,

$$y'_{2n-1}(y_{2n} - y_{2n-2}) = 2n,$$

which proves the theorem when n is odd.

$$\begin{aligned} \bigl[A_{1,n+1}^{(n+1)}\bigr]^2 y'_{2n} &= A_{11}^{(n+1)} D(A_{n+1}) - A_{n+1} D(A_{11}^{(n+1)}) \\ &= (2n+1)\bigl[A_{n+1}A_{1,n+2;1,n+1}^{(n+2)} - A_{11}^{(n+1)} A_{n+2,n+1}^{(n+2)}\bigr] \\ &= -(2n+1)A_{1,n+1}^{(n+1)} A_{1,n+2}^{(n+2)}. \end{aligned}$$

Hence, referring to the first equation in (4.5.10),

$$\begin{aligned}\left[B_{1,n+1}^{(n+1)}\right]^2 y'_{2n} &= (2n+1)B_n B_{n+1},\\ B_n B_{n+1}(y_{2n-1} - y_{2n+1}) &= B_n B_{11}^{(n+1)} - B_{n+1} B_{11}^{(n)}\\ &= B_{n+1,n+1}^{(n+1)} B_{11}^{(n+1)} - B_{n+1} B_{1,n+1;1,n+1}^{(n+1)}\\ &= \left[B_{1,n+1}^{(n+1)}\right]^2.\end{aligned}$$

Hence,

$$y'_{2n}(y_{2n-1} - y_{2n+1}) = 2n+1,$$

which proves the theorem when n is even. □

6.6 The Matsukidaira–Satsuma Equations

6.6.1 A System With One Continuous and One Discrete Variable

Let $A^{(n)}(r)$ denote the Turanian–Wronskian of order n defined as follows:

$$A^{(n)}(r) = \left|f_{r+i+j-2}\right|_n, \tag{6.6.1}$$

where $f_s = f_s(x)$ and $f'_s = f_{s+1}$. Then,

$$\begin{aligned}A_{11}^{(n)}(r) &= A^{(n-1)}(r+2),\\ A_{1n}^{(n)}(r) &= A^{(n-1)}(r+1).\end{aligned}$$

Let

$$\tau_r = A^{(n)}(r). \tag{6.6.2}$$

Theorem 6.7.

$$\begin{vmatrix}\tau_{r+1} & \tau_r\\ \tau_r & \tau_{r-1}\end{vmatrix}\begin{vmatrix}\tau''_r & \tau'_r\\ \tau'_r & \tau_r\end{vmatrix} = \begin{vmatrix}\tau'_{r+1} & \tau_{r+1}\\ \tau'_r & \tau_r\end{vmatrix}\begin{vmatrix}\tau'_r & \tau_r\\ \tau'_{r-1} & \tau_{r-1}\end{vmatrix}$$

for all values of n *and all differentiable functions* $f_s(x)$.

PROOF. Each of the functions

$$\tau_{r\pm 1},\ \tau_{r+2},\ \tau'_r,\ \tau''_r,\ \tau'_{r\pm 1}$$

can be expressed as a cofactor of $A^{(n+1)}$ with various parameters:

$$\begin{aligned}\tau_r &= A_{n+1,n+1}^{(n+1)}(r),\\ \tau_{r+1} &= (-1)^n A_{1,n+1}^{(n+1)}(r)\\ &= (-1)^n A_{n+1,1}^{(n+1)}(r)\\ \tau_{r+2} &= A_{11}^{(n+1)}(r).\end{aligned}$$

Hence applying the Jacobi identity (Section 3.6),

$$\begin{aligned}\begin{vmatrix} \tau_{r+2} & \tau_{r+1} \\ \tau_{r+1} & \tau_r \end{vmatrix} &= \begin{vmatrix} A_{11}^{(n+1)}(r) & (-1)^n A_{1,n+1}^{(n+1)}(r) \\ (-1)^n A_{n+1,1}^{(n+1)}(r) & A_{n+1,n+1}^{(n+1)}(r) \end{vmatrix} \\ &= A^{(n+1)}(r)A^{(n-1)}(r+2).\end{aligned}$$

Replacing r by $r-1$,

$$\begin{vmatrix} \tau_{r+1} & \tau_r \\ \tau_r & \tau_{r-1} \end{vmatrix} = A^{(n+1)}(r-1)A^{(n-1)}(r+1) \tag{6.6.3}$$

$$\begin{aligned}\tau_r' &= -A_{n,n+1}^{(n+1)}(r) \\ &= -A_{n+1,n}^{(n+1)}(r) \\ \tau_r'' &= A_{nn}^{(n+1)}(r).\end{aligned}$$

Hence,

$$\begin{aligned}\begin{vmatrix} \tau_r'' & \tau_r' \\ \tau_r' & \tau_r \end{vmatrix} &= \begin{vmatrix} A_{nn}^{(n+1)}(r) & A_{n,n+1}^{(n+1)}(r) \\ A_{n+1,n}^{(n+1)}(r) & A_{n+1,n+1}^{(n+1)}(r) \end{vmatrix} \\ &= A^{(n+1)}(r)A_{n,n+1;n,n+1}^{(n+1)}(r) \\ &= A^{(n+1)}(r)A^{(n-1)}(r). \end{aligned} \tag{6.6.4}$$

Similarly,

$$\begin{aligned}\tau_{r+1} &= -A_{1,n+1}^{(n+1)}(r) \\ &= -A_{n+1,1}^{(n+1)}(r), \\ \tau_{r+1}' &= (-1)^{n+1}A_{1n}^{(n+1)}(r),\end{aligned}$$

$$\begin{vmatrix} \tau_{r+1}' & \tau_{r+1} \\ \tau_r' & \tau_r \end{vmatrix} = A^{(n+1)}(r)A^{(n-1)}(r+1). \tag{6.6.5}$$

Replacing r by $r-1$,

$$\begin{vmatrix} \tau_r' & \tau_r \\ \tau_{r-1}' & \tau_{r-1} \end{vmatrix} = A^{(n+1)}(r-1)A^{(n-1)}(r). \tag{6.6.6}$$

Theorem 6.7 follows from (6.6.3)–(6.6.6). □

Theorem 6.8.

$$\begin{vmatrix} \tau_r & \tau_{r+1} & \tau_{r+1}' \\ \tau_{r-1} & \tau_r & \tau_r' \\ \tau_{r-1}' & \tau_r' & \tau_r'' \end{vmatrix} = 0.$$

Proof. Denote the determinant by F. Then, Theorem 6.7 can be expressed in the form

$$F_{33}\ F_{11} = F_{31}\ F_{13}.$$

Applying the Jacobi identity,

$$\begin{aligned} F\,F_{13,13} &= \begin{vmatrix} F_{11} & F_{13} \\ F_{31} & F_{33} \end{vmatrix} \\ &= 0. \end{aligned}$$

But $F_{13,13} \neq 0$. The theorem follows. □

Theorem 6.9. *The Matsukidaira–Satsuma equations with one continuous independent variable, one discrete independent variable, and two dependent variables, namely*

a. $q'_r = q_r(u_{r+1} - u_r)$,

b. $\dfrac{u'_r}{u_r - u_{r-1}} = \dfrac{q'_r}{q_r - q_{r-1}}$,

where q_r *and* u_r *are functions of* x, *are satisfied by the functions*

$$\begin{aligned} q_r &= \frac{\tau_{r+1}}{\tau_r}, \\ u_r &= \frac{\tau'_r}{\tau_r} \end{aligned}$$

for all values of n *and all differentiable functions* $f_s(x)$.

PROOF.

$$\begin{aligned} q'_r &= -\frac{F_{31}}{\tau_r^2}, \\ q_r - q_{r-1} &= -\frac{F_{33}}{\tau_{r-1}\tau_r}, \\ u'_r &= \frac{F_{11}}{\tau_r^2}, \\ u_r - u_{r-1} &= \frac{F_{13}}{\tau_{r-1}\tau_r}, \\ u_{r+1} - u_r &= -\frac{F_{31}}{\tau_r\tau_{r+1}}. \end{aligned}$$

Hence,

$$\frac{q'_r}{u_{r+1} - u_r} = \frac{\tau_{r+1}}{\tau_r} = q_r,$$

which proves (a) and

$$\begin{aligned} \frac{u'_r(q_r - q_{r-1})}{q'_r(u_r - u_{r-1})} &= \frac{F_{11}\,F_{33}}{F_{31}\,F_{13}} \\ &= 1, \end{aligned}$$

which proves (b). □

6.6.2 A System With Two Continuous and Two Discrete Variables

Let $A^{(n)}(r,s)$ denote the two-way Wronskian of order n defined as follows:

$$A^{(n)}(r,s) = \left|f_{r+i-1,s+j-1}\right|_n, \tag{6.6.7}$$

where $f_{rs} = f_{rs}(x,y)$, $(f_{rs})_x = f_{r,s+1}$, and $(f_{rs})_y = f_{r+1,s}$.

Let

$$\tau_{rs} = A^{(n)}(r,s). \tag{6.6.8}$$

Theorem 6.10.

$$\begin{aligned}\begin{vmatrix}\tau_{r+1,s} & \tau_{r+1,s-1}\\ \tau_{rs} & \tau_{r,s-1}\end{vmatrix}\begin{vmatrix}(\tau_{rs})_{xy} & (\tau_{rs})_y\\ (\tau_{rs})_x & \tau_{rs}\end{vmatrix}\\ = \begin{vmatrix}(\tau_{rs})_y & (\tau_{r,s-1})_y\\ \tau_{rs} & \tau_{r,s-1}\end{vmatrix}\begin{vmatrix}(\tau_{r+1,s})_x & \tau_{r+1,s}\\ (\tau_{rs})_x & \tau_{rs}\end{vmatrix}\end{aligned}$$

for all values of n and all differentiable functions $f_{rs}(x,y)$.

PROOF.

$$\begin{aligned}\tau_{rs} &= A^{(n+1)}_{n+1,n+1}(r,s),\\ \tau_{r+1,s} &= -A^{(n+1)}_{1,n+1}(r,s),\\ \tau_{r,s+1} &= -A^{(n+1)}_{n+1,1}(r,s),\\ \tau_{r+1,s+1} &= A^{(n+1)}_{11}(r,s).\end{aligned}$$

Hence, applying the Jacobi identity,

$$\begin{aligned}\begin{vmatrix}\tau_{r+1,s+1} & \tau_{r+1,s+1}\\ \tau_{r,s+1} & \tau_{rs}\end{vmatrix} &= A^{(n+1)}(r,s)A^{(n+1)}_{1,n+1;1,n+1}(r,s)\\ &= A^{(n+1)}(r,s)A^{(n-1)}(r+1,s+1).\end{aligned}$$

Replacing s by $s-1$,

$$\begin{vmatrix}\tau_{r+1,s} & \tau_{r+1,s-1}\\ \tau_{rs} & \tau_{r,s-1}\end{vmatrix} = A^{(n+1)}(r,s-1)A^{(n-1)}(r+1,s), \tag{6.6.9}$$

$$\begin{aligned}(\tau_{rs})_x &= -A^{(n+1)}_{n+1,n}(r,s),\\ (\tau_{rs})_y &= -A^{(n+1)}_{n,n+1}(r,s),\\ (\tau_{rs})_{xy} &= A^{(n+1)}_{nn}(r,s).\end{aligned}$$

Hence, applying the Jacobi identity,

$$\begin{aligned}\begin{vmatrix}(\tau_{rs})_{xy} & (\tau_{rs})_y\\ (\tau_{rs})_x & \tau_{rs}\end{vmatrix} &= A^{(n+1)}(r,s)A^{(n+1)}_{n,n+1;n,n+1}(r,s)\\ &= A^{(n+1)}(r,s)A^{(n-1)}(r,s) \qquad (6.6.10)\\ (\tau_{r,s+1})_y &= -A^{(n+1)}_{n1}(r,s).\end{aligned}$$

Hence,

$$\begin{vmatrix} (\tau_{r,s+1})_y & (\tau_{rs})_y \\ \tau_{r,s+1} & \tau_{rs} \end{vmatrix} = A^{(n+1)}(r,s)A^{(n+1)}_{n,n+1;1,n+1}(r,s)$$
$$= A^{(n+1)}(r,s)A^{(n)}_{n1}(r,s)$$
$$= A^{(n+1)}(r,s)A^{(n-1)}(r,s+1).$$

Replacing s by $s-1$,

$$\begin{vmatrix} (\tau_{rs})_y & (\tau_{r,s-1})_y \\ \tau_{rs} & \tau_{r,s-1} \end{vmatrix} = A^{(n+1)}(r,s-1)A^{(n-1)}(r,s), \tag{6.6.11}$$
$$(\tau_{r+1,s})_x = A^{(n+1)}_{1n}(r,s).$$

Hence,

$$\begin{vmatrix} (\tau_{r+1,s})_x & \tau_{r+1,s} \\ (\tau_{rs})_x & \tau_{rs} \end{vmatrix} = A^{(n+1)}(r,s)A^{(n+1)}_{1,n+1;n,n+1}(r,s)$$
$$= A^{(n+1)}(r,s)A^{(n-1)}(r+1,s). \tag{6.6.12}$$

Theorem 6.10 follows from (6.6.9)–(6.6.12). □

Theorem 6.11.

$$\begin{vmatrix} \tau_{r+1,s-1} & \tau_{r,s-1} & (\tau_{r,s-1})_y \\ \tau_{r+1,s} & \tau_{rs} & (\tau_{rs})_y \\ (\tau_{r+1,s})_x & (\tau_{rs})_x & (\tau_{rs})_{xy} \end{vmatrix} = 0.$$

PROOF. Denote the determinant by G. Then, Theorem 6.10 can be expressed in the form

$$G_{33}\ G_{11} = G_{31}\ G_{13}. \tag{6.6.13}$$

Applying the Jacobi identity,

$$G\ G_{13,13} = \begin{vmatrix} G_{11} & G_{13} \\ G_{31} & G_{33} \end{vmatrix}$$
$$= 0.$$

But $G_{13,13} \neq 0$. The theorem follows. □

Theorem 6.12. *The Matsukidaira–Satsuma equations with two continuous independent variables, two discrete independent variables, and three dependent variables, namely*

a. $(q_{rs})_y = q_{rs}(u_{r+1,s} - u_{rs})$,

b. $\dfrac{(u_{rs})_x}{u_{rs} - u_{r,s-1}} = \dfrac{(v_{r+1,s} - v_{rs})q_{rs}}{q_{rs} - q_{r,s-1}}$,

where q_{rs}, u_{rs}, and v_{rs} are functions of x and y, are satisfied by the functions

$$q_{rs} = \frac{\tau_{r+1,s}}{\tau_{rs}},$$

$$u_{rs} = \frac{(\tau_{rs})_y}{\tau_{rs}},$$
$$v_{rs} = \frac{(\tau_{rs})_x}{\tau_{rs}},$$

for all values of n *and all differentiable functions* $f_{rs}(x, y)$.

Proof.

$$\begin{aligned}(q_{rs})_y &= \frac{1}{\tau_{rs}^2}\begin{vmatrix}(\tau_{r+1,s})_y & (\tau_{rs})_y \\ \tau_{r+1,s} & \tau_{rs}\end{vmatrix} \\ &= \frac{\tau_{r+1,s}}{\tau_{rs}}\left[\frac{(\tau_{r+1,s})_y}{\tau_{r+1,s}} - \frac{(\tau_{rs})_y}{\tau_{rs}}\right] \\ &= q_{rs}(u_{r+1,s} - u_{rs}),\end{aligned}$$

which proves (a).

$$\begin{aligned}(u_{rs})_x &= \frac{G_{11}}{\tau_{rs}^2}, \\ v_{r+1,s} - v_{rs} &= -\frac{G_{13}}{\tau_{r+1,s}\tau_{rs}}, \\ u_{rs} - u_{r,s-1} &= \frac{G_{31}}{\tau_{rs}\tau_{r,s-1}}, \\ q_{rs} - q_{r,s-1} &= -\frac{G_{33}}{\tau_{rs}\tau_{r,s-1}}.\end{aligned}$$

Hence, referring to (6.2.13),

$$\begin{aligned}\frac{(q_{rs} - q_{r,s-1})(u_{rs})_x}{q_{rs}(u_{rs} - u_{r,s-1})(v_{r+1,s} - v_{rs})} &= \frac{G_{11}\ G_{33}}{G_{31}\ G_{13}} \\ &= 1,\end{aligned}$$

which proves (b). □

6.7 The Korteweg–de Vries Equation

6.7.1 *Introduction*

The KdV equation is

$$u_t + 6uu_x + u_{xxx} = 0. \tag{6.7.1}$$

The substitution $u = 2v_x$ transforms it into

$$v_t + 6v_x^2 + v_{xxx} = 0. \tag{6.7.2}$$

Theorem 6.13. *The KdV equation in the form (6.7.2) is satisfied by the function*

$$v = D_x(\log A),$$

where

$$\begin{aligned}
A &= |a_{rs}|_n, \\
a_{rs} &= \delta_{rs} e_r + \frac{2}{b_r + b_s} = a_{sr}, \\
e_r &= \exp(-b_r x + b_r^3 t + \varepsilon_r).
\end{aligned}$$

The ε_r *are arbitrary constants and the* b_r *are constants such that the* $b_r + b_s \neq 0$ *but are otherwise arbitrary.*

Two independent proofs of this theorem are given in Sections 6.7.2 and 6.7.3. The method of Section 6.7.2 applies nonlinear differential recurrence relations in a function of the cofactors of A. The method of Section 6.7.3 involves partial derivatives with respect to the exponential functions which appear in the elements of A.

It is shown in Section 6.7.4 that A is a simple multiple of a Wronskian and Section 6.7.5 consists of an independent proof of the Wronskian solution.

6.7.2 The First Form of Solution

First Proof of Theorem 6.1.3. The proof begins by extracting a wealth of information about the cofactors of A by applying the double-sum relations (A)–(D) in Section 3.4 in different ways. Apply (A) and (B) with f' interpreted first as f_x and then as f_t. Apply (C) and (D) first with $f_r = g_r = b_r$, then with $f_r = g_r = b_r^3$. Later, apply (D) with $f_r = -g_r = b_r^2$.

Appling (A) and (B),

$$\begin{aligned}
v = D_x(\log A) &= -\sum_r \sum_s \delta_{rs} b_r e_r A^{rs} \\
&= -\sum_r b_r e_r A^{rr}, \qquad (6.7.3)
\end{aligned}$$

$$D_x(A^{ij}) = \sum_r b_r e_r A^{ir} A^{rj}. \qquad (6.7.4)$$

Applying (C) and (D) with $f_r = g_r = b_r$,

$$\sum_r \sum_s \big[\delta_{rs}(b_r + b_s) e_r + 2\big] A^{rs} = 2 \sum_r b_r,$$

which simplifies to

$$\sum_r b_r e_r A^{rr} + \sum_r \sum_s A^{rs} = \sum_r b_r, \qquad (6.7.5)$$

$$\sum_r b_r e_r A^{ir} A^{rj} + \sum_r \sum_s A^{is} A^{rj} = \tfrac{1}{2}(b_i + b_j) A^{ij}. \qquad (6.7.6)$$

Eliminating the sum common to (6.7.3) and (6.7.5) and the sum common to (6.7.4) and (6.7.6),

$$v = D_x(\log A) = \sum_r \sum_s A^{rs} - \sum_r b_r, \tag{6.7.7}$$

$$D_x(A^{ij}) = \tfrac{1}{2}(b_i + b_j)A^{ij} - \sum_r \sum_s A^{is}A^{rj}. \tag{6.7.8}$$

Returning to (A) and (B),

$$D_t(\log A) = \sum_r b_r^3 e_r A^{rr}, \tag{6.7.9}$$

$$D_t(A^{ij}) = -\sum_r b_r^3 e_r A^{ir} A^{rj}. \tag{6.7.10}$$

Now return to (C) and (D) with $f_r = g_r = b_r^3$.

$$\sum_r b_r^3 e_r A^{rr} + \sum_r \sum_s (b_r^2 - b_r b_s + b_s^2) A^{rs} = \sum_r b_r^3, \tag{6.7.11}$$

$$\begin{aligned}\sum_r b_r^3 e_r A^{ir} A^{rj} &+ \sum_r \sum_s (b_r^2 - b_r b_s + b_s^2) A^{is} A^{rj} \\ &= \tfrac{1}{2}(b_i^3 + b_j^3) A^{ij}.\end{aligned} \tag{6.7.12}$$

Eliminating the sum common to (6.7.9) and (6.7.11) and the sum common to (6.7.10) and (6.7.12),

$$D_t(\log A) = \sum_r b_r^3 - \sum_r \sum_s (b_r^2 - b_r b_s + b_s^2) A^{rs}, \tag{6.7.13}$$

$$D_t(A^{ij}) = \sum_r \sum_s (b_r^2 - b_r b_s + b_s^2) A^{is} A^{rj} - \tfrac{1}{2}(b_i^3 + b_j^3) A^{ij}. \tag{6.7.14}$$

The derivatives v_x and v_t can be evaluated in a convenient form with the aid of two functions ψ_{is} and ϕ_{ij} which are defined as follows:

$$\psi_{is} = \sum_r b_r^i A^{rs}, \tag{6.7.15}$$

$$\begin{aligned}\phi_{ij} &= \sum_s b_s^j \psi_{is}, \\ &= \sum_r \sum_s b_r^i b_s^j A^{rs} \\ &= \phi_{ji}.\end{aligned} \tag{6.7.16}$$

They are definitions of ψ_{is} and ϕ_{ij}. □

Lemma. *The function ϕ_{ij} satisfies the three nonlinear recurrence relations*:

a. $\phi_{i0}\phi_{j1} - \phi_{j0}\phi_{i1} = \frac{1}{2}(\phi_{i+2,j} - \phi_{i,j+2})$,
b. $D_x(\phi_{ij}) = \frac{1}{2}(\phi_{i+1,j} + \phi_{i,j+1}) - \phi_{i0}\phi_{j0}$,

c. $D_t(\phi_{ij}) = \phi_{i0}\phi_{j2} - \phi_{i1}\phi_{j1} + \phi_{i2}\phi_{j0} - \frac{1}{2}(\phi_{i+3,j} + \phi_{i,j+3})$.

PROOF. Put $f_r = -g_r = b_r^2$ in identity (D).

$$\begin{aligned}(b_i^2 - b_j^2)A^{ij} &= \sum_r \sum_s \big[\delta_{rs}(b_r^2 - b_s^2)e_r + 2(b_r - b_s)\big] A^{is}A^{rj} \\ &= 0 + 2\sum_r \sum_s (b_r - b_s)A^{is}A^{rj} \\ &= 2\sum_s A^{is} \sum_r b_r A^{rj} - 2\sum_r A^{rj} \sum_s b_s A^{is} \\ &= 2(\psi_{0i}\psi_{1j} - \psi_{0j}\psi_{1i}).\end{aligned}$$

It follows that if

$$F_{ij} = 2\psi_{0i}\psi_{1j} - b_i^2 A^{ij},$$

then

$$F_{ji} = F_{ij}.$$

Furthermore, if G_{ij} is any function with the property

$$G_{ji} = -G_{ij},$$

then

$$\sum_i \sum_j G_{ij}F_{ij} = 0. \tag{6.7.17}$$

The proof is trivial and is obtained by interchanging the dummy suffixes.

The proof of (a) can now be obtained by expanding the quadruple series

$$S = \sum_{p,q,r,s} (b_p^i b_r^j - b_p^j b_r^i) b_s A^{pq} A^{rs}$$

in two different ways and equating the results.

$$\begin{aligned}S &= \sum_{p,q} b_p^i A^{pq} \sum_{r,s} b_r^j b_s A^{rs} - \sum_{p,q} b_p^j A^{pq} \sum_{r,s} b_r^i b_s A^{rs} \\ &= \phi_{i0}\phi_{j1} - \phi_{j0}\phi_{i1},\end{aligned}$$

which is identical to the left side of (a). Also, referring to (6.7.17) with $i, j \to p, r$,

$$\begin{aligned}S &= \sum_{p,r} (b_p^i b_r^j - b_p^j b_r^i) \sum_q A^{pq} \sum_s b_s A^{rs} \\ &= \sum_{p,r} (b_p^i b_r^j - b_p^j b_r^i)\psi_{0p}\psi_{1r} \\ &= \frac{1}{2}\sum_{p,r} (b_p^i b_r^j - b_p^j b_r^i)(F_{pr} + b_p^2 A^{pr})\end{aligned}$$

$$= 0 + \frac{1}{2}\sum_{p,r}(b_p^{i+2}b_r^j - b_p^{j+2}b_r^i)A^{pr}$$

$$= \frac{1}{2}(\phi_{i+2,j} - \phi_{i,j+2}),$$

which is identical with the right side of (a). This completes the proof of (a).

Referring to (6.7.8) with $r, s \to p, q$ and $i, j \to r, s$,

$$\begin{aligned}
D_x(\phi_{ij}) &= \sum_r \sum_s b_r^i b_s^j D_x(A^{rs}) \\
&= \sum_r \sum_s b_r^i b_s^j \left[\tfrac{1}{2}(b_r + b_s)A^{rs} - \sum_p \sum_q A^{rq}A^{ps} \right] \\
&= \frac{1}{2}\sum_r \sum_s b_r^i b_s^j (b_r + b_s)A^{rs} - \sum_{q,r} b_r^i A^{rq} \sum_{p,s} b_s^j A^{ps} \\
&= \frac{1}{2}(\phi_{i+1,j} + \phi_{i,j+1}) - \phi_{i0}\phi_{j0},
\end{aligned}$$

which proves (b). Part (c) is proved in a similar manner. □

Particular cases of (a)–(c) are

$$\phi_{00}\phi_{11} - \phi_{10}^2 = \tfrac{1}{2}(\phi_{21} - \phi_{03}), \tag{6.7.18}$$

$$\begin{aligned}
D_x(\phi_{00}) &= \phi_{10} - \phi_{00}^2, \\
D_t(\phi_{00}) &= 2\phi_{00}\phi_{20} - \phi_{10}^2 - \phi_{30}.
\end{aligned} \tag{6.7.19}$$

The preparations for finding the derivatives of v are now complete. The formula for v given by (6.7.7) can be written

$$v = \phi_{00} - \text{constant}.$$

Differentiating with the aid of parts (b) and (c) of the lemma,

$$\begin{aligned}
v_x &= \phi_{10} - \phi_{00}^2, \\
v_{xx} &= \tfrac{1}{2}(\phi_{20} + \phi_{11} - 6\phi_{00}\phi_{10} + 4\phi_{00}^3), \\
v_{xxx} &= \tfrac{1}{4}(\phi_{30} + 3\phi_{21} - 8\phi_{00}\phi_{20} - 14\phi_{10}^2, \\
&\qquad +48\phi_{00}^2\phi_{10} - 6\phi_{00}\phi_{11} - 24\phi_{00}^4) \\
v_t &= 2\phi_{00}\phi_{20} - \phi_{10}^2 - \phi_{30}.
\end{aligned} \tag{6.7.20}$$

Hence, referring to (6.7.18),

$$\begin{aligned}
4(v_t + 6v_x^2 + v_{xxx}) &= 3\big[(\phi_{21} - \phi_{30}) - 2(\phi_{00}\phi_{11} - \phi_{10}^2)\big] \\
&= 0,
\end{aligned}$$

which completes the verification of the first form of solution of the KdV equation by means of recurrence relations.

6.7.3 The First Form of Solution, Second Proof

Second Proof of Theorem 6.13. It can be seen from the definition of A that the variables x and t occur only in the exponential functions e_r, $1 \le r \le n$. It is therefore possible to express the derivatives A_x, v_x, A_t, and v_t in terms of partial derivatives of A and v with respect to the e_r.

The basic formulas are as follows.

If

$$y = y(e_1, e_2, \dots, e_n),$$

then

$$\begin{aligned} y_x &= \sum_r \frac{\partial y}{\partial e_r}\frac{\partial e_r}{\partial x} \\ &= -\sum_r b_r e_r \frac{\partial y}{\partial e_r}, \qquad (6.7.21) \\ y_{xx} &= -\sum_s b_s e_s \frac{\partial y_x}{\partial e_s} \\ &= \sum_s b_s e_s \sum_r b_r \frac{\partial}{\partial e_s}\left(e_r \frac{\partial y}{\partial e_r}\right) \\ &= \sum_{r,s} b_r b_s e_s \left[\delta_{rs}\frac{\partial y}{\partial e_r} + e_r \frac{\partial^2 y}{\partial e_r \partial e_s}\right] \\ &= \sum_r b_r^2 e_r \frac{\partial y}{\partial e_r} + \sum_{r,s} b_r b_s e_r e_s \frac{\partial^2 y}{\partial e_r \partial e_s}. \qquad (6.7.22) \end{aligned}$$

Further derivatives of this nature are not required. The double-sum relations (A)–(D) in Section 3.4 are applied again but this time f' is interpreted as a partial derivative with respect to an e_r.

The basic partial derivatives are as follows:

$$\begin{aligned} \frac{\partial e_r}{\partial e_s} &= \delta_{rs}, \qquad (6.7.23) \\ \frac{\partial a_{rs}}{\partial e_m} &= \delta_{rs}\frac{\partial e_r}{\partial e_m} \\ &= \delta_{rs}\delta_{rm}. \qquad (6.7.24) \end{aligned}$$

Hence, applying (A) and (B),

$$\begin{aligned} \frac{\partial}{\partial e_m}(\log A) &= \sum_{r,s} \frac{\partial a_{rs}}{\partial e_m} A^{rs} \\ &= \sum_{r,s} \delta_{rs}\delta_{rm} A^{rs} \\ &= A^{mm} \qquad (6.7.25) \end{aligned}$$

$$\frac{\partial}{\partial e_m}(A^{ij}) = -A^{im}A^{mj}. \tag{6.7.26}$$

Let

$$\psi_p = \sum_s A^{sp}. \tag{6.7.27}$$

Then, (6.7.26) can be written

$$\sum_r b_r e_r A^{ir}A^{rj} = \tfrac{1}{2}(b_i + b_j)A^{ij} - \psi_i\psi_j. \tag{6.7.28}$$

From (6.7.27) and (6.7.26),

$$\begin{aligned}\frac{\partial\psi_p}{\partial e_q} &= -A^{pq}\sum_s A^{sq}\\ &= -\psi_q A^{pq}.\end{aligned} \tag{6.7.29}$$

Let

$$\theta_p = \psi_p^2. \tag{6.7.30}$$

Then,

$$\frac{\partial\theta_p}{\partial e_q} = -2\psi_p\psi_q A^{pq} \tag{6.7.31}$$

$$= \frac{\partial\theta_q}{\partial e_p}, \tag{6.7.32}$$

$$\begin{aligned}\frac{\partial^2\theta_r}{\partial e_p\partial e_q} &= -2\frac{\partial}{\partial e_p}(\psi_q\psi_r A^{qr})\\ &= 2(\psi_p\psi_q A^{pr}A^{qr} + \psi_q\psi_r A^{qp}A^{rp} + \psi_r\psi_p A^{rq}A^{pq}),\end{aligned}$$

which is invariant under a permutation of p, q, and r. Hence, if G_{pqr} is any function with the same property,

$$\sum_{p,q,r} G_{pqr}\frac{\partial^2\theta_r}{\partial e_p\partial e_q} = 6\sum_{p,q,r} G_{pqr}\psi_p\psi_q A^{pr}A^{qr}. \tag{6.7.33}$$

The above relations facilitate the evaluation of the derivatives of v which, from (6.7.7) and (6.7.27) can be written

$$v = \sum_m(\psi_m - b_m).$$

Referring to (6.7.29),

$$\begin{aligned}\frac{\partial v}{\partial e_r} &= -\psi_r\sum_m A^{mr}\\ &= -\psi_r^2\\ &= -\theta_r.\end{aligned} \tag{6.7.34}$$

Hence,

$$\begin{aligned} v_x &= -\sum_r b_r e_r \frac{\partial v}{\partial e_r} \\ &= \sum_r b_r e_r \theta_r. \end{aligned} \tag{6.7.35}$$

Similarly,

$$v_t = -\sum_r b_r^3 e_r \theta_r. \tag{6.7.36}$$

From (6.7.35) and (6.7.23),

$$\begin{aligned} \frac{\partial v_x}{\partial e_q} &= \sum_r b_r \left(\delta_{qr}\theta_r + e_r \frac{\partial \theta_r}{\partial e_q} \right) \\ &= b_q \theta_q + \sum_r b_r e_r \frac{\partial \theta_r}{\partial e_q}. \end{aligned} \tag{6.7.37}$$

Referring to (6.7.32),

$$\begin{aligned} \frac{\partial^2 v_x}{\partial e_p \partial e_q} &= b_q \frac{\partial \theta_q}{\partial e_p} + \sum_r b_r \left(\delta_{pr} \frac{\partial \theta_r}{\partial e_q} + e_r \frac{\partial^2 \theta_r}{\partial e_p \partial e_q} \right) \\ &= (b_p + b_q) \frac{\partial \theta_p}{\partial e_q} + \sum_r b_r e_r \frac{\partial^2 \theta_r}{\partial e_p \partial e_q}. \end{aligned} \tag{6.7.38}$$

To obtain a formula for v_{xxx}, put $y = v_x$ in (6.7.22), apply (6.7.37) with $q \to p$ and $r \to q$, and then apply (6.7.38):

$$\begin{aligned} v_{xxx} &= \sum_p b_p^2 e_p \frac{\partial v_x}{\partial e_p} + \sum_{p,q} b_p b_q e_p e_q \frac{\partial^2 v_x}{\partial e_p \partial e_q} \\ &= \sum_p b_p^2 e_p \left[b_p \theta_p + \sum_q b_q e_q \frac{\partial \theta_q}{\partial e_p} \right] \\ &\quad + \sum_{p,q} b_p b_q e_p e_q \left[(b_p + b_q) \frac{\partial \theta_p}{\partial e_q} + \sum_r b_r e_r \frac{\partial^2 \theta_r}{\partial e_p \partial e_q} \right] \\ &= Q + R + S + T \end{aligned} \tag{6.7.39}$$

where, from (6.7.36), (6.7.32), and (6.7.31),

$$\begin{aligned} Q &= \sum_p b_p^3 e_p \theta_p \\ &= -v_t \\ R &= \sum_{p,q} b_p^2 b_q e_p e_q \frac{\partial \theta_p}{\partial e_q} \end{aligned}$$

$$= -2\sum_{p,q} b_p^2 b_q e_p e_q \psi_p \psi_q A^{pq},$$

$$S = 2R. \tag{6.7.40}$$

Referring to (6.7.33), (6.7.28), and (6.7.35),

$$\begin{aligned}
T &= \sum_{p,q,r} b_p b_q b_r e_p e_q e_r \frac{\partial^2 \theta_r}{\partial e_p \partial e_q} \\
&= 6\sum_{p,q} b_p b_q e_p e_q \psi_p \psi_q \sum_r b_r e_r A^{pr} A^{qr} \\
&= 6\sum_{p,q} b_p b_q e_p e_q \psi_p \psi_q \left[\tfrac{1}{2}(b_p + b_q)A^{pq} - \psi_p \psi_q\right] \\
&= 6\sum_{p,q} b_p^2 b_q e_p e_q \psi_p \psi_q A^{pq} - 6\sum_p b_p e_p \theta_p \sum_q b_q e_q \theta_q \\
&= -(3R + 6v_x^2).
\end{aligned}$$

Hence,

$$\begin{aligned}
v_{xxx} &= -v_t + R + 2R - (3R + 6v_x^2) \\
&= -(v_t + 6v_x^2),
\end{aligned}$$

which completes the verification of the first form of solution of the KdV equation by means of partial derivatives with respect to the exponential functions.

6.7.4 The Wronskian Solution

Theorem 6.14. *The determinant A in Theorem 6.7.1 can be expressed in the form*

$$A = k_n(e_1\ e_2 \cdots e_n)^{1/2} W,$$

where k_n is independent of x and t, and W is the Wronskian defined as follows:

$$W = \left|D_x^{j-1}(\phi_i)\right|_n, \tag{6.7.41}$$

where

$$\phi_i = \lambda_i e_i^{1/2} + \mu_i e_i^{-1/2}, \tag{6.7.42}$$

$$e_i = \exp(-b_i x + b_i^3 t + \varepsilon_i), \tag{6.7.43}$$

$$\lambda_i = \frac{1}{2}\prod_{p=1}^{n}(b_p + b_i),$$

$$\mu_i = \prod_{\substack{p=1 \\ p\neq i}}^{n}(b_p - b_i). \tag{6.7.44}$$

PROOF.

$$D_x^j(\phi_i) = \left(\tfrac{1}{2}b_i\right)^j e_i^{-1/2}[(-1)^j \lambda_i e_i + \mu_i]$$

so that every element in row i of W contains the factor $e_i^{-1/2}$. Removing all these factors from the determinant,

$$\begin{aligned}(e_1\ e_2 \ldots e_n)^{1/2}W \\ = \begin{vmatrix} \lambda_1 e_1 + \mu_1 & \frac{1}{2}b_1(-\lambda_1 e_1 + \mu_1) & (\frac{1}{2}b_1)^2(\lambda_1 e_1 + \mu_1) & \cdots \\ \lambda_2 e_2 + \mu_2 & \frac{1}{2}b_2(-\lambda_2 e_2 + \mu_2) & (\frac{1}{2}b_2)^2(\lambda_2 e_2 + \mu_2) & \cdots \\ \lambda_3 e_3 + \mu_3 & \frac{1}{2}b_3(-\lambda_3 e_3 + \mu_3) & (\frac{1}{2}b_3)^2(\lambda_3 e_3 + \mu_3) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n\end{aligned} \tag{6.7.45}$$

Now remove the fractions from the elements of the determinant by multiplying column j by 2^{j-1}, $1 \le j \le n$, and compensate for the change in the value of the determinant by multiplying the left side by

$$2^{1+2+3\cdots+(n-1)} = 2^{n(n-1)/2}.$$

The result is

$$2^{n(n-1)/2}(e_1\ e_2 \cdots e_n)^{1/2}W = |\alpha_{ij}e_i + \beta_{ij}|_n, \tag{6.7.46}$$

where

$$\begin{aligned}\alpha_{ij} &= (-b_i)^{j-1}\lambda_i, \\ \beta_{ij} &= b_i^{j-1}\mu_i.\end{aligned} \tag{6.7.47}$$

The determinants $|\alpha_{ij}|_n$, $|\beta_{ij}|_n$ are both Vandermondians. Denote them by U_n and V_n, respectively, and use the notation of Section 4.1.2:

$$\begin{aligned}U_n = |\alpha_{ij}|_n &= (\lambda_1\ \lambda_2 \cdots \lambda_n)\left|(-b_i)^{j-1}\right|_n, \\ &= (\lambda_1\ \lambda_2 \cdots \lambda_n)[X_n]_{x_i=-b_i}, \\ V_n &= |\beta_{ij}|_n.\end{aligned} \tag{6.7.48}$$

The determinant on the right-hand side of (6.7.46) is identical in form with the determinant $|a_{ij}x_i + b_{ij}|_n$ which appears in Section 3.5.3. Hence, applying the theorem given there with appropriate changes in the symbols,

$$|\alpha_{ij}e_i + \beta_{ij}|_n = U_n|E_{ij}|_n,$$

where

$$E_{ij} = \delta_{ij}e_i + \frac{K_{ij}^{(n)}}{U_n} \tag{6.7.49}$$

and where $K_{ij}^{(n)}$ is the hybrid determinant obtained by replacing row i of U_n by row j of V_n. Removing common factors from the rows of the determinant,

$$K_{ij}^{(n)} = (\lambda_1\ \lambda_2 \cdots \lambda_n)\frac{\mu_j}{\lambda_i}\left[H_{ij}^{(n)}\right]_{y_i=-x_i=b_i}.$$

Hence, from (6.7.48),

$$\begin{aligned}\frac{K_{ij}^{(n)}}{U_n} &= \frac{\mu_j}{\lambda_i}\left[\frac{H_{ij}^{(n)}}{X_n}\right]_{y_i=-x_i=b_i}\\ &= \frac{\mu_j}{\lambda_i}\,\frac{\prod_{p=1}^{n}(b_p+b_j)}{(b_i+b_j)\prod_{\substack{p=1\\p\neq i}}^{n}(b_p-b_j)}\\ &= \frac{2\lambda_j\mu_j}{(b_i+b_j)\lambda_i\mu_i}\,. \end{aligned} \tag{6.7.50}$$

Hence,

$$|E_{ij}|_n = \left|\delta_{ij}e_i + \frac{2\lambda_j\mu_j}{(b_i+b_j)\lambda_i\mu_i}\right|_n. \tag{6.7.51}$$

Multiply row i of this determinant by $\lambda_i\mu_i$, $1 \le i \le n$, and divide column j by $\lambda_j\mu_j$, $1 \le j \le n$. These operations do not affect the diagonal elements or the value of the determinant but now

$$\begin{aligned}|E_{ij}|_n &= \left|\delta_{ij}e_i + \frac{2}{b_i+b_j}\right|_n\\ &= A. \end{aligned} \tag{6.7.52}$$

It follows from (6.7.46) and (6.7.49) that

$$2^{n(n-1)/2}(e_1\ e_2\cdots e_n)^{1/2}W = U_nA, \tag{6.7.53}$$

which completes the proof of the theorem since U_n is independent of x and t. □

It follows that

$$\log A = \text{constant} + \frac{1}{2}\sum_i(-b_ix + b_i^3t) + \log W. \tag{6.7.54}$$

Hence,

$$u = 2D_x^2(\log A) = 2D_x^2(\log W) \tag{6.7.55}$$

so that solutions containing A and W have equally valid claims to be determinantal solutions of the KdV equation.

6.7.5 Direct Verification of the Wronskian Solution

The substitution

$$u = 2D_x^2(\log w)$$

into the KdV equation yields

$$u_t + 6uu_x + u_{xxx} = 2D_x\left(\frac{F}{w^2}\right), \tag{6.7.56}$$

where

$$F = ww_{xt} - w_x w_t + 3w_{xx}^2 - 4w_x w_{xxx} + ww_{xxxx}.$$

Hence, the KdV equation will be satisfied if

$$F = 0. \tag{6.7.57}$$

Theorem 6.15. *The KdV equation in the form (6.7.56) and (6.7.57) is satisfied by the Wronskian w defined as follows:*

$$w = \left|D_x^{j-1}(\psi_i)\right|_n,$$

where

$$\begin{aligned}\psi_i &= \exp\left(\tfrac{1}{4}b_i^2 z\right)\phi_i,\\ \phi_i &= p_i e_i^{1/2} + q_i e_i^{-1/2},\\ e_i &= \exp(-b_i x + b_i^3 t).\end{aligned}$$

z is independent of x and t but is otherwise arbitrary. b_i, p_i, and q_i are constants.

When $z = 0$, $p_i = \lambda_i$, and $q_i = \mu_i$, then $\psi_i = \phi_i$ and $w = W$ so that this theorem differs little from Theorem 6.14 but the proof of Theorem 6.15 which follows is direct and independent of the proofs of Theorems 6.13 and 6.14. It uses the column vector notation and applies the Jacobi identity.

PROOF. Since

$$(D_t + 4D_x^3)\phi_i = 0,$$

it follows that

$$(D_t + 4D_x^3)\psi_i = 0. \tag{6.7.58}$$

Also

$$(D_z - D_x^2)\psi_i = 0. \tag{6.7.59}$$

Since each row of w contains the factor $\exp\left(\frac{1}{4}b_i^2 z\right)$,

$$w = e^{Bz}W,$$

where

$$W = \left|D_x^{j-1}(\phi_i)\right|_n$$

and is independent of z and

$$B = \tfrac{1}{4}\sum_i b_i^2.$$

Hence, $ww_{zz} - w_z^2 = 0$,

$$\begin{aligned} F &= ww_{xt} - w_x w_t + 3w_{xx}^2 - 4w_x w_{xxx} + ww_{xxxx} + 3(ww_{zz} - w_z^2) \\ &= w\big[(w_t + 4w_{xxx})_x - 3(w_{xxxx} - w_{zz})\big] \\ &\quad - w_x(w_t + 4w_{xxx}) + 3(w_{xx}^2 - w_z^2). \end{aligned} \tag{6.7.60}$$

The evaluation of the derivatives of a Wronskian is facilitated by expressing it in column vector notation.

Let

$$W = \Big|\overset{\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_{n-3}\ \mathbf{C}_{n-2}\ \mathbf{C}_{n-1}\Big|_n, \tag{6.7.61}$$

where

$$\mathbf{C}_j = \big[D_x^j(\psi_1)\ D_x^j(\psi_2) \cdots D_x^j(\psi_n)\big]^T.$$

The significance of the row of dots above the $(n-3)$ columns $\mathbf{C}_0$ to $\mathbf{C}_{n-4}$ will emerge shortly. It follows from (6.7.58) and (6.7.59) that

$$\begin{aligned} D_x(\mathbf{C}_j) &= \mathbf{C}_{j+1}, \\ D_z(\mathbf{C}_j) &= D_x^2(\mathbf{C}_j) = \mathbf{C}_{j+2}, \\ D_t(\mathbf{C}_j) &= -4D_x^3(\mathbf{C}_j) = -4\mathbf{C}_{j+3}. \end{aligned} \tag{6.7.62}$$

Hence, differentiating (6.7.61) and discarding determinants with two identical columns,

$$\begin{aligned} w_x &= \Big|\overset{\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_{n-3}\ \mathbf{C}_{n-2}\ \mathbf{C}_n\Big|_n, \\ w_{xx} &= \Big|\overset{\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_{n-3}\ \mathbf{C}_{n-1}\ \mathbf{C}_n\Big|_n \\ &\quad + \Big|\overset{\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_{n-3}\ \mathbf{C}_{n-2}\ \mathbf{C}_{n+1}\Big|_n, \\ w_z &= \Big|\overset{\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_{n-3}\ \mathbf{C}_n\ \mathbf{C}_{n-1}\Big|_n \\ &\quad + \Big|\overset{\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_{n-3}\ \mathbf{C}_{n-2}\ \mathbf{C}_{n+1}\Big|_n, \end{aligned}$$

etc. The significance of the row of dots above columns $\mathbf{C}_0$ to $\mathbf{C}_{n-4}$ is beginning to emerge. These columns are common to all the determinants which arise in all the derivatives which appear in the second line of (6.7.60). They can therefore be omitted without causing confusion.

Let

$$V_{pqr} = \Big|\overset{\cdots\cdots\cdots\cdots}{\mathbf{C}_0\ \mathbf{C}_1 \cdots \mathbf{C}_{n-4}}\ \ \mathbf{C}_p\ \mathbf{C}_q\ \mathbf{C}_r\Big|_n. \tag{6.7.63}$$

Then, $V_{pqr} = 0$ if p, q, and r are not distinct and $V_{qpr} = -V_{pqr}$, etc. In this notation,

$$\begin{aligned} w &= V_{n-3,n-2,n-1}, \\ w_x &= V_{n-3,n-2,n}, \\ w_{xx} &= V_{n-3,n-1,n} + V_{n-3,n-2,n+1}, \\ w_{xxx} &= V_{n-2,n-1,n} + 2V_{n-3,n-1,n+1} + V_{n-3,n-2,n+2}, \end{aligned}$$

$$\begin{aligned}
w_{xxxx} &= 2V_{n-3,n,n+1} + 3V_{n-3,n-1,n+2} + 3V_{n-2,n-1,n+1} + V_{n-3,n-2,n+3},\\
w_z &= -V_{n-3,n-1,n} + V_{n-3,n-2,n+1},\\
w_{zz} &= 2V_{n-3,n,n+1} - V_{n-3,n-1,n+2} - V_{n-2,n-1,n+1},\\
w_t &= -4(V_{n-2,n-1,n} - V_{n-3,n-1,n+1} + V_{n-3,n-2,n+2}),\\
w_{xt} &= 4(V_{n-3,n,n+1} - V_{n-3,n-2,n+3}).
\end{aligned} \tag{6.7.64}$$

Each of the sections in the second line of (6.7.60) simplifies as follows:

$$\begin{aligned}
w_t + 4w_{xxx} &= 12V_{n-3,n-1,n+1},\\
(w_t + 4w_{xxx})_x &= 12(V_{n-2,n-1,n+1} + V_{n-3,n,n+1} + V_{n-3,n-1,n+2}),\\
w_{xxxx} - w_{zz} &= 4(V_{n-2,n-1,n+1} + V_{n-3,n-1,n+2}),\\
(w_t + 4w_{xxx})_x - 3(w_{xxxx} - w_{zz}) &= 12V_{n-3,n,n+1}\\
w_{xx}^2 - w_z^2 &= 4V_{n-3,n-1,n}V_{n-3,n-2,n+1}.
\end{aligned} \tag{6.7.65}$$

Hence,

$$\begin{aligned}
\frac{1}{12}F = V_{n-3,n-2,n-1}V_{n-3,n,n+1} &+ V_{n-3,n-2,n}V_{n-3,n-1,n+1}\\
&+ V_{n-3,n-1,n}V_{n-3,n-2,n+1}.
\end{aligned} \tag{6.7.66}$$

Let

$$\begin{aligned}
\mathbf{C}_{n+1} &= \big[\alpha_1\ \alpha_2 \ldots \alpha_n\big]^T,\\
\mathbf{C}_{n+2} &= \big[\beta_1\ \beta_2 \ldots \beta_n\big]^T,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_r &= D_x^n(\psi_r)\\
\beta_r &= D_x^{n+1}(\psi_r).
\end{aligned}$$

Then

$$\begin{aligned}
V_{n-3,n-2,n-1} &= A_n,\\
V_{n-3,n-2,n} &= \sum_r \alpha_r A_{rn}^{(n)},\\
V_{n-3,n-1,n+1} &= -\sum_s \beta_s A_{r,n-1}^{(n)},\\
V_{n-3,n-2,n+1} &= \sum_s \beta_s A_{sn}^{(n)},\\
V_{n-3,n-1,n} &= -\sum_r \alpha_r A_{r,n-1}^{(n)},\\
V_{n-3,n,n+1} &= \sum_r \sum_s \alpha_r \beta_s A_{rs;n-1,n}^{(n)}.
\end{aligned} \tag{6.7.67}$$

Hence, applying the Jacobi identity,

$$\frac{1}{12}F = A_n \sum_r \sum_s \alpha_r \beta_s A_{rs;n-1,n}^{(n)} + \sum_r \alpha_r A_{rn}^{(n)} \sum_s \beta_s A_{s,n-1}^{(n)}$$

$$
\begin{aligned}
&-\sum_r \alpha_r A^{(n)}_{r,n-1} \sum_s \beta_s A^{(n)}_{sn} \\
&= \sum_r \sum_s \alpha_r \beta_s \left[A_n A^{(n)}_{rs;n-1,n} - \begin{vmatrix} A^{(n)}_{r,n-1} & A^{(n)}_{rn} \\ A^{(n)}_{s,n-1} & A^{(n)}_{sn} \end{vmatrix} \right] \\
&= 0,
\end{aligned}
$$

which completes the proof of the theorem. □

Exercise. Prove that

$$\log w = k + \log W,$$

where k is independent of x and, hence, that w and W yield the same solution of the KdV equation.

6.8 The Kadomtsev–Petviashvili Equation

6.8.1 The Non-Wronskian Solution

The KP equation is

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{6.8.1}$$

The substitution $u = 2v_x$ transforms it into

$$(v_t + 6v_x^2 + v_{xxx})_x + 3v_{yy} = 0. \tag{6.8.2}$$

Theorem 6.16. *The KP equation in the form (6.8.2) is satisfied by the function*

$$v = D_x(\log A),$$

where

$$
\begin{aligned}
A &= |a_{rs}|_n, \\
a_{rs} &= \delta_{rs} e_r + \frac{1}{b_r + c_s}, \\
e_r &= \exp\bigl[-(b_r + c_r)x + (b_r^2 - c_r^2)y + 4(b_r^3 + c_r^3)t + \varepsilon_r\bigr] \\
&= \exp\bigl[-\lambda_r x + \lambda_r \mu_r y + 4\lambda_r(b_r^2 - b_r c_r + c_r^2)t + \varepsilon_r\bigr], \\
\lambda_r &= b_r + c_r, \\
\mu_r &= b_r - c_r.
\end{aligned}
$$

The ε_r are arbitrary constants and the b_r and c_s are constants such that $b_r + c_s \neq 0$, $1 \le r, s \le n$, but are otherwise arbitrary.

PROOF. The proof consists of a sequence of relations similar to those which appear in Section 6.7 on the KdV equation. Those identities which arise from the double-sum relations (A)–(D) in Section 3.4 are as follows:

Applying (A),

$$v = D_x(\log A) = -\sum_r \lambda_r e_r A^{rr}, \tag{6.8.3}$$

$$D_y(\log A) = \sum_r \lambda_r \mu_r e_r A^{rr}, \tag{6.8.4}$$

$$D_t(\log A) = 4\sum_r \lambda_r (b_r^2 - b_r c_r + c_r^2) e_r A^{rr}. \tag{6.8.5}$$

Applying (B),

$$D_x(A^{ij}) = \sum_r \lambda_r e_r A^{ir} A^{rj}, \tag{6.8.6}$$

$$D_y(A^{ij}) = -\sum_r \lambda_r \mu_r e_r A^{ir} A^{rj}, \tag{6.8.7}$$

$$D_t(A^{ij}) = -4\sum_r \lambda_r (b_r^2 - b_r c_r + c_r^2) e_r A^{ir} A^{rj}. \tag{6.8.8}$$

Applying (C) with

i. $f_r = b_r, \qquad g_r = c_r;$
ii. $f_r = b_r^2, \qquad g_r = -c_r^2;$
iii. $f_r = b_r^3, \qquad g_r = c_r^3;$

in turn,

$$\sum_r \lambda_r e_r A^{rr} + \sum_{r,s} A^{rs} = \sum_r \lambda_r, \tag{6.8.9}$$

$$\sum_r \lambda_r \mu_r e_r A^{rr} + \sum_{r,s} (b_r - c_s) A^{rs} = \sum_r \lambda_r \mu_r, \tag{6.8.10}$$

$$\sum_r \lambda_r (b_r^2 - b_r c_r + c_r^2) e_r A^{rr} + \sum_{r,s} (b_r^2 - b_r c_s + c_s^2) A^{rs} = \sum_r \lambda_r (b_r^2 - b_r c_r + c_r^2). \tag{6.8.11}$$

Applying (D) with (i)–(iii) in turn,

$$\sum_r \lambda_r e_r A^{ir} A^{rj} + \sum_{r,s} A^{is} A^{rj} = (b_i + c_j) A^{ij}, \tag{6.8.12}$$

$$\sum_r \lambda_r \mu_r e_r A^{ir} A^{rj} + \sum_{r,s} (b_r - c_s) A^{is} A^{rj} = (b_i^2 - c_j^2) A^{ij}, \tag{6.8.13}$$

$$\sum_r \lambda_r (b_r^2 - b_r c_r + c_r^2) e_r A^{ir} A^{rj} + \sum_{r,s} (b_r^2 - b_r c_s + c_s^2) A^{is} A^{rj} = (b_i^3 + c_j^3) A^{ij}. \tag{6.8.14}$$

Eliminating the sum common to (6.8.3) and (6.8.9), the sum common to (6.8.4) and (6.8.10) and the sum common to (6.8.5) and (6.8.11), we find new formulae for the derivatives of $\log A$:

$$v = D_x(\log A) = \sum_{r,s} A^{rs} - \sum_r \lambda_r, \tag{6.8.15}$$

$$D_y(\log A) = -\sum_{r,s}(b_r - c_s)A^{rs} + \sum_r \lambda_r\mu_r, \tag{6.8.16}$$

$$\begin{aligned} D_t(\log A) &= -4\sum_{r,s}(b_r^2 - b_r c_s + c_s^2)A^{rs} \\ &\quad +4\sum_r \lambda_r(b_r^2 - b_r c_r + c_r^2). \end{aligned} \tag{6.8.17}$$

Equations (6.8.16) and (6.8.17) are not applied below but have been included for their interest.

Eliminating the sum common to (6.8.6) and (6.8.12), the sum common to (6.8.7) and (6.8.13), and the sum common to (6.8.8) and (6.8.14), we find new formulas for the derivatives of A^{ij}:

$$\begin{aligned} D_x(A^{ij}) &= (b_i + c_j)A^{ij} - \sum_{r,s} A^{is}A^{rj}, \\ D_y(A^{ij}) &= -(b_i^2 - c_j^2)A^{ij} + \sum_{r,s}(b_r - c_s)A^{is}A^{rj}, \\ D_t(A^{ij}) &= -4(b_i^3 + c_j^3)A^{ij} + 4\sum_{r,s}(b_r^2 - b_r c_s + c_s^2)A^{is}A^{rj}. \end{aligned} \tag{6.8.18}$$

Define functions h_{ij}, H_{ij}, and $\overline{H}_{ij}$ as follows:

$$\begin{aligned} h_{ij} &= \sum_{r=1}^{n}\sum_{s=1}^{n} b_r^i c_s^j A^{rs}, \\ H_{ij} &= h_{ij} + h_{ji} = H_{ji}, \\ \overline{H}_{ij} &= h_{ij} - h_{ji} = -\overline{H}_{ji}. \end{aligned} \tag{6.8.19}$$

The derivatives of these functions are found by applying (6.8.18):

$$\begin{aligned} D_x(h_{ij}) &= \sum_{r,s} b_r^i c_s^j \left[(b_r + c_s)A^{rs} - \sum_{p,q} A^{rq}A^{ps}\right] \\ &= \sum_{r,s} b_r^i c_s^j (b_r + c_s)A^{rs} - \sum_{r,q} b_r^i A^{rq} \sum_{p,s} c_s^j A^{ps} \\ &= h_{i+1,j} + h_{i,j+1} - h_{i0}h_{0j}, \end{aligned}$$

which is a nonlinear differential recurrence relation. Similarly,

$$\begin{aligned} D_y(h_{ij}) &= h_{i0}h_{1j} - h_{i1}h_{0j} - h_{i+2,j} + h_{i,j+2}, \\ D_t(h_{ij}) &= 4(h_{i0}h_{2j} - h_{i1}h_{1j} + h_{i2}h_{0j} - h_{i+3,j} - h_{i,j+3}), \\ D_x(H_{ij}) &= H_{i+1,j} + H_{i,j+1} - h_{i0}h_{0j} - h_{0i}h_{j0}, \\ D_y(\overline{H}_{ij}) &= (h_{i0}h_{1j} + h_{0i}h_{j1}) - (h_{i1}h_{0j} + h_{1i}h_{j0}) \end{aligned}$$

$$-H_{i+2,j} + H_{i,j+2}. \tag{6.8.20}$$

From (6.8.15),

$$v = h_{00} - \text{constant}.$$

The derivatives of v can now be found in terms of the h_{ij} and H_{ij} with the aid of (6.8.20):

$$\begin{aligned}
v_x &= H_{10}h_{00}^2, \\
v_{xx} &= H_{20} + H_{11} - 3h_{00}H_{10} + 2h_{00}^3, \\
v_{xxx} &= 12h_{00}^2H_{10} - 3H_{10}^2 - 4h_{00}H_{20} - 3h_{00}H_{11} + 3H_{21} \\
&\quad + H_{30} - 2h_{10}h_{01} - 6h_{00}^4, \\
v_y &= h_{00}\bar{H}_{10} - \bar{H}_{20} \\
v_{yy} &= 2(h_{10}h_{20} + h_{01}h_{02}) - (h_{10}h_{02} + h_{01}h_{20}) \\
&\quad - h_{00}(h_{10}^2 - h_{10}h_{01} + h_{01}^2) + 2h_{00}^2h_{11} \\
&\quad - 2h_{00}H_{21} + H_{22} + h_{00}H_{30} - H_{40}, \\
v_t &= 4(h_{00}H_{20} - h_{10}h_{01} - H_{30}).
\end{aligned} \tag{6.8.21}$$

Hence,

$$v_t + 6v_x^2 + v_{xxx} = 3(h_{10}^2 + h_{01}^2 - h_{00}H_{11} + H_{21} - H_{30}). \tag{6.8.22}$$

The theorem appears after differentiating once again with respect to x. □

6.8.2 The Wronskian Solution

The substitution

$$u = 2D_x^2(\log w)$$

into the KP equation yields

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 2D_x^2\left(\frac{G}{w^2}\right), \tag{6.8.23}$$

where

$$G = ww_{xt} - w_xw_t + 3w_{xx}^2 - 4w_xw_{xxx} + ww_{xxxx} + 3(ww_{yy} - w_y^2).$$

Hence, the KP equation will be satisfied if

$$G = 0. \tag{6.8.24}$$

The function G is identical in form with the function F in the first line of (6.7.60) in the section on the KdV equation, but the symbol y in this section and the symbol z in the KdV section have different origins. In this section, y is one of the three independent variables x, y, and t in the KP equation whereas x and t are the only independent variables in the KdV section and z is introduced to facilitate the analysis.

Theorem. *The KP equation in the form (6.8.2) is satisfied by the Wronskian w defined as follows:*

$$w = \left|D_x^{j-1}(\psi_i)\right|_n,$$

where

$$\begin{aligned}\psi_i &= \exp\left(\tfrac{1}{4}b_i^2 y\right)\phi_i,\\ \phi_i &= p_i e_i^{1/2} + q_i e_i^{-1/2},\\ e_i &= \exp(-b_i x + b_i^3 t)\end{aligned}$$

and b_i, p_i, and q_i are arbitrary functions of i.

The proof is obtained by replacing z by y in the proof of the first line of (6.7.60) with $F = 0$ in the KdV section. The reverse procedure is invalid. If the KP equation is solved first, it is not possible to solve the KdV equation by putting $y = 0$.

6.9 The Benjamin–Ono Equation

6.9.1 Introduction

The notation $\omega^2 = -1$ is used in this section, as i and j are indispensable as row and column parameters.

Theorem. *The Benjamin–Ono equation in the form*

$$A_x A_x^* - \tfrac{1}{2}\left[A^*(A_{xx} + \omega A_t) + A(A_{xx} + \omega A_t)^*\right] = 0, \tag{6.9.1}$$

where A^* is the complex conjugate of A, is satisfied for all values of n by the determinant

$$A = |a_{ij}|_n,$$

where

$$a_{ij} = \begin{cases} \frac{2c_i}{c_i - c_j}, & j \neq i \\ 1 + \omega\theta_i, & j = i \end{cases} \tag{6.9.2}$$

$$\theta_i = c_i x - c_i^2 t - \lambda_i, \tag{6.9.3}$$

and where the c_i are distinct but otherwise arbitrary constants and the λ_i are arbitrary constants.

The proof which follows is a modified version of the one given by Matsuno. It begins with the definitions of three determinants B, P, and Q.

6.9.2 Three Determinants

The determinant A and its cofactors are closely related to the Matsuno determinant E and its cofactor (Section 5.4)

$$\begin{aligned} A &= K_n E, \\ 2c_r A_{rs} &= K_n E_{rs}, \\ 4c_r c_s A_{rs,rs} &= K_n E_{rs,rs}, \end{aligned}$$

where

$$K_n = 2^n \prod_{r=1}^{n} c_r. \tag{6.9.4}$$

The proofs are elementary. It has been proved that

$$\begin{aligned} \sum_{r=1}^{n} E_{rr} &= \sum_{r=1}^{n} \sum_{s=1}^{n} E_{rs}, \\ \sum_{r=1}^{n} \sum_{s=1}^{n} E_{rs,rs} &= -2 \sum_{r=1}^{n} \sum_{s-1}^{n}{}^{\dagger} c_s E_{rs}. \end{aligned}$$

It follows that

$$\sum_{r=1}^{n} c_r A_{rr} = \sum_{r=1}^{n} \sum_{s=1}^{n} c_r A_{rs} \tag{6.9.5}$$

$$\sum_{r=1}^{n} \sum_{s=1}^{n} c_r c_s A_{rs,rs} = -\sum_{r=1}^{n} \sum_{s=1}^{n}{}^{\dagger} c_r c_s A_{rs}. \tag{6.9.6}$$

Define the determinant B as follows:

$$B = |b_{ij}|_n,$$

where

$$b_{ij} = \begin{cases} a_{ij} - 1 \\ \frac{c_i + c_j}{c_i - c_j}, & j \neq i \\ \omega\theta_i, & j = i \quad (\omega^2 = -1). \end{cases} \tag{6.9.7}$$

It may be verified that, for all values of i and j,

$$\begin{aligned} b_{ji} &= -b_{ij}, \qquad j \neq i, \\ b_{ij} - 1 &= -a_{ji}^*, \\ a_{ij}^* - 1 &= -b_{ji}. \end{aligned} \tag{6.9.8}$$

When $j \neq i$, $a_{ij}^* = a_{ij}$, etc.

Notes on bordered determinants are given in Section 3.7. Let P denote the determinant of order $(n+2)$ obtained by bordering A by two rows and two columns as follows:

$$P = \begin{vmatrix} & & & & c_1 & 1 \\ & & & & c_2 & 1 \\ & & & & \cdots & \cdots \\ & & [a_{ij}]_n & & \cdots & \cdots \\ & & & & \cdots & \cdots \\ & & & & c_n & 1 \\ -c_1 & -c_2 & \cdots & -c_n & 0 & 0 \\ -1 & -1 & \cdots & -1 & 0 & 0 \end{vmatrix}_{n+2} \tag{6.9.9}$$

and let Q denote the determinant of order $(n+2)$ obtained by bordering B in a similar manner. Four of the cofactors of P are

$$P_{n+1,n+1} = \begin{vmatrix} & & & & 1 \\ & & & & 1 \\ & & & & \cdots \\ & & [a_{ij}]_n & & \cdots \\ & & & & \cdots \\ & & & & 1 \\ -1 & -1 & \cdots & -1 & 0 \end{vmatrix}_{n+1}, \tag{6.9.10}$$

$$\begin{aligned} P_{n+1,n+2} &= -\begin{vmatrix} & & & & c_1 \\ & & & & c_2 \\ & & & & \cdots \\ & & [a_{ij}]_n & & \cdots \\ & & & & \cdots \\ & & & & c_n \\ -1 & -1 & \cdots & -1 & 0 \end{vmatrix}_{n+1} \\ &= \sum_r \sum_s c_r A_{rs}, \end{aligned} \tag{6.9.11}$$

$$P_{n+2,n+1} = -\begin{vmatrix} & & & & 1 \\ & & & & 1 \\ & & & & \cdots \\ & & [a_{ij}]_n & & \cdots \\ & & & & \cdots \\ & & & & 1 \\ -c_1 & -c_2 & \cdots & -c_n & 0 \end{vmatrix}_{n+1}, \tag{6.9.12}$$

$$P_{n+2,n+2} = \begin{vmatrix} & & & & c_1 \\ & & & & c_2 \\ & & & & \cdots \\ & & [a_{ij}]_n & & \cdots \\ & & & & \cdots \\ & & & & c_n \\ -c_1 & -c_2 & \cdots & -c_n & 0 \end{vmatrix}_{n+1}$$

$$= \sum_r \sum_s c_r c_s A_{rs}. \tag{6.9.13}$$

The determinants A, B, P, and Q, their cofactors, and their complex conjugates are related as follows:

$$\begin{aligned}
B &= Q_{n+1,n+2;n+1,n+2}, && (6.9.14)\\
A &= B + Q_{n+1,n+1}, && (6.9.15)\\
A^* &= (-1)^n (B - Q_{n+1,n+1}), && (6.9.16)\\
P_{n+1,n+2} &= Q_{n+1,n+2}, && (6.9.17)\\
P^*_{n+1,n+2} &= (-1)^{n+1} Q_{n+2,n+1}, && (6.9.18)\\
P_{n+2,n+2} &= Q_{n+2,n+2} + Q, && (6.9.19)\\
P^*_{n+2,n+2} &= (-1)^{n+1}(Q_{n+2,n+2} - Q). && (6.9.20)
\end{aligned}$$

The proof of (6.9.14) is obvious. Equation (6.9.15) can be proved as follows:

$$B + Q_{n+1,n+1} = \begin{vmatrix} & & & & 1 \\ & & & & 1 \\ & & & & \cdots \\ & & [b_{ij}]_n & & \cdots \\ & & & & \cdots \\ & & & & 1 \\ -1 & -1 & \cdots & -1 & 1 \end{vmatrix}_{n+1}. \tag{6.9.21}$$

Note the element 1 in the bottom right-hand corner. The row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \mathbf{R}_{n+1}, \quad 1 \le i \le n, \tag{6.9.22}$$

yield

$$B + Q_{n+1,n+1} = \begin{vmatrix} & & & & 0 \\ & & & & 0 \\ & & & & \cdots \\ & & [b_{ij}+1]_n & & \cdots \\ & & & & \cdots \\ & & & & 0 \\ -1 & -1 & \cdots & -1 & 1 \end{vmatrix}_{n+1}. \tag{6.9.23}$$

Equation (6.9.15) follows by applying (6.9.7) and expanding the determinant by the single nonzero element in the last column. Equation (6.9.16) can be proved in a similar manner. Express $Q_{n+1,n+1} - B$ as a bordered determinant similar to (6.9.21) but with the element 1 in the bottom right-hand corner replaced by -1. The row operations

$$\mathbf{R}'_i = \mathbf{R}_i + \mathbf{R}_{n+1}, \quad 1 \le i \le n, \tag{6.9.24}$$

leave a single nonzero element in the last column. The result appears after applying the second line of (6.9.8).

To prove (6.9.17), perform the row operations (6.9.24) on $P_{n+1,n+2}$ and apply (6.9.7). To prove (6.9.18), perform the same row operations on $P^*_{n+1,n+2}$, apply the third equation in (6.9.8), and transpose the result.

To prove (6.9.19), note that

$$Q + Q_{n+2,n+2} = \begin{vmatrix} & & & & c_1 & 1 \\ & & & & c_2 & 1 \\ & & & & \cdots & \cdots \\ & & [b_{ij}]_n & & \cdots & \cdots \\ & & & & \cdots & \cdots \\ & & & & c_n & 1 \\ -c_1 & -c_2 & \cdots & -c_n & 0 & 0 \\ -1 & -1 & \cdots & -1 & 0 & 1 \end{vmatrix}_{n+2}. \tag{6.9.25}$$

The row operations

$$\mathbf{R}'_i = \mathbf{R}_i - \mathbf{R}_{n+2}, \quad 1 \le i \le n,$$

leave a single nonzero element in the last column. The result appears after applying the second equation in (6.9.7).

To prove (6.9.20), note that $Q - Q_{n+2,n+2}$ can be expressed as a determinant similar to (6.9.25) but with the element 1 in the bottom right-hand corner replaced by -1. The row operations

$$\mathbf{R}'_i = \mathbf{R}_i + \mathbf{R}_{n+2}, \quad 1 \le i \le n,$$

leave a single nonzero element in the last column. The result appears after applying the second equation of (6.9.8) and transposing the result.

6.9.3 Proof of the Main Theorem

Denote the left-hand side of (6.9.1) by F. Then, it is required to prove that $F = 0$. Applying (6.9.3), (6.9.5), (6.9.11), and (6.9.17),

$$\begin{aligned} A_x &= \sum_r \frac{\partial A}{\partial \theta_r} \frac{\partial \theta_r}{\partial x} \\ &= \omega \sum_r c_r A_{rr} \qquad (6.9.26) \\ &= \omega \sum_r \sum_s c_r A_{rs} \\ &= \omega P_{n+1,n+2} \\ &= \omega Q_{n+1,n+2}. \qquad (6.9.27) \end{aligned}$$

Taking the complex conjugate of (6.9.27) and referring to (6.9.18),

$$\begin{aligned} A^*_x &= -\omega P^*_{n+1,n+2} \\ &= (-1)^n \omega Q_{n+2,n+1}. \end{aligned}$$

Hence, the first term of F is given by

$$A_x A_x^* = (-1)^{n+1} Q_{n+1,n+2} Q_{n+2,n+1}. \tag{6.9.28}$$

Differentiating (6.9.26) and referring to (6.9.6),

$$\begin{aligned}
A_{xx} &= \omega \sum_r c_r \frac{\partial A_{rr}}{\partial x} \\
&= \omega \sum_r c_r \sum_s \frac{\partial A_{ss}}{\partial \theta_s} \frac{\partial \theta_s}{\partial x} \\
&= -\sum_r \sum_s c_r c_s A_{rs,rs} \\
&= \sum_r \sum_s{}^{\dagger} c_r c_s A_{rs}, \qquad (6.9.29)\\
A_t &= \sum_r \frac{\partial A}{\partial \theta_r} \frac{\partial \theta_r}{\partial t} \\
&= -\omega \sum_r c_r^2 A_{rr}. \qquad (6.9.30)
\end{aligned}$$

Hence, applying (6.9.13) and (6.9.19),

$$\begin{aligned}
A_{xx} + \omega A_t &= \sum_r \sum_s{}^{\dagger} c_r c_s A_{rs} + \sum_r c_r^2 A_{rr} \\
&= \sum_r \sum_s c_r c_s A_{rs} \\
&= P_{n+2,n+2} \\
&= Q_{n+2,n+2} + Q. \qquad (6.9.31)
\end{aligned}$$

Hence, the second term of F is given by

$$A^*(A_{xx} + \omega A_t) = (-1)^n (B - Q_{n+1,n+1})(Q_{n+2,n+2} + Q). \tag{6.9.32}$$

Taking the complex conjugate of (6.9.31) and applying (6.9.20) and (6.9.15),

$$\begin{aligned}
(A_{xx} + \omega A_t)^* &= P_{n+2,n+2}^* \\
&= (-1)^{n+1}(Q_{n+2,n+2} - Q). \qquad (6.9.33)
\end{aligned}$$

Hence, the third term of F is given by

$$A(A_{xx} + \omega A_t)^* = (-1)^{n+1}(B + Q_{n+1,n+1})(Q_{n+2,n+2} - Q). \tag{6.9.34}$$

Referring to (6.9.14),

$$\begin{aligned}
&\tfrac{1}{2}(-1)^n \left[A^*(A_{xx} + \omega A_t) + A(A_{xx} + \omega A_t)^*\right] \\
&\quad = BQ - Q_{n+1,n+1} Q_{n+2,n+2} \\
&\quad = Q Q_{n+1,n+2;n+1,n+2} - Q_{n+1,n+1} Q_{n+2,n+2}.
\end{aligned}$$

Hence, referring to (6.9.28) and applying the Jacobi identity,

$$\begin{aligned}(-1)^n F &= \begin{vmatrix} Q_{n+1,n+1} & Q_{n+1,n+2} \\ Q_{n+2,n+1} & Q_{n+2,n+2} \end{vmatrix} - QQ_{n+1,n+2;n+1,n+2} \\ &= 0,\end{aligned}$$

which completes the proof of the theorem.

6.10 The Einstein and Ernst Equations

6.10.1 Introduction

This section is devoted to the solution of the scalar Einstein equations, namely

$$\phi\left(\phi_{\rho\rho} + \frac{1}{\rho}\phi_\rho + \phi_{zz}\right) - \phi_\rho^2 - \phi_z^2 + \psi_\rho^2 + \psi_z^2 = 0, \tag{6.10.1}$$

$$\phi\left(\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz}\right) - 2(\phi_\rho\psi_\rho + \phi_z\psi_z) = 0, \tag{6.10.2}$$

but before the theorems can be stated and proved, it is necessary to define a function u_r, three determinants A, B, and E, and to prove some lemmas. The notation $\omega^2 = -1$ is used again as i and j are indispensable as row and column parameters, respectively.

6.10.2 Preparatory Lemmas

Let the function $u_r(\rho, z)$ be defined as any real solution of the coupled equations

$$\frac{\partial u_{r+1}}{\partial \rho} + \frac{\partial u_r}{\partial z} = -\frac{r u_{r+1}}{\rho}, \quad r = 0, 1, 2, \ldots, \tag{6.10.3}$$

$$\frac{\partial u_{r-1}}{\partial \rho} - \frac{\partial u_r}{\partial z} = \frac{r u_{r-1}}{\rho}, \quad r = 1, 2, 3 \ldots, \tag{6.10.4}$$

which are solved in Appendix A.11.

Define three determinants A_n, B_n, and E_n as follows.

$$A_n = |a_{rs}|_n$$

where

$$a_{rs} = \omega^{|r-s|} u_{|r-s|}, \quad (\omega^2 = -1). \tag{6.10.5}$$

$$B_n = |b_{rs}|_n,$$

where

$$b_{rs} = \begin{cases} u_{r-s}, & r \geq s \\ (-1)^{s-r} u_{s-r}, & r \leq s \end{cases}$$

$$b_{rs} = \omega^{s-r} a_{rs}. \tag{6.10.6}$$

$$E_n = |e_{rs}|_n = (-1)^n A^{(n+1)}_{1,n+1} = (-1)^n A^{(n+1)}_{n+1,1}. \tag{6.10.7}$$

In some detail,

$$A_n = \begin{vmatrix} u_0 & \omega u_1 & -u_2 & -\omega u_3 & \cdots \\ \omega u_1 & u_0 & \omega u_1 & -u_2 & \cdots \\ -u_2 & \omega u_1 & u_0 & \omega u_1 & \cdots \\ -\omega u_3 & -u_2 & \omega u_1 & u_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n \quad (\omega^2 = -1), \tag{6.10.8}$$

$$B_n = \begin{vmatrix} u_0 & -u_1 & u_2 & -u_3 & \cdots \\ u_1 & u_0 & -u_1 & u_2 & \cdots \\ u_2 & u_1 & u_0 & -u_1 & \cdots \\ u_3 & u_2 & u_1 & u_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n, \tag{6.10.9}$$

$$E_n = \begin{vmatrix} \omega u_1 & u_0 & \omega u_1 & -u_2 & \cdots \\ -u_2 & \omega u_1 & u_0 & \omega u_1 & \cdots \\ -\omega u_3 & -u_2 & \omega u_1 & u_0 & \cdots \\ u_4 & -\omega u_3 & -u_2 & \omega u_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n \quad (\omega^2 = -1), \tag{6.10.10}$$

$$A_n = (-1)^n E^{(n+1)}_{n+1,1}. \tag{6.10.11}$$

A_n is a symmetric Toeplitz determinant (Section 4.5.2) in which $t_r = \omega^r u_r$. All the elements on and below the principal diagonal of B_n are positive. Those above the principal diagonal are alternately positive and negative.

The notation is simplified by omitting the order n from a determinant or cofactor where there is no risk of confusion. Thus A_n, $A^{(n)}_{ij}$, A^{ij}_n, etc., may appear as A, A_{ij}, A^{ij}, etc. Where the order is not equal to n, the appropriate order is shown explicitly.

A and E, and their simple and scaled cofactors are related by the following identities:

$$\begin{aligned} A_{11} = A_{nn} &= A_{n-1}, \\ A_{1n} = A_{n1} &= (-1)^{n-1} E_{n-1}, \\ E_{p1} &= (-1)^{n-1} A_{pn}, \\ E_{nq} &= (-1)^{n-1} A_{1q}, \\ E_{n1} &= (-1)^{n-1} A_{n-1}, \end{aligned} \tag{6.10.12}$$

$$\left(\frac{A}{E}\right)^2 = \left(\frac{E^{n1}}{A^{11}}\right)^2, \tag{6.10.13}$$

$$E^2 E^{p1} E^{nq} = A^2 A^{pn} A^{1q}. \tag{6.10.14}$$

Lemma 6.17.

$$A = B.$$

Proof. Multiply the rth row of A by ω^{-r}, $1 \le r \le n$ and the sth column by ω^s, $1 \le s \le n$. The effect of these operations is to multiply A by the factor 1 and to multiply the element a_{rs} by ω^{s-r}. Hence, by (6.10.6), A is transformed into B and the lemma is proved. □

Unlike A, which is real, the cofactors of A are not all real. An example is given in the following lemma.

Lemma 6.18.

$$A_{1n} = \omega^{n-1} B_{1n} \quad (\omega^2 = -1).$$

Proof.

$$A_{1n} = (-1)^{n+1} |e_{rs}|_{n-1},$$

where

$$\begin{aligned} e_{rs} &= a_{r+1,s} \\ &= \omega^{|r-s+1|} u_{|r-s+1|} \\ &= a_{r,s-1} \end{aligned}$$

and

$$B_{1n} = (-1)^{n+1} |\beta_{rs}|_{n-1},$$

where

$$\begin{aligned} \beta_{rs} &= b_{r+1,s} \\ &= b_{r,s-1}, \end{aligned}$$

that is,

$$\beta_{rs} = \omega^{s-r-1} e_{rs}.$$

Multiply the rth row of $A_{1n}^{(n)}$ by ω^{-r-1}, $1 \le r \le n-1$ and the sth column by ω^s, $1 \le s \le n-1$. The effect of these operations is to multiply $A_{1n}^{(n)}$ by the factor

$$\omega^{-(2+3+\cdots+n)+(1+2+3+\cdots+\overline{n-1})} = \omega^{1-n}$$

and to multiply the element e_{rs} by ω^{s-r-1}. The lemma follows. □

Both A and B are persymmetric (Hankel) about their secondary diagonals. However, A is also symmetric about its principal diagonal, whereas B is neither symmetric nor skew-symmetric about its principal diagonal. In the analysis which follows, advantage has been taken of the fact that A with its complex elements possesses a higher degree of symmetry than B with its real elements. The expected complicated analysis has been avoided by replacing B and its cofactors by A and its cofactors.

Lemma 6.19.

a. $\dfrac{\partial e_{pq}}{\partial \rho} + \omega \dfrac{\partial a_{pq}}{\partial z} = \left(\dfrac{q-p}{\rho}\right) e_{pq},$

b. $\dfrac{\partial a_{pq}}{\partial \rho} + \omega \dfrac{\partial e_{pq}}{\partial z} = \left(\dfrac{p-q+1}{\rho}\right) a_{pq} \quad (\omega^2 = -1).$

PROOF. If $p \geq q-1$, then, applying (6.10.3) with $r \to p-q$,

$$\begin{aligned}
\left(\frac{\partial}{\partial \rho} + \frac{p-q}{\rho}\right) e_{pq} &= \left(\frac{\partial}{\partial \rho} + \frac{p-q}{\rho}\right) (\omega^{p-q+1} u_{p-q+1}) \\
&= -\frac{\partial}{\partial z} (\omega^{p-q+1} u_{p-q}) \\
&= -\omega \frac{\partial a_{pq}}{\partial z}.
\end{aligned}$$

If $p < q-1$, then, applying (6.10.4) with $r \to q-p$,

$$\begin{aligned}
\left(\frac{\partial}{\partial \rho} + \frac{p-q}{\rho}\right) e_{pq} &= \left(\frac{\partial}{\partial \rho} - \frac{q-p}{\rho}\right) (\omega^{q-p-1} u_{q-p-1}) \\
&= \frac{\partial}{\partial z} (\omega^{q-p-1} u_{q-p}) \\
&= -\omega \frac{\partial a_{pq}}{\partial z},
\end{aligned}$$

which proves (a). To prove (b) with $p \geq q-1$, apply (6.10.4) with $r \to p-q+1$. When $p < q-1$, apply (6.10.3) with $r \to q-p-1$. □

Lemma 6.20.

a. $E^2 \dfrac{\partial E^{n1}}{\partial \rho} + \omega A^2 \dfrac{\partial A^{n1}}{\partial z} = \dfrac{(n-1)E^2 E^{n1}}{\rho},$

b. $A^2 \dfrac{\partial A^{n1}}{\partial \rho} + \omega E^2 \dfrac{\partial E^{n1}}{\partial z} = \dfrac{(n-2)A^2 A^{n1}}{\rho} \quad (\omega^2 = -1).$

PROOF.

$$A = |a_{pq}|_n, \qquad \sum_{p=1}^{n} a_{pq} A^{pr} = \delta_{qr},$$

$$E = |e_{pq}|_n, \qquad \sum_{p=1}^{n} e_{pq} E^{pr} = \delta_{qr}.$$

Applying the double-sum identity (B) (Section 3.4) and (6.10.12),

$$\frac{\partial E^{n1}}{\partial \rho} = -\sum_p \sum_q \frac{\partial e_{pq}}{\partial \rho} E^{p1} E^{nq},$$

$$\frac{\partial A^{n1}}{\partial z} = -\sum_p \sum_q \frac{\partial a_{pq}}{\partial z} A^{pn} A^{1q}$$

$$= -\left(\frac{E}{A}\right)^2 \sum_p \sum_q \frac{\partial a_{pq}}{\partial z} E^{p1} E^{nq}.$$

Hence, referring to Lemma 6.19,

$$\begin{aligned}
\frac{\partial E^{n1}}{\partial \rho} + \omega \left(\frac{A}{E}\right)^2 \frac{\partial A^{n1}}{\partial z} &= -\sum_p \sum_q \left(\frac{\partial e_{pq}}{\partial \rho} + \omega \frac{\partial a_{pq}}{\partial z}\right) E^{pq} E^{nq} \\
&= \frac{1}{\rho} \sum_p \sum_q (p-q) e_{pq} E^{p1} E^{nq} \\
&= \frac{1}{\rho} \left[\sum_p p E^{p1} \sum_q e_{pq} E^{nq} - \sum_q q E^{nq} \sum_p e_{pq} E^{p1}\right] \\
&= \frac{1}{\rho} \left[\sum_p p E^{p1} \delta_{pn} - \sum_q q E^{nq} \delta_{q1}\right] \\
&= \frac{1}{\rho} (nE^{n1} - E^{n1}),
\end{aligned}$$

which is equivalent to (a).

$$\begin{aligned}
\frac{\partial A^{n1}}{\partial \rho} = \frac{\partial A^{1n}}{\partial \rho} &= -\sum_p \sum_q \frac{\partial a_{pq}}{\partial \rho} A^{pn} A^{1q} \\
\frac{\partial E^{n1}}{\partial z} &= -\sum_p \sum_q \frac{\partial e_{pq}}{\partial z} E^{p1} E^{nq} \\
&= -\left(\frac{A}{E}\right)^2 \sum_p \sum_q \frac{\partial e_{pq}}{\partial z} A^{pn} A^{1q}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\partial A^{n1}}{\partial \rho} + \omega \left(\frac{E}{A}\right)^2 \frac{\partial E^{n1}}{\partial z} \\
&\quad = -\sum_p \sum_q \left(\frac{\partial a_{pq}}{\partial \rho} + \omega \frac{\partial e_{pq}}{\partial z}\right) A^{pn} A^{1q} \\
&\quad = -\frac{1}{\rho} \sum_p \sum_q (p-q+1) a_{pq} A^{pn} A^{1q} \\
&\quad = \frac{1}{\rho} \left[\sum_q q A^{1q} \sum_p a_{pq} A^{pn} - \sum_p (p+1) A^{pn} \sum_q a_{pq} A^{1q}\right] \\
&\quad = \frac{1}{\rho} \left[\sum_q q A^{1q} \delta_{qn} - \sum_p (p+1) A^{pn} \delta_{p1}\right] \\
&\quad = \frac{1}{\rho} (nA^{1n} - 2A^{1n}) \quad (A^{1n} = A^{n1}),
\end{aligned}$$

which is equivalent to (b). This completes the proof of Lemma 6.20. □

Exercise. Prove that

$$\left(\omega\frac{\partial}{\partial\rho} - \frac{p-q-1}{\rho}\right) A_n^{pq} = -\frac{A_{n+1,q}^{(n+1)}}{A_n}\frac{\partial A_n^{pn}}{\partial z} + A_n^{1q}\frac{\partial}{\partial z}\left(\frac{A_{p+1,1}^{(n+1)}}{A_n}\right) + \frac{\partial A_n^{p,q-1}}{\partial z},$$

$$\omega\frac{\partial A_n^{pq}}{\partial z} = \frac{A_{n+1,q}^{(n+1)}}{A_n}\left(\frac{\partial}{\partial\rho} - \frac{n}{\rho}\right) A_n^{pq} - A_n^{1q}\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\frac{A_{p+1,1}^{(n+1)}}{A_n}$$
$$-\left(\frac{\partial}{\partial\rho} - \frac{q-1}{\rho}\right) A_n^{p,q-1} - \left(\frac{p+1}{\rho}\right) A_n^{p+1,q}$$
$$(\omega^2 = -1).$$

Note that some cofactors are scaled but others are unscaled. Hence, prove that

$$\left(\omega\frac{\partial}{\partial\rho} - \frac{n-2}{\rho}\right)\frac{E_{n-1}}{A_n} = \frac{E_n}{A_n}\frac{\partial}{\partial z}\left(\frac{A_{n-1}}{A_n}\right) - \frac{A_{n-1}}{A_n}\frac{\partial}{\partial z}\left(\frac{E_n}{A_n}\right),$$

$$\omega\frac{\partial}{\partial z}\left(\frac{E_{n-1}}{A_n}\right) = (-1)^n\frac{E_n}{A_n}\left(\frac{\partial}{\partial\rho} - \frac{n}{\rho}\right)\frac{E_{n-1}}{A_n}$$
$$+\frac{A_{n-1}}{A_n}\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\frac{E_n}{A_n}.$$

6.10.3 The Intermediate Solutions

The solutions given in this section are not physically significant and are called intermediate solutions. However, they are used as a starting point in Section 6.10.5 to obtain physically significant solutions.

Theorem. *Equations (6.10.1) and (6.10.2) are satisfied by the function pairs* $P_n(\phi_n, \psi_n)$ *and* $P_n'(\phi_n', \psi_n')$, *where*

a. $\phi_n = \dfrac{\rho^{n-2}A_{n-1}}{A_{n-2}} = \dfrac{\rho^{n-2}}{A_{n-1}^{11}}$,

b. $\psi_n = \dfrac{\omega\rho^{n-2}E_{n-1}}{A_{n-2}} = \dfrac{(-1)^n\omega\rho^{n-2}}{E_{n-1}^{n-1,1}} = \dfrac{(-1)^{n-1}\omega\rho^{n-2}A_{1n}}{A_{n-2}}$,

c. $\phi_n' = \dfrac{A^{11}}{\rho^{n-2}}$,

d. $\psi_n' = \dfrac{(-1)^n\omega A^{1n}}{\rho^{n-2}} \quad (\omega^2 = -1).$

The first two formulas are equivalent to the pair $P_{n+1}(\phi_{n+1}, \psi_{n+1})$, *where*

e. $\phi_{n+1} = \dfrac{\rho^{n-1}}{A^{11}}$,

f. $\psi_{n+1} = \dfrac{(-1)^{n+1}\omega\rho^{n-1}}{E^{n1}}$.

PROOF. The proof is by induction and applies the Bäcklund transformation theorems which appear in Appendix A.12 where it is proved that if $P(\phi, \psi)$ is a solution and

$$\begin{aligned}\phi' &= \frac{\phi}{\phi^2+\psi^2},\\ \psi' &= -\frac{\psi}{\phi^2+\psi^2},\end{aligned} \tag{6.10.15}$$

then $P'(\phi', \psi')$ is also a solution. Transformation β states that if $P(\phi, \psi)$ is a solution and

$$\begin{aligned}\phi' &= \frac{\rho}{\phi},\\ \frac{\partial \psi'}{\partial \rho} &= -\frac{\omega\rho}{\phi^2}\frac{\partial \psi}{\partial z},\\ \frac{\partial \psi'}{\partial z} &= \frac{\omega\rho}{\phi^2}\frac{\partial \psi}{\partial \rho} \quad (\omega^2 = -1),\end{aligned} \tag{6.10.16}$$

then $P'(\phi', \psi')$ is also a solution. The theorem can therefore be proved by showing that the application of transformation γ to P_n gives P'_n and that the application of Transformation β to P'_n gives P_{n+1}.

Applying the Jacobi identity (Section 3.6) to the cofactors of the corner elements of A,

$$A_{n+1}^2 - A_{1n}^2 = A_n A_{n-2}. \tag{6.10.17}$$

Hence, referring to (6.10.15),

$$\begin{aligned}\phi_n^2 + \psi_n^2 &= \left(\frac{\rho^{n-2}}{A_{n-2}}\right)^2 \left(A_{n-1}^2 - E_{n-1}^2\right)\\ &= \left(\frac{\rho^{n-2}}{A_{n-2}}\right)^2 \left(A_{n-1}^2 - A_{1n}^2\right)\\ &= \frac{\rho^{2n-4}A_n}{A_{n-2}},\end{aligned} \tag{6.10.18}$$

$$\begin{aligned}\frac{\phi_n}{\phi_n^2+\psi_n^2} &= \frac{A_{n-1}}{\rho^{2n-2}A_n} \quad (A_{n-1} = A_{11})\\ &= \frac{A^{11}}{\rho^{2n-2}}\\ &= \phi'_n,\\ \frac{\psi_n}{\phi_n^2+\psi_n^2} &= \frac{\omega E_{n-1}}{\rho^{2n-2}A_n}\\ &= \frac{(-1)^{n-1}\omega A^{1n}}{\rho^{n-2}}\\ &= -\psi'_n.\end{aligned}$$

Hence, the application of transformation γ to P_n gives P_n'.

In order to prove that the application of transformation β to P_n' gives P_{n+1}, it is required to prove that

$$\phi_{n+1} = \frac{\rho}{\phi_n'},$$

which is obviously satisfied, and

$$\begin{aligned}\frac{\partial \psi_{n+1}}{\partial \rho} &= -\frac{\omega\rho}{(\phi_n')^2}\frac{\partial \psi_n'}{\partial z}\\ \frac{\partial \psi_{n+1}}{\partial z} &= \frac{\omega\rho}{(\phi_n')^2}\frac{\partial \psi_n'}{\partial \rho},\end{aligned} \tag{6.10.19}$$

that is,

$$\begin{aligned}\frac{\partial}{\partial \rho}\left[\frac{(-1)^{n+1}\omega\rho^{n-1}}{E^{n1}}\right] &= -\omega\rho\left[\frac{\rho^{n-2}}{A^{11}}\right]^2\frac{\partial}{\partial z}\left[\frac{(-1)^n\omega A^{1n}}{\rho^{n-2}}\right],\\ \frac{\partial}{\partial z}\left[\frac{(-1)^{n+1}\omega\rho^{n-1}}{E^{n1}}\right] &= \omega\rho\left[\frac{\rho^{n-2}}{A^{11}}\right]^2\frac{\partial}{\partial \rho}\left[\frac{(-1)^n\omega A^{1n}}{\rho^{n-2}}\right]\\ &\qquad (\omega^2 = -1).\end{aligned} \tag{6.10.20}$$

But when the derivatives of the quotients are expanded, these two relations are found to be identical with the two identities in Lemma 6.10.4 which have already been proved. Hence, the application of transformation β to P_n' gives P_{n+1} and the theorem is proved. □

The solutions of (6.10.1) and (6.10.2) can now be expressed in terms of the determinant B and its cofactors. Referring to Lemmas 6.17 and 6.18,

$$\begin{aligned}\phi_n &= \frac{\rho^{n-2}B_{n-1}}{B_{n-2}},\\ \psi_n &= -\frac{(-\omega)^n\rho^{n-2}B_{1n}}{B_{n-2}} \quad (\omega^2=-1),\ n \ge 3,\end{aligned} \tag{6.10.21}$$

$$\begin{aligned}\phi_n' &= \frac{B_{n-1}}{\rho^{n-2}B_n},\\ \psi_n' &= \frac{(-\omega)^n B_{1n}}{\rho^{n-2}B_n}, \quad n \ge 2.\end{aligned} \tag{6.10.22}$$

The first few pairs of solutions are

$$\begin{aligned}P_1'(\phi,\psi) &= \left(\frac{\rho}{u_0}, \frac{-\omega\rho}{u_0}\right),\\ P_2(\phi,\psi) &= (u_0, -u_1),\\ P_2'(\phi,\psi) &= \left(\frac{u_0}{u_0^2+u_1^2}, \frac{u_1}{u_0^2+u_1^2}\right),\\ P_3(\phi,\psi) &= \left(\frac{\rho(u_0^2+u_1^2)}{u_0}, \frac{\omega\rho(u_0u_2-u_1^2)}{u_0}\right).\end{aligned} \tag{6.10.23}$$

Exercise. The one-variable Hirota operators H_x and H_{xx} are defined in Section 5.7 and the determinants A_n and E_n, each of which is a function of ρ and z, are defined in (6.10.8) and (6.10.10). Apply Lemma 6.20 to prove that

$$\begin{aligned}
H_\rho(A_{n-1}, E_n) - \omega H_z(A_n, E_{n-1}) &= \left(\frac{n-1}{\rho}\right) A_{n-1}E_n, \\
H_\rho(A_n, E_{n-1}) - \omega H_z(A_{n-1}, E_n) &= -\left(\frac{n-2}{\rho}\right) A_n E_{n-1} \quad (\omega^2 = -1).
\end{aligned}$$

Using the notation

$$K^2(f,g) = \left(H_{\rho\rho} + \frac{1}{\rho}H_\rho + H_{zz}\right)(f,g),$$

where $f = f(\rho, z)$ and $g = g(\rho, z)$, prove also that

$$\begin{aligned}
K^2(E_n, A_n) &= \frac{n(n-2)}{\rho^2} E_n A_n, \\
\left\{K^2 + \frac{2n-4}{\rho}\right\}(A_n, A_{n-1}) &= -\frac{1}{\rho^2} A_n A_{n-1}, \\
K^2\left\{\rho^{n(n-2)/2}E_n, \rho^{n(n-2)/2}A_n\right\} &= 0, \\
K^2\left\{\rho^{(n^2-4n+2)/2}A_{n-1}, \rho^{n(n-2)/2}A_n\right\} &= 0, \\
K^2\left\{\rho^{(n^2-2)/2}A_{n+1}, \rho^{n(n-2)/2}A_n\right\} &= 0.
\end{aligned}$$

(Sasa and Satsuma)

6.10.4 Preparatory Theorems

Define a Vandermondian (Section 4.1.2) $V_{2n}(\mathbf{x})$ as follows:

$$\begin{aligned}
V_{2n}(\mathbf{x}) &= \left|x_i^{j-1}\right|_{2n} \\
&= V(x_1, x_2, \ldots, x_{2n}),
\end{aligned} \tag{6.10.24}$$

and let the (unsigned) minors of $V_{2n}(\mathbf{c})$ be denoted by $M_{ij}^{(2n)}(\mathbf{c})$. Also, let

$$\begin{aligned}
M_i(\mathbf{c}) &= M_{i,2n}^{(2n)}(\mathbf{c}) = V(c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{2n}), \\
M_{2n}(\mathbf{c}) &= M_{2n,2n}^{(2n)}(\mathbf{c}) = V_{2n-1}(\mathbf{c}).
\end{aligned} \tag{6.10.25}$$

$$\begin{aligned}
x_j &= \frac{z + c_j}{\rho}, \\
\varepsilon_j &= e^{\omega\theta_j}\sqrt{1 + x_j^2} \quad (\omega^2 = -1) \\
&= \frac{\tau_j}{\rho},
\end{aligned} \tag{6.10.26}$$

where τ_j is a function which appears in the Neugebauer solution and is defined in (6.2.20).

$$w_r = \sum_{j=1}^{2n} \frac{(-1)^{j-1} M_j(\mathbf{c}) x_j^r}{\varepsilon_j^*}. \tag{6.10.27}$$

Then,

$$\begin{aligned} x_i - x_j &= \frac{c_i - c_j}{\rho}, \quad \text{independent of } z, \\ \varepsilon_j \varepsilon_j^* &= 1 + x_j^2. \end{aligned} \tag{6.10.28}$$

Now, let $H_{2n}^{(m)}(\varepsilon)$ denote the determinant of order $2n$ whose column vectors are defined as follows:

$$\mathbf{C}_j^{(m)}(\varepsilon) = \left[\varepsilon_j \;\; c_j\varepsilon_j \;\; c_j^2\varepsilon_j \cdots c_j^{m-1}\varepsilon_j \;\; 1 \;\; c_j \;\; c_j^2 \cdots c_j^{2n-m-1}\right]_{2n}^T, \quad 1 \le j \le 2n. \tag{6.10.29}$$

Hence,

$$\begin{aligned} \mathbf{C}_j^{(m)}\left(\frac{1}{\varepsilon}\right) &= \left[\frac{1}{\varepsilon_j} \;\; \frac{c_j}{\varepsilon_j} \;\; \frac{c_j^2}{\varepsilon_j} \cdots \frac{c_j^{m-1}}{\varepsilon_j} \;\; 1 \;\; c_j \;\; c_j^2 \cdots c_j^{2n-m-1}\right]_{2n}^T \qquad (6.10.30) \\ &= \frac{1}{\varepsilon_j}\left[1 \;\; c_j \;\; c_j^2 \cdots c_j^{m-1} \;\; \varepsilon_j \;\; c_j\varepsilon_j \;\; c_j^2\varepsilon_j \cdots c_j^{2n-m-1}\varepsilon_j\right]_{2n}^T. \end{aligned}$$

But,

$$\mathbf{C}_j^{(2n-m)}(\varepsilon) = \left[\varepsilon_j \;\; c_j\varepsilon_j \;\; c_j^2\varepsilon_j \cdots c_j^{2n-m-1}\varepsilon_j \;\; 1 \; c_j \; c_j^2 \cdots c_j^{m-1}\right]_{2n}^T. \tag{6.10.31}$$

The elements in the last column vector are a cyclic permutation of the elements in the previous column vector. Hence, applying Property (c(i)) in Section 2.3.1 on the cyclic permutation of columns (or rows, as in this case),

$$\begin{aligned} H_{2n}^{(m)}\left(\frac{1}{\varepsilon}\right) &= (-1)^{m(2n-1)} \left(\prod_{j=1}^{2n} \varepsilon_j\right)^{-1} H_{2n}^{(2n-m)}(\varepsilon), \\ \frac{H_{2n}^{(n+1)}(1/\varepsilon)}{H_{2n}^{(n)}(1/\varepsilon)} &= -\frac{H_{2n}^{(n-1)}(\varepsilon)}{H_{2n}^{(n)}(\varepsilon)}. \end{aligned} \tag{6.10.32}$$

Theorem.

a. $|w_{i+j-2} + w_{i+j}|_m = (-\rho^2)^{-m(m-1)/2}\{V_{2n}(\mathbf{c})\}^{m-1} H_{2n}^{(m)}(\varepsilon)$,

b. $|w_{i+j-2}|_m = (-\rho^2)^{-m(m-1)/2}\{V_{2n}(\mathbf{c})\}^{m-1} H_{2n}^{(m)}\left(\frac{1}{\varepsilon^*}\right)$.

The determinants on the left are Hankelians.

PROOF. *Proof of (a).* Denote the determinant on the left by W_m.

$$w_{i+j-2} + w_{i+j} = \sum_{k=1}^{2n} y_k x_k^{i+j-2},$$

where

$$y_k = (-1)^{k+1}\varepsilon_k M_k(\mathbf{c}). \tag{6.10.33}$$

Hence, applying the lemma in Section 4.1.7 with $N \to 2n$ and $n \to m$,

$$\begin{aligned} W_m &= \left| \sum_{k=1}^{2n} y_k x_k^{i+j-2} \right| \\ &= \sum_{k_1,k_2,\ldots,k_m=1}^{2n} Y_m \left(\prod_{r=2}^{m} x_{k_r}^{r-1} \right) \left| x_{k_i}^{j-1} \right|_m, \end{aligned}$$

where

$$Y_m = \prod_{r=1}^{m} y_{k_r}. \tag{6.10.34}$$

Hence, applying Identity 4 in Appendix A.3,

$$W_m = \frac{1}{m!} \sum_{k_1,k_2,\ldots,k_m=1}^{2n} Y_m \sum_{j_1,j_2,\ldots,j_m}^{k_1,k_2,\ldots k_m} \left(\prod_{r=2}^{m} x_{j_r}^{r-1} \right) V(x_{j_1}, x_{j_2}, \ldots, x_{j_m}). \tag{6.10.35}$$

Applying Theorem (b) in Section 4.1.9 on Vandermondian identities,

$$W_m = \frac{1}{m!} \sum_{k_1,k_2,\ldots,k_m=1}^{2n} Y_m \big\{ V(x_{k_1}, x_{k_2}, \ldots, x_{k_m}) \big\}^2. \tag{6.10.36}$$

Due to the presence of the squared Vandermondian factor, the conditions of Identity 3 in Appendix A.3 with $N \to 2n$ are satisfied. Also, eliminating the x's using (6.10.26) and (6.10.28) and referring to Exercise 3 in Section 4.1.2,

$$\big\{ (V(x_{k_1}, x_{k_2}, \ldots, x_{k_m}) \big\}^2 = \rho^{-m(m-1)} \big\{ V(c_{k_1}, c_{k_2}, \ldots, c_{k_m}) \big\}^2. \tag{6.10.37}$$

Hence,

$$W_m = \rho^{-m(m-1)} \sum_{1 \le k_1 < k_2 < \ldots < k_m \le 2n} Y_m \big\{ V(c_{k_1}, c_{k_2}, \ldots, c_{k_m}) \big\}^2. \tag{6.10.38}$$

From (6.10.33) and (6.10.34),

$$Y_m = (-1)^K E_m \prod_{r=1}^{m} M_{k_r}(\mathbf{c}),$$

where

$$E_m = \prod_{r=1}^{m} \varepsilon_{k_r},$$

$$K = \sum_{r=1}^{m} (k_r - 1). \tag{6.10.39}$$

Applying Theorem (c) in Section 4.1.8 on Vandermondian identities,

$$Y_m = (-1)^K E_m \frac{V(c_{k_{m+1}}, c_{k_{m+2}}, \ldots, c_{k_{2n}})\{V_{2n}(\mathbf{c})\}^{m-1}}{V(c_{k_1}, c_{k_2}, \ldots, c_{k_m})}. \tag{6.10.40}$$

Hence,

$$W_m = \frac{(-1)^K \{V_{2n}(\mathbf{c})\}^{m-1}}{\rho^{m(m-1)}} \sum_{1 \le k_1 < k_2 < \ldots < k_m \le 2n} \cdot E_m V(c_{k_1}, c_{k_2}, \ldots, c_{k_m}) V(c_{k_{m+1}}, c_{k_{m+2}}, \ldots, c_{k_{2n}}). \tag{6.10.41}$$

Using the Laplace formula (Section 3.3) to expand $H_{2n}^{(m)}(\varepsilon)$ by the first m rows and the remaining $(2n - m)$ rows and referring to the exercise at the end of Section 4.1.8,

$$H_{2n}^{(m)}(\varepsilon) = \sum_{1 \le k_1 < k_2 < \cdots < k_m \le 2n} N_{12\cdots m; k_1, k_2, \ldots, k_m} A_{12\cdots m; k_1, k_2, \ldots, k_m}, \tag{6.10.42}$$

where

$$\begin{aligned} N_{12\cdots m; k_1, k_2, \ldots, k_m} &= E_m V(c_{k_1}, c_{k_2}, \ldots, c_{k_m}), \\ A_{12\cdots m; k_1, k_2, \ldots, k_m} &= (-1)^R M_{12\cdots m; k_1, k_2, \ldots, k_m} \\ &= (-1)^R V(c_{k_{m+1}}, c_{k_{m+2}}, \ldots, c_{k_{2n}}), \end{aligned} \tag{6.10.43}$$

where M is the unsigned minor associated with the cofactor A and R is the sum of their parameters. Referring to (6.10.39),

$$\begin{aligned} R &= \tfrac{1}{2} m(m+1) + \sum_{r=1}^{m} k_r \\ &= K + \tfrac{1}{2} m(m-1). \end{aligned} \tag{6.10.44}$$

Hence,

$$\begin{aligned} H_{2n}^{(m)}(\varepsilon) &= (-1)^R \sum_{1 \le k_1 < k_2 < \cdots < k_m \le 2n} E_m V(c_{k_1}, c_{k_2}, \ldots, c_{k_m}) \\ &\quad \cdot V(c_{k_{m+1}}, c_{k_{m+2}}, \ldots, c_{k_{2n}}) \\ &= \frac{(-\rho^2)^{m(m-1)/2}}{\{V_{2n}(\mathbf{c})\}^{m-1}} W_m, \end{aligned} \tag{6.10.45}$$

which proves part (a) of the theorem. Part (b) can be proved in a similar manner. □

6.10.5 Physically Significant Solutions

From the theorem in Section 6.10.2 on the intermediate solution,

$$\begin{aligned}\phi_{2n+1} &= \frac{\rho^{2n-1}A_{2n}}{A_{2n-1}},\\ \psi_{2n+1} &= \frac{\omega\rho^{2n-1}A_{1,2n+1}^{(2n+1)}}{A_{2n-1}} \quad (\omega^2=-1).\end{aligned} \tag{6.10.46}$$

Hence the functions ζ_+ and ζ_- introduced in Section 6.2.8 can be expressed as follows:

$$\begin{aligned}\zeta_+ &= \phi_{2n+1}+\omega\psi_{2n+1}\\ &= \frac{\rho^{2n-1}(A_{2n}-A_{1,2n+1}^{(2n+1)})}{A_{2n-1}},\end{aligned} \tag{6.10.47}$$

$$\begin{aligned}\zeta_- &= \phi_{2n+1}-\omega\psi_{2n+1}\\ &= \frac{\rho^{2n-1}(A_{2n}+A_{1,2n+1}^{(2n+1)})}{A_{2n-1}}.\end{aligned} \tag{6.10.48}$$

It is shown in Section 4.5.2 on symmetric Toeplitz determinants that if $A_n = |t_{|i-j|}|_n$, then

$$\begin{aligned}A_{2n-1} &= 2P_{n-1}Q_n,\\ A_{2n} &= P_nQ_n + P_{n-1}Q_{n+1},\\ A_{1,2n+1}^{(2n+1)} &= P_nQ_n - P_{n-1}Q_{n+1},\end{aligned} \tag{6.10.49}$$

where

$$\begin{aligned}P_n &= \tfrac{1}{2}\left|t_{|i-j|}-t_{i+j}\right|_n\\ Q_n &= \tfrac{1}{2}\left|t_{|i-j|}+t_{i+j-2}\right|_n.\end{aligned} \tag{6.10.50}$$

Hence,

$$\begin{aligned}\zeta_+ &= \frac{\rho^{2n-1}Q_{n+1}}{Q_n},\\ \zeta_- &= \frac{\rho^{2n-1}P_n}{P_{n-1}}.\end{aligned} \tag{6.10.51}$$

In the present problem, $t_r = \omega^r u_r$ $(\omega^2=-1)$, where u_r is a solution of the coupled equations (6.10.3) and (6.10.4). In order to obtain the Neugebauer solutions, it is necessary first to choose the solution given by equations (A.11.8) and (A.11.9) in Appendix A.11, namely

$$u_r = (-1)^r \sum_{j=1}^{2n} \frac{e_j f_r(x_j)}{\sqrt{1+x_j^2}}, \quad x_j = \frac{z+c_j}{\rho}, \tag{6.10.52}$$

and then to choose

$$e_j = (-1)^{j-1}M_j(\mathbf{c})e^{\omega\theta_j}. \tag{6.10.53}$$

Denote this particular solution by U_r. Then,

$$t_r = (-\omega)^r U_r,$$

where

$$U_r = \sum_{j=1}^{2n} \frac{(-1)^{j-1} M_j(\mathbf{c}) f_r(x_j)}{\varepsilon_j^*} \tag{6.10.54}$$

and the symbol $*$ denotes the complex conjugate. This function is of the form (4.13.3), where

$$a_j = \frac{(-1)^{j-1} M_j(\mathbf{c})}{\varepsilon_j^*} \tag{6.10.55}$$

and $N = 2n$. These choices of a_j and N modify the function k_r defined in (4.13.5). Denote the modified k_r by w_r, which is given explicitly in (6.10.3).

Since the results of Section 4.13.2 are unaltered by replacing ω by $(-\omega)$, it follows from (4.13.22) and (4.13.23) with $n \to m$ that

$$\begin{aligned} P_m &= (-1)^{m(m-1)/2} 2^{m^2-1} \left| w_{i+j} + w_{i+j-2} \right|_m, \\ Q_m &= (-1)^{m(m-1)/2} 2^{(m-1)^2} \left| w_{i+j-2} \right|_m. \end{aligned} \tag{6.10.56}$$

Applying the theorem in Section 6.10.4,

$$\begin{aligned} P_m &= 2^{m^2-1} \rho^{-m(m-1)} \left\{ V_{2n}(\mathbf{c}) \right\}^{m-1} H_{2n}^{(m)}(\varepsilon), \\ Q_m &= 2^{(m-1)^2} \rho^{-m(m-1)} \left\{ V_{2n}(\mathbf{c}) \right\}^{m-1} H_{2n}^{(m)} \left(\frac{1}{\varepsilon^*} \right). \end{aligned} \tag{6.10.57}$$

Hence,

$$\frac{P_n}{P_{n-1}} = 2^{2n-1} \rho^{-2(n-1)} V_{2n}(\mathbf{c}) \frac{H_{2n}^{(n)}(\varepsilon)}{H_{2n}^{(n-1)}(\varepsilon)}. \tag{6.10.58}$$

Also, applying (6.10.32),

$$\begin{aligned} \frac{Q_{n+1}}{Q_n} &= 2^{2n-1} \rho^{-2n} V_{2n}(\mathbf{c}) \frac{H_{2n}^{(n+1)}\left(1/\varepsilon^*\right)}{H_{2n}^{(n)}\left(1/\varepsilon^*\right)} \\ &= -2^{2n-1} \rho^{-2n} V_{2n}(\mathbf{c}) \frac{H_{2n}^{(n-1)}(\varepsilon^*)}{H_{2n}^{(n)}(\varepsilon^*)}. \end{aligned} \tag{6.10.59}$$

Since $\tau_j = \rho\varepsilon_j$, (the third line of (6.10.26)), the functions F and G defined in Section 6.2.8 are given by

$$\begin{aligned} F &= H_{2n}^{(n-1)}(\rho\varepsilon) = \rho^{n-1} H_{2n}^{(n-1)}(\varepsilon), \\ G &= H_{2n}^{(n)}(\rho\varepsilon) = \rho^n H_{2n}^{(n)}(\varepsilon). \end{aligned} \tag{6.10.60}$$

Hence,

$$\frac{P_n}{P_{n-1}} = \left\{\frac{2}{\rho}\right\}^{2n-1} V_{2n}(\mathbf{c}) \left\{\frac{G}{F}\right\},$$
$$\frac{Q_{n+1}}{Q_n} = -\left\{\frac{2}{\rho}\right\}^{2n-1} V_{2n}(\mathbf{c}) \left\{\frac{F^*}{G^*}\right\} \tag{6.10.61}$$
$$\zeta_+ = -2^{2n-1}V_{2n}(\mathbf{c})\left\{\frac{F^*}{G^*}\right\},$$
$$\zeta_- = 2^{2n-1}V_{2n}(\mathbf{c})\left\{\frac{G}{F}\right\}. \tag{6.10.62}$$

Finally, applying the Bäcklund transformation ε in Appendix A.12 with $b = 2^{2n-1}V_{2n}(\mathbf{c})$,

$$\begin{aligned}\zeta'_+ &= \frac{\zeta_- - 2^{2n-1}V_{2n}(\mathbf{c})}{\zeta_- + 2^{2n-1}V_{2n}(\mathbf{c})}\\ &= \frac{1-(F/G)}{1+(F/G)}.\end{aligned}$$

Similarly,

$$\zeta'_- = \frac{1-(F^*/G^*)}{1+(F^*/G^*)}. \tag{6.10.63}$$

Discarding the primes, $\zeta_- = \zeta_+^*$. Hence, referring to (6.2.13),

$$\begin{aligned}\phi &= \tfrac{1}{2}(\zeta_+ + \zeta_-) = \tfrac{1}{2}(\zeta_+ + \zeta_+^*),\\ \psi &= \frac{1}{2\omega}(\zeta_+ - \zeta_-) = \frac{1}{2\omega}(\zeta_+ - \zeta_+^*) \quad (\omega^2 = -1),\end{aligned} \tag{6.10.64}$$

which are both real. It follows that these solutions are physically significant.

Exercise. Prove the following identities:

$$\begin{aligned}A_{2n} &= \alpha_n(GG^* - FF^*),\\ A_{2n+1} &= \beta_n F^*G,\\ A_{2n-1} &= \beta_{n-1}FG^*,\\ A_{1,2n+1}^{(2n+1)} &= \alpha_n(GG^* + FF^*),\end{aligned}$$

where

$$\alpha_n = \frac{(-1)^n 2^{2n(n-1)} V_{2n}^{2(n-1)}}{\rho^{2n(n-1)} \prod_{i=1}^{2n} \varepsilon_i},$$

$$\beta_n = \frac{(-1)^{n-1} 2^{2n^2} V_{2n}^{2n-1}}{\rho^{2n^2-1} \prod_{i=1}^{2n} \varepsilon_i^*}.$$

6.10.6 The Ernst Equation

The Ernst equation, namely

$$(\xi\xi^* - 1)\nabla^2\xi = 2\xi^*(\nabla\xi)^2,$$

is satisfied by each of the functions

$$\xi_n = \frac{pU_n(x) - \omega q U_n(y)}{U_n(1)} \quad (\omega^2 = -1), \quad n = 1, 2, 3, \ldots,$$

where $U_n(x)$ is a determinant of order $(n+1)$ obtained by bordering an nth-order Hankelian as follows:

$$U_n(x) = \begin{vmatrix} & & x \\ & & x^3/3 \\ & [a_{ij}]_n & x^5/5 \\ & & \cdots \\ & & x^{2n-1}/(2n-1) \\ 1\ 1\ 1 \cdots 1 & & \bullet \end{vmatrix}_{n+1},$$

where

$$a_{ij} = \frac{1}{i+j-1}[p^2 x^{2(i+j-1)} + q^2 y^{2(i+j-1)} - 1],$$
$$p^2 + q^2 = 1,$$

and x and y are prolate spheriodal coordinates. The argument x in $U_n(x)$ refers to the elements in the last column, so that $U_n(1)$ is the determinant obtained from $U_n(x)$ by replacing the x in the last column *only* by 1. A note on this solution is given in Section 6.2 on brief historical notes. Some properties of $U_n(x)$ and a similar determinant $V_n(x)$ are proved in Section 4.10.3.

6.11 The Relativistic Toda Equation — A Brief Note

The relativistic Toda equation in a function R_n and a substitution for R_n in terms of U_{n-1} and U_n are given in Section 6.2.9. The resulting equation can be obtained by eliminating V_n and W_n from the equations

$$H_x^{(2)}(U_n, U_n) = 2(V_n W_n - U_n^2), \tag{6.11.1}$$
$$aH_x^{(2)}(U_n, U_{n-1}) = aU_n U_{n-1} + V_n W_{n-1}, \tag{6.11.2}$$
$$V_{n+1} W_{n-1} - U_n^2 = a^2(U_{n+1}U_{n-1} - U_n^2), \tag{6.11.3}$$

where $H_x^{(2)}$ is the one-variable Hirota operator (Section 5.7),

$$a = \frac{1}{\sqrt{1+c^2}},$$

$$x = t\sqrt{1-a^2} = \frac{ct}{\sqrt{1+c^2}}\,. \tag{6.11.4}$$

Equations (6.11.1)–(6.11.3) are satisfied by the functions

$$\begin{aligned} U_n &= |u_{i,n+j-1}|_m, \\ V_n &= |v_{i,n+j-1}|_m, \\ W_n &= |w_{i,n+j-1}|_m, \end{aligned} \tag{6.11.5}$$

where the determinants are Casoratians (Section 4.14) of arbitrary order m whose elements are given by

$$\begin{aligned} u_{ij} &= F_{ij} + G_{ij}, \\ v_{ij} &= a_i F_{ij} + \frac{1}{a_i} G_{ij}, \\ w_{ij} &= \frac{1}{a_i} F_{ij} + a_i G_{ij}, \end{aligned} \tag{6.11.6}$$

where

$$\begin{aligned} F_{ij} &= \left(\frac{1}{a_i - a}\right)^j \exp(\xi_i), \\ G_{ij} &= \left(\frac{a_i}{1 - a a_i}\right)^j \exp(\eta_i), \\ \xi_i &= \frac{x}{a_i} + b_i, \\ \eta_i &= a_i x + c_i, \end{aligned} \tag{6.11.7}$$

and where the a_i, b_i, and c_i are arbitrary constants.

Appendix A

A.1 Miscellaneous Functions

The Kronecker Delta Function

$$\delta_{ij} = \begin{cases} 1, & j = i \\ 0, & j \neq i. \end{cases}$$

$$\sum_{j=p}^{q} x_j \delta_{jr} = \begin{cases} 0, & p \leq r \leq q, \\ 0, & \text{otherwise.} \end{cases}$$

$$\sum_{j=p}^{q} x_j \delta_{jr}(1 - \delta_{jr}) = x_r.$$

$$\mathbf{I}_n = [\delta_{ij}]_n, \quad \text{the unit matrix,}$$

$$\sum_{r=1}^{n} \begin{vmatrix} f_r & \delta_{ir} \\ g_r & \delta_{jr} \end{vmatrix} = \begin{vmatrix} f_j & 1 \\ g_i & 1 \end{vmatrix}, \quad 1 \leq i, j \leq n,$$

$$\sum_{r=1}^{n} \begin{vmatrix} a_{ip} & a_{iq} & \delta_{ir} \\ a_{jp} & a_{jq} & \delta_{jr} \\ a_{kp} & a_{kq} & \delta_{kr} \end{vmatrix} = \begin{vmatrix} a_{ip} & a_{iq} & 1 \\ a_{jp} & a_{jq} & 1 \\ a_{kp} & a_{kq} & 1 \end{vmatrix}, \quad 1 \leq i, j, k \leq n.$$

$$\delta_{i,\text{even}} = \begin{cases} 1, & i \ \text{even,} \\ 0, & i \ \text{odd.} \end{cases}$$

$$\delta_{i,\mathrm{odd}} = \begin{cases} 1, & i \ \text{odd}, \\ 0, & i \ \text{even}. \end{cases}$$

$$\delta_{i_1 i_2; j_1 j_2} = \begin{cases} 1, & (j_1, j_2) = (i_1, i_2) \\ 0, & \text{otherwise}. \end{cases}$$

The Binomial Coefficient and Gamma Function

$$\binom{n}{r} = \begin{cases} \frac{n!}{r!(n-r)!}, & 0 \le r \le n \\ 0, & \text{otherwise}. \end{cases}$$

$$\binom{n}{n-r} = \binom{n}{r},$$

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

The lower or upper limit $r = i\ (\to j)$ in a sum denotes that the limit was originally i, but i can be replaced by j without affecting the sum since the additional or rejected terms are all zero. For example,

$$\sum_{r=0(\to n)}^{\infty} \frac{a_r}{(r-n)!} \text{ denotes that } \sum_{r=0}^{\infty} \frac{a_r}{(r-n)!} \text{ can be replaced by } \sum_{r=n}^{\infty} \frac{a_r}{(r-n)!};$$

$$\sum_{r=0}^{n(\to\infty)} \binom{n}{r} a_r \text{ denotes that } \sum_{r=0}^{n} \binom{n}{r} a_r \text{ can be replaced by } \sum_{r=0}^{\infty} \binom{n}{r} a_r.$$

This notation has applications in simplifying multiple sums by changing the order of summation. For example,

$$\sum_{n=0}^{q} \sum_{p=0}^{n} \binom{n}{p} a_p = \sum_{p=0}^{q} \binom{q+1}{p+1} a_p.$$

PROOF. Denote the sum on the left by S_q and apply the well-known identity

$$\sum_{n=p}^{q} \binom{n}{p} = \binom{q+1}{p+1}$$

$$\begin{aligned} S_q = \sum_{n=0}^{q} \sum_{p=0}^{n(\to\infty)} \binom{n}{p} a_p &= \sum_{p=0}^{\infty} a_p \sum_{n=0(\to p)}^{q} \binom{n}{p} \\ &= \sum_{p=0}^{\infty(\to q)} a_p \binom{q+1}{p+1}. \end{aligned}$$

The result follows. □

Other applications are found in Appendix A.4 on Appell polynomials.

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt,$$
$$\Gamma(x+1) = x\Gamma(x)$$
$$\Gamma(n+1) = n!, \quad n = 1, 2, 3, \dots .$$

The Legendre duplication formula is

$$\sqrt{\pi}\,\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma\left(x + \tfrac{1}{2}\right),$$

which is applied in Appendix A.8 on differences.

Stirling Numbers

The Stirling numbers of the first and second kinds, denoted by s_{ij} and S_{ij}, respectively, are defined by the relations

$$(x)_r = \sum_{k=0}^{r} s_{rk} x^k, \qquad s_{r0} = \delta_{r0},$$

$$x^r = \sum_{k=0}^{r} S_{rk} (x)_k, \quad S_{r0} = \delta_{r0},$$

where $(x)_r$ is the falling factorial function defined as

$$(x)_r = x(x-1)(x-2)\cdots(x-r+1), \quad r = 1, 2, 3, \dots .$$

Stirling numbers satisfy the recurrence relations

$$s_{ij} = s_{i-1,j-1} - (i-1)s_{i-1,j}$$
$$S_{ij} = S_{i-1,j-1} + jS_{i-1,j}.$$

Some values of these numbers are given in the following short tables:

s_{ij}

i \ j	1	2	3	4	5
1	1				
2	−1	1			
3	2	−3	1		
4	−6	11	−6	1	
5	24	−50	35	−10	1

S_{ij}

i \ j	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

Further values are given by Abramowitz and Stegun. Stirling numbers appear in Section 5.6.3 on distinct matrices with nondistinct determinants and in Appendix A.6.

The matrices $\mathbf{s}_n(x)$ and $\mathbf{S}_n(x)$ are defined as follows:

$$\mathbf{s}_n(x) = \left[s_{ij}x^{i-j}\right]_n = \begin{bmatrix} 1 \\ -x & 1 \\ 2x^2 & -3x & 1 \\ -6x^3 & 11x^2 & -6x & 1 \\ 24x^4 & -50x^3 & 35x^2 & -10x & 1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \end{bmatrix}_n,$$

$$\mathbf{S}_n(x) = \left[S_{ij}x^{i-j}\right]_n = \begin{bmatrix} 1 \\ x & 1 \\ x^2 & 3x & 1 \\ x^3 & 7x^2 & 6x & 1 \\ x^4 & 15x^3 & 25x^2 & 10x & 1 \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{bmatrix}_n.$$

A.2 Permutations

Inversions, the Permutation Symbol

The first n positive integers $1, 2, 3, \dots, n$, can be arranged in a linear sequence in $n!$ ways. For example, the first three integers can be arranged in $3! = 6$ ways, namely

$$\begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{matrix}$$

Let N_n denote the set of the first n integers arranged in ascending order of magnitude,

$$N_n = \{1\ 2\ 3 \cdots n\},$$

and let I_n and J_n denote arrangements or permutations of the same n integers

$$I_n = \{i_1\ i_2\ i_3 \cdots i_n\},$$
$$J_n = \{j_1\ j_2\ j_3 \cdots j_n\}.$$

There are $n!$ possible sets of the form I_n or J_n including N_n. The numbers within the set are called elements. The operation which consists of interchanging any two elements in a set is called an inversion. Assuming that $J_n \neq I_n$, that is, $j_r \neq i_r$ for at least two values of r, it is possible to transform J_n into I_n by means of a sequence of inversions. For example, it is possible to transform the set $\{3\ 5\ 2\ 1\ 4\}$ into the set N_5 in four steps, that is, by means of four inversions, as follows:

	3	5	2	1	4
1 :	1	5	2	3	4
2 :	1	2	5	3	4
3 :	1	2	3	5	4
4 :	1	2	3	4	5

The choice of inversions is clearly not unique for the transformation can also be accomplished as follows:

	3	5	2	1	4
1 :	3	4	2	1	5
2 :	3	1	2	4	5
3 :	2	1	3	4	5
4 :	1	2	3	4	5

No steps have been wasted in either method, that is, the methods are efficient and several other efficient methods can be found. If steps are wasted by, for example, removing an element from its final position at any stage of the transformation, then the number of inversions required to complete the transformation is increased.

However, it is known that if the number of inversions required to transform J_n into I_n is odd by one method, then it is odd by all methods, and J_n is said to be an odd permutation of I_n. Similarly, if the number of inversions required to transform J_n into I_n is even by one method, then it is even by all methods, and J_n is said to be an even permutation of I_n.

The permutation symbol is an expression of the form

$$\left\{ \begin{matrix} I_n \\ J_n \end{matrix} \right\} = \left\{ \begin{matrix} i_1 & i_2 & i_3 & \cdots & i_n \\ j_1 & j_2 & j_3 & \cdots & j_n \end{matrix} \right\},$$

which enables I_n to be compared with J_n.

The sign of the permutation symbol, denoted by σ, is defined as follows:

$$\sigma = \operatorname{sgn} \left\{ \begin{matrix} I_n \\ J_n \end{matrix} \right\} = \operatorname{sgn} \left\{ \begin{matrix} i_1 & i_2 & i_3 & \cdots & i_n \\ j_1 & j_2 & j_3 & \cdots & j_n \end{matrix} \right\} = (-1)^m,$$

where m is the number of inversions required to transform J_n into I_n, or vice versa, by any method. $\sigma = 0$ if J_n is not a permutation of I_n.

Examples.

$$\operatorname{sgn}\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{Bmatrix} = -1,$$
$$\operatorname{sgn}\begin{Bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{Bmatrix} = 1.$$

Permutations Associated with Pfaffians

Let the $2n$-set $\{i_1\ j_1\ i_2\ j_2 \cdots i_n\ j_n\}_{2n}$ denote a permutation of N_{2n} subject to the restriction that $i_s < j_s$, $1 \le s \le n$. However, if one permutation can be transformed into another by repeatedly interchanging two pairs of parameters of the form $\{i_r j_r\}$ and $\{i_s j_s\}$ then the two permutations are not considered to be distinct in this context. The number of distinct permutations is $(2n)!/(2^n n!)$.

Examples.

a. Put $n = 2$. There are three distinct permitted permutations of N_4, including the identity permutation, which, with their appropriate signs, are as follows: Omitting the upper row of integers,

$$\operatorname{sgn}\{1\ 2\ 3\ 4\} = 1,\quad \operatorname{sgn}\{1\ 3\ 2\ 4\} = -1,\quad \operatorname{sgn}\{1\ 4\ 2\ 3\} = 1.$$

The permutation $P_1\{2\ 3\ 1\ 4\}$, for example, is excluded since it can be transformed into $P\{1\ 4\ 2\ 3\}$ by interchanging the first and second pairs of integers. P_1 is therefore not distinct from P in this context.

b. Put $n = 3$. There are 15 distinct permitted permutations of N_6, including the identity permutation, which, with their appropriate signs, are as follows:

$$\begin{array}{lll}
\operatorname{sgn}\{1\ 2\ 3\ 4\ 5\ 6\} = 1, & \operatorname{sgn}\{1\ 2\ 3\ 5\ 4\ 6\} = -1, & \operatorname{sgn}\{1\ 2\ 3\ 6\ 4\ 5\} = 1, \\
\operatorname{sgn}\{1\ 3\ 2\ 4\ 5\ 6\} = -1, & \operatorname{sgn}\{1\ 3\ 2\ 5\ 4\ 6\} = 1, & \operatorname{sgn}\{1\ 3\ 2\ 6\ 4\ 5\} = -1, \\
\operatorname{sgn}\{1\ 4\ 2\ 3\ 5\ 6\} = 1, & \operatorname{sgn}\{1\ 4\ 2\ 5\ 3\ 6\} = -1, & \operatorname{sgn}\{1\ 4\ 2\ 6\ 3\ 5\} = 1, \\
\operatorname{sgn}\{1\ 5\ 2\ 3\ 4\ 6\} = -1, & \operatorname{sgn}\{1\ 5\ 2\ 4\ 3\ 6\} = 1, & \operatorname{sgn}\{1\ 5\ 2\ 6\ 3\ 4\} = -1, \\
\operatorname{sgn}\{1\ 6\ 2\ 3\ 4\ 5\} = 1, & \operatorname{sgn}\{1\ 6\ 2\ 4\ 3\ 5\} = -1, & \operatorname{sgn}\{1\ 6\ 2\ 5\ 3\ 4\} = 1.
\end{array}$$

The permutations $P_1\{1\ 4\ 3\ 6\ 2\ 5\}$ and $P_2\{3\ 6\ 1\ 4\ 2\ 5\}$, for example, are excluded since they can be transformed into $P\{1\ 4\ 2\ 5\ 3\ 6\}$ by interchanging appropriate pairs of integers. P_1 and P_2 are therefore not distinct from P in this context.

Lemma.

$$\operatorname{sgn}\begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & m \\ i & m & r_3 & r_4 & \ldots & r_m \end{Bmatrix}_m$$

$$= (-1)^{i+m+1} \operatorname{sgn} \begin{Bmatrix} 1 & 2 & \dots & (i-1)(i+1) & \dots & (m-1) \\ r_3 & r_4 & \dots & \dots & \dots & r_m \end{Bmatrix}_{m-2},$$

where $1 \le r_k \le m-2$, $r_k \ne i$, and $3 \le k \le m$.

PROOF. The cases $i = 1$ and $i > 1$ are considered separately. When $i = 1$, then $2 \le r_k \le m-1$. Let p denote the number of inversions required to transform the set $\{r_3\ r_4 \dots r_m\}_{m-2}$ into the set $\{2\ 3 \dots (m-1)\}_{m-2}$, that is,

$$(-1)^p = \operatorname{sgn} \begin{Bmatrix} 2 & 3 & \dots & (m-1) \\ r_3 & r_4 & \dots & r_m \end{Bmatrix}_{m-2}.$$

Hence

$$\begin{aligned} &\operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \dots & m \\ i & m & r_3 & r_4 & \dots & r_m \end{Bmatrix}_m \\ &\quad = (-1)^p \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \dots & m \\ i & m & 2 & 3 & \dots & (m-1) \end{Bmatrix}_m \\ &\quad = (-1)^{p+m-2} \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \dots & (m-1)m \\ 1 & 2 & 3 & 4 & \dots & (m-1)m \end{Bmatrix}_m \\ &\quad = (-1)^{p+m-2} \\ &\quad = (-1)^{m-2} \operatorname{sgn} \begin{Bmatrix} 2 & 3 & \dots & (m-1) \\ r_3 & r_4 & \dots & r_m \end{Bmatrix}_{m-2}, \end{aligned}$$

which proves the lemma when $i = 1$.

When $i > 1$, let q denote the number of inversions required to transform the set $\{r_3\ r_4 \cdots r_m\}_{m-2}$ into the set $\{1\ 2 \cdots (i-1)(i+1) \cdots (m-1)\}_{m-2}$. Then,

$$(-1)^q = \operatorname{sgn} \begin{Bmatrix} 1 & 2 & \dots & (i-1)(i+1) & \dots & (m-1) \\ r_3 & r_4 & \dots & \dots & \dots & r_m \end{Bmatrix}_{m-2}.$$

Hence,

$$\begin{aligned} &\operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \cdots & m \\ i & m & r_3 & r_4 & \cdots & r_m \end{Bmatrix}_m \\ &\quad = (-1)^q \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \cdots & \cdots & \cdots & (m-1) & m \\ i & m & 1 & 2 & \cdots & (i-1)(i+1) & \cdots & (m-2) & (m-1) \end{Bmatrix}_m \\ &\quad = (-1)^{q+m} \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \cdots & \cdots & \cdots & (m-1) & m \\ i & 1 & 2 & 3 & \cdots & (i-1)(i+1) & \cdots & (m-1) & m \end{Bmatrix}_m \\ &\quad = (-1)^{q+m+i-1} \operatorname{sgn} \begin{Bmatrix} 1 & 2 & 3 & 4 & \cdots & m \\ 1 & 2 & 3 & 4 & \cdots & m \end{Bmatrix}_m \\ &\quad = (-1)^{q+m+i-1} \\ &\quad = (-1)^{m+i-1} \operatorname{sgn} \begin{Bmatrix} 1 & 2 & \cdots & (i-1)(i+1) & \cdots & (m-1) \\ r_3 & r_4 & \cdots & \cdots & \cdots & r_m \end{Bmatrix}_{m-2}, \end{aligned}$$

which proves the lemma when $i > 1$. $\square$

Cyclic Permutations

The cyclic permutations of the r-set $\{i_1\ i_2\ i_3\ \ldots\ i_r\}$ are alternately odd and even when r is even, and are all even when r is odd. Hence, the signs associated with the permutations alternate when r is even but are all positive when r is odd.

Examples. If

$$\operatorname{sgn}\{i\ j\} = 1,$$

then

$$\operatorname{sgn}\{j\ i\} = -1.$$

If

$$\operatorname{sgn}\{i\ j\ k\} = 1,$$

then

$$\operatorname{sgn}\{j\ k\ i\} = 1,$$
$$\operatorname{sgn}\{k\ i\ j\} = 1.$$

If

$$\operatorname{sgn}\{i\ j\ k\ m\} = 1,$$

then

$$\operatorname{sgn}\{j\ k\ m\ i\} = -1,$$
$$\operatorname{sgn}\{k\ m\ i\ j\} = 1,$$
$$\operatorname{sgn}\{m\ i\ j\ k\} = -1.$$

Cyclic permutations appear in Section 3.2.4 on alien second and higher cofactors and in Section 4.2 on symmetric determinants.

Exercise. Prove that

$$\underset{1\le i,j\le n}{|\delta_{r_i s_j}|_n} = \operatorname{sgn}\begin{Bmatrix} r_1 & r_2 & r_3 & \cdots & r_n \\ s_1 & s_2 & s_3 & \cdots & s_n \end{Bmatrix}.$$

A.3 Multiple-Sum Identities

1. If

$$f_i = \sum_{j=1}^{n} c_{i+1-j,j}, \quad 1 \le i \le 2n-1,$$

where $c_{ij} = 0$ when $i, j < 1$ or $i, j > n$, then

$$\sum_{i=1}^{2n-1} f_i g_i = \left[\sum_{i=1}^{n} g_i \sum_{j=1}^{i} + \sum_{i=n+1}^{2n-1} g_i \sum_{j=i+1-n}^{n}\right] c_{i+1-j,j}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} g_{i+j-1}.$$

The last step can be checked by writing out the terms in the last double sum in a square array and collecting them together again along diagonals parallel to the secondary diagonal.

2. The interval $(1, 2n+1-i-j)$ can be split into the two intervals $(1, n+1-j)$ and $(n+2-j,\ 2n+1-i-j)$. Let

$$S = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{2n+1-i-j} F_{ijs}.$$

Then, splitting off the $i = n$ term temporarily,

$$S = \sum_{i=1}^{n-1}\sum_{j=1}^{n}\sum_{s=1}^{2n+1-i-j} F_{ijs} + \sum_{j=1}^{n}\sum_{s=1}^{n+1-j} F_{njs}$$
$$= \sum_{i=1}^{n-1}\sum_{j=1}^{n}\left[\sum_{s=1}^{n+1-j} + \sum_{s=n+2-j}^{2n+1-i-j}\right] F_{ijs} + \sum_{j=1}^{n}\sum_{s=1}^{n+1-j} F_{njs}.$$

The first and third sums can be recombined. Hence,

$$S = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{n+1-j} F_{ijs} + \sum_{i=1}^{n-1}\sum_{j=1}^{n}\sum_{s=n+2-j}^{2n+1-i-j} F_{ijs}.$$

The identities given in 1 and 2 are applied in Section 5.2 on the generalized Cusick identities.

3. If $F_{k_1k_2\ldots k_m}$ is invariant under any permutation of the parameters k_r, $1 \le r \le m$, and is zero when the parameters are not distinct, then

$$\sum_{k_1\ldots k_m=1}^{N} F_{k_1k_2\ldots k_m} = m! \sum_{1\le k_1<k_2<\cdots<k_m\le N} F_{k_1k_2\ldots k_m}, \quad m \le N.$$

PROOF. Denote the sum on the left by S and the sum on the right by T. Then, S consists of all the terms in which the parameters are distinct, whereas T consists only of those terms in which the parameters are in ascending order of magnitude. Hence, to obtain all the terms in S, it is necessary to permute the m parameters in each term of T. The number of these permutations is $m!$. Hence, $S = m!T$, which proves the identity. □

4. If $F_{k_1k_2\ldots k_m}$ is invariant under any permutation of the parameters k_r, $1 \le r \le m$, then

$$\sum_{k_1,k_2,\ldots,k_m} F_{k_1k_2\ldots k_m} G_{k_1k_2\ldots k_m} = \frac{1}{m!} \sum_{k_1,k_2,\ldots,k_m} F_{k_1k_2\ldots k_m} \sum_{j_1,j_2,\ldots,j_m}^{k_1,k_2,\ldots,k_m} G_{j_1j_2\ldots j_m},$$

where the sum on the left ranges over the $m!$ permutations of the parameters and, in the inner sum on the right, the parameters j_r, $1 \le r \le m$, range over the $m!$ permutations of the k_r.

PROOF. Denote the sum on the left by S. The $m!$ permutations of the parameters k_r give $m!$ alternative formulae for S, which differ only in the order of the parameters in $G_{k_1k_2\ldots k_m}$. The identity appears after summing these $m!$ formulas. □

Illustration. Put $m = 3$ and use a simpler notation. Let

$$S = \sum_{i,j,k} F_{ijk} G_{ijk}.$$

Then,

$$S = \sum_{i,k,j} F_{ikj} G_{ikj} = \sum_{i,j,k} F_{ijk} G_{ikj}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S = \sum_{k,j,i} F_{kji} G_{kji} = \sum_{i,j,k} F_{ijk} G_{kji}.$$

Summing these 3! formulas for S,

$$3!\,S = \sum_{i,j,k} F_{ijk}(G_{ijk} + G_{ikj} + \cdots + G_{kji}),$$

$$S = \frac{1}{3!} \sum_{i,j,k} F_{ijk} \sum_{p,q,r}^{i,j,k} G_{pqr}.$$

5.

$$\sum_{k_1,k_2,\ldots,k_m=1}^{n} F_{k_1k_2\ldots k_m} G_{k_1k_2\ldots k_m}$$

$$= \frac{1}{m!} \sum_{k_1,k_2,\ldots,k_m=1}^{n} F_{k_1k_2\ldots k_m} \sum_{j_1,j_2,\ldots,j_m}^{k_1,k_2,\ldots,k_m} G_{j_1j_2\ldots j_m}.$$

The inner sum on the right is identical with the inner sum on the right of Identity 4 and the proof is similar to that of Identity 4. In this case, the number of terms in the sum on the left is m^n, but the number of alternative formulas for this sum remains at $m!$.

The identities given in 3–5 are applied in Section 6.10.3 on the Einstein and Ernst equations.

A.4 Appell Polynomials

Appell polynomials $\phi_m(x)$ may be defined by means of the generating function relation

$$\begin{aligned} e^{xt}G(t) &= \sum_{m=0}^{\infty} \frac{\phi_m(x)t^m}{m!} \\ &= \sum_{m=1}^{\infty} \frac{m\phi_{m-1}(x)t^{m-1}}{m!}, \end{aligned} \tag{A.4.1}$$

where

$$G(t) = \sum_{r=0}^{\infty} \frac{\alpha_r t^r}{r!}\,. \tag{A.4.2}$$

Differentiating the first line of (A.4.1) with respect to x and dividing the result by t,

$$\begin{aligned} e^{xt}G(t) &= \sum_{m=0}^{\infty} \frac{\phi'_m(x)t^{m-1}}{m!} \\ &= \frac{\phi'_0}{t} + \sum_{m=1}^{\infty} \frac{\phi'_m(x)t^{m-1}}{m!}\,. \end{aligned} \tag{A.4.3}$$

Comparing the last relation with the second line of (A.4.1), it is seen that

$$\phi_0 = \text{constant}, \tag{A.4.4}$$

$$\phi'_m = m\phi_{m-1}, \tag{A.4.5}$$

which is a differential–difference equation known as the Appell equation.

Substituting (A.4.2) into the first line of (A.4.1) and using the upper and lower limit notation introduced in Appendix A.1,

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{\phi_m(x)t^m}{m!} &= \sum_{r=0}^{\infty} \frac{\alpha_r t^r}{r!} \sum_{m=r(\to 0)}^{\infty} \frac{(xt)^{m-r}}{(m-r)!} \\ &= \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{r=0}^{\infty(\to m)} \binom{m}{r} \alpha_r x^{m-r}. \end{aligned}$$

Hence,

$$\begin{aligned} \phi_m(x) &= \sum_{r=0}^{m} \binom{m}{r} \alpha_r x^{m-r} \\ &= \sum_{r=0}^{m} \binom{m}{r} \alpha_{m-r} x^r, \\ \phi_m(0) &= \alpha_m. \end{aligned} \tag{A.4.6}$$

TABLE A.1. Particular Appell Polynomials and Their Generating Functions

	α_r	$G(t)=\sum_{r=0}^{\infty}\frac{\alpha_r t^r}{r!}$	$\phi_m(x)=\sum_{r=0}^{m}\binom{m}{r}\alpha_r x^{m-r}$
1	δ_r	1	x^m
2	1	e^t	$(1+x)^m$
3	r	te^t	$m(1+x)^{m-1}$
4	$\frac{1}{r+1}$	$\frac{e^t-1}{t}$	$\frac{(1+x)^{m+1}-x^{m+1}}{m+1}$
5	$\frac{(-1)^r}{r!}$	$J_0(2\sqrt{t})$ (Bessel)	$x^m L_m\left(\frac{1}{x}\right)$ (Laguerre)
6	$\left.\begin{aligned}\alpha_{2r} &= \frac{(-1)^r(2r)!}{2^{2r}r!}\\ \alpha_{2r+1} &= 0\end{aligned}\right\}$	e^{-t^2}	$2^{-m}H_m(x)$ (Hermite)
7		$\frac{t}{e^t-1}$	$B_m(x)$ (Bernoulli)
8		$\frac{2}{e^t+1}$	$E_m(x)$ (Euler)

Note: Further examples are given by Carlson.

The first four polynomials are

$$\begin{aligned}
\phi_0(x) &= \alpha_0,\\
\phi_1(x) &= \alpha_0 x + \alpha_1,\\
\phi_2(x) &= \alpha_0 x^2 + 2\alpha_1 x + \alpha_2,\\
\phi_3(x) &= \alpha_0 x^3 + 3\alpha_1 x^2 + 3\alpha_2 x + \alpha_3.
\end{aligned} \tag{A.4.7}$$

Particular cases of these polynomials and their generating functions are given in Table 1. When expressed in matrix form, equations (A.4.7) become

$$\begin{bmatrix}\phi_0(x)\\ \phi_1(x)\\ \phi_2(x)\\ \phi_3(x)\\ \cdots\end{bmatrix} = \begin{bmatrix}1 & & & \\ x & 1 & & \\ x^2 & 2x & 1 & \\ x^3 & 3x^2 & 3x & 1\\ \cdots & \cdots & \cdots & \cdots\end{bmatrix}\begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \cdots\end{bmatrix}. \tag{A.4.8}$$

The infinite triangular matrix in (A.4.8) can be expressed in the form $e^{x\mathbf{Q}}$, where

$$\mathbf{Q} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & 3 & 0 & \\ & & & & \cdots \end{bmatrix}.$$

Identities among this and other triangular matrices have been developed by Vein. The triangular matrix in (8) with its columns arranged in reverse order appears in Section 5.6.2.

Denote the column vector on the left of (A.4.8) by $\Phi(x)$. Then,

$$\Phi(x) = e^{x\mathbf{Q}}\Phi(0).$$

Hence,

$$\Phi(0) = e^{-x\mathbf{Q}}\Phi(x)$$

that is,

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdots \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -x & 1 & & \\ x^2 & -2x & 1 & \\ -x^3 & 3x^2 & -3x & 1 \\ \multicolumn{4}{c}{\dotfill} \end{bmatrix} \begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \cdots \end{bmatrix},$$

which yields the relation which is inverse to the first line of (A.4.6), namely

$$\alpha_m = \sum_{r=0}^{m} \binom{m}{r} \phi_r(x)(-x)^{m-r} \tag{A.4.9}$$

$\phi_m(x)$ is also given by the following formulas but with a lower limit for m in each case:

$$\begin{aligned} \phi_m(x) &= \sum_{r=0}^{m-1} \begin{vmatrix} \alpha_r & \alpha_{r+1} \\ -1 & x \end{vmatrix} x^{m-r-1}, \quad m \geq 1, \\ \phi_m(x) &= \sum_{r=0}^{m-2} \begin{vmatrix} \alpha_r & 2\alpha_{r+1} & \alpha_{r+2} \\ -1 & x & \\ & -1 & x \end{vmatrix} x^{m-r-2}, \quad m \geq 2, \end{aligned} \tag{A.4.10}$$

etc. The polynomials ϕ_m and the constants α_m are related by the two-parameter identity

$$\sum_{r=0}^{p}(-1)^r \binom{p}{r} \phi_{p+q-r}x^r = \sum_{r=0}^{q} \binom{q}{r} \alpha_{p+r}x^{q-r}, \quad p,q = 0,1,2,\ldots. \tag{A.4.11}$$

Appell Sets

Any sequence of polynomials $\{\phi_m(x)\}$ where $\phi_m(x)$ is of exact degree m and satisfies the Appell equation (A.4.5) is known as an Appell set.

The sequence in which

$$\phi_m(x) = \binom{m+s}{s}^{-1} (x+c)^{m+s}, \quad s = 1, 2, 3, \ldots,$$

satisfies (A.4.5), but its members are not of degree m. The sequence in which

$$\phi_m(x) = \frac{2^{2m+1} m!\,(m+1)!\,(x+c)^{m+(1/2)}}{(2m+2)!}$$

satisfies (A.4.5), but its members are not polynomials. Hence, neither sequence is an Appell set.

Carlson proved that if $\{\phi_m\}$ and $\{\psi_m\}$ are each Appell sets and

$$\theta_m = 2^{-m} \sum_{r=0}^{m} \binom{m}{r} \phi_r \psi_{m-r},$$

then $\{\theta_m\}$ is also an Appell set.

In a paper on determinants with hypergeometric elements, Burchnall proved that if $\{\phi_m\}$ and $\{\psi_m\}$ are each Appell sets and

$$\theta_m = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \phi_{m+n-r} \psi_r, \quad n = 0, 1, 2, \ldots,$$

then $\{\theta_m\}$ is also an Appell set for each value of n. Burchnall's formula can be expressed in the form

$$\theta_m = \sum_{r=0}^{n} (-1)^r \begin{vmatrix} \psi_r & \phi_{m+n-r} \\ \psi_{r+1} & \phi_{m+n-r+1} \end{vmatrix}, \quad n = 0, 1, 2, \ldots .$$

The generalized Appell equation

$$\theta'_m = m f' \theta_{m-1}, \quad f = f(x),$$

is satisfied by

$$\theta_m = \phi_m(f),$$

where $\phi_m(x)$ is any solution of (A.4.5). For example, the equation

$$\theta'_m = \frac{m\theta_{m-1}}{(1+x)^2}$$

is satisfied by

$$\theta_m = \phi_m\left(-\frac{1}{1+x}\right).$$

If

$$\phi_m = \frac{(1+x)^{m+1} - cx^{m+1}}{m+1},$$

then

$$\theta_m = \frac{x^{m+1} + (-1)^m c}{(m+1)(1+x)^{m+1}},$$
$$\theta_0 = \frac{x+c}{1+x}.$$

The Taylor Series Solution

Functions $\phi_m(x)$ which satisfy the Appell equation (A.4.5) but are not Appell sets according to the strict definition given above may be called Appell functions, but they should not be confused with the four Appell hypergeometric series in two variables denoted by F_1, F_2, F_3, and F_4, which are defined by Whittaker and Watson and by Erdelyi et al.

The most general Taylor series solution of (A.4.5) for given ϕ_0 which is valid in the neighborhood of the origin is expressible in the form

$$\phi_m = \sum_{r=1}^{m} \binom{m}{r} \alpha_r x^{m-r} + m \int_0^x \phi_0(u)(x-u)^{m-1}\,du, \quad m = 1, 2, 3, \ldots.$$

A proof is given by Vein and Dale. Hildebrand obtained a similar result by means of the substitution $\phi_m = m!\,f_m$, which reduces (A.4.5) to

$$f'_m = f_{m-1}.$$

Multiparameter and Multivariable Appell Polynomials

The Appell equation (A.4.5) can be generalized in several ways. The two-parameter equation

$$u'_{ij} = iu_{i-1,j} + ju_{i,j-1} \tag{A.4.12}$$

is a differential partial difference equation whose general polynomial solution is

$$u_{ij}(x) = \sum_{r=0}^{i} \sum_{s=0}^{j} \binom{i}{r} \binom{j}{s} \alpha_{rs} x^{i+j-r-s}, \quad i, j = 0, 1, 2, \ldots,$$

where the α_{rs} are arbitrary constants.

$$u_{00} = \alpha_{00},$$
$$u_{ij}(0) = \alpha_{ij}.$$

A proof can be constructed by applying the identity

$$i\binom{i-1}{r}\binom{j}{s} + j\binom{i}{r}\binom{j-1}{s} = (i+j-r-s)\binom{i}{r}\binom{j}{s}.$$

These polynomials can be displayed in matrix form as follows:

Let

$$\mathbf{U}(x) = \begin{bmatrix} u_{00} & u_{01} & u_{02} & \cdots \\ u_{10} & u_{11} & u_{12} & \cdots \\ u_{20} & u_{21} & u_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Then,

$$\mathbf{U}(x) = e^{x\mathbf{Q}}\mathbf{U}(0)\big(e^{x\mathbf{Q}}\big)^T.$$

Hence,

$$\mathbf{U}(0) = e^{-x\mathbf{Q}}\mathbf{U}(x)\big(e^{-x\mathbf{Q}}\big)^T,$$

that is,

$$\alpha_{ij} = \sum_{r=0}^{i}\sum_{s=0}^{j}\binom{i}{r}\binom{j}{s}u_{rs}(-x)^{i+j-r-s}, \quad i,j = 0,1,2,\ldots.$$

Other solutions of (A.4.12) can be expressed in terms of simple Appell polynomials; for example,

$$\begin{aligned} u_{ij} &= \phi_i\phi_j, \\ u_{ij} &= \begin{vmatrix} \phi_i & \phi_j \\ \phi_{i+1} & \phi_{j+1} \end{vmatrix}. \end{aligned}$$

Solutions of the three-parameter Appell equation, namely

$$u'_{ijk} = iu_{i-1,j,k} + ju_{i,j-1,k} + ku_{ij,k-1},$$

include

$$\begin{aligned} u_{ijk} &= \phi_i\phi_j\phi_k, \\ u_{ijk} &= \begin{vmatrix} \phi_i & \phi_j & \phi_k \\ \phi_{i+1} & \phi_{j+1} & \phi_{k+1} \\ \phi_{i+2} & \phi_{j+2} & \phi_{k+2} \end{vmatrix}. \end{aligned}$$

Carlson has studied polynomials $\phi_m(x,y,z,\ldots)$ which satisfy the relation

$$(D_x + D_y + D_z + \cdots)\phi_m = m\phi_{m-1}, \quad D_x = \frac{\partial}{\partial x}, \text{ etc.},$$

and Carlitz has studied polynomials $\phi_{mnp\ldots}(x,y,z,\ldots)$ which satisfy the relations

$$\begin{aligned} D_x(\phi_{mnp\ldots}) &= m\phi_{m-1,np\ldots}, \\ D_y(\phi_{mnp\ldots}) &= n\phi_{m,n-1,p\ldots}, \\ D_z(\phi_{mnp\ldots}) &= p\phi_{mn,p-1,\ldots}. \end{aligned}$$

The polynomial

$$\psi_{mn}(x) = \sum_{r=0}^{m} \binom{m}{r} \alpha_{n+r} x^r$$

satisfies the relations

$$\begin{aligned} \psi'_{mn} &= m\psi_{m-1,n+1}, \\ \psi_{mn} - \psi_{m-1,n} &= x\psi'_{mn} \\ &= mx\psi_{m-1,n+1}. \end{aligned}$$

Exercises

1. Prove that

$$\begin{aligned} \phi_m(x-h) &= \sum_{r=0}^{m} \binom{m}{r} (-h)^r \phi_{m-r}(x) \\ &= \Delta_h^m \phi_0. \end{aligned}$$

2. If

$$\begin{aligned} S_m(x) &= \sum_{r+s=m} \phi_r \phi_s, \\ T_m(x) &= \sum_{r+s+t=m} \phi_r \phi_s \phi_t, \end{aligned}$$

prove that

$$\begin{aligned} S'_m &= (m+1)S_{m-1}, \\ S_m(x+h) &= \sum_{r=0}^{m} \binom{m+1}{r} h^r S_{m-r}(x), \\ T'_m &= (m+2)T_{m-1}, \\ T_m(x+h) &= \sum_{r=0}^{m} \binom{m+2}{r} h^r T_{m-r}(x). \end{aligned}$$

3. Prove that

$$\phi_m^{-1} = \frac{1}{\alpha_m} \sum_{n=0}^{\infty} (-1)^n c_{mn} x^n,$$

where

$$\begin{aligned} c_{m0} &= 1, \\ c_{mn} &= \frac{1}{\alpha_m^n} \left| \binom{m}{m-i+j-1} \alpha_{m-i+j-1} \right|_n, \quad n \geq 1. \end{aligned}$$

This determinant is of Hessenberg form, is symmetric about its secondary diagonal, and contains no more than $(m+1)$ nonzero diagonals parallel to and including the principal diagonal.

4. Prove that the vector Appell equation, namely

$$\mathbf{C}_j' = j\mathbf{C}_{j-1}, \quad j > 0,$$

is satisfied by the column vector

$$\mathbf{C}_j = \left[\binom{j}{0}^{-1} \phi_j \;\; \binom{p+1}{1}^{-1} \phi_{j+1} \;\; \binom{j+2}{2}^{-1} \phi_{j+2} \right.$$
$$\left. \cdots \binom{j+n-1}{n-1}^{-1} \phi_{j+n-1} \right]_n^T, \quad n \geq 1.$$

5. If

$$f_{nm} = \sum_{r=0}^{m} (-1)^r \binom{m}{r} \phi_r \phi_{n-r}, \quad n \geq m,$$

prove that

$$f_{nm}' = (n-m) f_{n-1,m}.$$

A.5 Orthogonal Polynomials

The following brief notes relate to the Laguerre, Hermite, and Legendre polynomials which appear in the text.

Laguerre Polynomials $L_n^{(\alpha)}(x)$ and $L_n(x)$

Definition.

$$L_n^{(\alpha)}(x) = (n+\alpha)! \sum_{r=0}^{n} \frac{(-1)^r x^r}{r!\,(n-r)!\,(r+\alpha)!},$$
$$L_n(x) = L_n^{(0)}(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{x^r}{r!}.$$

Rodrigues formula.

$$L_n(x) = \frac{e^x}{n!} D^n (e^{-x} x^n); \quad D = \frac{d}{dx}.$$

Generating function relation.

$$(1-t)^{-1} e^{-xt/(1-t)} = \sum_{n=0}^{\infty} L_n(x) t^n;$$

Recurrence relations.

$$(n+1)L_{n+1}(x) - (2n+1-x)L_n(x) = +nL_{n-1}(x) = 0,$$
$$xL_n'(x) = n[L_n(x) - L_{n-1}(x)];$$

Differential equation.

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0;$$

Appell relation. If

$$\phi_n(x) = x^n L_n\left(\frac{1}{x}\right),$$

then

$$\phi_n'(x) = n\phi_{n-1}(x).$$

$\phi_n(x)$ is the Laguerre polynomial with its coefficients arranged in reverse order.

Hermite Polynomial $H_n(x)$

Definition.

$$H_n(x) = n!\sum_{r=0}^{N} \frac{(-1)^r (2x)^{n-2r}}{r!\,(n-2r)!}, \quad N = \left[\tfrac{1}{2}n\right].$$

Rodrigues formula.

$$H_n(x) = (-1)^n e^{x^2} D^n\left(e^{-x^2}\right), \quad D = \frac{d}{dx};$$

Generating function relation.

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!};$$

Recurrence relation.

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0;$$

Differential equation.

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0;$$

Appell relation.

$$H_n'(x) = 2nH_{n-1}(x).$$

Legendre Polynomials $P_n(x)$

Definition.

$$P_n(x) = \frac{1}{2^n}\sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!\, x^{n-2r}}{r!\,(n-r)!\,(n-2r)!}, \quad N = \left[\tfrac{1}{2}n\right].$$

Rodrigues formula.

$$P_n(x) = \frac{1}{2^n n!} D^n (x^2 - 1)^n, \quad D = \frac{d}{dx};$$

Generating function relation.

$$(1 - 2xh + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) h^n;$$

Recurrence relations.

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0,$$
$$(x^2 - 1)P_n'(x) = n[xP_n(x) - P_{n-1}(x)];$$

Differential equation.

$$(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0;$$

Appell relation. If

$$\phi_n(x) = (1 - x^2)^{-n/2} P_n(x),$$

then

$$\phi_n'(x) = nF\phi_{n-1}(x),$$

where

$$F = (1 - x^2)^{-3/2}.$$

A.6 The Generalized Geometric Series and Eulerian Polynomials

The generalized geometric series $\phi_m(x)$ and the closely related function $\psi_m(x)$ are defined as follows:

$$\phi_m(x) = \sum_{r=0}^{\infty} r^m x^r, \tag{A.6.1}$$

$$\psi_m(x) = \sum_{r=1}^{\infty} r^m x^r. \tag{A.6.2}$$

The two sums differ only in their lower limits:

$$\begin{aligned}
\phi_m(x) &= \psi_m(x), \quad m > 0, \\
\phi_0(x) &= \frac{1}{1-x}, \\
\psi_0(x) &= \frac{x}{1-x} \\
&= x\phi_0(x) \\
&= \phi_0(x) - 1.
\end{aligned} \tag{A.6.3}$$

It follows from (A.6.2) that

$$x\psi_m' = \psi_{m+1}, \quad m \ge 0. \tag{A.6.4}$$

The formula

$$\Delta^m \psi_0 = x\psi_m, \quad m > 0,$$

is proved in the section on differences in Appendix A.8.

Other formulas for ψ_m include the following:

$$\psi_m = \sum_{r=0}^{m} \frac{(-1)^{m+r} r!\, S_{m+1,r+1}}{(1-x)^{r+1}}, \quad m \ge 0 \ \ \text{(Comtet)}, \tag{A.6.5}$$

$$\psi_m = \frac{x}{1-x} \sum_{r=1}^{m} \frac{(-1)^{m+r} r!\, S_{mr}}{(1-x)^r}, \quad m \ge 0, \tag{A.6.6}$$

where the S_{mr} are Stirling numbers of the second kind (Appendix A.1).

$$\psi_m = \left[D^r \left(\frac{1}{1 - xe^u} \right) \right]_{u=0}, \quad D = \frac{\partial}{\partial u} \ \ \text{(Zeitlin)}. \tag{A.6.7}$$

Let

$$t = \phi_0 = \frac{1}{1-x}\,.$$

Then,

$$\begin{aligned}
\psi_0 &= -(1-t), \\
\psi_1 &= -t + t^2 \\
&= -t(1-t), \\
\psi_2 &= t - 3t^2 + 2t^3 \\
&= t(1-t)(1-2t), \\
\psi_3 &= -t + 7t^2 - 12t^3 + 6t^4 \\
&= -t(1-t)(1-6t+6t^2), \\
\psi_4 &= t - 15t^2 + 50t^3 - 60t^4 + 24t^5 \\
&= t(1-t)(1-14t+36t^2-24t^3).
\end{aligned}$$

The function ψ_m satisfies the linear recurrence relations

$$\psi_m = x\left[1 + \sum_{r=0}^{m} \binom{m}{r} \psi_r \right], \quad m \ge 0 \tag{A.6.8}$$

$$= \frac{x}{1-x}\left[1 + \sum_{r=0}^{m-1} \binom{m}{r} \psi_r \right], \quad m \ge 1 \tag{A.6.9}$$

$$\begin{aligned}
x\sum_{r=0}^{m} \binom{m}{r} \psi_{m+r} &= \sum_{r=0}^{m} (-1)^{m+r} \binom{m}{r} \psi_{m+r} \\
&= \Delta^m \psi_m.
\end{aligned} \tag{A.6.10}$$

Lawden's function $S_m(x)$ is defined as follows:

$$S_m(x) = (1-x)^{m+1}\psi_m(x), \quad m \geq 0. \tag{A.6.11}$$

It follows from (A.6.5) that S_m is a polynomial of degree m in $(1-x)$ and hence is also a polynomial of degree m in x. Lawden's investigation into the properties of ψ_m and S_m arose from the application of the z-transform to the solution of linear difference equations in the theory of sampling servomechanisms.

The Eulerian polynomial $A_m(x)$, not to be confused with the Euler polynomial $E_m(x)$, is defined as follows:

$$\begin{aligned}
A_m(x) &= (1-x)^{m+1}\phi_m(x), \quad m \geq 0, && \text{(A.6.12)}\\
A_m(x) &= S_m(x), \quad m > 0,\\
A_0(x) &= 1,\\
S_0(x) &= x, && \text{(A.6.13)}\\
A_m(x) &= \sum_{n=1}^{m} A_{mn}x^n, && \text{(A.6.14)}
\end{aligned}$$

where the coefficients A_{mn} are the Eulerian numbers which are given by the formula

$$\begin{aligned}
A_{mn} &= \sum_{r=0}^{n-1}(-1)^r \binom{m+1}{r}(n-r)^m, \quad m \geq 0,\ n \geq 1,\\
&= A_{m,m+1-n}. && \text{(A.6.15)}
\end{aligned}$$

These numbers satisfy the recurrence relation

$$A_{mn} = (m-n+1)A_{m-1,n-1} + nA_{m-1,n}. \tag{A.6.16}$$

The first few Eulerian polynomials are

$$\begin{aligned}
A_1(x) &= S_1(x) = x,\\
A_2(x) &= S_2(x) = x + x^2,\\
A_3(x) &= S_3(x) = x + 4x^2 + x^3,\\
A_4(x) &= S_4(x) = x + 11x^2 + 11x^3 + x^4,\\
A_5(x) &= S_5(x) = x + 26x^2 + 66x^3 + 26x^4 + x^5.
\end{aligned}$$

S_m satisfies the linear recurrence relation

$$(1-x)S_m = (-1)^{m-1}\sum_{r=0}^{m-1}(-1)^r \binom{m}{r}(1-x)^{m-r}S_r$$

and the generating function relation

$$V = \frac{x(x-1)}{x - e^{u(x-1)}} = \sum_{m=0}^{\infty} \frac{S_m(x)u^m}{m!},$$

$$\frac{\partial V}{\partial u} = V(V + 1 - x)$$

from which it follows that S_m satisfies the nonlinear recurrence relation

$$S_{m+1} = (1 - x)S_m + \sum_{r=0}^{m} \binom{m}{r} S_r S_{m-r}.$$

It then follows that

$$\Delta\psi_m = \psi_{m+1} - \psi_m = \sum_{r=0}^{m} \binom{m}{r} \psi_r \psi_{m-r}.$$

A.7 Symmetric Polynomials

Let the function $f_n(x)$ and the polynomials $\sigma_p^{(n)}$ in the n variables x_i, $1 \le i \le n$, be defined as follows:

$$f_n(x) = \prod_{i=1}^{n}(x - x_i) = \sum_{p=0}^{n}(-1)^p \sigma_p^{(n)} x^{n-p}. \tag{A.7.1}$$

Examples

$$\begin{aligned}
\sigma_0^{(n)} &= 1, \\
\sigma_1^{(n)} &= \sum_{r=1}^{n} x_r, \\
\sigma_2^{(n)} &= \sum_{1 \le r < s \le n} x_r x_s, \\
\sigma_3^{(n)} &= \sum_{1 \le r < s < t \le n} x_r x_s x_t, \\
\cdots \quad & \cdots\cdots\cdots\cdots \\
\sigma_n^{(n)} &= x_1 x_2 x_3 \ldots x_n.
\end{aligned}$$

These polynomials are known as symmetric polynomials.

Let the function $g_{nr}(x)$ and the polynomials $\sigma_{rs}^{(n)}$ in the $(n-1)$ variables x_i, $1 \le i \le n$, $i \ne r$, be defined as follows:

$$g_{nr}(x) = \frac{f_n(x)}{x - x_r} = \sum_{s=0}^{n-1}(-1)^s \sigma_{rs}^{(n)} x^{n-1-s}, \tag{A.7.2}$$

$$g_{nn}(x) = f_{n-1}(x) \tag{A.7.3}$$

for all values of x. Hence,

$$\sigma_{ns}^{(n)} = \sigma_s^{(n-1)}. \tag{A.7.4}$$

Also,

$$g_{nj}(x_i) = 0, \quad j \neq i. \tag{A.7.5}$$

Examples

$$\begin{aligned}
\sigma_2^{(3)} &= x_1x_2 + x_1x_3 + x_2x_3, \\
\sigma_{r0}^{(n)} &= 1, \quad 1 \leq r \leq n, \\
\sigma_{21}^{(3)} &= x_1 + x_3, \\
\sigma_{22}^{(3)} &= x_1x_3, \\
\sigma_{31}^{(4)} &= x_1 + x_2 + x_4, \\
\sigma_{32}^{(4)} &= x_1x_2 + x_1x_4 + x_2x_4, \\
\sigma_{33}^{(4)} &= x_1x_2x_4.
\end{aligned}$$

Lemma.

$$\sigma_{rs}^{(n)} = \sum_{p=0}^{s} \sigma_p^{(n)}(-x_r)^{s-p}.$$

PROOF. Since

$$\begin{aligned}
g_r(x) &= -\frac{1}{x_r}\left(1 - \frac{x}{x_r}\right)^{-1} f(x) \\
&= -\frac{f(x)}{x_r}\sum_{q=0}^{\infty}\left(\frac{x}{x_r}\right)^q,
\end{aligned}$$

it follows that

$$\sum_{s=0}^{n-1}(-1)^{s+1}\sigma_{rs}^{(n)}x^{n-1-s} = \sum_{p=0}^{n}\sum_{q=0}^{\infty}\frac{(-1)^p\sigma_p^{(n)}x^{n-p+q}}{x_r^{q+1}}.$$

Equating coefficients of x^{n-1-s},

$$(-1)^{s+1}\sigma_{rs}^{(n)} = \sum_{p=s+1}^{n}(-1)^p\sigma_p^{(n)}x_r^{s-p}.$$

Hence

$$\begin{aligned}
(-1)^{s+1}\sigma_{rs}^{(n)} + \sum_{p=0}^{s}(-1)^p\sigma_p^{(n)}x_r^{s-p} &= \sum_{p=0}^{n}(-1)^p\sigma_p^{(n)}x_r^{s-p} \\
&= x_r^{s-n}f(x_r) \\
&= 0.
\end{aligned}$$

The lemma follows. □

Symmetric polynomials appear in Section 4.1.2 on Vandermondians.

A.8 Differences

Given a sequence $\{u_r\}$, the nth h-difference of u_0 is written as $\Delta_h^n u_0$ and is defined as

$$\Delta_h^n u_0 = \sum_{r=0}^{n} \binom{n}{r} (-h)^{n-r} u_r$$
$$= \sum_{r=0}^{n} \binom{n}{r} (-h)^r u_{n-r}.$$

The first few differences are

$$\Delta_h^0 u_0 = u_0,$$
$$\Delta_h^1 u_0 = u_1 - hu_0,$$
$$\Delta_h^2 u_0 = u_2 - 2hu_1 + h^2 u_0,$$
$$\Delta_h^3 u_0 = u_3 - 3hu_2 + 3h^2 u_1 - h^3 u_0.$$

The inverse relation is

$$u_n = \sum_{r=0}^{n} \binom{n}{r} (\Delta_h^r u_0) h^{n-r},$$

which is an Appell polynomial with $\alpha_r = \Delta_h^r u_0$. Simple differences are obtained by putting $h = 1$ and are denoted by $\Delta^r u_0$.

Example A.1. If

$$u_r = x^r,$$

then

$$\Delta_h^n u_0 = (x - h)^n.$$

The proof is elementary.

Example A.2. If

$$u_r = \frac{1}{2r+1}, \quad r \geq 1, 0$$

then

$$\Delta^n u_0 = \frac{(-1)^n 2^{2n} n!^2}{(2n+1)!}.$$

Proof.

$$\Delta^n u_0 = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} u_r$$
$$= (-1)^n \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{1}{2r+1}$$
$$= (-1)^n f(1),$$

where

$$
\begin{aligned}
f(x) &= \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{x^{2r+1}}{2r+1}, \\
f'(x) &= \sum_{r=0}^{n} (-1)^r \binom{n}{r} x^{2r} \\
&= (1-x^2)^n. \\
f(x) &= \int_0^x (1-t^2)^n \, dt, \\
f(1) &= \int_0^1 (1-t^2)^n \, dt \\
&= \int_0^{\pi/2} \cos^{2n+1} \theta \, d\theta \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{2\Gamma\left(n+\frac{3}{2}\right)}.
\end{aligned}
$$

The proof is completed by applying the Legendre duplication formula for the Gamma function (Appendix A.1). This result is applied at the end of Section 4.10.3 on bordered Yamazaki–Hori determinants. □

Example A.3. If

$$
u_r = \frac{x^{2r+2} - c}{r+1},
$$

then

$$
\Delta^n u_0 = \frac{(x^2-1)^{n+1} - (-1)^n (c-1)}{n+1}.
$$

PROOF.

$$
\begin{aligned}
\Delta^n u_0 &= \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \left[\frac{x^{2r+2}-1}{r+1} - \frac{c-1}{r+1} \right] \\
&= (-1)^n [S(x) + (c-1)S(0)],
\end{aligned}
$$

where

$$
\begin{aligned}
S(x) &= \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{x^{2r+2}-1}{r+1} \\
&= \frac{1}{n+1} \sum_{r=0}^{n} (-1)^r \binom{n+1}{r+1} (x^{2r+2} - 1) \\
&= \frac{1}{n+1} \sum_{r=0}^{n+1} (-1)^{r+1} \binom{n+1}{r} (x^{2r} - 1), \quad \text{(The } r=0 \text{ term is zero)}
\end{aligned}
$$

$$= -\frac{1}{n+1}\left[\sum_{r=0}^{n+1}(-1)^r\binom{n+1}{r}x^{2r} - \sum_{r=0}^{n+1}(-1)^r\binom{n+1}{r}\right]$$
$$= -\frac{1}{n+1}[(1-x^2)^{n+1} - 0]$$
$$S(0) = -\frac{1}{n+1}\,.$$

The result follows. It is applied with $c = 1$ in Section 4.10.4 on a particular case of the Yamazaki–Hori determinant. □

Example A.4. If

$$\psi_m = \sum_{r=1}^{\infty} r^m x^r,$$

then

$$\Delta^m \psi_0 = x\psi_m.$$

ψ_m is the generalized geometric series (Appendix A.6).

PROOF.

$$(r-1)^m = \sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}r^s.$$

Multiply both sides by x^r and sum over r from 1 to ∞. (In the sum on the left, the first term is zero and can therefore be omitted.)

$$x\sum_{r=2}^{\infty}(r-1)^m x^{r-1} = \sum_{r=1}^{\infty}x^r\sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}r^s,$$
$$x\sum_{s=1}^{\infty}s^m x^s = \sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}\sum_{r=1}^{\infty}r^s x^r,$$
$$x\psi_m = \sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}\psi_s$$
$$= \Delta^m\psi_0.$$

This result is applied in Section 5.1.2 to prove Lawden's theorem. □

A.9 The Euler and Modified Euler Theorems on Homogeneous Functions

The two theorems which follow concern two distinct kinds of homogeneity of the function

$$f = f(x_0, x_1, x_2, \ldots, x_n). \tag{A.9.1}$$

The first is due to Euler. The second is similar in nature to Euler's and can be obtained from it by means of a change of variable.

The function f is said to be homogeneous of degree s in its variables if

$$f(\lambda x_0, \lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^s f. \tag{A.9.2}$$

Theorem A.5 (Euler). *If the variables are independent and f is differentiable with respect to each of its variables and is also homogeneous of degree s in its variables, then*

$$\sum_{r=0}^{n} x_r \frac{\partial f}{\partial x_r} = sf.$$

The proof is well known.

The function f is said to be homogeneous of degree s in the *suffixes* of its variables if

$$f(x_0, \lambda x_1, \lambda^2 x_2, \dots, \lambda^n x_n) = \lambda^s f. \tag{A.9.3}$$

Theorem A.6 (Modified Euler). *If the variables are independent and f is differentiable with respect to each of its variables and is also homogeneous of degree s in the suffixes of its variables, then*

$$\sum_{r=1}^{n} r x_r \frac{\partial f}{\partial x_r} = sf.$$

Proof. Put

$$u_r = \lambda^r x_r, \quad 0 \le r \le n \ \ \text{[in (A.9.3)]}.$$

Then,

$$f(u_0, u_1, u_2, \dots, u_n) \equiv \lambda^s f.$$

Differentiating both sides with respect to λ,

$$\sum_{r=0}^{n} \frac{\partial f}{\partial u_r} \frac{du_r}{d\lambda} = s\lambda^{s-1} f,$$

$$\sum_{r=0}^{n} \frac{\partial f}{\partial u_r} r\lambda^{r-1} x_r = s\lambda^{s-1} f.$$

Put $\lambda = 1$. Then, $u_r = x_r$ and the theorem appears. □

A proof can also be obtained from Theorem A.5 with the aid of the change of variable

$$v_r = x_r^r.$$

Both these theorems are applied in Section 4.8.7 on double-sum relations for Hankelians.

Illustration. The function

$$f = Ax_0x_2x_4x_6 + \frac{Bx_0x_2x_3^2x_5}{x_1} + \frac{Cx_0^2x_1x_3^5 + Dx_2^8}{Ex_0^3x_4 + Fx_1^4} \tag{A.9.4}$$

is homogeneous of degree 4 in its variables and homogeneous of degree 12 in the suffixes of its variables. Hence,

$$\sum_{r=0}^{6} x_r \frac{\partial f}{\partial x_r} = 4f,$$
$$\sum_{r=1}^{6} rx_r \frac{\partial f}{\partial x_r} = 12f.$$

A.10 Formulas Related to the Function $(x + \sqrt{1+x^2})^{2n}$

Define functions λ_{nr} and μ_{nr} as follows. If n is a positive integer,

$$(x + \sqrt{1+x^2})^{2n} = \sum_{r=0}^{n} \lambda_{nr}x^{2r} + \sqrt{1+x^2}\sum_{r=1}^{n} \mu_{nr}x^{2r-1}, \tag{A.10.1}$$

where

$$\lambda_{nr} = \frac{n}{n+r}\binom{n+r}{2r}2^{2r}, \tag{A.10.2}$$
$$\mu_{nr} = \frac{r\lambda_{nr}}{n}. \tag{A.10.3}$$

Define the function ν_i as follows:

$$(1+z)^{-1/2} = \sum_{i=0}^{\infty} \nu_i z^i. \tag{A.10.4}$$

Then

$$\begin{aligned} \nu_i &= \frac{(-1)^i}{2^{2i}}\binom{2i}{i} \\ &= P_{2i}(0), \\ \nu_0 &= 1, \end{aligned} \tag{A.10.5}$$

where $P_n(x)$ is the Legendre polynomial.

Theorem A.7.

$$\sum_{j=1}^{n} \lambda_{n-1,j-1}\nu_{i+j-2} = \frac{\delta_{in}}{2^{2(n-1)}}, \quad 1 \le i \le n.$$

PROOF. Replace x by $-x^{-1}$ in (A.10.1), multiply by x^{2n}, and put $x^2 = z$. The result is

$$(-1+\sqrt{1+z})^{2n} = \left[z^n + \sum_{i=1}^{n} \lambda_{ni} z^{n-i}\right] - (1+z)^{\frac{1}{2}} \sum_{i=1}^{n} \mu_{ni} z^{n-i}$$
$$= \sum_{j=1}^{n+1} \lambda_{n,n-j+1} z^{j-1} - (1+z)^{\frac{1}{2}} \sum_{j=1}^{n} \mu_{n,n-j+1} z^{j-1}.$$

Rearrange, multiply by $(1+z)^{-1/2}$ and apply (A.10.4):

$$\sum_{i=0}^{\infty} \nu_i z^i \sum_{j=1}^{n+1} \lambda_{n,n-j+1} z^{j-1} = \sum_{j=1}^{n} \mu_{n,n-j+1} z^{j-1} + (1+z)^{-1/2}(-1+\sqrt{1+z})^{2n}.$$

In some detail,

$$(1+\nu_1 z + \nu_2 z^2 + \cdots)(\lambda_{nn} + \lambda_{n,n-1} z + \cdots + \lambda_{n1} z^{n-1} + \lambda_{n0} z^n)$$
$$= (\mu_{nn} + \mu_{n,n-1} z + \cdots + \mu_{n1} z^{n-1}) + \left(\frac{z}{2}\right)^{2n} (1+z)^{-1/2} \left(1 - \frac{1}{4} z + \cdots\right)^{2n}.$$

Note that there are no terms containing z^n, $z^{n+1}, \ldots, z^{2n-1}$ on the right-hand side and that the coefficient of z^{2n} is 2^{-2n}. Hence, equating coefficients of z^{n-1+i}, $1 \le i \le n+1$,

$$\sum_{j=1}^{n+1} \lambda_{n,j-1} \nu_{i+j-2} = \begin{cases} 0, & 1 \le i \le n \\ 2^{-2n}, & i = n+1. \end{cases}$$

The theorem appears when n is replaced by $(n-1)$ and is applied in Section 4.11.3 in connection with a determinant with binomial elements. □

It is convenient to redefine the functions λ_{nr} and μ_{nr} for an application in Section 4.13.1, which, in turn, is applied in Section 6.10.5 on the Einstein and Ernst equations.

If n is a positive integer,

$$(x+\sqrt{1+x^2})^{2n} = g_n + h_n \sqrt{1+x^2}, \tag{A.10.6}$$

where

$$\begin{aligned} g_n &= \sum_{r=0}^{n} \lambda_{nr} (2x)^{2r}, \quad \text{an even function,} \\ g_0 &= 1; \\ h_n(x) &= \sum_{r=1}^{n} \mu_{nr} (2x)^{2r-1}, \quad \text{an odd function,} \\ h_0 &= 0. \end{aligned} \tag{A.10.7}$$

$$\begin{aligned}
\lambda_{nr} &= \frac{n}{n+r}\binom{n+r}{2r}, \quad 1 \le r \le n,\\
\lambda_{n0} &= 1, \quad n \ge 0;\\
\mu_{nr} &= \frac{2r\lambda_{nr}}{n}, \quad 1 \le r \le n,\\
\mu_{n0} &= 0, \quad n \ge 0.
\end{aligned} \tag{A.10.8}$$

Changing the sign of x in (A.10.6),

$$(x - \sqrt{1+x^2})^{2n} = g_n - h_n\sqrt{1+x^2}\,. \tag{A.10.9}$$

Hence,

$$\begin{aligned}
g_n &= \tfrac{1}{2}\{(x+\sqrt{1+x^2})^{2n} + (x-\sqrt{1+x^2})^{2n}\},\\
h_n &= \tfrac{1}{2}\{(x+\sqrt{1+x^2})^{2n} - (x-\sqrt{1+x^2})^{2n}\}(\sqrt{1+x^2})^{-1}.
\end{aligned} \tag{A.10.10}$$

These functions satisfy the recurrence relations

$$\begin{aligned}
g_{n+1} &= (1+2x^2)g_n + 2x(1+x^2)h_n,\\
h_{n+1} &= (1+2x^2)h_n + 2xg_n.
\end{aligned} \tag{A.10.11}$$

Let

$$f_n = \tfrac{1}{2}\{(x+\sqrt{1+x^2})^n + (x-\sqrt{1+x^2})^n\}. \tag{A.10.12}$$

Lemmas.

a. $f_{2n} = g_n$

b. $f_{2n+1} = \dfrac{g_{n+1} - g_n}{2x}$.

PROOF. The proof of (a) is trivial. To prove (b), note that

$$\begin{aligned}
f_{2n+1} &= \tfrac{1}{2}\{(x+\sqrt{1+x^2})(g_n + h_n\sqrt{1+x^2})\\
&\quad +(x-\sqrt{1+x^2})(g_n - h_n\sqrt{1+x^2})\}\\
&= xg_n + (1+x^2)h_n.
\end{aligned} \tag{A.10.13}$$

The result is obtained by eliminating h_n from the first line of (A.10.11). □

In the next theorem, Δ is the finite-difference operator (Appendix A.8).

Theorem A.8.

a. $g_{m+n} + g_{m-n} = 2g_m g_n$,

b. $\Delta(g_{m+n-1} + g_{m-n-1}) = 2g_n \Delta g_{m-1}$,

c. $2x^2(g_{m+n-1} - g_{m-n}) = \Delta g_{m-1}\Delta g_{n-1}$,

d. $\Delta(g_{m+n-1} - g_{m-n}) = 2g_m \Delta g_{n-1}$,

e. $g_{m+n+1} + g_{m-n} = 2(1+x^2)(g_m + xh_m)(g_n + xh_n)$,

f. $\Delta(g_{m-n} + g_{m-n}) = 4x(1+x^2)h_m(g_n - xh_n)$,

g. $g_{m+n} - g_{m-n} = 2(1+x^2)h_m h_n$,

h. $\Delta(g_{m+n-1} - g_{m-n-1}) = 4x(1+x^2)h_n(g_m - xh_m)$.

PROOF OF (A). Put $x = \text{sh}\ \theta$. Then,

$$\begin{aligned} g_n &= \tfrac{1}{2}(e^{2n\theta} + e^{-2n\theta}) \\ &= \text{ch}\ 2n\theta, \\ g_{m+n} + g_{m-n} &= \text{ch}(2m+2n)\theta + \text{ch}(2m-2n)\theta \\ &= 2\ \text{ch}\ 2m\theta\ \text{ch}\ 2n\theta \\ &= 2g_m g_n. \end{aligned}$$

The other identities can be verified in a similar manner. □

It will be observed that

$$g_n(x) = i^{2n} T_n(ix),$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind (Abramowitz and Stegun), but this relation has not been applied in the text.

A.11 Solutions of a Pair of Coupled Equations

The general solution of the coupled equations which appear in Section 6.10.2 on the Einstein and Ernst equations, namely,

$$\frac{\partial u_{r+1}}{\partial \rho} + \frac{\partial u_r}{\partial z} = -\frac{r u_{r+1}}{\rho}, \quad r = 0, 1, 2, \ldots, \tag{A.11.1}$$

$$\frac{\partial u_{r-1}}{\partial \rho} - \frac{\partial u_r}{\partial z} = \frac{r u_{r-1}}{\rho}, \quad r = 1, 2, 3, \ldots, \tag{A.11.2}$$

can be obtained in the form of a contour integral by applying the theory of the Laurent series. The solution is

$$u_r = \frac{\rho^{1-r}}{2\pi i} \int_C f\left(\frac{\rho^2 w^2 - 2zw - 1}{w}\right) \frac{dw}{w^{1+r}}, \tag{A.11.3}$$

where C is a contour embracing the origin in the w-plane and $f(v)$ is an arbitrary function of v.

The particular solution corresponding to $f(v) = v^{-1}$ is

$$\begin{aligned} u_r &= \frac{\rho^{1-r}}{2\pi i} \int_C \frac{dw}{w^r(\rho^2 w^2 - 2zw - 1)} \\ &= \frac{\rho^{-1-r}}{2\pi i} \int_C \frac{dw}{w^r(w-\alpha)(w-\beta)}, \end{aligned} \tag{A.11.4}$$

where

$$\begin{aligned} \alpha &= \frac{1}{\rho^2}\left\{z + \sqrt{\rho^2 + z^2}\right\}, \\ \beta &= \frac{1}{\rho^2}\left\{z - \sqrt{\rho^2 + z^2}\right\}. \end{aligned} \tag{A.11.5}$$

This solution can be particularized still further using Cauchy's theorem. First, allow C to embrace α but not β and then allow C to embrace β but not α. This yields the solutions

$$\frac{\rho^{-1-r}}{\beta^r(\alpha-\beta)}, \quad \frac{-\rho^{-1-r}}{\alpha^r(\alpha-\beta)},$$

but since the coupled equations are linear, the difference between these two solutions is also a solution. This solution is

$$\frac{\rho^{-1-r}(\alpha^r+\beta^r)}{(\alpha\beta)^r(\alpha-\beta)} = \frac{(-1)^r f_r(z/\rho)}{\sqrt{1+z^2/\rho^2}}, \tag{A.11.6}$$

where

$$f_n(x) = \tfrac{1}{2}\{(x+\sqrt{1+x^2})^n + (x-\sqrt{1+x^2})^n\}. \tag{A.11.7}$$

Since z does not appear in the coupled equations except as a differential operator, another particular solution is obtained by replacing z by $z+c_j$, where c_j is an arbitrary constant. Denote this solution by u_{rj}:

$$u_{rj} = \frac{(-1)^r f_r(x_j)}{\sqrt{1+x_j^2}}, \quad x_j = \frac{z+c_j}{\rho}. \tag{A.11.8}$$

Finally, a linear combination of these solutions, namely

$$u_r = \sum_{j=1}^{2n} e_j u_{rj}, \tag{A.11.9}$$

where the e_j are arbitrary constants, can be taken as a more general series solution of the coupled equations.

A highly specialized series solution of (A.11.1) and (A.11.2) can be obtained by replacing r by $(r-1)$ in (A.11.1) and then eliminating u_{r-1} using (A.11.2). The result is the equation

$$\frac{\partial^2 u_r}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial u_r}{\partial \rho} - \frac{(r^2-1)u_r}{\rho^2} + \frac{\partial^2 u_r}{\partial z^2} = 0, \tag{A.11.10}$$

which is satisfied by the function

$$u_r = \rho\sum_n \{a_n J_r(n\rho) + b_n Y_r(n\rho)\}e^{\pm nz}, \tag{A.11.11}$$

where J_r and Y_r are Bessel functions of order r and the coefficients a_n and b_n are arbitrary. This solution is not applied in the text.

A.12 Bäcklund Transformations

It is shown in Section 6.2.8 on brief historical notes on the Einstein and Ernst equations that the equations

$$\tfrac{1}{2}(\zeta_+ + \zeta_-)\nabla^2\zeta_\pm = (\nabla\zeta_\pm)^2,$$

where

$$\zeta_\pm = \phi \pm \omega\psi \quad (\omega^2 = -1), \tag{A.12.1}$$

are equivalent to the coupled equations

$$\phi\nabla^2\phi - (\nabla\phi)^2 + (\nabla\psi)^2 = 0, \tag{A.12.2}$$

$$\phi\nabla^2\psi - 2\nabla\phi\cdot\nabla\psi = ,0 \tag{A.12.3}$$

which, in turn, are equivalent to the pair

$$\phi\left(\phi_{\rho\rho} + \frac{1}{\rho}\phi_\rho + \phi_{zz}\right) - \phi_\rho^2 - \phi_z^2 + \psi_\rho^2 + \psi_z^2 = 0, \tag{A.12.4}$$

$$\frac{\partial}{\partial\rho}\left(\frac{\rho\psi_\rho}{\phi^2}\right) + \frac{\partial}{\partial z}\left(\frac{\rho\psi_z}{\phi^2}\right) = 0. \tag{A.12.5}$$

Given one pair of solutions of (A.12.1), it is possible to construct other solutions by means of Bäcklund transformations.

Transformation δ

If ζ_+ and ζ_- are solutions of (A.12.1) and

$$\zeta'_+ = a\zeta_- - b,$$
$$\zeta'_- = a\zeta_+ + b,$$

where a, b are arbitrary constants, then ζ'_+ and ζ'_- are also solutions of (A.12.1). The proof is elementary.

Transformation γ

If ζ_+ and ζ_- are solution of (A.12.1) and

$$\zeta'_+ = \frac{c}{\zeta_+} + d,$$
$$\zeta'_- = \frac{c}{\zeta_-} - d,$$

where c and d are arbitrary constants, then ζ'_+ and ζ'_- are also solutions of (A.12.1).

PROOF.

$$\tfrac{1}{2}(\zeta'_+ + \zeta'_-) = \frac{c(\zeta_+ + \zeta_-)}{2\zeta_+\zeta_-},$$

$$\nabla \zeta'_+ = -\frac{c}{\zeta_+^2} \nabla \zeta_+,$$
$$\nabla^2 \zeta'_+ = -\frac{c}{\zeta_+^2} \left[\nabla^2 \zeta_+ - \frac{2}{\zeta_+} (\nabla \zeta_+)^2 \right].$$

Hence,

$$\tfrac{1}{2}(\zeta'_+ + \zeta'_-)\nabla^2 \zeta'_+ - (\nabla^2 \zeta'_+)^2 = -\frac{c^2}{\zeta_+^3 \zeta_-} \left[\tfrac{1}{2}(\zeta_+ + \zeta_-)\nabla^2 \zeta_+ - (\nabla \zeta_+)^2 \right]$$
$$= 0.$$

This identity remains valid when ζ'_+ and ζ'_- are interchanged, which proves the validity of transformation γ. It follows from the particular case in which $c = 1$ and $d = 0$ that if the pair $P(\phi, \psi)$ is a solution of (A.12.4) and (A.12.5) and

$$\phi' = \frac{\phi}{\phi^2 + \psi^2},$$
$$\psi' = -\frac{\psi}{\phi^2 + \psi^2},$$

then the pair $P'(\phi', \psi')$ is also a solution of (A.12.4) and (A.12.5). This relation is applied in Section 6.10.2 on the intermediate solution of the Einstein equations. □

Transformation ε

Combining transformation γ and δ with $a = d = 1$ and $c = -2b$, it is found that if ζ_+ and ζ_- are solutions of (A.12.1) and

$$\zeta'_+ = \frac{\zeta_- - b}{\zeta_- + b},$$
$$\zeta'_- = \frac{b + \zeta_+}{b - \zeta_+},$$

then ζ'_+ and ζ'_- are also solutions of (A.12.1). This transformation is applied in Section 6.10.4 on physically significant solutions of the Einstein equations.

The following formulas are well known and will be applied later. (ρ, z) are cylindrical polar coordinates:

$$\nabla V = \left(\frac{\partial V}{\partial \rho}, \frac{\partial V}{\partial z} \right), \tag{A.12.6}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial F_z}{\partial z}, \tag{A.12.7}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial z^2}, \tag{A.12.8}$$

$$\nabla \cdot (V\mathbf{F}) = V \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla V, \tag{A.12.9}$$

$$\nabla \cdot \left(\frac{\nabla\psi}{\phi}\right) = \frac{1}{\phi^2}(\phi\nabla^2\psi - \nabla\phi\cdot\nabla\psi), \tag{A.12.10}$$

$$\nabla \cdot \left(\frac{\nabla\psi}{\phi^2}\right) = \frac{1}{\phi^3}(\phi\nabla^2\psi - 2\nabla\phi\cdot\nabla\psi), \tag{A.12.11}$$

$$\nabla^2(\log\phi) = \frac{1}{\phi^2}[\phi\nabla^2\phi - (\nabla\phi)^2], \tag{A.12.12}$$

$$\nabla^2(\log\rho) = 0. \tag{A.12.13}$$

Applying (A.12.12) and (A.12.11), the coupled equations (A.12.2) and (A.12.3) become

$$\phi^2\nabla^2(\log\phi) + (\nabla\psi)^2 = 0, \tag{A.12.14}$$

$$\nabla \cdot \left(\frac{\nabla\psi}{\phi^2}\right) = 0. \tag{A.12.15}$$

Transformation β (Ehlers)

If the pair $P(\phi,\psi)$ is a solution of (A.12.4) and (A.12.5), and ϕ' and ψ' are functions which satisfy the relations

a. $\phi' = \dfrac{\rho}{\phi}$,

b. $\dfrac{\partial\psi'}{\partial\rho} = -\dfrac{\omega\rho}{\phi^2}\dfrac{\partial\psi}{\partial z}$,

c. $\dfrac{\partial\psi'}{\partial z} = \dfrac{\omega\rho}{\phi^2}\dfrac{\partial\psi}{\partial\rho}$, $(\omega^2 = -1)$,

then the pair $P'(\phi',\psi')$ is also a solution.

PROOF. Applying (A.12.6) and (A.12.7) to (A.12.15),

$$\nabla \cdot \left(\frac{1}{\phi^2}\frac{\partial\psi}{\partial\rho}, \frac{1}{\phi^2}\frac{\partial\psi}{\partial z}\right) = 0,$$

$$\frac{\partial}{\partial\rho}\left(\frac{\rho}{\phi^2}\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial}{\partial z}\left(\frac{\rho}{\phi^2}\frac{\partial\psi}{\partial z}\right) = 0,$$

which is satisfied by (b) and (c). Eliminating ψ from (b) and (c),

$$\frac{\partial}{\partial\rho}\left(\frac{\phi^2}{\rho}\frac{\partial\psi'}{\partial\rho}\right) + \frac{\partial}{\partial z}\left(\frac{\phi^2}{\rho}\frac{\partial\psi'}{\partial z}\right) = 0,$$

$$\frac{\partial^2\psi'}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial\psi'}{\partial\rho} + \frac{\partial^2\psi'}{\partial z^2} = -\frac{2}{\phi}\left(\frac{\partial\phi}{\partial\rho}\frac{\partial\psi'}{\partial\rho} + \frac{\partial\phi}{\partial z}\frac{\partial\psi'}{\partial z}\right).$$

Hence, referring to (A.12.8) and (a),

$$\begin{aligned}\nabla^2\psi' &= \frac{2\phi}{\rho}\left[\left(\frac{1}{\phi} - \frac{\rho}{\phi^2}\frac{\partial\phi}{\partial\rho}\right)\frac{\partial\psi'}{\partial\rho} - \frac{\rho}{\phi^2}\frac{\partial\phi}{\partial z}\frac{\partial\psi'}{\partial z}\right]\\ &= \frac{2\phi}{\rho}\left[\frac{\partial}{\partial\rho}\left(\frac{\rho}{\phi}\right)\frac{\partial\psi'}{\partial\rho} + \frac{\partial}{\partial z}\left(\frac{\rho}{\phi}\right)\frac{\partial\psi'}{\partial z}\right]\end{aligned}$$

$$= \frac{2}{\phi'}\left(\frac{\partial\phi'}{\partial\rho}\frac{\partial\psi'}{\partial\rho} + \frac{\partial\phi'}{\partial z}\frac{\partial\psi'}{\partial z}\right).$$

Hence,

$$\phi'\nabla^2\psi' - 2\nabla\phi'\cdot\nabla\psi' = 0. \tag{A.12.16}$$

Referring to (a) and applying (A.12.13),

$$\log\phi' = \log\rho - \log\phi,$$
$$\nabla^2(\log\phi') = -\nabla^2(\log\phi).$$

From (b) and (c),

$$(\nabla\psi')^2 = -\frac{\rho^2}{\phi^4}(\nabla\psi)^2.$$

Hence, referring to (A.12.14),

$$\begin{aligned}\phi'\nabla^2\phi' - (\nabla\phi')^2 + (\nabla\psi')^2 &= \phi'^2\nabla^2(\log\phi') + (\nabla\psi')^2\\ &= -\frac{\rho^2}{\phi^4}[\phi^2\nabla^2(\log\phi) + (\nabla\psi)^2]\\ &= 0. \end{aligned} \tag{A.12.17}$$

Equations (A.12.17) and (A.12.16) are respectively identical in form with (A.12.2) and (A.12.3), which proves the validity of transformation β. □

A third transformation denoted by α is merely γ (with $c = 1$ and $d = 0$) followed by β:

$$\alpha = \beta \circ \gamma.$$

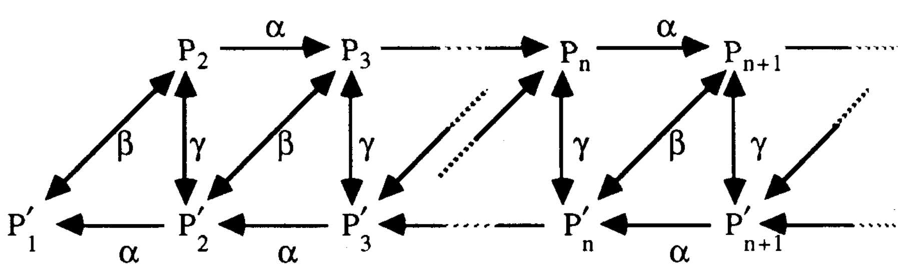
Note that

$$(\beta\circ\gamma)P_n = P_{n+1},$$

whereas

$$(\gamma\circ\beta)P_n = P_{n-1}.$$

The solutions P'_n, $n \geq 1$, and P_n, $n \geq 2$, and their relationships with transformations α, β, and γ (with $c = 1$ and $d = 0$) are displayed in the following diagram:



A.13 Muir and Metzler, A Treatise on the Theory of Determinants

A Table of Contents

This page intentionally left blank

Bibliography

MR = Mathematical Reviews
Zbl = Zentralblatt für Mathematik
PA = Physics Abstracts

A. Abian, A direct proof of Taylor theorems. *Nieuw Arch. Wisk IV*, Ser. 3 (1985), 281–283. [Zbl **589** (1986), 41026.]

M. Abramowitz, I.A. Stegun (Eds.), *Handbook of Mathematical Functions*, Dover Pub. Inc., New York, 1965.

H. Aden, B. Carl, On realizations of solutions of the KdV equation by determinants on operator ideals. *J. Math. Phys.* **37** (1996), 1833–1857. [MR 96m: 58099].

A.C. Aitken, Note on a special persymmetric determinant. *Ann. Math.* Ser. 2, **32** (1931), 461–462.

A.C. Aitken, *Determinants and Matrices*, Oliver and Boyd, London, 1951.

N.I. Akhiezer, *The Classical Moment Problem*, Oliver and Boyd, London, 1965.

W.A. Al-Salam, L. Carlitz, Some determinants of Bernoulli, Euler and related numbers. *Port. Math.* **18** (1959), 91–99. [Zbl **93** (1962), 15.]

American Mathematical Monthly, Index for the years 1894–1949. Determinants, 19–20.

A. Anderson, An elegant solution of the n-body Toda problem. *J. Math. Phys.* **37** (1996), 1349–1355. [PA (1996), 59183.]

T.W. Anderson, Evaluation of the expected value of a determinant. *Statist. Probab. Lett.* **8** (1989), 25–26. [MR 90h: 62121.]

T.H. Andres, W.D. Hoskins, R.G. Stanton, The determinant of a class of skew-symmetric Toeplitz matrices. *Linear. Alg. Applic.* **14** (1976), 179–186. [MR **58** (1979), 5713; Zbl **416** (1980), 15005.]

G.E. Andrews, W.H. Burge, Determinantal identities. *Pacific J. Math.* **158** (1993), 1–14.

J.W. Archbold, *Algebra*, 4th edition, Pitman, London, 1970.

N. Asano, Y. Kato, Fredholm determinant solution for the inverse scattering transform of the $N \times N$ Zakharov-Shabat equation. *Prog. Theor. Phys.* **83** (1990), 1090–1107. [MR 91k: 34133.]

N.B. Backhouse, A.G. Fellouris, On the superdeterminant function for supermatrices. *J. Phys. A* **17** (1984), 1389–1395. [MR 86c: 58014.]

G.A. Baker, P.R. Graves-Morris, Padé Approximants, Parts 1, 2. *Encyclopedia of Mathematics and Its Applications*, Vols. 13 and 14, Addison-Wesley, Reading, MA, 1981. [MR 83a: 41009a, b.]

M. Barnebei, A. Brini, Symmetrized skew-determinants. *Commun. Alg.* **15** (1987), 1455–1468. [MR 89a: 20010.]

W.W. Barrett, C.R. Johnson, Determinantal formulae for matrices with sparse inverses. *Linear Alg. Applic.* **56** (1984), 73–88. [Zbl **523** (1984), 15008.]

E. Barton, Multiplication of determinants. *Math. Gaz.* **47** (1963), 54–55.

E. Basor, Asymptotic formulas for Toeplitz determinants. *Trans. Am. Math. Soc.* **239** (1978), 33–65. [MR **58** (1979), 12484.]

E.L. Basor, A localization theorem for Toeplitz determinants. *Indiana Univ. Math. J.* **28** (1979), 975–983. [MR 81e: 47029.]

E. Basor, J.W. Helton, A new proof of the Szegö limit theorem and new results for Toeplitz operators with discontinuous symbol, *J. Operator Theory* **3** (1980), 23–39. [MR 81m: 47042.]

E. Basor, H. Widom, Toeplitz and Wiener-Hopf determinants with piecewise continuous symbols. *J. Funct. Anal.* **50** (1983), 387–413. [MR 85d: 47026.]

W. Bauhardt, C. Pöppe, The Fredholm determinant method for discrete integrable evolution equations. *Lett. Math. Phys.* **13** (1987), 167–178. [MR 88g: 35165.]

M Bautz, Uber $(0, 1)$-determinanten mit grossen absolutwerten. *Wiss. Z. Tech. Hochsch. Ilmenau* **27** (1981), 39–55. [MR 82m: 15010; Zbl **481** (1982), 15008.]

G. Baxter, P. Schmidt, Determinants of a certain class of non-Hermitian Toeplitz matrices. *Math. Scand.* **9** (1961), 122–128. [MR **23A** (1962), 3949.]

N. Bebiano, J.K. Merikoski, J. da Providencia, On a conjecture of G.N. de Oliveira on determinants. *Linear Multilinear Alg.* **20** (1987), 167–170. [MR 88b: 15005.]

E.F. Bechenbach, R. Bellman, On the positivity of circulant and skew-circulant determinants. General Inequalities. *Proc. First Internat. Conf. Math. Res. Inst. Oberwalfach 1976*, Birkhäuser, Bessel, 1978, Vol. 1, pp. 39–48. [MR **58** (1979), 16715.]

E.F. Beckenbach, W. Seidel, O. Szasz, Recurrent determinants of Legendre and ultaspherical polynomials. *Duke Math. J.* **18** (1951), 1–10. [MR **12** (1951), 702.]

B. Beckermann, G. Muhlbach, A general determinant identity of Sylvester type and some applications. *Linear Alg. Applic.* **197** (1994), 93–112. [MR 95e: 15005].

R. Bellman, A note on the mean value of a random determinant. *Q. Appl. Math.* **13** (1955), 322–324. [MR **17** (1956), 274.]

R. Bellman, On the positivity of determinants with dominant main diagonal. *J. Math. Anal. Applic.* **59** (1977), 210. [MR **56** (1978), 380.]

R. Bellman, On the evaluation of determinants. *Nonlinear Anal. Theory Meth. Applic.* **4** (1980), 733–734. [Zbl **445** (1981), 65025; MR 81g: 65054.]

L.Berg, Über eine identität von W.F. Trench, zwischen der Toeplitzschen und eine verallgemeinerten Vandermondeschen determinante. *Z. Angew. Math. Mech.* **66** (1986), 314–315. [Zbl **557** (1985), 15005; MR 87m: 15019.]

A. Berlinet, Sequence transformations as statistical tools. *Appl. Numer. Math.* **1** (1985), 531–544. [Zbl **628** (1988), 62092.]

S. Beslin, S. Ligh, Another generalization of Smith's determinant. *Bull. Austral. Math. Soc.* **40** (1989), 413–415. [Zbl **675** (1990), 10002.]

P. Binding, Another positivity result for determinantal operators. *Proc. Roy. Soc. Edin.* **86A** (1980), 333–337. [MR 82e: 47002.]

P. Binding, P.J. Browne, Positivity results for determinantal operators. *Proc. Roy. Soc. Edin.* **81A** (1978), 267–271. [MR 80f: 47015.]

P. Binding, P.J. Browne, A definiteness result for determinantal operators. In *Lecture Notes in Mathematics*, *Ordinary Differential Equations and Operators*, Proc. Dundee 1982. [ed. W.N. Everitt, R.T. Lewis.] [MR 86h:47055.]

W.D. Blair, Goldman determinant and a theorem of Frobenius. *Linear Multilinear Alg.* **5** (1977/78), 181–182. [MR **56** (1978), 12031.]

Z. Bohte, Calculation of the derivative of a determinant. *Obzornick Mat. Fiz.* **28** (1981), 33–50 (Slovenian). [MR 82d: 65041.]

Z.I. Borevich, N.A. Vavilov, On determinants in net subgroups. *Zap. Naucln. Semin. Leningr. Otd. Mat. Inst. Steklova* **114** (1982), 37–49. [Zbl **496** (1983), 15008.]

A. Böttcher, Toeplitz determinants with piecewise continuous generating function. *Z. Anal. Anwend* **1** (1982), 23–39. [Zbl **519** (1984), 47015.]

A. Böttcher, B. Silbermann, Notes on the asymptotic behaviour of block Toeplitz matrices and determinants. *Math. Nachr.* **98** (1980), 183–210. [MR 82j: 47042; Zbl **468** (1982), 47016.]

A. Böttcher, B. Silbermann, The asymptotic behaviour of Toeplitz determinants for generating functions with zeros of integral orders. *Math. Nachr.* **102** (1981), 79–105. [Zbl **479** (1982), 47025.]

A. Böttcher, B. Silbermann, Toeplitz determinants with symbols from the Fisher–Hartwig class. *Soviet Math. Dokl.* **30** (1984), 301–304. [Zbl **588** (1986), 15006.]

A. Böttcher, B. Silbermann, Toeplitz matrices and determinants with Fisher–Hartwig symbols. *J. Funct. Anal.* **63** (1985), 178–214. [Zbl **592** (1987), 47016; MR 87b: 47026.]

D.W. Boyd, The asymptotic behaviour of the binomial circulant determinant. *J. Math. Anal. Applic.* **86** (1982), 30–38. [MR 83f: 10007.]

H.W. Braden, Periodic functional determinants. *J. Phys. A Math. Gen.* **18** (1985), 2127–2140. [Zbl **594** (1987), 34026.]

L. Brand, *Differential and Difference Equations*, Wiley, New York, 1966.

J.P. Brennan, J. Wolfskill, Remarks on the probability the determinant of an $n \times n$ matrix over a finite field vanishes. *Discrete Math.* **67** (1987), 311–313. [MR 86a: 15012.]

J. Brenner, L. Cummings, The Hadamard maximum determinant problem. *Am. Math. Monthly* **79** (1972), 626–630. [MR **46** (1973), 190.]

R.P. Brent, B.D. McKay, Determinants and ranks of random matrices over Z_m. *Discrete Math.* **66** (1987), 35–50. [MR 88h: 15042.]

C. Brezinski, Some determinantal identities in a vector space, with applications. In *Padé Approximation and its Applications*, Bad Honnef 1983, 1–11, Lecture Notes in Math. **1071**, Springer, Berlin-New York, 1984. [Zbl **542** (1985), 41016; MR 85m: 15004.]

P.L. Britten, D.M. Collins, Information theory as a basis for the maximum determinant. *Acta Cryst. A* **38** (1982), 129–132. [PA **85** (1982), 44957.]

J. Browkin Determinants revisited (Polish). *Ann. Soc. Math. Pol. Ser. II, Wiad. Mat.* **27** (1986), 47–58. [Zbl **617** (1988), 15020.]

A. Brown, Solution of a linear difference equation. *Bull. Austral. Math. Soc.* **11** (1974), 325–331. [MR **50** (1975), 13935.]

P.J. Browne, R.V. Nillsen, On difference operators and their factorization. *Can. J. Math.* **35** (1983), 873–897. [MR 85j: 39005.]

D. Bruce, E. Corrigan, D. Olive, Group theoretical calculation of traces and determinants occurring in dual theories. *Nucl. Phys.* **B95** (1975), 427–433.

R.C. Brunet, Pyramidal composition rules for Wronskians upon Wronskians. *J. Math. Phys.* **16** (1975), 1112–1116. [MR **51** (1976), 5858.]

M.W. Buck, R.A. Coley, D.P. Robbins, A generalized Vandermonde determinant. *J. Alg. Combin.* **1** (1992), 105–109. [Zbl **773** (1993/23), 05008.]

A. Bultheel, The asymptotic behaviour of Toeplitz determinants generated by Laurent coefficients of a meromorphic function. *SIAM J. Alg. Discrete Methods* **6** (1985), 624–629. [Zbl **569** (1986), 41026.]

D.J. Buontempo, The determinant of a skew-symmetric matrix. *Math. Gaz.* **66** (1982), 67–69.

J.L. Burchnall, Some determinants with hypergeometric elements. *Q.J. Math.* (Oxford) (2) **3** (1952), 151–157. [MR **14** (1953), 44.]

J.L. Burchnall, A method of evaluating certain determinants. *Proc. Edin. Math. Soc.* (2) **9** (1954), 100–104. [MR **16** (1955), 557.]

J.W. Burgmeier, See A.A. Jagers.

P.J. Burillo Lopez, J. Aguilella Almer, Geometry of Gram's determinants in Hilbert space. *Stochastica* **5** (1981), 5–24. [Zbl **475** (1982), 46019.]

I.W. Busbridge, On skew-symmetric determinants. *Math. Gaz.* **33** (1949), 292–294.

M. Buys, A. Finkel, The inverse periodic problem for Hill's equation with a finite gap potential. *J. Diff. Eqns.* **55** (1984), 257–275. [MR 86a: 34052.]

L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for the degenerate Monge–Ampère equation. *Rev. Mat. Iberoam.* **2** (1987), 19–27. [Zbl **611** (1987), 35029.]

E.R. Caianiello, Theory of coupled quantized fields. *Nuovo Cimento Suppl.* **14** (1959), 177–191. [PA 63 (1960), 3912.]

F. Calogero, Matrices, differential operators and polynomials. *J. Math. Phys.* **22** (1981), 919–934. [PA 84 (1981), 99012.]

F. Calogero, Determinantal representations of the classical polynomials. *Boll. Unione Mat. Ital. VI Ser A* **4** (1985), 407–414. [MR 87i: 33022; Zbl **581** (1986), 33007.]

F. Calogero, Ji Xiaoda, Solvable (nonrelativistic, classical) n-body problems on the line. I, *J. Math. Phys.* **34** (1993), 5659–5670. [PA (1994), 25257.]

F. Calogero, Ji Xiaoda, Solvable n-body problems in multidimensions. *J. Math. Phys.* **35** (1994), 710–733. [PA (1994), 93634.]

L. Carlitz, Hankel determinants and Bernoulli numbers. *Tohoku Math. J.* **5** (1954), 272–276. [MR **15** (1954), 777.]

L. Carlitz, A special determinant. *Proc. Am. Math. Soc.* **6** (1955), 270–272. [MR **16** (1955), 999.]

L. Carlitz, A special determinant. *Am. Math. Monthly* **62** (1955), 242–243. [MR **16** (1955), 989.]

L. Carlitz, A determinant. *Am. Math. Monthly* **64** (1957), 186–188. [MR **19** (1958), 7.]

L. Carlitz, Some cyclotomic determinants. *Bull. Calcutta Math. Soc.* **49** (1957), 49–51. [MR **20** (1959), 3812.]

L. Carlitz, A special determinant. [Problem 1333, 1958, 627] proposed by V.F. Ivanoff, *Am. Math. Monthly.* **66** (1959), 314–315.

L. Carlitz, A determinant connected with Fermat's last theorem. *Proc. Am. Math. Soc.* **10** (1959), 686–690. [MR **21** (1960), 7182.] Also, *Proc. Am. Math. Soc.* **11** (1960), 730–733. [MR **22** (1961), 7974.].

L. Carlitz, A generalization of Maillet's determinant and a bound for the first factor of the class number. *Proc. Am. Math. Soc.* **12** (1961), 256–261. [MR **22** (1961), 12093.]

L. Carlitz, Some operational equations for symmetric polynomials. *Duke Math. J.* **28** (1961), 355–368. [MR **24**A (1962), 20.]

L. Carlitz, Some determinants of q-binomial coefficients. *J. Reine Angew. Math.* **226** (1967), 216–220. [Zbl **162** (1969), 30.]

L. Carlitz, F.R. Olsen, Maillet's determinant. *Proc. Am. Math. Soc.* **6** (1955), 265–269. [MR **16** (1955), 999.]

D. Carlson, On some determinantal inequalitites. *Proc. Am. Math. Soc.* **19** (1968), 462–466.

D. Carlson, Polynomials satisfying a binomial theorem. *J. Math. Anal. Applic.* **32** (1970), 543–548. [MR **42** (1971), 6288.]

P. Cartier, *A Course on Determinants, Conformal Invariance and String Theory*, (Poiana Brasov 1987), Academic Press, Boston, 1989, pp. 443–557. [MR 91a: 58200.]

K.M. Case, Fredholm determinants and multiple solitions. *J. Math. Phys.* **17** (1976), 1703–1706. [MR **54** (1977), 1942.]

P.J. Caudrey, The inverse problem for a general $N \times N$ spectral equation. *Physica* **6D** (1982), 51–66. [PA 86 (1983), 16153.]

P.J. Caudrey, R.K. Dodd, J.D. Gibbon, A new hierarchy of Korteweg–de Vries equations. *Proc. Roy. Soc. Lond. A* **351** (1976), 407–422. [MR 58 (1979), 6760.]

I. Cederbaum, Matrices all of whose elements and subdeterminants are 1, −1 or 0. *J. Math. Phys.* **36** (1957), 351–361. [MR **19** (1958), 1033.]

I. Cederbaum, On the cofactors of zero-sum matrices of higher nullity. *Int. J. Eng. Sci.* **19** (1981), 1639–1641. [Zbl **481** (1982), 15008.]

R. Chalkley, A persymmetric determinant. *J. Math. Anal. Applic.* **187** (1994), 107–117.

S. Chapman, Some ratios of infinite determinants occurring in the kinetic theory of gases. *J. Lond. Math. Soc.* **8** (1933), 266–272.

R.N. Chaudhuri, The Hill determinant: An application to a class of confinement potentials. *J. Phys. A. Math. Gen.* **16** (1983), 209–211. [PA 86 (1983), 30663.]

S-Y. Cheng, S-T. Yau, On the regularity of the Monge–Ampère equation $\det(\partial^2 u/\partial x_i \partial x_j) = F(x,u)$, *Commun. Pure Appl. Math.* **30** (1977), 41–68. [MR **55** (1978), 10727.]

I.Y. Cherdantsev, R.A. Shapirov, Solitons on a finite-gap background in Bullough–Dodd–Jiber–Shabat model. *Int. J. Mod. Phys. A* **5** (1990), 3021–3027. [MR 91f: 35221.]

Z.X. Chi, S.H. Shen, Y.S. Yang, A topological evaluation of determinants (Chinese, English summary). *J. Dakian Inst. Technol.* **21** (1982), 137–141. [MR 84k: 65041.]

J.G. Christiano, J.E. Hall, On the n'th derivative of a determinant of the j'th order. *Math. Mag.* **37** (1964), 215–217.

W.C. Chu, On the evaluation of some determinants with Gaussian q-binomial coefficients (Chinese, English summary). *J. Syst. Sci. Math. Sci.* **8** (1988), 361–366. [MR 89m: 05014.]

D.V. Chudnovsky, G.V. Chudnovsky, The Wronskian formalism for linear differential equations and Padé approximations. *Adv. Math.* **53** (1984), 28–54. [MR 86i: 11038.]

G. Claessens, Bigradiants, Hankel determinants and the Newton–Padé table. *Aequations Math.* **19** (1979), 104–112. [Zbl **418** (1980), 41009.]

R.J. Clarke, The general Fibonacci series. *Int. J. Math. Educ. Sci. Technol.* **18** (1987), 149–151.

J.H.E. Cohn, On the value of determinants. *Proc. Am. Math. Soc.* **14** (1963), 581–588. [Zbl **129** (1967), 267.]

J.H.E. Cohn, On determinants with elements ±1, I. *J. Lond. Math. Soc.* **42** (1967), 436–442. [MR **35** (1968), 5344; II. *Bull. Lond. Math. Soc.* **21** (1989), 36–42. [MR 90d: 05051.]

G.E. Collins, Polynomial remainder sequences and determinants. *Am. Math. Monthly* **73** (1966), 708–712. [MR **33** (1967), 7365.]

L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

P.C. Consul, Some factorable determinants. *Fibonnacci Quart.* **14** (1976), 171–172. [MR **53** (1977), 5626.]

C.M. Cordes and D.P. Roselle, Generalized frieze patterns. *Duke Math. J.* **39** (1972), 637–648. [MR **47** (1974), 3209.]

A. Corduneanu, Natural generalization of Vandermonde determinants (Romanian). *Gaz. Mat. Perfect. Metod. Metodol. Nat. Inf.* **11** (1990), 31–34. [Zbl **722** (1991), 15008.]

H. Cornille, Differential equations satisfied by Fredholm determinants and application to the inversion formalism for parameter dependent potentials. *J. Math. Phys.* **17** (1976), 2143–2157. [PA 80 (1977), 22437.]

H. Cornille, Generalization of the inversion equations and application to nonlinear partial differential equations. I. *J. Math. Phys.* **18** (1977), 1855–1869. [PA 81 (1978), 8223.]

C.M. Cosgrove, New family of exact stationary axisymmetric gravitational fields generalizing the Tomimatsu–Sato solutions. *J. Phys. A* **10** (1977), 1481–1524. [MR **58** (1979), 20168.]

T.M. Cover, J.A. Thomas, Determinant inequalities via information theory. *SIAM J. Matrix Anal. Applic.* **9** (1988), 384–392. [MR 89k: 15028.]

W.B. Craggs, The Padé table and its relations to certain algorithms of numerical analysis. *SIAM Rev.* **14** (1972), 1–62.

T. Crilly, Half-determinants: An historical note [Pfaffians]. *Math. Gaz.* **66** (1982), 316.

T.W. Cusick, Identities involving powers of persymmetric determinants. *Proc. Camb. Phil. Soc.* **65** (1969), 371–376. [MR **38** (1969), 5802.]

P. Dale, Axisymmetric gravitational fields: A nonlinear differential equation that admits a series of exact eigenfunction solutions. *Proc. Roy. Soc. London A* **362** (1978) 463–468. [MR 80b: 83011.]

P. Dale, Functional determinants containing Vein's numbers. Tech. Rep. TR 91001, Dept. Comp. Sci. & Appl. Math., Aston University, Birmingham, U.K, 1991.

P. Dale, P.R. Vein, Determinantal identities applicable to the solution of the Benjamin–Ono and Kadomtsev–Petviashvili equations. Tech. Rep. TR 90004, Dept. Comp. Sci. & Appl. Math., Aston University, Birmingham, U.K, 1990.

M.K. Das, On certain determinants for the classical polynomials. *J. Annamalai Univ. Part B. Sci.* **26** (1965), 127-135. [MR **32** (1966), 1385.]

M.K. Das, On some determinants whose elements are orthogonal polynomials. *Math. Japan.* **11** (1966), 19–25. [MR **34** (1967), 2965.]

M.K. Das, Determinants related to the generalized Laguerre polynomials. *J. Natur. Sci. Math.* **9** (1969), 67–70. [Zbl **186** (1970), 105.]

K.M. Day, Toeplitz matrices generated by a Laurent series expansion of an arbitrary rational function. *Trans. Am. Math.Soc.* **206** (1975), 224–245. [MR **52** (1976), 708; Zbl **324** (1976), 47016.]

H. Dette, New identities for orthogonal polynomials on a compact interval. *J. Math. Anal. Applic.* **179** (1993), 547–573.

C. De Concini, D. Eisenbud, C. Procesi, Young diagrams and determinantal varieties. *Invent. Math.* **56** (1980), 129–165. [Zbl **435** (1981), 14015.]

A. De Luca, L.M. Ricciardi, R. Vasudevan, Note on relating Pfaffians and Hafnians with determinants and permanents. *J. Math. Phys.* **11** (1970), 530–535.

W. Dietz, C. Hoensalaers, A new representation of the HKX transformation. *Phys. Lett.* **90A** (1982), 218–219. [MR 83i:83019.]

J. Dieudonné, Une proprieté des racines de l'unité. *Rev. Un. Mat. Argentina* **25** (1970), 1–3. [Zbl **229** (1972), 12002.]

P. Dillen, Polynomials with constant Hessian determinant. *J. Pure Appl. Alg.* **71** (1991), 13–18. [MR 92b: 13011.]

J.D. Dixon, How good is Hadamard's inequality for determinants? *Can. Math. Bull.* **27** (1984), 260–264. [MR 85e: 15009; Zbl **554** (1985), 15011.]

C.L. Dodgson, [Lewis Carroll] Condensation of determinants. *Proc. Roy. Soc. London* **15** (1866), 150–155.

A. Dold, B. Eckmann (eds.), *Backlund Transformations, Nashville, Tennessee*, *Lecture Notes in Mathematics* **515**, Springer-Verlag, Berlin, 1974.

R.Z. Dordevic, G.V. Milananovic, V. Gradimir, On a functional equation having determinant form. *Pub. Fac. Electrotechn. Univ. Belgrade. Ser. Math. Phys.* 498–541, 85–90 (1975). [Zbl **316** (1976), 39011.]

H. Dörrie, *Determinanten* (in German), Oldenbourg, Munich, 1940.

M.P. Drazin, A note on skew-symmetric matrices. *Math. Gaz.* **36** (1952), 253–255. [MR **15** (1954), 497.]

A. Dresden, *Solid Analytical Geometry and Determinants*, Dover, New York, 1965. [Reviewed in *Math. Gaz.* **50** (1966), 96–97.]

A.W.M. Dress, W. Wenzel, A simple proof of an identity concerning Pfaffians of skew-symmetric matrices. *Adv. Math.* **112** (1995), 120–134. [MR 96b: 15018.]

F. Dress, Déterminants de Hankel du quotient de deux séries entières à coefficients entiers. *C.R. Acad. Sci. Paris* **256** (1963), 4338–4340. [Zbl **113** (1965), 34.]

F. Dress, Déterminants de Hankel du quotient de deux séries entières. *Théorie des Nombres, Sém. Delange–Pisot* **4** (1962/63). [Zbl **234** (1972), 15011.]

F. Dress, Déterminants de Hankel et quotient de séries formelles. *Théorie des Nombres, Sém. Delange–Pisot–Poitou* **7** (1965/66). [Zbl **234** (1972), 15012.]

H.J.A. Duparc, On some determinants. *Nederl. Akad. Wetensch, Proc.* **50** (1947), 157–165. *Indagations Math.* **9** (1947), 120–128. [Zbl **29** (1948), 337; MR **8** (1947), 431.]

F.J. Dyson, Fredholm determinants and inverse scattering problems. *Commun. Math. Phys.* **47** (1976), 171–183. [Zbl **323** (1976), 33008.]

A. Edrei, Sur les déterminants récurrents et les singularités d'une fonction donnée pas son development de Taylor. *Compos. Math.* **7** (1939), 20–88. [MR **1** (1940), 210.]

N.V. Efimov, The symmetrization of a determinant. *Tr. Tbilis. Mat. Inst. Razmadze* **64** (1980), 54–55. [Zbl **493** (1983), 15005.]

U. Elias, Minors of the Wronskian of the differential equation $L_n y + p(x)y = 0$. *Proc. Roy. Soc. Edin.* **106A** (1987), 341–359. [Zbl **632** (1988), 34027.]

U. Elias, Minors of the Wronskian of the differential equation $L_n y + p(x)y = 0$. II. Dominance of solutions. *Proc. Roy. Soc. Edin.* **108** (1988), 229–239. [Zbl **655** (1989), 34038.]

A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, McGraw-Hill, New York, 1953.

F.J. Ernst, Complex potential formulation of the axially symmetric gravitational field problem. *J. Math. Phys.* **15** (1974), 1409–1412. [MR **49** (1975), 12028.]

R.J. Evans, I.M. Isaacs, Generalized Vandermonde determinants and roots of unity of prime order. *Proc. Am. Math. Soc.* **58** (1976), 51–54. [MR **54** (1977), 332.]

T. Fack, Proof of the conjecture of A. Grothendieck on the Fuglede–Kasidon determinant. *J. Func. Anal.* **50** (1983), 215–218. [MR 84m: 46081.]

K. Fan, J. Todd, A determinantal inequality. *J. Lond. Math. Soc.* **30** (1955), 58–64. [MR **16** (1955), 664.]

D.R. Farenik, P.J. Browne, Geršgorin theory and the definiteness of determinantal operators. *Proc. Roy. Soc. Edin.* **106A** (1987), 1–10. [MR 88m: 47004.]

J. Farges, Solution to Problem 83-11, A nonnegative determinant (Proposed G.N. Lewis). *SIAM Rev.* **26** (1984), 431.

C. Farmer, A certain sequence of infinite determinants. *J. Math. Anal. Applic.* **34** (1971), 228–232. [Zbl **188** (1970), 76.]

A.B. Farnell, A special Vandermondian determinant. *Am. Math. Monthly* **66** (1959), 564–569. [MR **21** (1960), 6383.]

D. Fearnley-Sander, A characterization of the determinant. *Am. Math. Monthly* **82** (1975), 838–840. [MR **53** (1977), 2977.]

M. Fiedler, Remarks on the Schur complement. *Linear Alg. Applic.* **39** (1981), 189–196. [Zbl **465** (1982), 15004.]

M. Fiedler, On a conjecture of P.R. Vein and its generalization. *Linear Multilinear Alg.* **16** (1984), 147–154. [Zbl **554** (1985), 15002; MR 86g: 15004.]

H. Filesta, A new method for solving a class of ballot problems. *J. Comb. Theory A* **39** (1985), 102–111. [Zbl **579** (1986), 05005.]

N.J. Fine, Solution of Problem 5165 (1964, 99). Alternant type determinant, proposed by A.S. Galbraith. *Am. Math. Monthly* **72** (1965), 91–92.

A.M. Fink, Certain determinants related to the Vandermonde. *Proc. Am. Math. Soc.* **38** (1973), 483–488. [MR **47** (1974), 239.]

M.E. Fisher, R.E. Hartwig, Toeplitz determinants: Some applications, theorems and conjectures. *Adv. Chem. Phys.* **15** (1968), 333–353.

D. Foata, The noncommutative version of the matrix inversion formula. *Adv. Math.* **31** (1979), 330–349. [MR 80f: 15006.]

P.J. Forrester, T.D. Hughes, Complex Wishart matrices and conductance in mesoscopic systems. Exact results. *J. Math. Phys.* **35** (1994), 6736–6747.

A. Frankel, Solution of Problem 71-2 on Bigradients, proposed by A.S. Householder. *SIAM Rev.* **14** (1972), 495–496.

J.S. Frame, Factors of the binomial circulant determinant. *Fibonacci Quart.* **18** (1980), 9–23. [Zbl **432** (1981), 10005.]

N.C. Freeman, Soliton solutions of non-linear evolution equations, *IMA J. Appl. Math.* **32** (1984), 125–145. [Zbl **542** (1985), 35079.]

N.C. Freeman, J.J.C. Nimmo, Soliton solutions of the Korteweg–de Vries and the Kadomtsev–Petviashvili equations: The Wronskian technique. *Proc. Roy. Soc. London* **389** (1983), 319–329. *Sci. Abstr. A* **86** (1983), 117002. [MR 85f: 35175; Zbl **588** (1986), 35077.]

N.C. Freeman, C.R. Wilson, J.J.C. Nimmo, Two-component KP hierarchy and the classical Boussinesq equation. *J. Phys. A Math. Gen.* **23** (1990), 4793–4803. [PA **94** (1991), 20961.]

M.G. Frost, A. Sackfield, Polynomials of the form Fas determinants. *J. Inst. Math. Applic.* **16** (1975), 389–392. [MR **53** (1977), 13670.]

B. Fuglede, R.V. Kadison, Determinant theory in finite factors. *Ann. Math.* **55** (1952), 520–530. [MR **13** (1952), 255; **14** (1953), 660.]

S Fujii, F. Kako, N. Mugibayashi, Inverse method applied to the solution of nonlinear network equations describing a Volterra system. *J. Phys. Soc. Japan* **42** (1977), 335–340.

C.M. Fulton, Product of determinants by induction. *Am. Math. Monthly* **61** (1954), 344–345.

R.E. Gamboa-Savari, M.A. Muschietti, J.E. Solomin, On perturbation theory for regularized determinants of differential operators. *Commun. Math. Phys.* **89** (1983), 363–373. [MR 85i: 81046.]

F.R. Gantmacher, *The Theory of Matrices*, 2 vols., Chelsea, New York, 1960.

M. Gasca, A. Lopez-Carmona, V. Ramirez, A generalized Sylvester's identity on determinants and its application to interpolation problems. Multivariate Approx. Theory II, Proc. Conf. Oberwolfach 1982, ISNM 61, 171–184. [Zbl **496** (1983), 41002; MR 86h: 15006.]

A.G. Gasparjan, Application of multi-dimensional matrices to the investigation of polynomials (Russian). *Akad. Nauk. Armjan. SSR Dokl.* **70** (1980), 133–141. [MR 82c: 15035.]

A.S. Gasparyan, An analogue of the Binet–Cauchy formula for multidimensional matrices. *Soviet Math. Dokl.* **28** (1983), 605–610. [Zbl **554** (1985), 15001.]

G. Gasper, On two conjectures of Askey concerning normalized Hankel determinants for the classical polynomials. *SIAM J. Math. Anal.* [Zbl **229** (1972), 33016.]

I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, Hyperdeterminants. *Adv. Math.* **96** (1992), 226–263. [Zbl **774** (1993), 15002.]

B. Germano, P.E. Ricci, General systems of orthogonal functions (Italian, English summary). *Rend. Mat.* **3** (1983), 427–445. [MR 85g: 42025.]

J.S. Geronimo, Szegö's theorem for Hankel determinants. *J. Math. Phys.* **20** (1979), 484–491. [Zbl **432** (1981), 33008.]

J. Geronimus, On some persymmetric determinants. *Proc. Roy. Soc. Edin.* **50** (1929/1930), 304–309.

J. Geronimus, On some problems involving the persymmetric determinants. *Proc. Roy. Soc. Edin.* **51** (1931), 14–18. [Zbl **2** (1932), 113.]

J. Geronimus, On some persymmetric determinants formed by the polynomials of Appell. *J. Lond. Math. Soc.* **6** (1931), 55–59. [Zbl **1** (1931), 194.]

S.R. Ghorpade, Young multitableux and higher dimensional determinants. *Adv. Math.* **121** (1996), 167–195.

V.L. Girko, Theory of random determinants (in Russian) Višča Škola, Kiev, 1980. Review (in Russian). *Ukrainski Math. J.* **33** (1981), 575–576. [MR 82h: 60002; Zbl **628** (1988), 15021.]

V.L. Girko, The central limit theorem for random determinants. *SIAM Theory Prob. Applic.* **26** (1981), 521–531. [MR 83a: 60037.]

V.L. Girko, V.V. Vasilev, Limit theorems for determinants of random Jacobi matrices. *Theory Prob. Math. Statist.* **24** (1982), 19–29. [MR 83a: 60038.]

I. Gissell, G. Viennot, Binomial determinants, paths and hook length formulae. *Adv. Math.* **58** (1985), 300–321. [Zbl **579** (1986), 05004.]

M. Goldberg, The derivative of a determinant. *Am. Math. Monthly* **79** (1972), 1124–1126. [MR **47** (1974), 3732.]

R.I. Gonzalez, R.V. Castellanos, Generalized Vandermonde determinant, (Spanish). *Gac. Mat.* **1**, Ser. 31, (1979), 122–125. [Zbl **448** (1981), 15007.]

I.J. Good, A note on positive determinants. *J. Lond. Math. Soc.* **22** (1947), 92–95. [MR **9** (1948), 273.]

I.J. Good, A short proof of MacMahon's Master Theorem. *Proc. Camb. Phil. Soc.* **58** (1962), 160. [MR **25** (1963), 1109; Zbl **108** (1964), 251.]

I.J. Good, T.N. Tideman, Generalized determinants and generalized variance. *J. Statist. Comput. Simul.* **12** (1980/81), 311–315.

R.L. Goodstein, The generalized Vandermonde determinant. *Math. Gaz.* **54** (1970), 264–266.

E.T. Goodwin, Note on the evaluation of complex determinants. *Proc. Camb. Phil. Soc.* **46** (1950), 450–452. [MR **12** (1951), 3.]

J.G. de Gooijer, On a recurrence relation for a special type of determinant. *Matrix Tensor Q.* **30** (1979), 55–60, 61, 80. [Zbl **434** (1981), 15001.]

J.A. Gordon, On the non-vanishing of determinants of matrices of confluent Vandermonde type. *Contributions to Probability and Mathematical Statistics, Teaching of Mathematics and Analysis*, Dedicated to D. Alfonso Guiraun Martin on the occasion of his Academic Jubilee, (in Spanish), (1979), 290–295.] [Zbl **472** (1982), 41003.]

I.P. Goulden, D.M. Jackson, Ballot sequences and a determinant of Good's. *J. Comb. Theory A* **40** (1985), 55–61. [MR 86m: 05011.]

A. Granville, On a class of determinants. *Fibonacci Quart.* **27** (1989), 253–256. [MR 90d: 15006.]

P.R. Graves-Morris, (Ed.), *Padé Approximants and Their Applications*, Academic Press, New York, 1973. [Zbl **277** (1974), 00009.]

P.R. Graves-Morris, Vector valued rational interpolants II. *IMA J. Num. Anal.* **4** (1984), 209–224. [MR 85h: 65023.]

U. Grenander, G. Szegö, *Toeplitz Forms and Their Applications*, University of California Press, Berkely, 1958. [MR **20** (1959), 1349.]

B. Grone, C. Johnson, E. Marques de Sá, H. Wolkowicz, Improving Hadamard's inequality. *Linear Multilinear Alg.* **16** (1984), 305–322. [Zbl **548** (1985) 15014.]

S. Guerra, Su un determinante collegato ad un sistema di polinomi ortogonali. *Rend. Ist. Mat. Univ. Trieste* **10** (1978), 66–79. [Zbl **412** (1980), 33012; MR **53** (1977), 13264.]

E.N. Güichal, A relation between Gram and Wronsky determinants. *Linear Alg. Applic.* **72** (1985), 59–72. [Zbl **588** (1986), 41021.]

F. Guil, M. Manas, Finite rank constraints on linear flows and the Davey–Stewartson equation. *J. Phys. A: Math. Gen.* **28** (1995), 1713–1726.

B. Gyires, Über determinanten, deren elemente integrale von funktionen sind. *Acta Univ. Debrecen. Ludovico Kossuth* **3** (1957), 41–48. [MR **19** (1958), 945; Zbl **85** (1960), 10.]

T.T. Ha, J.A. Gibson, A note on the determinant of a functional confluent Vandermonde matrix and controllability. *Linear Algebra Applic.* **30** (1980), 69–75. [Zbl **445** (1981), 15009.]

L. Halada, A recursive computation of the determinant of a band triangular matrix. Numerical methods (Third Colloq. Keszthely, 1977), 325–333. [MR 82d: 65042.]

J.H. Halton, A combinatorial proof of Cayley's theorem on Pfaffians. *J. Comb. Theory* **1** (1966), 224–232. [MR **33** (1967), 5647.]

J.H. Halton, An identity of the Jacobi type for Pfaffians. *J. Comb. Theory* **1** (1966), 333–337. [MR **34** (1967), 75.]

M.A. Hamdan, A type of determinant related to the Laguerre polynomials. *J. Natur. Sci. Math.* **6** (1966), 237–239. [Zbl **154** (1968), 64.]

W.L. Hart, Summable infinite determinants. *Bull. Am. Math. Soc.* **28** (1922), 171–178.

D.J. Hartfiel, R. Loewy, A determinantal version of the Frobenius–König theorem. *Linear Multilinear Alg.* **16** (1984), 155–166. [Zbl **551** (1985), 15002.]

R. Hartwig, M.E. Fisher, Asymptotic behaviour of Toeplitz matrices and determinants. *Arch. Rat. Mech. Anal.* **32** (1969), 190–225. [MR **38** (1969), 4888.]

Y. Hase, R. Hirota, Y. Ohta, J. Satsuma, Soliton solutions of the Melnikov equations. *J. Phys. Soc. Japan* **58** (1989), 2713–2720. [PA **92** (1989), 139048.]

P. Haukkanen, Higher dimensional ged matrices. *Linear Alg. Applic.* **170** (1992), 53–63. [Zbl **749** (1992), 15013.]

A. Hautot, On the Hill-determinant method. *Phys. Rev. D* **33** (1986), 437–443. [MR 87h: 81027.]

P. Henrici, The quotient difference algorithm. *NBS Appl. Math. Ser.* **49** (1958), 23–46. [MR **20** (1959), 1410.]

J. Hietarinta, R. Hirota, Multidromian solutions of the Davey–Stewartson equation. *Phys. Lett. A* **145** (1990), 237–244. [PA **93** (1990), 78590.]

R. Hirota, Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.* **27** (1971), 1192–1194. [PA **74** (1971), 81290.]

R. Hirota, Solutions of the classical Boussinesq equation and the spherical Boussinesq equation: The Wronskian technique. *J. Phys. Soc. Japan* **55** (1986), 2137–2150. [PA **89** (1986), 120052.]

R. Hirota, Discrete two-dimensional Toda molecule equation. *J. Phys. Soc. Japan* **56** (1987), 4285–4288. [MR 88k: 39004.]

R. Hirota, Exact solutions of the spherical Toda molecule equation. *J. Phys. Soc. Japan* **57** (1988), 66–70. [MR 89d: 35157.]

R. Hirota, Soliton solutions of the BKP equations. I. The Pfaffian technique. *J. Phys. Soc. Japan* **58** (1989), 2285–2296. [MR 90i: 58072.]

R. Hirota, Soliton solutions of the BKP equations. II. The integral equation. *J. Phys. Soc. Japan* **58** (1989), 2703–2712. [PA **92** (1989), 138997; MR 91b: 35101.]

R. Hirota, M. Ito, F. Kako, Two-dimensional Toda lattice equations. *Prog. Theoret. Phys. Suppl.* **94** (1988), 42–58. [MR 89k: 35211.]

R. Hirota, A. Nakamura, Exact solutions of the cylindrical Toda molecule equation. *J. Phys. Soc. Japan* **56** (1987), 3055–3061. [MR 89b: 35144.]

R. Hirota, Y. Ohta, Hierarchies of coupled soliton equations I. *J. Phys. Soc. Japan* **60** (1991), 789–809. [PA **94** (1991), 71670.]

R. Hirota, Y. Ohta, J. Satsuma, Solutions of the Kadomtsev–Petviashvili equation and the two-dimensional Toda equations, *J. Phys. Soc. Japan* **57** (1988), 1901–1904. [MR 89i: 58049.]

K.A. Hirsch, On the generalized Vandermonde determinant. *Math. Gaz.* **34** (1950), 118–120.

H. Hochstadt, On the evaluation of the Wronskian determinant. *Am. Math. Monthly* **75** (1968), 767–772. [Zbl **167** (1969), 370.]

T. Hoholdt, J. Justesen, Determinants of a class of Toeplitz matrices. *Math. Scand.* **43** (1978), 250–258. [MR 82m: 30005.]

E. Honig, Intrinsic geometry of Killing trajectories. *J. Math. Phys.* **17** (1976), 2169–2174. [PA **80** (1977), 22393.]

S. Hori, On the exact solution of the Tomimatsu–Sato family for an arbitrary integral value of the deformation parameter. *Prog. Theoret. Phys.* **59** (1978), 1870–1891. [PA **81** (1978), 89377.]

A.S. Householder, The Padé table, the Frobenius identities and the qd algorithm. *Linear Alg. Applic.* **4** (1971), 161–174. [Zbl **213** (1971), 164.]

J.L. Howland, A method for computing the real roots of determinantal equations. *Am. Math. Monthly* **68** (1961), 235–239. [MR **26** (1963), 3182.]

P.-L. Hsu, *Collected Papers.* (editor, K.L. Chung). (Two papers on determinantal equations). Springer-Verlag, New York, 1983. [MR 85e: 01062.]

J. Hylán, Asymptotic properties of the Wronski determinant of a certain class of linear differential equations of the second order. *Cas. Pestování Mat.* **110** (1985), 13–18. [Zbl **594** (1987), 34055.]

K. Iguchi, A class of new invariant surfaces under the trace maps for n ary Fibonacci lattices. *J. Math. Phys.* **35** (1994), 1008–1019. [MR 95a: 82055.]

T. Imai, T. Fukuyama, Aitken acceleration and stationary axially symmetric solutions of Einstein's equation. *J. Phys. Soc. Japan* **64** (1995), 3682–3687.

E. Inönü, Orthogonality of a set of polynomials encountered in neutron-transport and radiative-transfer problems. *J. Math. Phys.* **11** (1970), 568–577.

A. Inselberg, On determinants of Toeplitz–Hessenberg matrices arising in power series. *J. Math. Anal. Applic.* **63** (1978), 347–353. [MR **58** (1979), 718.]

D.V. Ionescu, Une identité importante et la décomposition d'une forme bilinéaire en une somme de produits. *Gaz. Mat. Fiz.* **A7** (1955), 303–312. [MR **17** (1956), 229.]

I.S. Iohvidov, *Hankel and Toeplitz Matrices and Forms. Algebraic Theory*, Birkhäuser, Boston, 1982. [MR 83k: 15021. [MR **51** (1976), 11172.]

A.G. Izergin, D.A. Coker, V.E. Korepin, Determinantal formula for the six-vertex model. *J. Phys. A: Math. Gen.* **25** (1992), 4315–4334. [PA (1992), 127496.]

P.N. Izvercianu, Appell functions of $n-1$ real arguments. *Bul. Sti. Tehn. Inst. Politehn., Timisoara* **13** (1968), 13–19. [Zbl **217** (1972), 111.]

E. Jacobsthal, Über eine determinante. *Norske Vid. Selsk. Forh. Trondheim* **23** (1951), 127–129. [MR **13** (1952), 98.]

A.A. Jagers, Solution of Problem E 2769 [1979, 307] proposed by J.W. Burgmeier [A determinant involving derivatives]. *Am. Math. Monthly* **87** (1980), 490.

A.A. Jagers, Solution of Problem 82-8, proposed by N.J. Boynton. [Zeros of a determinant]. *SIAM Rev.* **25** (1983), 271–273.

D.V. Jaiswal, On determinants involving generalized Fibonacci numbers. *Fibonacci Quart.* **7** (1969), 319–330. [Zbl **191** (1970), 45.]

D.G. James, On the automorphisms of $\det(x_{ij})$. *Math. Chron.* **9** (1980), 35–40. [MR 81m: 10044.]

G.D. James, G.E. Murphy, The determinant of the Gram matrix for a Specht module. *J. Alg.* **59** (1979), 222–235. [MR 82j: 20025.]

C.R. Johnson, W.W. Barrett, Determinantal inequalities for positive definite matrices. *Discrete Math.* **119** (1993), 97–106.

V.N. Joshi, A determinant for rectangular matrices. *Bull. Austral. Math. Soc.* **21** (1980), 137–146. [MR 81k: 15005; Zbl **421** (1980), 15007.]

T. Jozefiak, P. Pragacz, A determinantal formula for skew Q-functions. *J. Lond. Math. Soc.* **43** (1991), 76–90. [MR 92d: 05175.]

D.G. Kabe, Solution of Problem 5312 [1965, 795] proposed by D.S. Mitrinovic [A Vandermonde operator]. *Am. Math. Monthly* **73** (1966), 789.

T. Kaczorek, Extension of the method of continuants for n-order linear difference equations with variable coefficients. *Bull. Polish Acad. Sci. Technol. Sci.* **33** (1985), 395–400. [MR 89b: 39006.]

K. Kajiwara, J. Satsuma, q-difference version of the two-dimensional Toda lattice equation. *J. Phys. Soc. Japan* **60** (1991), 3986–3989. [PA (1992), 39224.]

K. Kajiwara, Y. Ohta, J. Satsuma, B. Grammaticos, Casorati determinant solutions for the discrete Painlevé II equation. *J. Phys. A: Math. Gen.* **27** (1994), 915–922.

K. Kajiwara, Y. Ohta, J. Satsuma, Cesorati determinant solutions for the discrete Painlevé III equation. *J. Math. Phys.* **36** (1995), 4162–4174. [MR 97j: 39012.]

K. Kajiwara, Y. Ohta, Determinant structure of the rational solutions for the Painlevé IV equation. *J. Phys. A: Math. Gen.* **31** (1998), 2431–2446.

S. Kakei, Toda lattice hierarchy and Zamolodchikov's conjecture. *J. Phys. Soc. Japan* **65** (1996), 337–339.

S. Kakei, J. Satsuma, Multi-soliton solutions of a coupled system of the nonlinear Schrödinger equation and the Maxwell–Bloch equations. *J. Phys. Soc. Japan* **63** (1994), 885–894.

S.A. Kamal, Determinant of a general tensor. *Matrix Tensor Quart.* **31** (1980/81), 64–66. [MR 82j: 15025; Zbl **524** (1984), 15005.]

F. Kaminski, On a method of order reduction for a tridiagonal determinant. *Bull. Acad. Pol. Sci. Ser. Sci. Technol.* **29** (1981), 75–81, 419–425. [Zbl **487** (1983), 15005.]

T. Kanzaki, Y Watanabe, Determinants of r-fold symmetric multilinear forms. *J. Alg.* **124** (1989), 219–229. [MR 90g: 11052.]

W. Kaplan, *On Circulants. Complex Analysis*, Birkhäuser, Boston, 1988, pp. 121–130. [MR 90i: 15008.]

S. Karlin, Determinants and eigenfunctions of Sturm–Liouville equations. *J. Anal. Math.* **9** (1961/62), 365–397. [MR **24**A (1962), 3321.]

S. Karlin, J. McGregor, Determinants of orthogonal polynomials. *Bull. Am. Math. Soc.* **68** (1962), 204–209. [MR **25** (1963), 2250.]

S. Karlin, J. McGregor, Some properties of determinants of orthogonal polynomials. *Theory and Applications of Special Functions* (editor: R.A. Askey), Academic Press, New York, 1975, pp. 521–550.

S. Karlin, G. Szegö, On certain determinants whose elements are orthogonal polynomials. *J. Anal. Math.* **8** (1960/1961), 1–157. [MR **26** (1963), 539.]

I. Katai, E. Rahmy, Computation of the determinant of five-diagonal symmetric Toeplitz matrices. *Ann. Univ. Sci. Budapest Sect. Comput. 1979*, no. 2, (1981), 13–31. [MR 83e: 65081.]

Y. Kato, Fredholm determinants and the solution of the Korteweg–de Vries equation. *Prog. Theoret. Phys.* **76** (1986), 768–783. [MR 88c: 35143.]

Y. Kato, Fredholm determinants and the Cauchy problem of a class of nonlinear evolution equations. *Prog. Theoret. Phys.* **78** (1987), 198–213. [MR 89b: 35145.]

H. Kaufman, A bibliographical note on higher order sine functions. *Scripta Math.* **28** (1967), 29–36. [MR **35** (1968), 6871.]

T. Kawashima, On $\det\left[\frac{1}{i+j-1}\right] = \frac{\{(n-1)!\,(n-2)!\cdots 1!\}}{(2n-1)!\,(2n-2)!\cdots n!}$. *Res. Ref. Ashikaga Inst. Technol.* **9** (1983), 149–154 (Japanese, English summary). [Zbl **529** (1984), 15003.]

I. Kay, H.E. Moses, Reflectionless transmission through dielectrics and scattering potentials. *J. Appl. Phys.* **27** (1956), 1503–1508. [PA **60** (1957), 4570.]

D. Kershaw, A note on orthogonal polynomials. *Proc. Edin. Math. Soc.* **17** (1970–71), 83–93. [MR **42** (1971), 3326.]

J.B. Kim, J.E. Dowdy, Determinants of n-dimensional matrices. *J. Korean Math. Soc.* **17** (1980/81), 141–146. [MR 81k: 15006.]

R.C. King, Generalised Vandermonde determinants and Schur functions. *Proc. Am. Math. Soc.* **48** (1975), 53–56. [Zbl **303** (1976), 15004.]

M.S. Klamkin, On the generalization of the geometric series. *Am. Math. Monthly* **64** (1957), 91–93. [Zbl **80** (1959), 43.]

C. Kliorys, Fibonacci number identities from algebraic units. *Fibonacci Quart.* **19** (1981), 149–153. [MR 82j: 15015.]

E. Knobloch, On the prehistory of the theory of determinants. Theoria cum frasi: On the relationship of theory and prascis in the Seventeeth and Eighteenth Centuries, Vol. IV (Hanover, 1977), 96–118, *Studia Leibnitiana Suffl.* **XXII**, Steiner, Wisbanon, 1982. [MR 84a: 01019.]

A. Korsak, C. Schubert, A determinant expression of Tchebyshev polynomials. *Can. Math. Bull.* **3** (1960), 243–246.

G. Kowalewski, *Determinantentheorie* (in German). Chelsea, New York, 1948.

A.M. Krall, On the moments of orthogonal polynomials. *Rev. Roum. Math. Pures Appl.* **27** (1982), 359–362. [Zbl **497** (1983), 33010.]

N. Kravitsky, On determinantal representations of algebraic curves. *Mathematical Theory of Networks and Systems*, Proc. Int. Symp. Beer Sheva, Israel, 1983. *Sect. Notes Control Inf. Sci.* **58** (1984), 584–590. [Zbl **577** (1986), 14041.]

A. Kuniba, S. Nakamura, R. Hirota, Pfaffian and determinantal solutions to a discretized Toda equation for B_r, C_r, D_r. *J. Phys. A: Math. Gen.* **29** (1996), 1759–1766.

M. Kun Kuti, The applications of the determinant K_n (Hungarian, German summary). *Mat. Lapok* **23** (1972), 337–348 (1974) [MR **51** (1976), 540.]

E. Kyriakopoulos, Solutions of the Ernst equation. *Class, Quantum Grav.* **3** (1986), L13. [PA **89** (1986), 61376.]

E. Kyriakopoulos, A general method for deriving Bäcklund transformations for the Ernst equation. *J. Phys. A* **20** (1987), 1669–1676. [PA **90** (1987), 102237.]

J.S. Lamont, M.S. Cheema, A multilinearity property of determinant functions. *Linear Multilinear Alg.* **14** (1983), 199–223. [Zbl **521** (1984), 15018; MR 85f: 15006.]

J.S. Lamont, M.S. Cheema, Properties of Pfaffians. *Rocky Mountain J. Math.* **15** (1985), 493–512. [MR 87c: 15016.]

N.D. Lane, Solution to Problem 4362 [1949, 557] proposed by D.J. Newman [The Hilbert determinant]. *Am. Math. Monthly* **58** (1951), 273.

H.T. Laquer, *Value of Circulants with Integer Entries. A Collection of Manuscripts Related to the Fibonacci Sequence*, Fibonacci Assoc., Santa Clara, CA, 1980, pp. 212–217. [MR 82g: 15011; Zbl **524** (1984), 15007.]

A Lascoux, Inversion des matrices de Hankel. *Linear Alg. Applic.* **129** (1990), 77–102. [MR 91d: 15064.]

J.L. Lavoie, On the evaluation of certain determinants. *Math. Compos.* **18** (1964), 653–659. [MR **30** (1965), 703.]

D.F. Lawden, The function and associated polynomials. *Proc. Camb. Phil. Soc.* **47** (1951), 309–314. [MR **12** (1951), 605; Zbl **42** (1952), 15.]

B. Leclerc, On identities satisfied by minors of a matrix. *Adv. Math.* **100** (1993), 101–132.

J. de Leeuw, A note on partitioned determinants. *Psychometrika* **47** (1982), 531–534. [MR 84h: 15009.]

A. Lenard, Some remarks on large Toeplitz determinants. *Pacific J. Math.* **42** (1972), 137–145. [MR **48** (1974), 9440.]

F.A. Lewis, On a determinant of Sylvester. *Am. Math. Monthly* **57** (1950), 324–326. [MR **11** (1950), 710.]

D.C. Lewis, The multiplication of determinants. *Am. Math. Monthly* **83** (1976), 268–270. [MR **53** (1977), 472.]

C. Li, The maximum determinant of an $n \times n$ lower Hessenberg matrix. *Linear Alg. Applic.* **183** (1993), 147–153. [Zbl **769** (1993/19), 15008.]

Q.-M. Liu, Casorati determinant form of the superposition formula of the two-dimensional Toda lattice. *Phys. Lett. A* **135** (1989), 443–446. [PA **92** (1989), 68413.]

Q.-M. Liu, Transformation formulae of the solutions of the two-dimensional Toda lattices. *J. Phys. A: Math. Gen.* **22** (1989), 4737–4742. [PA **93** (1990), 11916.]

Q.-M. Liu, Double Wronskian solutions of the AKNS and the classical Boussinesq hierarchies. *J. Phys. Soc. Japan* **59** (1990), 3520–3527. [PA **94** (1991), 6983.]

R. Loewy, Determinants of nonprincipal submatrices of positive semidefinite matrices. *Linear Alg. Applic.* **56** (1984), 1–16. [Zbl **523** (1984), 15009.]

J.S. Lomont, M.S. Cheema, Properties of Pfaffians. *Rocky Mountain J. Math.* **15** (1985), 493–512. [MR 87c: 15016.]

L. Lorch, Turanians and Wronskians for the zeros of Bessel functions. *SIAM J. Math. Anal.* **11** (1980), 223–227. [Zbl **446** (1981), 33011.]

I. Loris, R. Willox, Soliton solutions of Wronskian form to the nonlocal Boussinesq equation. *J. Phys. Soc. Japan* **65** (1996), 383–388.

I. Loris, R. Willox, On the solution of the constrained KP equation. *J. Math. Phys.* **38** (1997), 283–291.

O.P. Lossers, Solution of Problem 74-14 (1974) proposed by S. Venit [A generalization of the Vandermonde determinant]. *SIAM Rev.* **17** (1975), 694–695.

P.J. McCarthy, A generalization of Smith's determinant. *Can. Math. Bull.* **29** (1986), 109–113. [Zbl **588** (1986), 10005.]

B.M. McCoy, Physical applications and extensions of Szegö's theorem on Toeplitz determinants. *Abstr. AMS* **3** (1982), 344.

B.R. McDonald, A characterization of the determinant. *Linear Multilinear Alg.* **12** (1982), 31–36. [MR 83j: 15009.]

P.A. Macmahon, On an x-determinant which includes as particular cases both determinants and permanents. *Proc. Roy. Soc. Edin.* **44** (1923/1924), 21–22.

P.A. Macmahon, Researches in the theory of determinants. *Trans. Camb. Phil. Soc.* **23** (1924), 89–135.

P.A. Macmahon, The symmetric functions of which the general determinant is a particular case. *Proc. Camb. Phil. Soc.* **22** (1925), 633–654.

P.A. Macmahon, The structure of a determinant. *J. Lond. Math. Soc.* **2** (1927), 273–286.

P.A. Macmahon, The expansion of determinants and permanents in terms of symmetric functions. Proc. Int. Congress, Toronto, 1928, pp. 319–330.

W. Magnus, Infinite determinants associated with Hill's equation. *Pacific J. Math.* **5** (1955), 941–951. [MR **16** (1955), 261.]

V.L. Makarov, Estimates for the zeros of a certain determinant whose entries are Bessel functions of the first and second kinds (in Russian). *Ukrain. Mat. Z.* **24** (1972), 249–254. [MR **45** (1973), 3808.]

V.L. Makarov, Inequalities of Turán type for nonnormalized ultraspherical polynomials and Jacobi functions of the second kind. *Dopovidl Akad. Nauk. Ukrain. RSR. A* **187** (1972), 124–127. [MR **46** (1973), 418.]

I. Manning, A theorem on the determinantal solution of the Fredholm equation. *J. Math. Phys.* **5** (1964), 1223–1225. [PA **68** (1965), 14.]

M. Marcus, W.R. Gordon, An extension of the Minkowski determinant theorem. *Proc. Edin. Math. Soc.* **17** (1970–71), 321–324. [MR **46** (1973), 7271.]

M. Marcus, E. Wilson, Partial derivations and B. Iverson's linear determinants. *Aequationes Math.* **11** (1974), 123–137. [MR **51** (1976), 5635.]

A.W. Marshall, I. Olkin, A convexity proof of Hadamard's inequality. *Am. Math. Monthly* **89** (1982), 687–688. [MR 84f: 15023.]

S. Maruyama, On representations by determinants of $P(n)$ and $P_m(n)$. *J. Comb. Theory A* **45** (1987), 316–322. [Zbl **619** (1988), 10110.]

J. Matsukidaira, J. Satsuma, Integrable four-dimensional nonlinear lattice expressed by trilinear form. *J. Phys. Soc. Japan* **59** (1990), 3413–3416. [MR 92d: 58082.]

Y. Matsuno, Exact multi-soliton solution of the Benjamin-Ono equation. *J. Phys. A* **12** (1979), 619–621. [PA **82** (1979), 40379.]

Y. Matsuno *Bilinear Transformation Method*, Academic Press, New York, 1984. [MR 86f: 35163.]

Y. Matsuno, New integrable nonlinear integrodifferential equations and related solvable finite-dimensional dynamical systems. *J. Math. Phys.* **29** (1988), 49–56. [PA **91** (1988), 46498; MR 89a: 35192.] Erratum: *J. Math. Phys.* **30** (1989), 241. [MR 89k: 35223.]

Y. Matsuno, A direct proof of the N-soliton solution of the Benjamin–Ono equation by means of Jacobi's formula. *J. Phys. Soc. Japan* **57** (1988), 1924–1929. [MR 89h: 35309; PA **91** (1988), 115127.]

Y. Matsuno, A new proof of the rational N-soliton solution for the Kadomtsev–Petviashvili equation. *J. Phys. Soc. Japan* **58** (1989), 67–72. [PA **92** (1989), 63906.]

Y. Matsuno, New type of algebraic solitons expressed in terms of Pfaffians. *J. Phys. Soc. Japan* **58** (1989), 1948–1961. [PA **92** (1989), 118388.]

J.S. Maybee, D.D. Olegsky, P. van den Driessche, G. Wiener, Matrices, diagraphs and determinants. *Siam J. Matrix Anal. and Applic.* **10** (1989), 500–519. [MR 91b: 15012.]

L. Mejlbo, P. Schmidt, On the eigenvalues of generalized Toeplitz matrices. *Math. Scand.* **10** (1962), 5–16. [Zbl **117** (1965), 329.]

K.V. Menon, A determinant of symmetric forms. *Discrete Maths.* **48** (1984), 87–93. [MR 86b: 15007.]

K.V. Menon, Note on some determinants of q-binomial numbers. *Discrete Math.* **61** (1986), 337–341. [MR 87m: 05016.]

R. Merris, Extensions of the Minkowski determinant theorem. *Portugal Math.* **38** (1982), 149–153. [Zbl **497** (1983), 15004; MR 84c: 15009.]

R. Merris, Extension of the Hadamard determinant theorem. *Israel J. Math.* **46** (1983), 301–304. [MR 85g: 15014.]

L.M. Milne-Thomson, *Calculus of Finite Differences*, Macmillan, New York, 1933.

H. Minc, Bounds for permanents and determinants. *Linear Multilinear Alg.* **9** (1980), 5–16. [MR 81k: 15008; Zbl **435** (1981), 15006.]

J. Minkus, *Circulants and Horadam's Sequences. A Collection of Manuscripts Related to the Fibonacci Sequence*, 48–52. Fibonacci Assoc., Santa Clara, CA, 1980. [MR 82i: 15009; Zbl **524** (1984), 15006.]

L. Mirsky, *An Introduction to Linear Algebra*, Oxford University Press, Oxford, 1955.

L. Mirsky, On a generalization of Hadamard's determinantal inequality due to Szasz. *Arch. Math.* **8** (1957), 174–175. [MR **19** (1958), 936.]

P. Misra, R.V. Patel, A determinant identity and its application in evaluating frequency response matrices. *SIAM J. Matrix Anal. Applic.* **9** (1988), 248–255. [MR 88h: 00009.]

S. Miyake, Y. Ohta, J. Satsuma, A representation of solutions for the KP hierachy and its algebraic structure. *J. Phys. Soc. Japan* **59** (1990), 48–55. [PA **93** (1990), 65448.]

R. Mneimné, Formula de Taylor pour le déterminant et deux applications. *Linear Alg. Applic.* **112** (1989), 39–47. [MR 89m: 15004.]

B.L. Moiseiwitsch, *Integral Equations*, Longman, London, 1977, pp. 126–134.

J.W. Moon, Some determinant expansions and the matrix-tree theorem. *Discrete Math.* **124** (1994), 163–171.

D.A. Moran, Solution of Problem 3087 (1924), proposed by H.W. Bailey [Area of a plane polygon]. *Am. Math. Monthly* **68** (1961), 935–937.

M. Mori, Fredholm determinant for piecewise linear transformations. *Osaka J. Math.* **27** (1990), 81–116. [MR 90a: 28023.]

A.O. Morris, A note on symmetric functions. *Am. Math. Monthly* **71** (1964), 50–53.

K.R. Mount, A remark on determinantal loci. *J. Lond. Math. Soc.* **42** (1967), 595–598. [MR **35** (1968), 6674].

G. Mühlbach, On extending algebraic identities. *Linear Alg. Applic.* **132** (1990), 145–162. [MR 91f: 15022; Zbl **702** (1991), 15002.]

G. Mühlbach, M. Gasca, A generalization of Sylvester's identity on determinants and some applications. *Linear Alg. Applic.* **66** (1985), 221–234. [MR 86h: 15006; Zbl **576** (1986), 15005.]

T. Muir, *The History of Determinants in the Historical Order of Development.* Vol. I, *Origins–1840*, Macmillan, New York 1906; Vol II, *1840–1860*, Macmillan, New York, 1911; Vol. III, *1860–1880*, Macmillan, New York, 1920; Vol. IV, *1880–1900*, Macmillan, New York, 1923; Vol. V, *1900–1920*, Blackie, London, 1930.

T. Muir (revised and enlarged by W.H. Metzler), *A Treatise on the Theory of Determinants*, Dover, New York, 1960. [See Appendix 13 in this book.]

T. Muir, The theory of persymmetric determinants from 1894–1919. *Proc. Roy. Soc. Edin.* **47** (1926–1927), 11–33.

B. Murphy, Expansion of $(n-1)$-rowed sub-determinants. *Math. Z.* **147** (1976), 205–206. [MR **53** (1977), 2980.]

I.S. Murphy, A note on the product of complementary principal minors of a positive definite matrix. *Linear Alg. Applic.* **44** (1982), 169–172. [MR 83g: 15016.]

D. Mustard, Numerical integration over the n-dimensional spherical shell. *Math. Comput.* **18** (1964), 578–589. [MR **30** (1965), 712.]

A. Nagai, J. Satsuma, The Lotke–Volterra equations and the QR algorithm. *J. Phys. Soc. Japan* **64** (1995), 3669–3674. [MR 96h: 92014.]

K. Nagatomo, Explicit description of ansatz E_n for the Ernst equation in general relativity. *J. Math. Phys.* **30** (1989), 1100–1102. [PA **92** (1989), 97900.]

K. Nakamori, The theory of p-dimensional determinants. *Yokahama Math. J.* **6** (1958), 79–88. [MR **21** (1960), 678.]

A. Nakamura, A bilinear N-soliton formula for the KP equation. *J. Phys. Soc. Japan* **58** (1989), 412–422. [PA **92** (1989), 68150; MR 90i: 35257.]

A. Nakamura, Jacobi structures of the n-soliton solutions of the nonlinear Schroedinger, the Heisenberg spin and the cylindrical Heisenberg spin equations. *J. Phys. Soc. Japan* **58** (1989), 4334–4343. [MR 91b: 82015.]

A. Nakamura, The $3+1$ dimensional Toda molecule equation and its multiple soliton solutions. *J. Phys. Soc. Japan* **58** (1989), 2687–2693. [PA **92** (1989), 139233; MR 90i: 35258.]

A. Nakamura, Cylindrical multi-soliton solutions of the Toda molecule equation and their large molecule limit of the Toda lattice. *J. Phys. Soc. Japan* **59** (1990), 1553–1559. [PA **93** (1990), 93011.]

A. Nakamura, General cylindrical soliton solutions of the Toda molecule. *J. Phys. Soc. Japan* **59** (1990), 3101–3111. [MR 91h: 35277.]

A. Nakamura, Bilinear structures of the real $2N$-soliton solutions of the Ernst equation. *J. Phys. Soc. Japan* **63** (1994), 1214–1215.

A. Nakamura, Explicit N-soliton solutions of the $1+1$ dimensional Toda molecule equation. *J. Phys. Soc. Japan* **67** (1998), 791–798.

Y. Nakamura, Symmetries of stationary axially symmetric vacuum Einstein equations and the new family of exact solutions. *J. Math. Phys.* **24** (1983), 606–609. [PA **86** (1983), 49226.]

Y. Nakamura, On a linearisation of the stationary axially symmetric Einstein equations. *Class. Quantum Grav.* **4** (1987), 437–440. [PA 90 (1987), 67098; MR 88c: 83027.]

R. Narayan, R. Nityananda, The maximum determinant method and the maximum entropy method. *Acta Cryst. A* **38** (1982), 122–128. [MR 83m: 82050.]

K. Narita, New nonlinear difference–differential equation related to the Volterra equation. *J. Math. Anal. Applic.* **186** (1994), 120–131.

B. Nelson, B. Sheeks, Fredholm determinants associated with Wiener integrals. *J. Math. Phys.* **22** (1981), 2132–2136. [Zbl **473** (1983), 28005.]

G. Neugebauer, A general integral of the axially symmetric stationary Einstein equations. *J. Phys. A. Math. Gen.* **13** (1980), L19–21. [PA **83** (1980), 31622.]

M. Newman, Determinants of circulants of prime power order. *Linear Multilinear Alg.* **9** (1980), 187–191. [MR 82c: 15020.]

M. Newman, A result about determinantal divisors. *Linear Multilinear Alg.* **11** (1982), 363–366. [MR 83h: 15012.]

J.D. Niblett, A theorem of Nesbitt. *Am. Math. Monthly* **59** (1952), 171–174.

J.J.C. Nimmo, Soliton solutions of three differential–difference equations in Wronskian form. *Phys. Lett. A* **99** (1983), 281–286. [PA **87** (1984), 22595.]

J.J.C. Nimmo, Wronskian determinants, the KP hierarchy and supersymmetric polynomials. *J. Phys. A: Math. Gen.* **22** (1989), 3213–3221. [PA **92** (1989), 132225.]

J.J.C. Nimmo, Hall–Littlewood symmetric functions and the BKP equation. *J. Phys. A* **23** (1990), 751–760. [MR 91g: 05136.]

J.J.C. Nimmo, A class of solutions of the Konopelchenko–Rogers equations. *Phys. Lett. A* **168** (1992), 113–119. [MR 93f: 35206].

J.J.C. Nimmo, N.C. Freeman, A method of obtaining the N-soliton solution of the Boussinesq equation in terms of a Wronskian. *Phys. Lett.* **95A** (1983), 4–6. [PA **86** (1983), 64699.]

J.J.C. Nimmo, N.C. Freeman, Rational solutions of the KdV equation in Wronskian form. *Phys. Lett.* **96A** (1983), 443–446. [PA **86** (1983), 98781.]

J.J.C. Nimmo, N.C. Freeman, The use of Bäcklund transformations in obtaining N-soliton solutions in Wronskian form. *J. Phys. A* **17** (1984), 1415–1424. [PA **87** (1984), 68001.]

J.W. Noonan, D.K. Thomas, On the Hankel determinants of areally mean p-valent functions. *Proc. Lond. Math. Soc.* **25** (1972), 503–524. [MR **46** (1973), 5605.]

K.I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation. *Rev. Roumaine Math. Pures Appl.* **28** (1983), 731–739. [MR 85f: 30017.]

W. Oevel, W. Strampp, Wronskian solutions of the constrained KP hierarchy. *J. Math. Phys.* **37** (1996), 6213–6219.

S. Ogawa, S. Arioka, S. Kida, On linear independency of vector-valued mappings—an extension of Wronskian. *Math. Jap.* **31** (1986), 85–93. [Zbl **589** (1986), 15001; MR 87g: 15003.]

Y. Ohta, Pfaffian solutions for the Veselev–Novikov equation. *J. Phys. Soc. Japan* **61** (1992), 3928–3933. [PA **96** (1993), 19413.]

Y. Ohta, R. Hirota, A discrete KdV equation and its Casorati determinant solution. *J. Phys. Soc. Japan* **60** (1991), 2095. [PA **94** (1991), 113667.]

Y. Ohta, R. Hirota, S. Tsujimoto, T. Imai, Casorati and discrete Gram type determinant representation of solutions to the discrete KP hierarchy. *J. Phys. Soc. Japan* **62** (1993), 1872–1886. [PA (1994), 10461.]

Y. Ohta, K. Kajiwara, J. Matsukidaira, J. Satsuma, Casorati determinant solution for the relativistic Toda lattice equation. *J. Math. Phys.* **34** (1993), 5190–5204. [PA (1994), 10461; MR 95a: 35137].

K. Okamoto, Bäcklund transformations of classical orthogonal polynomials. *Algebraic Analysis* Vol. II, Academic Press, Boston, 1988, pp. 647–657. [MR 90j: 33012.]

R. Oldenburger, Higher dimensional determinants. *Am. Math. Monthly* **47** (1940), 25–33. [MR **1** (1940), 194.]

F.R. Olsen, Some determinants involving Bernoulli and Euler numbers of higher order. *Pacific J. Math.* **5** (1955), 259–268. [MR **16** (1955), 988.]

F.R. Olsen, Some special determinants. *Am. Math. Monthly* **63** (1956), 612.

P.J. Olver, Hyperjacobians, determinantal ideals and weak solutions to variational problems. *Proc. Roy. Soc. Edin.* **95** (1983), 317–340. [MR 85c: 58040.]

O. Ore, Some studies of cyclic determinants. *Duke Math. J.* **18** (1951), 343–354. [MR **13** (1952), 98.]

A. Ostrowski, *Collected Mathematical Papers*, Vol. 1, Birkhäuser, Boston, 1983–1984. [MR 86m: 01075.]

D. Pandres, On higher ordered differentiation *Am. Math. Monthly* **64** (1957), 566–572.

D. Pandres, A determinant representation of the classical orthogonal polynomials. *Am. Math. Monthly* **67** (1960), 658–659. [MR **24a** (1962), 3316.]

D.H. Pandya, A property in determinants. *Math. Educ. Sec.* **B9**, no.3, (1975), 56–57. [Zbl **341** (1977), 15008.]

S. Parameswaran, Skew-symmetric determinants. *Am. Math. Monthly* **61** (1954), 116.

M. Parodi, Sur les polynômes de Bessel. *C.R. Acad. Sci. Paris Sér. A–B* **274** (1972), A1153–1155. [MR **46** (1973), 416.]

E. Pascal, *Die Determinanten*, Druck und Verlag von B.G. Teubner, Leipzig, 1900.

D. Pelinovsky, Rational solutions of the Kadomtsev–Petviashvili hierarchy and the dynamics of their poles, 1. New form of a general rational solution. *J. Math. Phys.* **35** (1994), 5820–5830. [MR 95h: 58071.]

D. Piccini, Dieudonné determinant and invariant real polynomials on $\mathrm{gl}(n, H)$. *Rendiconte* **2** (1982), 31–45. [MR 83k: 55012.]

L.A. Pipes, Cyclical functions and permutation matrices. *J. Franklin Inst.* **287** (1969), 285–296. [MR **39** (1970), 7148.]

A.V. Pogorelov, A priori estimates for solutions of the equation $\det(z_{ij}) = \phi(z_1, z_2, \ldots, z_n,\ z,\ x_1, x_2, \ldots, x_n)$. *Dokl. Akad. Nauk SSSR* **272** (1983), 792–794. [MR 85i: 35022.]

G. Pólya, G. Szegö, *Problems and Theorems in Analysis*, Vol.2, Springer-Verlag, New York, Heidelberg, 1976. [MR **49** (1975), 8781; MR **53** (1977), 2.]

F.W. Ponting, A type of alternant. *Proc. Edin. Math. Soc.* II, **9**, (1953), 20–27. [MR **15** (1954), 498.]

C. Pöppe, Construction of solutions of the sine–Gordon equation by means of Fredholm determinants. *Physica* **9D** (1983), 103–139. [PA **87** (1984), 17856.]

C. Pöppe, The Fredholm determinant method for the KdV equation. *Physica* **13D** (1984), 137–160. [MR 86f: 35165.]

C. Pöppe, General determinants and the Tau-function for the Kadomtsev–Petviashvili hierarchy. *Inverse Problems* **5** (1989), 613–630. [PA **92** (1989), 124094.]

H. Porta, The determinants function. *Math. Notae* **30** (1983), 49–59. [MR 88i: 15020.]

P. Pragacz, Characteristic free resolution of $(n-2)$ order Pfaffians of an $n \times n$ antisymmetric matrix. *J. Alg.* **78** (1982), 386–396. [MR 85a: 13006.]

F.L. Pritchard, Polynomial mappings with Jacobian determinants of bounded degree. *Arch. Math.* **48** (1987), 495–504. [MR 88m: 32012.]

C.C. Puckette, Skew-symmetric determinants. *Math. Gaz.* **35** (1951), 254–255.

C. Radoux, Calcul effectif de certaines determinants de Hankel. *Bull. Soc. Math. Belg.* **31** (1979), 49–55. [MR 82j: 05016.]

A.K. Rajagopal, On some of the classical orthogonal polynomials. *Am. Math. Monthly* **67** (1960), 166–169. [MR **22** (1961), 11164.]

T.J. Randall, H. Sharples, A useful recurrence relation. *Math. Gaz.* **64** (1980), 279–280. [MR 82c: 15009.]

L. Rédei, Eine determinantenidentität für symmetriche functionen. *Acta Math. Acad. Sci. Hung.* **2** (1951), 105–107. [MR **13** (1952), 617.]

C. Reutenauer, M.-P. Schützenberger, A formula for the determinant of the sum of matrices. *Lett. Math. Phys.* **13** (1987), 299–302. [MR 88d: 15010.]

P.E. Ricci, Hankel–Gram determinants and recurrence calculations for systems of orthonormal polynomials. *Rend. Mat. Appl.* **6** (1986), 355–364. [MR 90c: 33024.]

P.E. Ricci, Computing Hankel determinants and orthonormal polynomial systems. *Rend. Mat. Appl.* **10** (1990), 987–994. [MR 92i: 65087.]

L.H. Rice, Some determinant expansions. *Am. J. Math.* **42** (1920), 237–242.

S.A.H. Rizvi, Solution of a generalized linear difference equation. *J. Austral. Math. Soc. B* **22** (1981), 314–317. [MR 82d: 39001; Zbl **621** (1988), 15005.]

D.P. Robbins, H. Rumsey, Determinants and alternating sign matrices. *Adv. Math.* **62** (1986), 169–184. [Zbl **611** (1987), 15008.]

F.E.G. Rodeja, Note on determinants of hyperbolic sines and cosines. *Rev. Mat. Hisp.-Am.* **14** (1954), 200–210. [MR **16** (1955), 557.]

T.G. Room, The freedoms of determinantal manifolds. *Proc. Lond. Math. Soc.* **36** (1933), 1–28. [Zbl **7** (1934), 227.]

R. Rösler, Erzeugung Gegenbaurscher polynome durch determinanten und eine neu rekursionsformel. *Z. Angew. Math. Mech.* **42** (1962), 416–417.

D.E. Rutherford, Some continuant determinants arising in physics and chemistry, I. *Proc. Roy. Soc. Edin.* **A62** (1945), 229–236; II. *Proc. Roy. Soc. Edin.* **A63** (1952), 232–241. [MR **15** (1954), 495.]

H.J. Ryser Maximal determinants in combinatorial investigations. *Can. J. Math.* **8** (1956), 245–249. [MR **18** (1957), 105.]

G. Salmon, *Modern Higher Algebra*, Hodges, Figgis, London, 1885.

H.E. Saltzer, A determinant form for nonlinear divided differences with applications. *Z. Angew. Math. Mech.* **66** (1986), 183–185. [Zbl **595** (1987), 41001.]

P. Sarnak, Determinants of Laplacians. *Commun. Math. Phys.* **110** (1987), 113–120. [MR 89e: 58116.]

N. Sasa, J. Satsuma, A series of exact solutions based on a bilinear form of the stationary axially symmetric vacuum Einstein equations. *J. Phys. Soc. Japan* **62** (1993), 1153–1158. [PA **96** (1993), 84963.]

T. Sasaki, Constructing Bezout's determinants from Sylvester's determinants. *J. Inf. Process. Japan* **6** (1983), 163–166. [PA **B87** (1984), 18799; Zbl **545** (1985), 65027.]

T. Sasamoto, M. Wadati, Determinantal form solutions for the derivative nonlinear Schrödinger type model. *J. Phys. Soc. Japan* **67** (1998), 784–790.

M. Sato, M. Kashiwara, The determinants of matrices of pseudo-differential operators. *Proc. Japan Acad.* **51** (1975), 17–19. [Zbl **337** (1977), 35067.]

J. Satsuma, A Wronskian representation of N-soliton solutions of nonlinear evolution equations. *Phys. Soc. Jap. Lett.* **46** (1979), 359–360. [PA **82** (1979), 25867.]

J. Satsuma, K. Kajiwara, J. Matsukidaira, J. Hietarinta, Solutions of the Broer–Kaup system through its trilinear form. *J. Phys. Soc. Japan* **61** (1992), 3096–3102. [PA (1992), 139300.]

J.W. Schleusner, J.P. Singhal, On determinantal representations for certain orthogonal polynomials. *J. Natur. Sci. Math.* **10** (1970), 287–291.

J. Schlitter, A. Metz, The Hessian in function minimization. *Int. J. Computer Math.* **24** (1988), 65–72. [Zbl **661** (1989), 65063.]

H. Schmidt, Über das additionstheorem der zyklishen funktionen. *Math. Z.* **76** (1961), 46–50. [MR **24A** (1962), 276.]

F. Schmittroth, Derivatives of a composite function $f[g(x)]$ [Solution to a problem proposed by V.F. Ivanoff]. *Am. Math. Monthly* **68** (1961), 69.

A. Schwartz, J.S. de Wet, The minors of a determinant in terms of Pfaffians. *Proc. Camb. Phil. Soc.* **46** (1950), 519–520. [MR **11** (1950), 710.]

R.F. Scott, G.B. Mathews, *The Theory of Determinants*, 2nd ed., Cambridge University Press, Cambridge, 1904.

W. Seidel, Note on a persymmetric determinant. *Quart. J. Math.* (Oxford), **4** (1953), 150–151. [MR **15** (1954), 3.]

E. Seiler, B. Simon, An inequality among determinants. *Proc. Nat. Acad. Sci. USA*, **72** (1975), 3277–3278. [MR **55** (1978), 6225.]

C. Shafroth, A generalization of the formula for computing the inverse of a matrix. *Am. Math. Monthly* **88** (1981), 614–616. [MR 82j: 15006.]

A.G. Shannon, Determinants and recurrences of arbitrary order. *Gac. Mat. I. Ser.* **31** (1979), 99–100. [Zbl **484** (1983), 15005.]

V.L. Shapiro, A theorem on a special class of near-Vandermonde determinants. *Am. Math. Monthly* **60** (1953), 697–699. [MR **15** (1954), 495.]

A. Sharma, E.G. Straus, On the irreducibility of a class of Euler–Frobenius polynomial. *Canad. Math. Bull.* **17** (1974), 265–273. [MR **50** (1975), 13990.]

Z.M. Shen, Y.W. Rong, A generalization of Hadamard's determinantal inequality. *J. China Univ. Sci. Technol.* **13** (1983), 272–274. [MR 85j: 15008.]

L.R. Shenton, A determinantal expansion of a class of definite integral. I. *Proc. Edin. Math. Soc.* **9** (1953), 43–52. [MR **15** (1954), 781]; II. **10** (1954), 78–91. [MR **16** (1955), 575]; III. **12** (1956), 134–140. [MR **17** (1956), 844].

O. Shisha, Tchebycheff systems and best partial bases. *Pacific J. Math.* **86** (1980), 579–592. [Zbl **437** (1981), 41032.]

B. Silbermann, Some remarks on the asymptotic behaviour of Toeplitz determinants. *Applicable Anal.* **11** (1980/81), 185–197. [MR 82j: 47048.]

J.A. Silva, A theorem on cyclic matrices. *Duke Math. J.* **18** (1951), 821–825. [MR **13** (1952), 424.]

G.D. Simionescu, Properties of determinants formed with similar powers of the terms of an arithmetic progression (in Romanian). *Gaz. Mat.* (Bucharest) **86** (1981), 406–411. [MR 83i: 15013.]

B. Simon, Notes on infinite determinants of Hilbert space operators. *Adv. Math.* **24** (1977), 244–273. [MR **58** (1979), 2401.]

S. Simons, Measurable relations, pentagonal arrays and the determinants of probabalistic integrals. *Linear Multilinear Alg.* **13** (1983), 41–66. [Zbl **511** (1984), 62034.]

V.N. Singh, Solution of a general homogeneous linear difference equation. *J. Austral. Math. Soc.* **B22** (1980), 53–57; Corrigendum, **B22** (1980), 254. [Zbl **441** (1981), 39002, 39003.]

J. St.-C.L. Sinnadurai, On Gram's and Hadamard's determinant inequalities. *Math. Gaz.* **47** (1963), 34–35.

S. Sirianunpiboon, S.D. Howard, S.K. Roy, A note on the Wronskian form of solutions of the KdV equation. *Phys. Lett. A* **134** (1988), 31–33. [PA **92** (1989), 39211; MR 89k: 35230.]

W. Solak, Z. Szydelko, Application of Gram's determinant for solution of linear matrix equation. *Zesz. Nauk. Akad. Górn.–Hutn. Stanislaw Staszic 730, Mat. Fiz. Chem.* **42** (1979), 63–66. [Zbl **432** (1981), 15009.]

R. Stalley, A generalization of the geometric series. *Am. Math. Monthly* **56** (1949), 325–327.

F.M. Stein, M.S. Henry, L.G. King, Determinantal representations of recursion relations for certain orthogonal polynomials. *Am. Math. Monthly* **73** (1966), 382–385. [Zbl **154** (1968), 65.]

P.R. Stein, On an identity from classical invariant theory. *Linear Multilinear Alg.* **11** (1982), 39–44. [Zbl **478** (1982), 15021.]

M. Stojakovic, Sur les déterminants de matrices rectangulaire. *Bull. Soc. Roy. Sci. Liège* **21** (1952), 303–305. [Zbl **48** (1953), 249; MR **14** (1953), 443.]

M. Stojakovic, Quelques applications des determinants rectangulaires aux produits intérieurs et extérieurs des matrices. *C.R. Acad. Sci. Paris* **237** (1953), 688–690.

K.B. Stolarsky Quintuple diagonal determinants related to a digital distribution problem. *Linear Multilinear Alg.* **8** (1979), 145–152. [Zbl **422** (1980), 15005.]

R.R. Stoll, Determinants and plane equations. *Am. Math. Monthly* **61** (1954), 255–256.

W. Strampp, The Kaup–Broer system and trilinear o.d.e.s. of Painlevé type. *J. Math. Phys.* **35** (1994), 5850–5861. [MR 96d: 58072.]

G. Strang, Inverse problems and derivatives of determinants. *Arch. Rat. Mech. Anal.* **114** (1991), 255–266. [MR 92i: 58203.]

E. Strickland, The symplectic group and determinants. *J. Alg.* **66** (1980), 511–533. [MR 83b: 20045.]

J.C. Sun, An identity for tridiagonal determinants and its applications. *Math. Numer. Sinica* **4** (1982), 323–327. [MR 85h: 15013.]

D.A. Supranenko, Determinants of matrices of a minimal irreducible group (in Russian). *Dokl. Akad. Nauk USSR* **274** (1984), 31–35. [MR 85k: 20147.]

O. Szasz Uber Hermitesche formen mit rekurrierender determinante und über rationale polynome. *Math. Z.* **11** (1921), 24–57.

G. Szegö, Solution to Problem 4771 [1958, 47], on Vogt's determinant, proposed by L. Carlitz. *Am. Math. Monthly* **65** (1958), 780.

G. Szegö, Hankel forms. *AMS Transl.* **108** (1977), 1–36. [Zbl **363** (1978), 15004.]

L. Tan, Some determinantal identities and the big cells. *Adv. Math.* **101** (1993), 1–9. [MR 94g: 30105.]

K. Tanaka, T. Morita, Asymptotic behaviours of modified block Toeplitz determinants. *Prog. Theoret. Phys.* **84** (1990), 392–409. [MR 92a: 15028a.]

K. Tanaka, T. Morita, Asymptotic forms of some block Toeplitz determinants. *Prog. Theoret. Phys.* **84** (1990), 410–414. [MR 92a: 15028b.]

H. Tango, Remarks on Schur polynomials (Japanese, English summary). *Bull. Kyoto Univ. Ed. Ser. B* **62** (1983), 1–14. [MR 85e: 14072.]

O. Taussky, A recurring theorem on determinants. *Am. Math. Monthly* **56** (1949), 672–676. [MR **11** (1950), 307.]

L. Tenca, Su una classe di matrici infinite. *Period. Mat.* **35** (1957), 219–223. [MR **19** (1958), 1034.]

M. Toda, Vibrations of a chain with nonlinear interaction. *J. Phys. Soc. Japan* **22** (1967), 431–436.

M. Toda, Development of the theory of a nonlinear lattice. *Suppl. of the Prog. of Theor. Phys.* **59** (1976), 1–35. [PA **80** (1977), 30105.]

A. Torgašev, Note on Gram determinant in Wachs spaces. *Math. Balk.* **8** (1978), 221–225. [Zbl **523** (1984), 15015; MR 85b: 15002.]

D.H. Trahan, Cyclic dislocations in determinants and a new expansion of a determinant. *Am. Math. Monthly* **70** (1963), 545. [Zbl **129** (1967), 267.]

G. Tsoucaris, A new method of phase determination: The "maximum determinant rule." *Acta Cryst. A* **26** (1970), 492–499. [PA **74** (1971), 5137.]

Z. Tsuboi, A. Kuniba, Solutions of a discretized Toda field equation for Dfrom analytic Bethe ansatz. *J. Phys. A: Math. Gen.* **29** (1996), 7785–7796.

P. Turan, On some questions concerning determinants. *Ann. Polon. Math.* **12** (1962), 49–53. [Zbl **106** (1964), 15.]

W.T. Tutte, The factorization of linear graphs. *J. Lond. Math. Soc.* **22** (1947), 107–111. [MR **9** (1948), 297.]

J.L. Ullman, Hankel determinants whose elements are sections of a Taylor series, Part 1. *Duke Math. J.* **18** (1951), 751–756; Part 2. *Duke Math. J.* **19** (1952), 155–164. [MR **13** (1952), 221, 926; Zbl **43** (1952), 78.]

R. Vaidyanathaswamy, A theory of multiplicative arithmetic functions VII, The theory of Smith's determinant. *Trans. Am. Math. Soc.* **33** (1931), 579–662. [Zbl **2** (1932), 113.]

Y. Vaklev, Soliton solutions and gauge-equivalence for the problem of Zakharov–Shabat and its generalizations. *J. Math. Phys.* **37** (1996), 1393–1413.

B.N. Valuev, Two definitions of the determinant and a proof of the Szegö–Kac theorem. *Teoret. Mat. Fiz.* **55** (1983), 475–480. [MR 85d: 47029.]

A.J. van der Poorten, Some determinants which should be better known. *J. Austral. Math. Soc. A* **21** (1976), 278–288. [MR **53** (1977), 10828.]

P.R. Vein, A lemma on cyclic dislocations in determinants and an application in the verification of an identity. *Am. Math. Monthly* **69** (1962), 120–124. [MR **24A** (1962), 1923.]

P.R. Vein, Nonlinear ordinary and partial differential equations associated with Appell functions. *J. Diff. Eqns.* **11** (1972), 221–244. [Zbl **227** (1972), 35014.]

P.R. Vein, Persymmetric determinants 1. The derivatives of determinants with Appell function elements. *Linear Multilinear Alg.* **11** (1982), 253–265. [MR 83m: 15007a; Zbl **457** (1982), 15004.]

P.R. Vein, Persymmetric determinants 2. Families of distinct submatrices with non-distinct determinants. *Linear Multilinear Alg.* **11** (1982), 267–276. [MR 83m: 15007b; Zbl **457** (1982), 15005.]

P.R. Vein, Persymmetric determinants 3. A basic determinant. *Linear Multilinear Alg.* **11** (1982), 305–315. [MR 83m: 15007c; Zbl **457** (1982), 15006.]

P.R. Vein, Persymmetric determinants 4. An alternative form of the Yamazaki–Hori determinantal solution of the Ernst equation. *Linear Multilinear Alg.* **12** (1983), 329–339. [MR 83m: 15007d; Zbl **457** (1982), 15007.]

P.R. Vein, Persymmetric determinants 5. Families of overlapping coaxial equivalent determinants. *Linear Multilinear Alg.* **14** (1983), 131–141. [MR 85e: 15011.]

P.R. Vein, Two related families of determinantal solutions of the stationary axially symmetric vacuum Einstein equations. *Class. Quantum Grav.* **2** (1985), 899–908. [MR 87m: 83024; Zbl **563** (1985), 35079; PA **89** (1986), 24174).]

P.R. Vein, Identities among certain triangular matrices. *Linear Alg. Applic.* **82** (1986), 27–79. [Zbl **598** (1987), 15009; PA **90** (1987), 24353; MR 88a: 05018.]

P.R. Vein, Determinantal solutions of two difference–differential equations. Tech. Rep. TR 90005, Dept. Comp. Sci. & Appl. Maths., Aston University, Birmingham, U.K., 1990.

P.R. Vein, P. Dale, Determinants, their derivatives and nonlinear differential equations. *J. Math. Anal. Applic.* **74** (1980), 599–634. [MR 81f: 15013; Zbl **459** (1982), 15009.]

P.R. Vein, P. Dale, Two verifications of the determinantal solutions of the Korteweg–de Vries and the Kadomtsev–Petviashvili equations. *Applic. Anal.* **25** (1987), 79–100. [Zbl **602** (1987), 35114; PA **90** (1987), 134523.]

R. Vermes, Hankel determinants formed from successive derivatives. *Duke Math. J.* **37** (1970), 255–259. [MR **41** (1971), 3843.]

V. Vinnikov, Determinantal representations of real cubics and canonical forms of corresponding triples of matrices. Math. theory of networks and systems. Proc. Int. Symp. Beer Sheva, Israel, 1983. *Lect. Notes Control Inf. Sci.* **58** (1984), 882–898. [Zbl **577** (1986), 14040.]

V.S. Vladimirov, I.V. Volovich, Superanalysis II. Integral calculus. *Theoret. Math. Phys.* **60** (1984), 743–765. [MR 86c: 58015b.]

H. Waadeland, Über eine determinante. *Norske Vid. Selsk. Forh. Trondheim* **24** (1952), 108–109. [MR **14** (1953), 1054.]

M. Wadati, The modified KdV equation. *J. Phys. Soc. Japan* **34** (1973), 1289–1296. [PA **76** (1973), 37905.]

M. Wadati, M. Toda, The exact n-soliton solution of the Korteweg–de Vries equation. *J. Phys. Soc. Japan* **32** (1972), 1403–1411.] [PA **75** (1972), 69553.]

A.W. Walker, A proof of the product rule for determinants. *Math. Gaz.* **33** (1949), 213–214.

C.R. Wall, Analogs of Smith's determinant. *Fibonacci Quart.* **25** (1987), 343–345. [MR 88i: 15022.]

C.L. Wang, Gramian expansions and their applications. *Util. Math.* **15** (1979), 97–111. [MR 80e: 15013.]

G. Wang, A Cramer's rule for minimum-norm (T) least-squares (S) solution of the inconsistent linear equations. *Linear Alg. Applic.* **74** (1986), 213–218. [Zbl **588** (1986), 15005.]

K. Wang On the generalization of circulants. *Linear Alg. Applic.* **25** (1979), 197–218. [MR 81i: 15022.]

K. Wang, On the generalization of a retrocirculant. *Linear Alg. Applic.* **37** (1981), 35–43. [MR 83i: 15026.]

T. Watanabe, On a determinant sequence in the lattice path counting. *J. Math. Anal. Applic.* **123** (1987), 401–414.

W.C. Waterhouse, Twisted forms of the determinant. *J. Alg.* **86** (1984), 60–75. [MR 85c: 11040.]

W.C. Waterhouse, How often do determinants over finite fields vanish? *Discrete Math.* **65** (1987), 103–104. [MR 88d: 11127.]

W.C. Waterhouse, Automorphisms of determinant (X_{ij}): The group scheme approach. *Adv. Math.* **65** (1987), 171–203. [Zbl **651** (1989), 4028.]

H. Wayland, Expansion of determinantal equations into polynomial form. *Quart. Appl. Math.* **2** (1945), 277–306. [MR **6** (1945), 218.]

A. Weinmann, On the square root of certain determinants. *J. Lond. Math. Soc.* **35** (1960), 265–272. [MR **24A** (1962), 124.]

H.-J. Werner, On extensions of Cramer's rule for solutions of restricted linear systems. *Linear Multilinear Alg.* **15** (1984), 319–330. [Zbl **544** (1985), 15002; MR 87b: 15004.]

R.R. Whitehead, A. Watt, On the avoidance of cancellations in the matrix moment problem. *J. Phys. A* **14** (1981), 1887–1892. [MR 82i: 15031.]

E.T. Whittaker, On the theory of continued fractions. *Proc. Roy. Soc. Edin.* **36** (1915–1916), 243–255.

E.T. Whittaker, G.N. Watson, *Modern Analysis*, Cambridge University Press, Cambridge, 1952.

H. Widom, Toeplitz determinants with singular generating functions. *Am. J. Math.* **95** (1973), 333–383. [MR **48** (1974), 9441.]

H. Widom, Asymptotic behaviour of block Toeplitz matrices and determinants. 1. *Adv. Math.* **13** (1974), 284–322; 2. *Adv. Math.* **21** (1976), 1–29. [MR **53** (1977), 13266a, 13266b.]

L. Wille, H. Verschelde, P. Phariseau, A new solvable one-dimensional crystal model. *J. Phys. A* **16** (1983), L771–775. [PA **87** (1984), 25201.]

J. Wilmet, Determinant de Hankel–Kronecker de la suite des factorielles. *Bull. Soc. Math. Belg.* **A30** (1978), 11–14. [Zbl **475** (1982), 05007; MR 84b: 05014.]

R. Wilson, Determinantal criteria for meromorphic functions. *Proc. Lond. Math. Soc.* **4** (1954), 357–374.

R. Wilson, Note on a previous paper. *Quart. J. Math.* **5** (1954), 99–101.

M. Wojtas, A property of certain symmetric determinants. *Prace Mat.* **5** (1961), 27–32. [MR **24**A (1962), 24.]

A.C.T. Wu, Cartan–Gram determinants for the simple Lie groups. *J. Math. Phys.* **23** (1982), 2019–2021. [PA **86** (1983), 21022.]

X.H. Xu, A new approach to Dale's equation. *Commun. Theoret. Phys.* **5** (1984), 639–642. [MR 87c: 83024.]

Q.A.M.M. Yahya, A new expression for Chebyshev polynomials. *Port. Math.* **24** (1965), 169–172. [Zbl **154** (1968), 65.]

O.S. Yakovlev, A generalization of determinant identities (Russian). *Ukrain. Mat. Z.* **30** (1978), 850–854, English translation: Plenum Press, New York, 1979. [MR 80b: 15014.]

M. Yamazaki, On the Kramer–Neugebauer spinning mass solutions. *Prog. Theoret. Phys.* **69** (1983), 503–515. [PA **86** (1983), 69943.]

S. Yamazaki, On the system of nonlinear differential equations $\dot{y}_k = y_k(y_{k+1} - y_{k-1})$. *J. Phys. A: Math. Gen.* **20** (1987), 6237–6241. [MR 89d: 34020; PA **91** (1988), 36330.]

F. Yuasa, Bäcklund transformations of the 2-dimensional Toda lattice and Casorati's determinants. *J. Phys. Soc. Japan* **56** (1987), 423–424. [MR 88c: 35145.]

V.E. Zakharov, Kinetic equation for solitons. *Soviet Phys. JETP* **33** (1971), 538–541.

D. Zeitlin, A Wronskian. *Am. Math. Monthly* **65** (1958), 345–349. [MR **20** (1959), 5835.]

D. Zeitlin, Two methods for the evaluation of $\sum_{k=1}^{\infty} k^n x^k$. *Am. Math. Monthly* **68** (1961), 986–989. [MR **24A** (1962), 1535.]

D. Zeitlin, On several applications of the first derivative of a determinant. *Am. Math. Monthly* **70** (1963), 58–60. [MR **26** (1963), 1328.]

F.J. Zhang, X.L. Li, A graph theoretic definition of determinants (in Chinese, English summary), *J. Xinjiang Univ. Natur. Sci.* **5** (1988), 6–8. [MR 90k: 15002.]

Y.-J. Zhang, Y. Cheng, The Wronskian structure for solutions of the k-constrained KP hierarchy. *J. Phys. A* **27** (1994), 4581–4588.

Z.J. Zhang, A class of determinants with value identically 1 (in Chinese). *Math. Pract. Theory* (1), (1985), 38–42.

Z.Z. Zhong, A double inverse scattering method and the family of the stationary axisymmetric dual gravitational soliton solutions. *Sci. Sinica Ser. A* **31** (1988), 436–450. [MR 90e: 83036.]

Index

Applied Mathematical Sciences

(continued from page ii)

61. *Sattinger/Weaver:* Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics.
62. *LaSalle:* The Stability and Control of Discrete Processes.
63. *Grasman:* Asymptotic Methods of Relaxation Oscillations and Applications.
64. *Hsu:* Cell-to-Cell Mapping: A Method of Global Analysis for Nonlinear Systems.
65. *Rand/Armbruster:* Perturbation Methods, Bifurcation Theory and Computer Algebra.
66. *Hlavácek/Haslinger/Necasl/Lovísek:* Solution of Variational Inequalities in Mechanics.
67. *Cercignani:* The Boltzmann Equation and Its Applications.
68. *Temam:* Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd ed.
69. *Golubitsky/Stewart/Schaeffer:* Singularities and Groups in Bifurcation Theory, Vol. II.
70. *Constantin/Foias/Nicolaenko/Temam:* Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations.
71. *Catlin:* Estimation, Control, and the Discrete Kalman Filter.
72. *Lochak/Meunier:* Multiphase Averaging for Classical Systems.
73. *Wiggins:* Global Bifurcations and Chaos.
74. *Mawhin/Willem:* Critical Point Theory and Hamiltonian Systems.
75. *Abraham/Marsden/Ratiu:* Manifolds, Tensor Analysis, and Applications, 2nd ed.
76. *Lagerstrom:* Matched Asymptotic Expansions: Ideas and Techniques.
77. *Aldous:* Probability Approximations via the Poisson Clumping Heuristic.
78. *Dacorogna:* Direct Methods in the Calculus of Variations.
79. *Hernández-Lerma:* Adaptive Markov Processes.
80. *Lawden:* Elliptic Functions and Applications.
81. *Bluman/Kumei:* Symmetries and Differential Equations.
82. *Kress:* Linear Integral Equations.
83. *Bebernes/Eberly:* Mathematical Problems from Combustion Theory.
84. *Joseph:* Fluid Dynamics of Viscoelastic Fluids.
85. *Yang:* Wave Packets and Their Bifurcations in Geophysical Fluid Dynamics.
86. *Dendrinos/Sonis:* Chaos and Socio-Spatial Dynamics.
87. *Weder:* Spectral and Scattering Theory for Wave Propagation in Perturbed Stratified Media.
88. *Bogaevski/Povzner:* Algebraic Methods in Nonlinear Perturbation Theory.
89. *O'Malley:* Singular Perturbation Methods for Ordinary Differential Equations.
90. *Meyer/Hall:* Introduction to Hamiltonian Dynamical Systems and the N-body Problem.
91. *Straughan:* The Energy Method, Stability, and Nonlinear Convection.
92. *Naber:* The Geometry of Minkowski Spacetime.
93. *Colton/Kress:* Inverse Acoustic and Electromagnetic Scattering Theory. 2nd ed.
94. *Hoppensteadt:* Analysis and Simulation of Chaotic Systems.
95. *Hackbusch:* Iterative Solution of Large Sparse Systems of Equations.
96. *Marchioro/Pulvirenti:* Mathematical Theory of Incompressible Nonviscous Fluids.
97. *Lasota/Mackey:* Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics, 2nd ed.
98. *de Boor/Höllig/Riemenschneider:* Box Splines.
99. *Hale/Lunel:* Introduction to Functional Differential Equations.
100. *Sirovich (ed):* Trends and Perspectives in Applied Mathematics.
101. *Nusse/Yorke:* Dynamics: Numerical Explorations, 2nd ed.
102. *Chossat/Iooss:* The Couette-Taylor Problem.
103. *Chorin:* Vorticity and Turbulence.
104. *Farkas:* Periodic Motions.
105. *Wiggins:* Normally Hyperbolic Invariant Manifolds in Dynamical Systems.
106. *Cercignani/Illner/Pulvirenti:* The Mathematical Theory of Dilute Gases.
107. *Antman:* Nonlinear Problems of Elasticity.
108. *Zeidler:* Applied Functional Analysis: Applications to Mathematical Physics.
109. *Zeidler:* Applied Functional Analysis: Main Principles and Their Applications.
110. *Diekmann/van Gils/Verduyn Lunel/Walther:* Delay Equations: Functional-, Complex-, and Nonlinear Analysis.
111. *Visintin:* Differential Models of Hysteresis.
112. *Kuznetsov:* Elements of Applied Bifurcation Theory, 2nd ed.
113. *Hislop/Sigal:* Introduction to Spectral Theory: With Applications to Schrödinger Operators.
114. *Kevorkian/Cole:* Multiple Scale and Singular Perturbation Methods.
115. *Taylor:* Partial Differential Equations I, Basic Theory.
116. *Taylor:* Partial Differential Equations II, Qualitative Studies of Linear Equations.
117. *Taylor:* Partial Differential Equations III, Nonlinear Equations.

(continued on next page)

Applied Mathematical Sciences

(continued from previous page)

118. *Godlewski/Raviart:* Numerical Approximation of Hyperbolic Systems of Conservation Laws.
119. *Wu:* Theory and Applications of Partial Functional Differential Equations.
120. *Kirsch:* An Introduction to the Mathematical Theory of Inverse Problems.
121. *Brokate/Sprekels:* Hysteresis and Phase Transitions.
122. *Gliklikh:* Global Analysis in Mathematical Physics: Geometric and Stochastic Methods.
123. *Le/Schmitt:* Global Bifurcation in Variational Inequalities: Applications to Obstacle and Unilateral Problems.
124. *Polak:* Optimization: Algorithms and Consistent Approximations.
125. *Arnold/Khesin:* Topological Methods in Hydrodynamics.
126. *Hoppensteadt/Izhikevich:* Weakly Connected Neural Networks.
127. *Isakov:* Inverse Problems for Partial Differential Equations.
128. *Li/Wiggins:* Invariant Manifolds and Fibrations for Perturbed Nonlinear Schrödinger Equations.
129. *Müller:* Analysis of Spherical Symmetries in Euclidean Spaces.
130. *Feintuch:* Robust Control Theory in Hilbert Space.
131. *Ericksen:* Introduction to the Thermodynamics of Solids, Revised ed.
132. *Ihlenburg:* Finite Element Analysis of Acoustic Scattering.
133. *Vorovich:* Nonlinear Theory of Shallow Shells.
134. *Vein/Dale:* Determinants and Their Applications in Mathematical Physics.
135. *Drew/Passman:* Theory of Multicomponent Fluids.