Business Mathematics and Statistics

SIXTH EDITION

Andre Francis

Business Mathematics and Statistics

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Sixth Edition

THOMSON

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Acknowledgements

The author would like to express thanks to the many students and teachers who have contributed to the text in various ways over the years.

In particular he would like to thank the following examining bodies for giving permission to reproduce selected past examination questions:

Chartered Association of Certified Accountants (ACCA)

Chartered Institute of Management Accountants (CIMA)

Institute of Chartered Secretaries and Administrators (ICSA)

Chartered Institute of Insurance (CII)

Association of Accounting Technicians (AAT)

Each question used is cross referenced to the appropriate Institute or Association.

Contents

Contents

Preface

1. Aims of the book

The general aim of the book is to give a thorough grounding in basic Mathematical and Statistical techniques to students of Business and Professional studies. No prior knowledge of the subject area is assumed.

2. Courses covered

a) The book is intended to support the courses of the following professional bodies:

Chartered Association of Certified Accountants

Chartered Institute of Management Accountants

Institute of Chartered Secretaries and Administrators

b) The courses of the following bodies which will be supported by the book to a large extent:

Chartered Institute of Insurance

Business and Technical Education Council (National level)

Association of Accounting Technicians

c) The book is also meant to cater for the students of any other courses who require a practical foundation of Mathematical and Statistical techniques used in Business, Commerce and Industry.

3. Format of the book

The book has been written in a standardised format as follows:

a) There are TEN separate parts which contain standard examination testing areas.

b) Numbered chapters split up the parts into smaller, identifiable segments, each of which have their own Summaries and Points to Note.

c) Numbered sections split the chapters up into smaller logical elements involving descriptions, definitions, formulae or examples.

At the end of each chapter, there is a Student Self Review section which contains questions that are meant to test general concepts, and a Student Exercise section which concentrates on the more practical numerical aspects covered in the chapter.

At the end of each part, there is

a) a separate section containing examination examples with worked solutions and

b) examination questions from various bodies. Worked solutions to these questions are given at the end of the book.

4. How to use the book

Chapters in the book should be studied in the order that they occur.

After studying each section in a chapter, the Summaries and Points to Note should be checked through. The Student Self Review Questions, which are cross-referenced to appropriate sections, should first be attempted unaided, before checking the answers with the text. Finally the Student Exercises should be worked through and the answers obtained checked with those given at the end of the book.

After completing a particular part of the book, the relevant section of the examination questions (at the end of the book) should be attempted. These questions should be considered as an integral part of the book, all the subject matter included having been covered in previous chapters and parts. Always make some attempt at the questions before reading the solution.

5. The use of calculators

Examining bodies permit electronic calculators to be used in examinations. It is therefore essential that students equip themselves with a calculator from the beginning of the course.

Essential facilities that the calculator should include are:

a)   a square root function, and

b)   an accumulating memory.

Very desirable extra facilities are:

c)   a power function (labelled 'x^y'),

d)   a logarithm function (labelled 'log x'), and

e)   an exponential function (labelled 'e^x').

Some examining bodies exclude the use (during examinations) of programmable calculators and/or calculators that provide specific statistical functions such as the mean or the standard deviation. Students are thus urged to check on this point before they purchase a calculator. Where relevant, this book includes sections which describe techniques for using calculators to their best effect.

<div align="right">Andre Francis, 2004</div>

Answers to examination questions

Part 1

Question 1

Simple random sampling. A method of sampling whereby each member of the population has an equal chance of being chosen. Normally, random sampling numbers are used to select individual items from some defined sampling frame.

Stratification. This is a process which splits a population up into as many groups and sub-groups (strata) as are of significance to the investigation. It can be used as a basis for quota sampling, but more often is associated with stratified (random) sampling. Stratified sampling involves splitting the total sample up into the same proportions and groups as that for the population stratification and then separately taking a simple random sample from each group. For example, employees of a company could be split into male/female, full-time/part-time and occupation category.

Quota sampling. A method of non-random sampling which is popular in market research. It uses street interviewers, armed with quotas of people to interview in a range of groups, to collect information from passers-by. For example, obtaining peoples' attitudes regarding the worth of secondary double glazing.

Sample frame. This is a listing of the members of some target population which needs to be used in order to select a random sample. An example of a sampling frame would be a stock list, if a random sample was required from current warehouse stock.

Cluster sampling. This is another non-random method of sampling, used where no sampling frame is in evidence. It consists of selecting (randomly) one or more areas, within which all relevant items or subjects are investigated. For example, a cluster sample could be taken in a large town to interview tobacconists.

Systematic sampling. A quasi-random method of sampling which involves examining or interviewing every n-th member of a population. Very useful method where no sampling frame exists, but population members are physically in evidence and ordered. For example, items coming off a production line. It is virtually as good as random sampling except where the items or members repeat themselves at regular intervals, which could lead to serious bias.

Question 2

(a) A postal questionnaire is a much cheaper and more convenient method of collecting data than the personal interview and often very large samples can be taken. However, much more care must be taken in the design of the questions, since there will be no help to hand if questions seem ambiguous or personal to the respondent. Also the response rate is very low, sometimes less than 20%, but this can sometimes be made larger by free gifts or financial incentives.

The personal interview has the particular advantage that difficult or ambiguous questions can be explained as well as the fact that an interviewer can make allowances or small adjustments according to the situation. Also, the questionnaire will be filled in as required. Disadvantages of this method include the cost,

the fact that large samples cannot generally be undertaken and the training of interviewers.

(b) Simple random sampling has the particular advantage that the method of selection (normally through the use of random sampling numbers) is free from bias. That is, each member of the population has an equal chance of being chosen as part of the sample. However, it cannot be guaranteed that the sample itself is truly representative of the population. For example, if a human population being sampled comprised 48% males, it is unlikely that the sample would reflect this percentage exactly.

Quota sampling is not a random sampling method and thus is generally at a disadvantage with regard to obtaining information that can claim to be representative. However, if the population has been stratified reasonably well, the street interviewer is experienced and conscientious and the questioning sites have been well thought out, it could be argued that, in certain localised situations, a quota sample could be very representative. For example, to gauge peoples opinions of a new shopping centre or to find out the views of theatregoers about a particular theatre.

Question 3

(a) (i) See pie chart.

Real consumers' relative expenditure in 1984 - component categories (1980 prices)
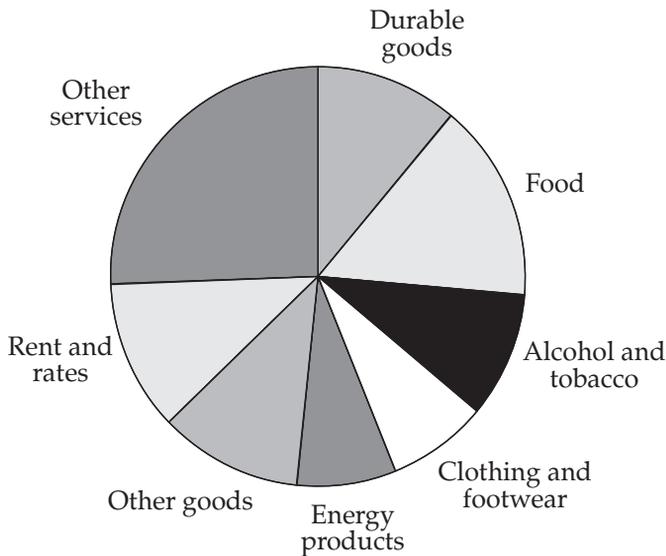
(ii) Other goods: books, toys, toiletries, transport. Other services: insurance, recreation, entertainment, (private) dental/health care.

(b) See line diagram.

**Comparative profit before tax (1980=100)
of six Scotch whisky companies**
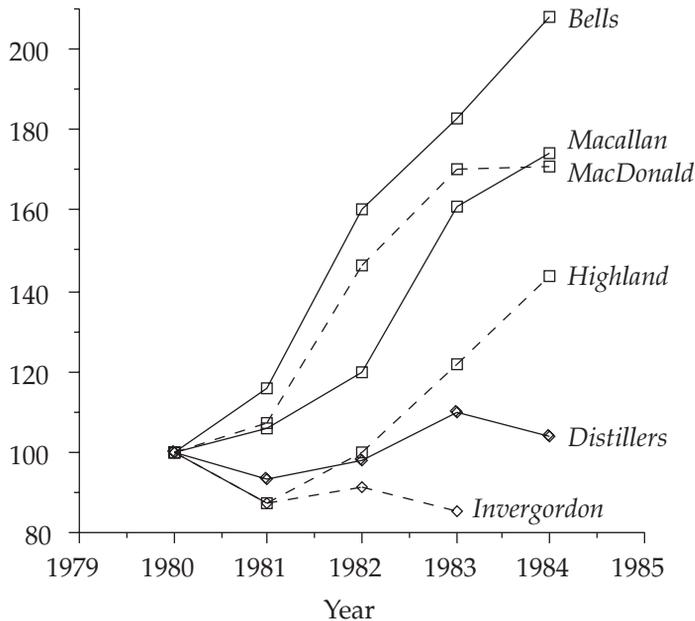
Question 4

(a) i. An absolute error is the difference between an estimated value and its true value. In most cases, only a maximum absolute error will be able to be calculated. For example, if a company's yearly profit was quoted as £252,000 (to the nearest £1000), the maximum absolute error would be £500.

ii. A relative error is an absolute error expressed as a percentage of the given estimated value. Thus in the example above, the maximum relative error in the company's yearly profit is:

$$\frac{500}{252,000} \times 100\% = 0.2\%$$

iii. A compensating error is an error that is made when 'fair' rounding has been carried out. For example, the numbers of people employed in each of a number of factories might well be rounded fairly, to the nearest 1000, say. When numbers, subject to compensating errors, are added, the total relative error should be approximately zero.

iv. Biased errors are made if rounding is always carried out in one direction. For example, when people's ages are quoted, they are normally rounded *down* to the lowest year. The error in the sum of numbers that are subject to biased errors is relatively high.

(b)

	Minimum	Estimate	Maximum
Time	145 hours	150 hours	155 hours
Wage rate	£4/hr	£4/hr	£4.40/hr
Labour cost	£580	£600	£682
Material cost	£2550	£2600	£2650
Total cost	£3130	£3200	£3332
Quote	£4000	£4000	£4000
PROFIT	£668	£800	£870

Question 5

(a) Smallest value = 347; largest value = 469. Thus, range = 122.
Since five classes are required, a class width of 122÷5 = 24.4, adjusted up to 25, seems appropriate.

Weekly production	Tally	Number of weeks
345 to 369	ИII ИII ИII I	16
370 to 394	ИII III	8
395 to 419	IIII	4
420 to 444	I	1
445 to 469	ИII ИII I	11
Total		40

(b) To construct the ogive, *cumulative frequency* needs to be plotted against *class upper bounds*.

Weekly production (upper bound)	Cumulative number of weeks
369.5	16
394.5	24
419.5	28
444.5	29
469.5	40

The ogive is shown in the figure following.

Figure 1

Weekly production over a 40 week period

Note: Since there are only a few plotted points and the distribution is not smooth, it is more appropriate to draw a polygon rather than attempt to draw a smooth curve.
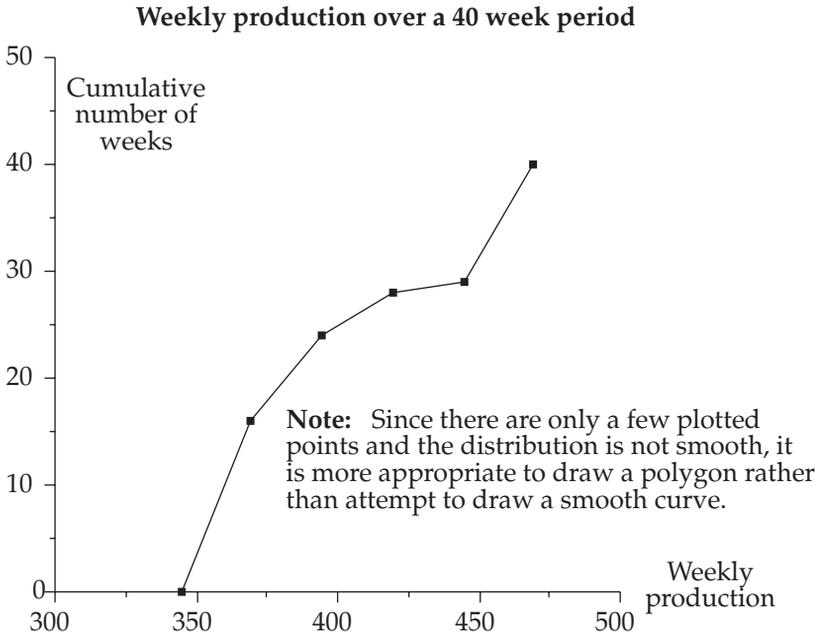
Question 6

(a) Although generally a component time series is best represented by a (cumulative) line diagram, in this case, since there are so few time points, a component bar chart has more impact. The chart is drawn in Figure 2.

Figure 2

Policies issued by an insurance company

585
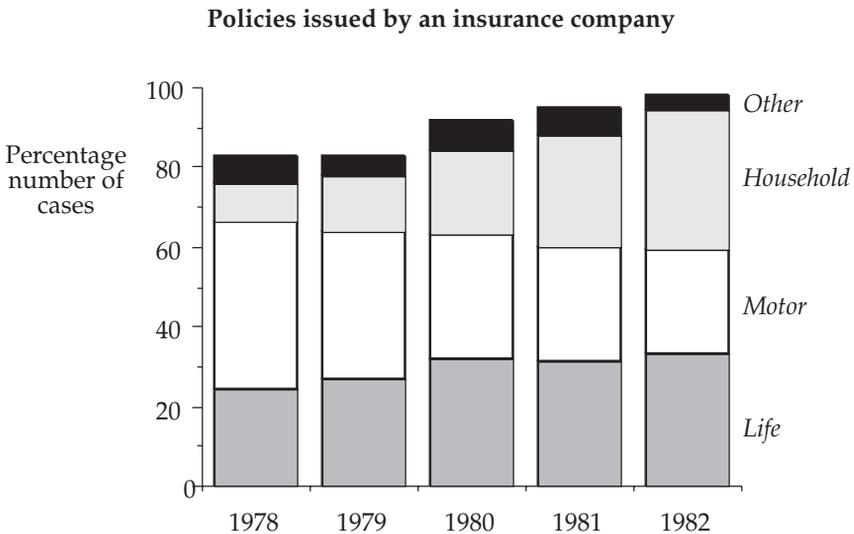
(b) Component bar charts enable comparisons between components across the years to be made easily, showing also yearly totals. The main disadvantage is the fact that actual values cannot easily be determined.

(c) Overall, there has been a steady increase in the number of policies issued each year. Household policies have shown a steady increase over the five year period at the expense of Motor, which have steadily decreased. Life has shown a very small increase over the period except for a small dip in 1981. Other policies have remained fairly steady, fluctuating only slightly around 6,000.

The information given concerns only numbers of new policies actually issued. No indication is given of premium values, cancellations or claims, therefore nothing can be said about the financial progress of the company.

Question 7

The standard calculations for the plotting of the two Lorenz curves are shown in Table 1 and the two corresponding Lorenz curves are plotted in Figure 3.

Figure 3

Identified personal wealth in the UK for 1967 and 1974
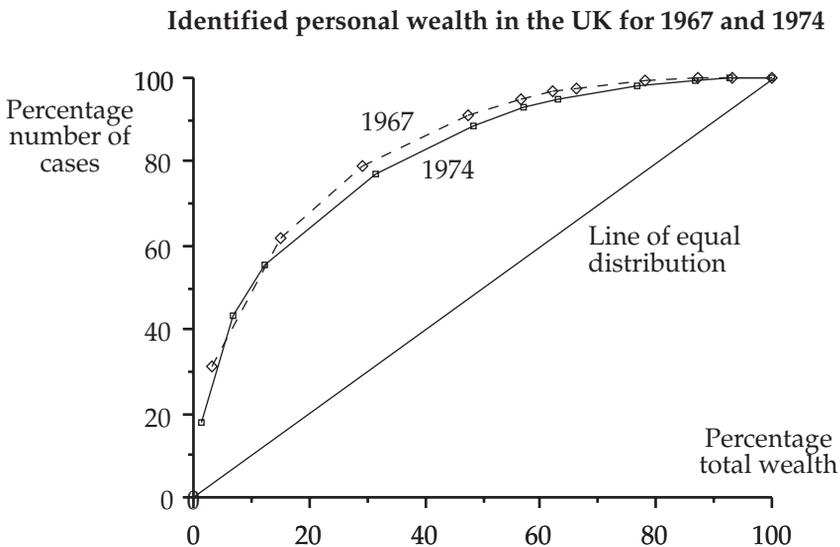
It can be seen from Figure 3 that the distribution of wealth in both years is similar, showing little change over the seven year period. There has been a very small redistribution towards equality, but this is not marked. In both years, the figures show that the least wealthy 50% of the population own only 10% of total wealth. However, 50% of all wealth was owned by the wealthiest 8% in 1967, while in 1974 it was shared between the wealthiest 10%.

Table 1

Range of wealth (£000)	Number of cases		Total wealth		Number of cases		Total wealth	
	%	cum %	%	cum %	%	cum %	%	cum %
0 to 1	31.2	31.2	3.4	3.4	18.1	18.1	1.3	1.3
1 to 3	30.5	61.7	11.7	15.1	25.4	43.5	5.5	6.8
3 to 5	17.1	78.8	13.9	29.0	11.8	55.3	5.5	12.3
5 to 10	12.6	91.4	18.3	47.3	21.9	77.2	19.3	31.6
10 to 15	3.6	95.0	9.1	56.4	11.5	88.7	16.9	48.5
15 to 20	1.6	96.6	5.7	62.1	4.0	92.7	8.5	57.0
20 to 25	0.9	97.5	4.1	66.2	2.2	94.9	6.1	63.1
25 to 50	1.6	99.1	11.8	78.0	3.4	98.3	13.8	76.9
50 to 100	0.6	99.7	9.0	87.0	1.2	99.5	9.8	86.7
100 to 200	0.2	99.9	6.1	93.1	0.4	99.9	5.8	92.5
over 200	0.1	100	6.9	100	0.1	100	7.5	100

Question 8

(i) The component bar chart for the given data is drawn in Figure 4.

Figure 4

Value of company assets by type
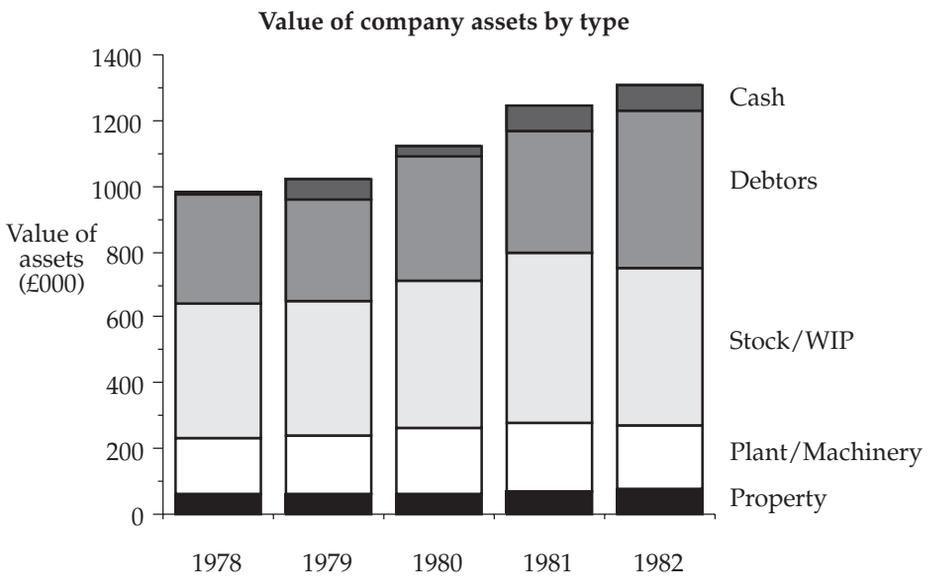
(ii) Overall, the total value of the given assets has increased steadily from just under £1m in 1978 to £1.3m in 1982. The most significant increase has been the debtors component, which has caught up with the stock and work-in-progress component, even though the latter has also increased. The property component shows very small increases, while plant and machinery shows small increases in the first four years and a decrease in the fifth year. Although the cash component has fluctuated over the five year period, it has shown an increase and is now comparable with property.

Answers to examination questions – part 1

Question 9

(a) (i) *Pictogram*. A representation that is easy to understand for a non-sophisti-cated audience. However, it cannot represent data accurately or be used for any further statistical work.

(ii) *Simple bar chart*. One of the most common forms of representing data which can be used for time series or qualitative frequency distributions. It is easy to understand and can represent data accurately. However, data values are not easily determined.

(iii) *Pie chart*. A type of chart which can have a lot of impact. Used mainly where the classes need to be compared in relative terms. However, they involve fairly technical calculations.

(iv) *Simple line diagram*. The simplest and most popular form of representing time series. They are easy to understand and represent data accurately. However, data values are not easily determined.

(b) A pie chart is one of the charts that could be drawn for the given data and is shown at Figure 5. Note however that a simple bar chart could equally well represent the data.

Figure 5

Shareholders owning more than 100,000 shares in Marks & Spencer plc
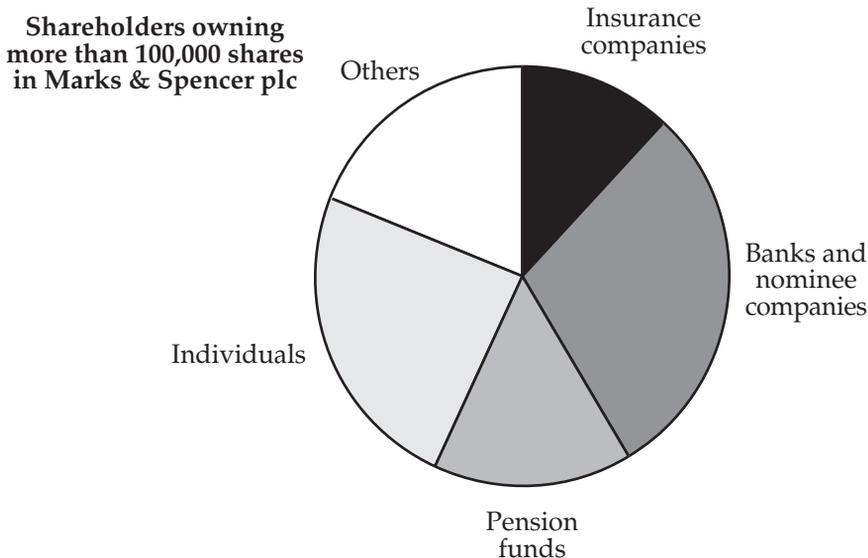
Question 10

The company can make and sell $10,000 \pm 2,000$ units in the year

The selling price will lie in the range $£50 \pm £5$ per unit

Thus the maximum revenue is $12,000 \times £55 = £660,000$

The minimum revenue is $8,000 \times £45 = £360,000$

The estimated revenue is $10,000 \times £50 = £500,000$

The maximum error from the estimated revenue is $£660,000 - £500,000 = £160,000$

and relative error $= \dfrac{150,000}{500,000} \times 100\% = 32\%$

The ranges of the various costs are:

	min		max
materials	£147,000	to	£153,000
wages	£95,000	to	£105,000
marketing	£45,000	to	£55,000
miscellaneous	£45,000	to	£55,000
Total:	£332,000	to	£368,000 est = £350,000

Maximum error from the estimated costs = £18,000

and relative error = $\dfrac{80,000}{350,000} \times 100\% = 5.1\%$

Maximum contribution = £660,000 – £332,000 = £328,000

Minimum contribution = £360,000 – £368,000 = –£8,000

the estimated contribution = £150,000

The maximum error from the estimated contribution is £328,000 – £150,000 = £178,000

Which gives the relative error $\dfrac{178,000}{150,000} \times 100\% = 118.7\%$

The maximum contribution of £328,000 arises when 12,000 units are made and sold

Therefore contribution/unit = $\dfrac{£328,000}{12,000} = £27.33/\text{unit}$

The minimum contribution of – £8,000 arises when 8,000 units are made and sold

Therefore contribution/unit = $\dfrac{-£8,000}{8,000} = -£1$

The estimated contribution /unit = £15

The maximum error from the estimated contribution/unit is £15 – (–£1) = £16

Therefore, relative error = relative error as $\dfrac{16}{15} \times 100\% = 106.7\%$.

Part 2

Question 1

	True limits							
Lower	Upper	Mid-point	f	fx	$x - \bar{x}$	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
0	5	2.5	39	97.5	–34.16	1,166.91	45,509.31
5	15	10	91	910	–26.66	710.75	64,678.75
15	30	22.5	122	2,745	–14.16	200.51	24,462.22
30	45	37.5	99	3,712.5	0.84	71	70.29
45	65	55	130	7,150	18.33	335.99	43,678.70
65	75	70	50	3,500	33.34	1,111.56	55,578.00
75	95	85	28	2,380	48.34	2,336.76	65,429.28
Total			559	20,495			299,406.55

The upper class limit of the final class is such that the class width is double that of the preceding class.

Mean, $\bar{x}$ = 20,495/559 = 36.66

Standard deviation = $\sqrt{\dfrac{299,406.55}{559}}$ = 23.14

For the histogram plot, since the class widths are uneven, we would need to scale the frequencies to plot heights for all those bars that are not standard width. This is due to the fact that the area of histogram bars (NOT height – except for classes that are all the same width) should represent frequency. Since most of the classes are of different widths, the calculations will be impractical and *not advisable in an examination*.

[AUTHOR NOTE: Clearly the examiner has made an error. This can be verified by reference to the suggested solution published by the examining board (ACCA) which shows the heights of bars representing frequencies. The correct histogram is too tedious to calculate and draw and thus is not represented here!]

Lower x	Upper x	f	%	Cumulative %
0	5	39	6.98	6.98
5	15	91	16.28	23.26
15	30	122	21.82	45.08
30	45	99	17.71	62.79
45	65	130	23.26	86.05
65	75	50	8.94	94.99
75	95	28	5.01	100

Median = 34. See following graph.

Age distribution
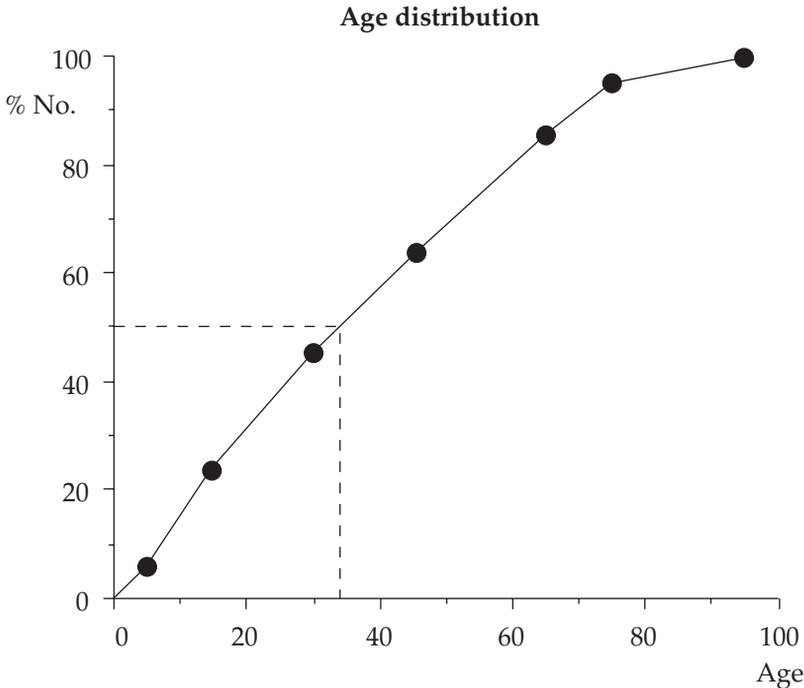
Question 2

(a) (i) Only $100 - 36.4 = 63.6\%$ received training. Hence, required percentage

$$= \frac{36.2}{63.6} \times 100 = 56.9\%$$

(ii) Similarly, $\dfrac{7.2}{100 - 48.2} = 13.9\%$ received training.

(b) *Non-apprentices receiving training*:

	Male		Female	
	%	cum %	%	cum %
1 to 2	9.9	9.9	9.8	9.8
3 to 8	28.1	38.1	43.3	53.1
9 to 26	30.0	68.1	29.5	82.6
27 to 52	12.3	80.4	7.9	90.6
53 to 104	11.1	91.5	6.7	97.3
105 or more	8.5	100.0	2.7	100.0

Note: Each male percent in the above is calculated using the given table % as a percentage of $100 - 57.7$.

For example, $9.9 = \dfrac{4.2}{100 - 57.7} \times 100$. Similarly for females.

Length of training of non-apprentices
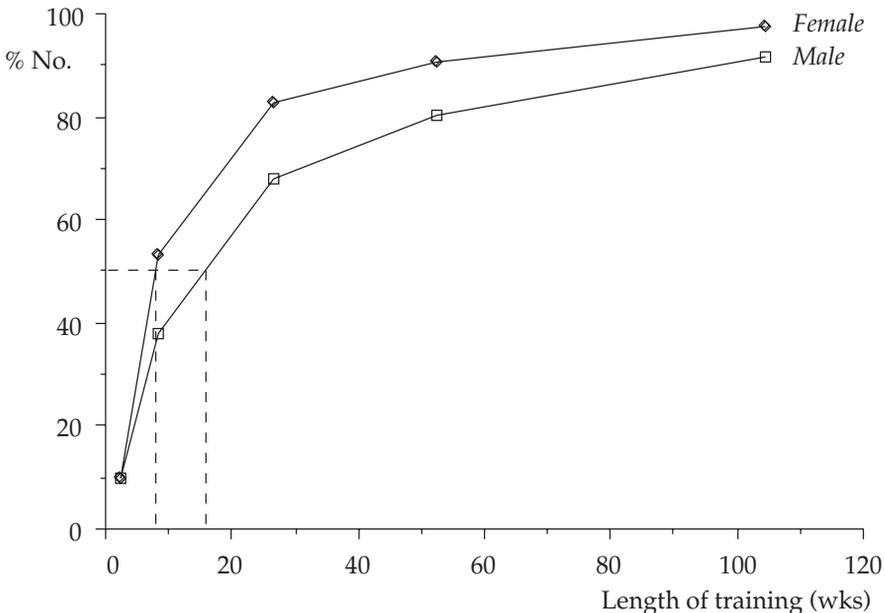
From the graph, the median for male non-apprentices is 16 weeks and the median for female non-apprentices is 8 wks.

(c) All apprentices receive at least 1 year's training. Non-apprentices receive 10 to 11 weeks training on average. Males receive more training than females in

general. Although a greater proportion of all females receive no training at all, more female non-apprentices receive *some* training than their male counter-parts. A much greater proportion of males are apprenticed (37%) than females (8%).

Question 3

Group	Mid-point			
	x	f	fx	fx^2
30 but less than 35	32.5	17	552.5	17956
35 but less than 40	37.5	24	900.0	33750
40 but less than 45	42.5	19	807.5	34319
45 but less than 50	47.5	28	1330.0	63175
50 but less than 55	52.5	19	997.5	52369
55 but less than 60	57.5	13	747.5	42981
		120	5335.0	244550

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{5335}{120} = 44.5 \text{ milliseconds}$$

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{244550}{559120} - \left(\frac{5335}{120}\right)^2} = 7.8 \text{ milliseconds}$$

Since the data are grouped, and thus the original access times are not known, both the measures above are estimates.

Question 4

(a) Smallest value = 3; largest value = 33; range = 30.

Seven classes will each have a class width of 30÷7 = 5 (approx). The formation of a cumulative frequency table is shown at Table 2.

Table 1

Number of rejects		Number of periods (f)	Cum f (F)	F%
0 to 4	II	2	2	4
5 to 9	III	3	5	10
10 to 14	IIII	4	9	18
15 to 19	ЖІІ II	7	16	32
20 to 24	ЖІ ЖІ ЖІ ЖІ	20	36	72
25 to 29	ЖІ ЖІ I	11	47	94
30 to 34	III	3	50	100

(b) Because the distribution is skewed, the median and quartile deviation are appropriate measures to describe the distribution.

Figure 1
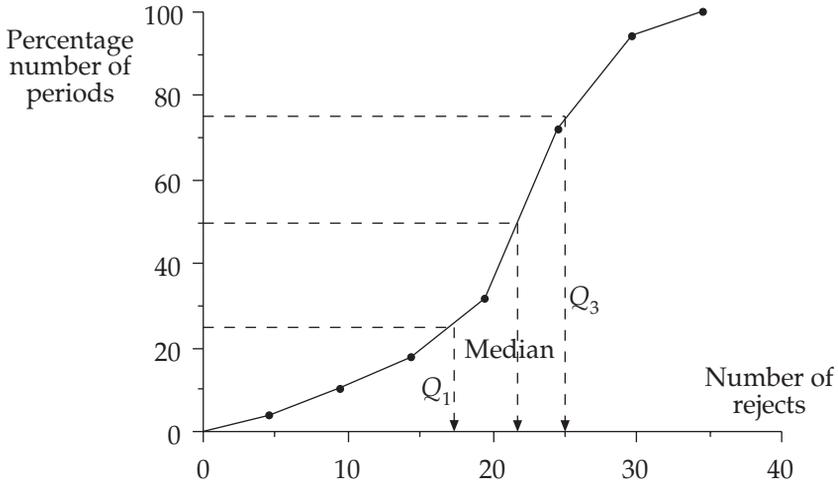
From the graph at Figure 2: $Q_1 = 17.5$; median $= 21.5$; $Q_3 = 25$.

Therefore, quartile deviation $= \dfrac{Q_3 - Q_1}{2} = \dfrac{150 + 120}{2} = 3.8$

(c) The median of 21.5 describes the average number of rejects in each five minute period $= 260/\text{hr}$ (approx). The quartile deviation measures the variability in the number of rejects from one five minute period to the next. In particular, we expect 50% of rejects to lie within 21.5 ± 3.8 in one five minute period.

Question 5

(a) Smallest value $= 510$; largest value $= 555$; range $= 45$. For 5 classes, each class should have width of $45 \div 5 = 9$ (but 10 is better!)

Number of components		f
510-519	ⅡⅢ II	7
520-529	ⅢⅢ ⅢⅢ	10
530-539	ⅢⅢ ⅢⅢ II	12
540-549	ⅢⅢ II	7
550-559	IIII	4

(b) (c) Figure 3 shows the histogram and, from it, the calculation of the mode.

Figure 2
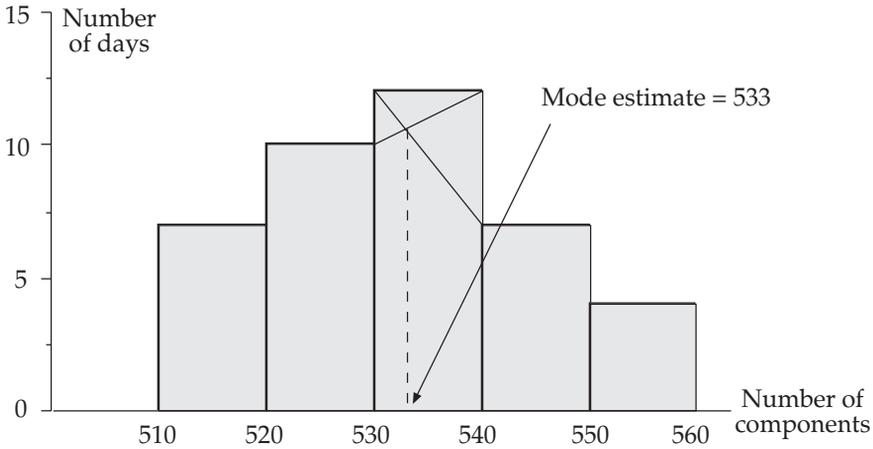
From the histogram, mode = 533.

(d) (e)

x	f	fx	fx^2
514.5	7	3601.5	1852971.7
524.5	10	5245	2751002.5
534.5	12	6414	3428283
544.5	7	3811.5	2075361.7
554.5	4	2218	1229881
	40	21290	11337499

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{21290}{40} = 532.25$$

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{11337499}{40} - \left(\frac{21290}{40}\right)^2} = 12.14 \ (2D)$$

(f) Mode = 533; mean = 532.25; sd = 12.14. Mode>mean implies *slight* left skew, which can just be made out from the frequency distribution.

Question 6

Income (£)	$f\%$	$F\%$	
Up to 30,000	5	5	
30,000-34,999	2	7	
35,000-35,999	3	10	1st F value to exceed 25
40,000-44,999	5	15	
45,000-49,999	10	25	
50,000-59,999	15	40	
60,000-69,999	18	58	1st F value to exceed 5(
70,000-99,999	21	79	
100,000-149,999	17	96	
150,000 and over	4	100	1st F value to exceed 75

Since the data is skewed and the first and last classes are open-ended, the median and quartile deviation are the most suitable measures of location and dispersion. The interpolation formula is used below to calculate the measures.

$$Q_1 = 49999.5 + \frac{25 - 25}{15} \times 10000 = 49{,}999.5 \ (50{,}000)$$

$$\text{Median} = 59999.5 + \frac{50 - 40}{18} \times 10000 = 65{,}555.1 \ (65{,}555)$$

$$Q_3 = 69999.5 + \frac{75 - 58}{21} \times 30000 = 94{,}285.1 \ (94{,}285)$$

$$\text{Quartile deviation} = \frac{94{,}285 - 50{,}000}{2} = 22{,}142.$$

Question 7

(a) Average rates of increase are usually found using the geometric mean. For example:

Year	1	2	3	4
Rate of increase	2.3%	3.8%	1.9%	4.2%

The appropriate multipliers are 1.023, 1.038, 1.019 and 1.042

Thus average multiplier = Geometric mean

$$= \sqrt{1.023 \times 1.038 \times 1.019 \times 1.042}$$

Therefore, average rate of increase = 3.05%

(b) In a skewed distribution, particularly where only a few values are contained at just one end, the median is the appropriate average to use since it largely ignores extremes and it would be giving the information that 50% of all values are less, and 50% more, than the median value.

(c) Since the speeds need to be averaged *over the same distance*, the harmonic mean is the appropriate average.

$$\text{hm} = \frac{2}{\dfrac{1}{30} + \dfrac{1}{60}} = 40 \text{ mph}$$

(However, if the speeds needed to be averaged *over the same time*, the arithmetic mean would be used, giving am $= \dfrac{30 + 60}{2} = 45$ mph.)

(d) An average would not be appropriate at all here, since clearly some of the ships would not be able to pass under a bridge built to this height. The height necessary needs to be (at least) the largest value in the distribution.

(e) A weighted mean would be appropriate here. If there were n_1 skilled and n_2 unskilled workers, the income for each of the workers could be calculated as:

$$\frac{4500 \times n_1 + 3500 \times n_2}{n_1 + n_2}$$

(f) A simple mean is all that is required.

i.e. mean amount $= \dfrac{\text{Total profits to be allocated}}{\text{Number of employees}}$

Question 8

The table showing calculations is given below.

	Mid-point (x)	(f)	(fx)	(fx^2)
40 to 60	50	5	250	12500
60 to 80	70	7	490	34300
80 to 100	90	7	630	56700
100 to 120	110	18	1980	217800
120 to 140	130	23	2990	388700
140 to 160	150	14	2100	315000
160 to 180	170	10	1700	289000
180 to 220	200	16	3200	640000
		100	13340	1954000

(i) $\bar{x} = \dfrac{\sum fx}{\sum f} = \dfrac{13340}{100} = 133.4$

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{1954000}{100} - \left(\frac{13340}{100}\right)^2} = 41.77 \text{ (2D)}$$

(ii) $\quad \text{cv}(1) = \dfrac{41.77}{133.4} \times 100 = 31.3\%; \ \text{cv}(2) = \dfrac{29.33}{88.0} \times 100 = 33.3\%.$

(iii) $\quad$ Distribution 2 is relatively more variable

Part 3
Question 1

(a)

Output and cost of standard size boxes

$(\bar{x}, \bar{y}) = (12, 40)$

Least squares regression line $y = 14.535 + 2.122x$
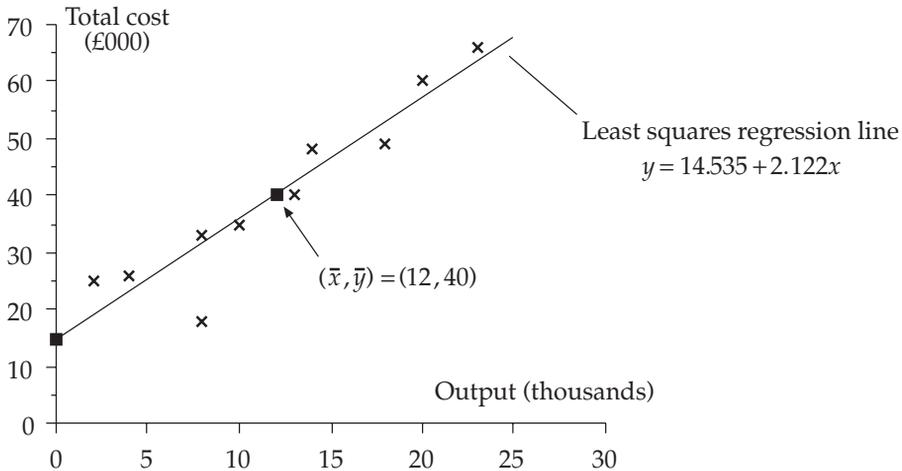
(b) Week 8's figures of 8000 output at a total cost of £18000 are distinctly out of line with the rest of the data. This is clearly due to special circumstances, perhaps a cheap off-loading of old stock.

(c) Any regression line fitted to a set of bivariate data must pass through the mean point $(\bar{x}, \bar{y})$.

In this case, $\bar{x} = \dfrac{120}{10} = 12$ and $\bar{y} = \dfrac{400}{10} = 40$.

(d) Let the regression line be in the form: $y = a + bx$. Using the least squares technique, we have:

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} = \frac{10(5704) - 120(400)}{10(1866) - (120)^2} = \frac{9040}{4260} = 2.122 \text{ (3D)}$$

$$a = \frac{\sum y}{n} - b\frac{\sum x}{n} = 40 - (2.122)12 = 14.535 \text{ (3D)}$$

i.e. least squares line of y (total cost) on x (output) is $y = 14.535 + 2.122x$

For the graph plot, y-intercept is 14.535 and the line must pass through the point (12,40) from part (c) above.

(e) The fixed costs of the factory is just the value of the y-intercept point of the regression line = 14.536 or £14536.

(f) If 25000 standard boxes are produced, then the regression line can be used to estimate the total costs as follows:

Estimated total costs = 14.535 + (2.122)(25) in £000 = £67585.

Question 2

(a) See the diagram below. Since both sets of data are close to their respective regression lines, correlation is quite good (and positive). The average turnover for multiples is higher than that for co-operatives, as evidenced by the higher figures, and, since the gradient of the multiple line is larger, multiples have also the higher marginal turnover.

(b) Putting $X=500$ into both regression lines gives:

multiples: $Y = -508.5 + (4.04)(500) = 1511.5$ i.e. a turnover of £1501m.

co-operatives: $Y = 22.73 + (0.67)(500) = 357.73$ i.e. a turnover of £350m.

Since correlation is high and both estimates have been interpolated, a good degree of accuracy might be expected.

(c) As mentioned in (a), the marginal turnover for multiples is higher than for co-operatives. Specifically, for multiples between 253 and 952 stores, each extra store generates a turnover of £4.04m.; for co-operatives between 210 and 575 stores, each extra store generates a turnover of £0.67m.
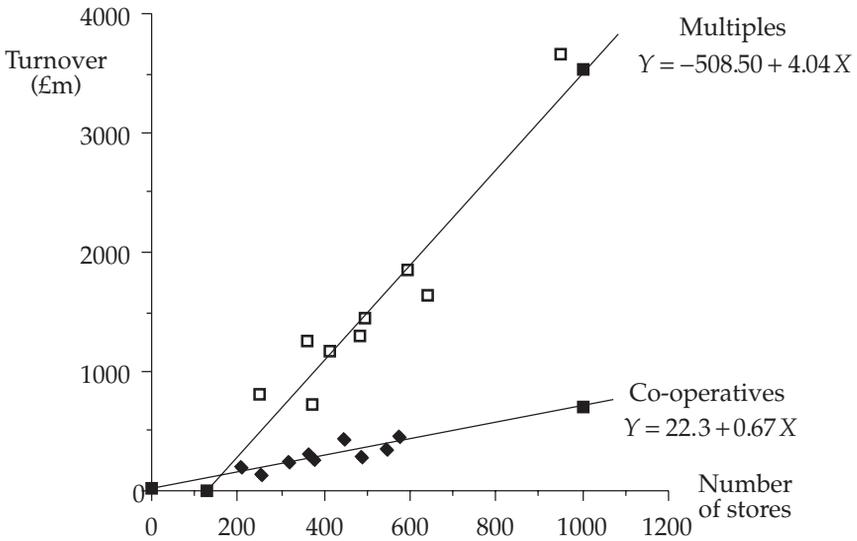
Question 3

(a) This statement is correct. Correlation does not attempt to measure the cause and effect that may exist between two variables, only the *strength of the mathematical relationship*. However, if a causal relationship exists between two variables, there should be a fairly high degree of correlation present.

Example 1: x = Milk consumption; y = Number of violent crimes. Clearly there will be high correlation due to higher population, but obviously no causation!

Example 2: x = Distance travelled by salesman; y = Number of sales made. Here a causal relationship is very probable with a resultant high correlation coefficient.

(b) Table for calculations:

Colour TV licences (millions)	Cinema admissions (millions)			
x	y	xy	x^2	y^2
5.0	134	670.0	25.0	17956
6.8	138	938.4	46.24	19044
8.3	116	962.8	68.89	13456
9.6	104	998.4	92.16	10816
10.7	103	1102.1	114.49	10609
12.0	126	1512.0	144.0	15876
12.7	112	1422.4	161.29	12544
12.9	96	1238.4	166.41	9216
78.0	929	8844.5	818.48	109517

$$r = \frac{n\sum xy - \sum x \sum y}{\sqrt{\left(n\sum x^2 - \left(\sum x\right)^2\right)\left(n\sum y^2 - \left(\sum y\right)^2\right)}}$$

$$= \frac{(8)(8844.5) - (78)(929)}{\sqrt{\left((8)(818.48) - 78^2\right)}\sqrt{\left((8)(109517) - 929^2\right)}}$$

$$= \frac{-1706}{(21.537)(114.433)} = -0.692$$

A moderately high degree of negative correlation, showing that as the number of colour licences increases so the number of cinema admissions decreases.

A causal relationship seems reasonable here, and with $r^2 = 0.48$ (2D), this demonstrates that approximately 50% of the variation in cinema attendances is explained by variations in the number of colour licences.

Question 4

(a) See the figure.

**Running costs and distance travelled
for 20 computer salesmen**
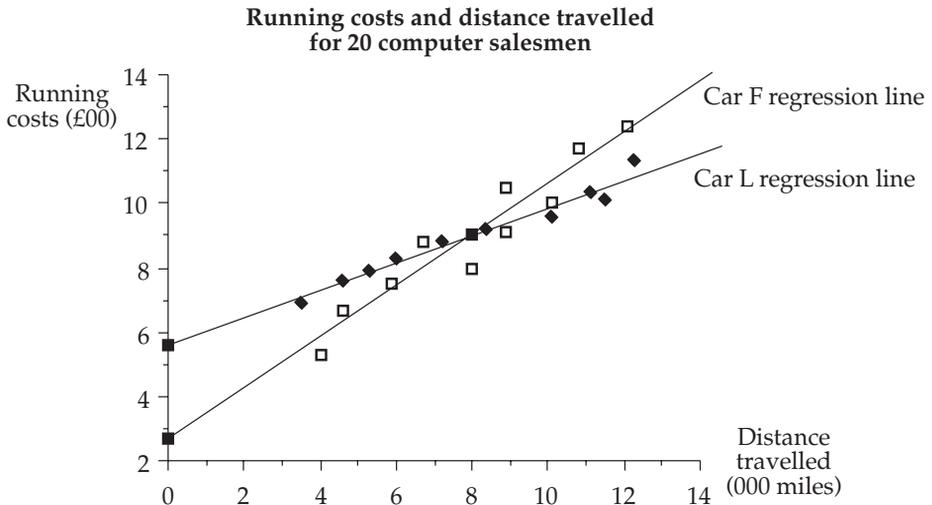
For the regression line plots in the figure,

Car F: intercept on *y*-axis is 2.65 and line must pass through (8,9).

Car L: intercept on *y*-axis is 5.585 and line must pass through (8,9).

(b) Car F: 2.65 is the initial (or fixed) running costs (£00) and 0.794 is the extra cost (£00) for each further one thousand miles travelled.

Car L: 5.585 is the initial cost and 0.427 is the extra cost for each further one thousand miles travelled.

(c) It is necessary to minimise the *average* cost per car for the two different types, taking into account the new average distance travelled = 1.5 x 8 = 12 (000 miles).

For type F: Average cost = 2.65 + (0.794)12 = 12.18 = £1218.
For type L: Average cost = 5.585 + (0.427)12 = 10.71 = £1071.
Therefore car L is cheaper on average.

(d) Using car L, the average cost for one car (with 10% extra costs) is given by (1.1)(5.585 + (0.427)12) = 11.7799 = £1177.99. Thus, expected total running costs for 5 cars is 5 × £1177.99 = £5889.95.

Question 5

(a)

Scatter diagram
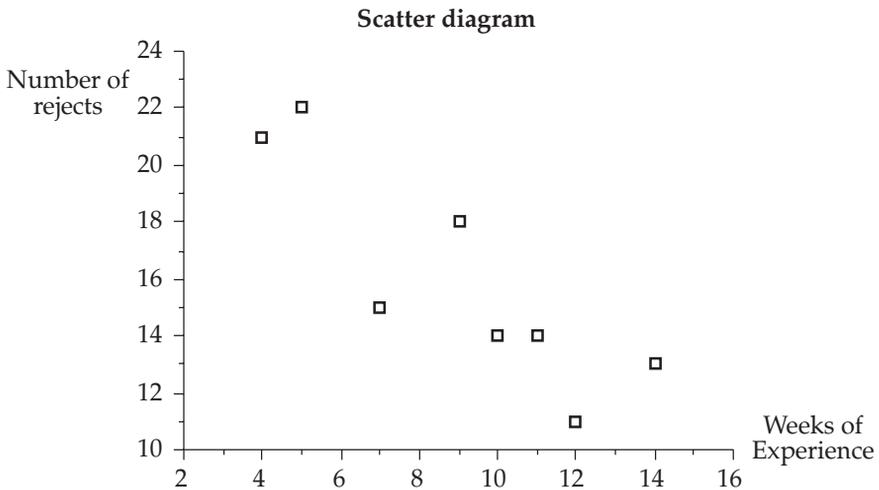
(b) For the product moment correlation coefficient:

$$\bar{x} = \frac{72}{8} = 9 \text{ and } \bar{y} = \frac{128}{8} = 16$$

$$r = \frac{1069 - 8(9)(16)}{\sqrt{\left(732 - 8(9)^2\right)}\sqrt{\left(2156 - 8(16)^2\right)}} = \frac{-83}{(9.165)(10.392)} = -0.87 \text{ (2D)}$$

The result shows high negative correlation and, since a causal relationship seems appropriate, this can be interpreted as the more experience an employee has in wiring components, the fewer the number of rejects to be expected.

c) Assuming a least squares line of the form $y = a + bx$, a and b are calculated as follows:

$$b = \frac{1069 - 8(9)(16)}{4 \times 3 \times 2 \times 1} = \frac{-83}{84} = -0.988 \text{ (3D) and } a = 16 - (-0.988) \times 9 = 24.892 \text{ (3D)}$$

The least squares regression line of y (rejects) on x (experience) is thus:

$$y = 24.892 - 0.988x.$$

After one week of experience ($x=1$), the expected number of rejects is given by:

$$y = 24.892 - 0.988(1) = 23.9(1D) \text{ or } 24 \text{ (to nearest whole number).}$$

Question 6

(i) Table for calculations:

Value	rank	Value	rank	d^2	Value	rank	d^2
15	3	13	2	1	16	2	1
19	5	25	5	0	19	3.5	2.25
30	7	23	4	9	26	6	1
12	2	26	6	16	14	1	1
58	8	48	8	0	65	8	0
10	1	15	3	4	19	3.5	6.25
23	6	28	7	1	27	7	1
17	4	10	1	9	22	5	1
				40			12.5

(1) Coefficient for actual and forecast 1:

$$r' = 1 - \frac{6(40)}{8(63)} = 0.52$$

(2) Coefficient for actual and forecast 2:

$$r' = 1 - \frac{6(12.5)}{8(63)} = 0.85$$

(ii) Clearly, forecasting method 2 is superior.

Question 7

(a) Briefly, regression describes the mathematical (linear) relationship between two variables while correlation describes the strength of this linear relationship.

(b) (i) See the figure on the following page.

The figure clearly shows that as the number of colour licences increases, so the number of cinema attendances decreases.

Scatter diagram
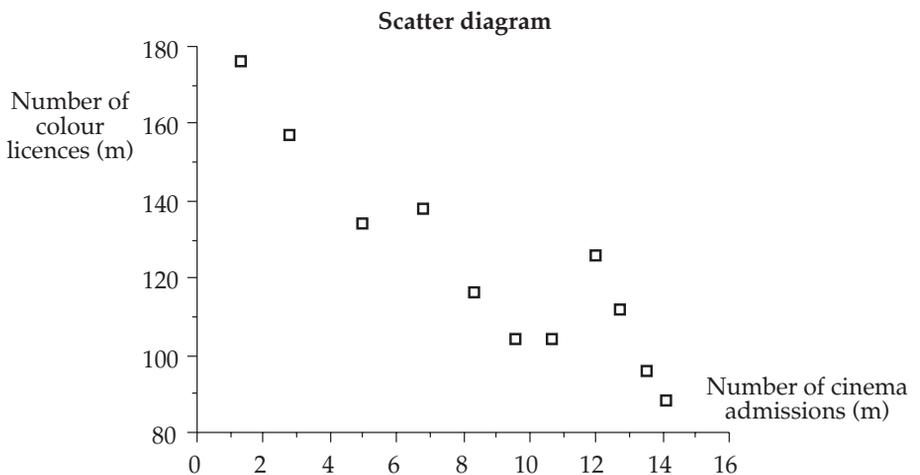
(ii)

Number of TV licences (m)	rank	Number of cinema admissions (m)	rank	d^2
1.3	1	176	11	100
2.8	2	157	10	64
5.0	3	134	8	25
6.8	4	138	9	25
8.3	5	116	6	1
9.6	6	104	3.5	6.25
10.7	7	104	3.5	12.25
12.0	8	126	7	1
12.7	9	112	5	16
13.5	10	96	2	64
14.1	11	88	1	100
				414.5

Rank correlation coefficient: $r' = 1 - \dfrac{6(414.5)}{11(120)}$

$$= -0.88$$

The above coefficient is showing strong inverse (or negative) correlation and, since there is every reason to believe that there is a causal relationship here, the hypothesis seems reasonable.

Question 8

(a) As the regression equation of profit on sales is required, put profit = y and sales = x

The equation is:

$y = a + bx$,

where: $b = \dfrac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$

$= \dfrac{12 \times 498{,}912.2 - 11{,}944 \times 462.1}{12 \times 12{,}763{,}470 - 11{,}944^2} = \dfrac{467{,}624}{10{,}502{,}504}$

$= 0.044525$

and: $a = \bar{y} - b\bar{x}$

$= \dfrac{462.1}{12} - 0.044525 \times \dfrac{11{,}994}{12} = 5.9944$

Hence the regression line is $y = -5.9944 + 0.044525x$

Profits (£m)
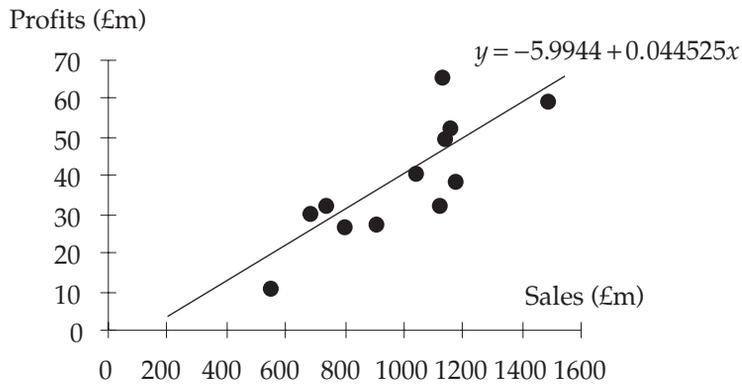
$$y = -5.9944 + 0.044525x$$

If the sales are £1,000 million, then $x = 1,000$ and the regression equation gives:
$$y = -5.9944 + 0.044525 \times 1000 = 38.5306$$
That is, profits = £38.5 million

(b) The regression line can be interpreted as indicating that, within the range of the data, each £1 million of sales generates £44,525 profit.

Assuming there are no changes in background circumstances, the forecast can be considered fairly reliable. First of all, the graph indicates a good correlation between profits and sales and so any forecasts produced by the regression line are likely to be reliable. (In fact the correlation coefficient, r, is approximately 0.8). Further, the forecast is an interpolation (the x-value line within the range of the given data), which is a further indication of reliability.

Part 4

Question 1

(a)(c)

		Account (£) (y)	Moving total	Moving average	Centred moving average (t)	Seasonal variation	Deseasonalised data
1982	Q2	662				-39	701
	Q3	712	2850	712.50	719.5	45	667
	Q4	790	2906	726.50	740.1	51	739
1983	Q1	686	3015	753.75	760.8	-56	742
	Q2	718	3071	767.75	774.9	-39	757
	Q3	821	3128	782.00	790.0	45	776
	Q4	846	3192	798.00	798.8	51	795
1984	Q1	743	3198	799.50	803.3	-56	799
	Q2	782	3228	807.00	814.8	-39	821
	Q3	827	3290	822.50	830.0	45	782
	Q4	876	3350	837.50	843.6	51	825
1985	Q1	805	3399	849.75		-56	861
	Q2	842				-39	881
	Q3	876				45	831

(b)

Quarterly electricity account
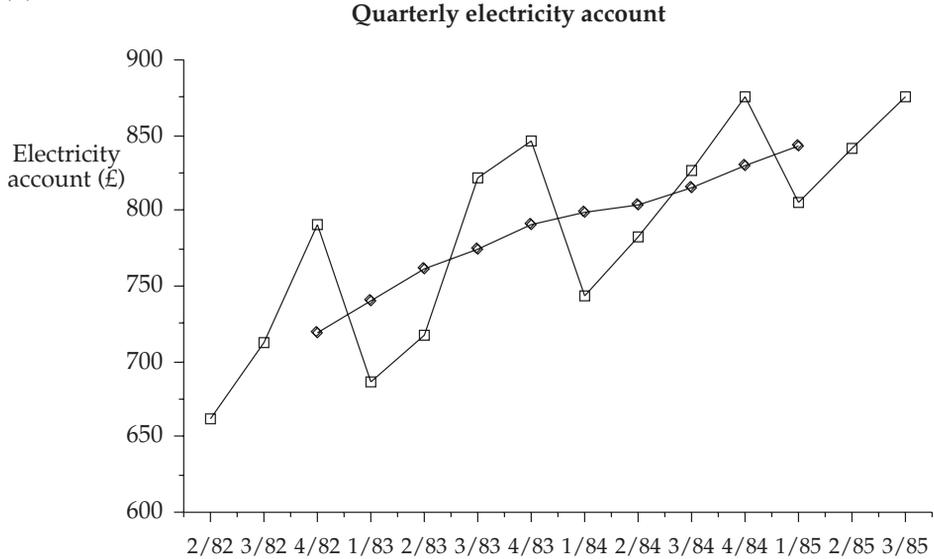
(d) The difference in values between the first and last trend values is:

843.6–719.5 = 124.1

Thus, the average increase between trend values is

$\dfrac{124.1}{9}$ = 14 approximately.

Estimated trend values

1985 Q4: 843.6+3(14) = 886

1986 Q1: 886+14 = 900.

Therefore, adding the respective seasonal variation value, we have:

Forecast values

1985 Q4: 886 + 51 = £937

1985 Q4: 900 – 56 = £844.

Question 2

(a) and (b) Main table of calculations:

		Moving total	Moving average	Centred moving average (t)	Deviation (y-t)	Seasonal variation (s)	Deseasonalised data (y-s)
73 1	100					-10.9	110.9
2	125					11.0	114.0
		454	113.50				
3	127			114.000	13.000	12.6	114.4
		458	114.50				
4	102			114.875	-12.875	-12.7	114.7
		461	115.25				
74 1	104			115.625	-11.625	-10.9	114.9
		464	116.00				
2	128			116.625	11.375	11.0	117.0
		469	117.25				
3	130			118.000	12.000	12.6	117.4
		475	118.75				
4	107			119.125	-12.125	-12.7	119.7
		478	119.50				
75 1	110			119.875	-9.875	-10.9	120.9
		481	120.25				
2	131			120.250	10.750	11.0	120.0
		481	120.25				
3	133			120.125	12.875	12.6	120.4
		480	120.00				
4	107			120.125	-13.125	-12.7	119.7
		481	120.25				
76 1	109					-10.9	119.9
2	132					11.0	121.0

Seasonal variation calculations

	Q1	Q2	Q3	Q4	
1973			13.000	−12.875	
1974	−11.625	11.375	12.000	−12.125	
1975	−9.875	10.750	12.875	−13.125	
Totals	−21.500	22.125	37.875	−38.125	
Averages (1D)	−10.8	11.1	12.6	−12.7	(Total=+0.2)
Adjustments	−0.1	−0.1	–	–	
Seasonal variation	−10.9	11.0	12.6	−12.7	

(c)

Department sales figures

(d) Both trend and seasonally adjusted values show a steady increase up to the beginning of 1975, when they levelled out. Seasonal patterns are well marked and continue throughout the whole period. Adjustments to seasonal averages were very small, leading to the conclusion that there was very little residual variation other than random factors.

Question 3

(a) Table of main calculations.

		Output (y)	5-day moving total	Trend (t)	Variation (y-t)
Week 1	Mon	187			
	Tue	203			
	Wed	208	1022	204.4	3.6
	Thu	207	1042	208.4	-1.4
	Fri	217	1047	209.4	7.6
Week 2	Mon	207	1049	209.8	-2.8
	Tue	208	1048	209.6	-1.6
	Wed	210	1043	208.6	1.4
	Thu	206	1038	207.6	-1.6
	Fri	212	1040	208.0	4.0
Week 3	Mon	202	1042	208.4	-6.4
	Tue	210	1041	208.2	1.8
	Wed	212	1043	208.6	3.4
	Thu	205	1049	209.8	-4.8
	Fri	214	1054	210.8	3.2
Week 4	Mon	208	1059	211.8	-3.8
	Tue	215	1071	214.2	0.8
	Wed	217	1070	214.0	3.0
	Thu	217			
	Fri	213			

(b) See graph on opposite page

(c)

	Mon	Tue	Wed	Thu	Fri	
Week 1			3.6	−1.4	7.6	
Week 2	−2.8	−1.6	1.4	−1.6	4.0	
Week 3	−6.4	1.8	3.4	−4.8	3.2	
Week 4	−3.8	0.8	3.0			
Totals	−13.0	1.0	11.4	−7.8	14.8	
Averages	−4.3	0.3	2.9	−2.6	4.9	(Total–1.2)
Adjustments	−0.3	−0.2	−0.2	−0.2	−0.3	
Daily variation	−4.6	0.1	2.7	−2.8	4.6	

Daily output of a company
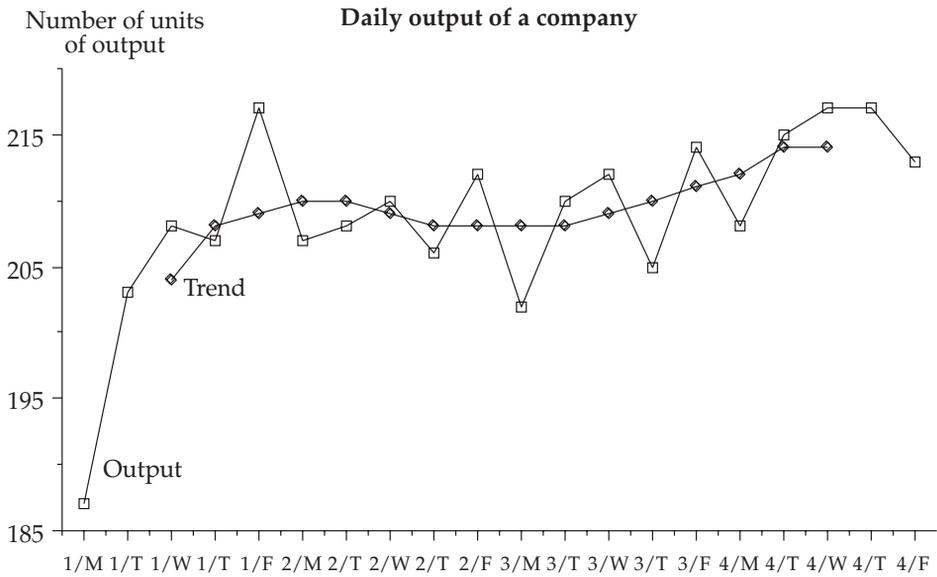
(d) Using the calculated (moving average) trend values, the average daily increase in trend can be calculated as: $\dfrac{214.0 - 204.4}{15} = 0.64$.

Trend value for Week 5 (Monday) = 214.0 + 3(0.64) = 215.9 (1D).

Trend value for Week 5 (Tuesday) = 215.9 + 0.64 = 216.6 (1D).

Forecast output for Week 5 (Monday) = 215.9 − 4.6 = 211 (to nearest unit).

Forecast output for Week 5 (Tuesday) = 216.6 + 0.1 = 217 (to nearest unit).

(e) No forecast can ever be confidently made, since it is based only on past evidence and there can be no guarantee that the trend projection is accurate or that the daily variation figures used will be valid for future time points. Only general experience and a particular knowledge of the given time series environment would help further in determining the accuracy of the given forecasts.

Question 4

(a) Trend, seasonal and residual variation. Residual variation contains both random and possible long-term cyclic variations.

(b) (i) Main table of calculations.

	Number of unemployed (y)	Totals of 4	Moving average	Centred moving average (t)	(y-t)
79 Jan	22				
Apr	12	175	43.75		
Jul	110	174	43.50	43.625	66.375
Oct	31	188	47.00	45.250	-14.250
80 Jan	21	228	57.00	52.000	-31.000
Apr	26	267	66.75	61.875	-35.875
Jul	150	296	74.00	70.350	79.625
Oct	70	306	76.50	75.250	-5.250
81 Jan	50	302	75.50	76.000	-26.000
Apr	36	342	85.50	80.500	-44.500
Jul	146				
Oct	110				

Calculations for seasonal variation:

	Jan	Apr	Jul	Oct
1979			66.375	−14.250
1980	−31.000	−35.875	79.625	−5.250
1981	−26.000	−44.500		
Totals	−57.000	−80.375	146.000	−19.500
Averages	−28.2	−40.2	73.0	−9.8 (Tot=−5.5)
Adjustments	+1.4	+1.4	+1.4	=1.3
Seasonal variation	−27.1	74.4	74.4	−8.5

(ii) Seasonally adjusted values: 1981, Jan = $y - s = 50 - (-27.1) = 77.1$

1981, Apr = $36 - (-38.3) = 74.3$

Part 5

Question 1

(i) An index number enables the value of some economic commodity to be compared over some defined time period. It is expressed in percentage terms, using a base of 100.

(ii) (iii) are shown in the following table:

Year	Average salary (£)	Year on year increase (%)	Retail Price Index (1975=100)	Year on year increase (%)	Revalued salary (1985 base)
1977	9500		135.1		19464.10
1978	10850	14.2	146.2	8.2	20542.27
1979	13140	21.1	165.8	13.4	21936.98
1980	14300	8.8	195.6	18.0	20236.40
1981	14930	4.4	218.8	11.9	18887.68
1982	15580	4.4	237.7	8.6	18142.80
1983	16200	4.0	248.6	4.6	18037.65
1984	16800	3.7	261.0	5.0	17817.01
1985	17500	4.2	276.8	6.1	17500.00

Note: $£17,817.01 = \dfrac{16800 \times 276.8}{261.0}$; $£18,037.65 = \dfrac{16200 \times 276.8}{248.6}$; etc

(iv) Except for the first two years, the increase in prices has outstripped the increase in salary. The revalued salary shows that (in real terms) the systems analysts are being paid less now than in any of the previous nine years.

Question 2

(a) The Retail Prices Index can be used by retailers to compare their own average price increases with those that consumers are subject to. Wage-earners often use the RPI (although the Tax and Price Index is more relevant) to compare their wage increases with the increases in prices. Trades Union use the value of the RPI to negotiate price increases with employers.

The Producer Price Indices can be used by retailers to compare the prices they are paying for their goods. It can also be used by consumers as a long term warning (nine months or so) of trends that will inevitably be felt in the RPI.

The Index of Output of the Production Industries is used as a general guide to measure the changes in the level of production in the UK.

(b) Putting July 1979 as year 0 etc, we have:

	$W_0 \times E_0$	$W_1 \times E_0$	$W_0 \times E_2$	$W_2 \times E_2$
Q	300	350	240	320
R	120	130	180	210
S	140	170	70	90
T	90	110	180	240
Total	650	760	670	860

(i) Laspeyre index for year $1 = L_1 = \dfrac{\sum W_1 E_0}{\sum W_0 E_0} = \dfrac{760}{650} \times 100 = 116.9$

(ii) Paasche index for year $2 = P_2 = \dfrac{\sum W_2 E_2}{\sum W_0 E_2} = \times 100 = 128.4$

(iii) The indices given can be base-changed to 1979. Thus:

$$L_{1/0} = \frac{187.4}{156.3} \times 100 = 119.9; \quad L_{2/0} = \frac{203.4}{156.3} \times 100 = 130.1$$

These indices, when compared with the company's indices, show that the wage rates of the company are lagging slightly behind the Chemical and Allied Industry's rates by about two points.

Question 3

(a) In 1974 (on average, per week) 4 hours overtime was worked, which is equivalent to 4 ×1.5=6 normal hours. Thus, dividing the average weekly earnings by 46 (the equivalent normal hours worked per week) will give the normal rate per hour as 40.19÷46=£0.87. Multiplying this by 40 will thus yield the normal weekly rate of 40× 0.87=£34.95. This must be done for each year.

i.e. average normal weekly hours = 40 + (ave hours worked − 40) × 1.5

and normal weekly rate = ave earnings × $\dfrac{40}{\text{average normal weekly hours}}$

Year	Average weekly earnings	Average hours worked	Equivalent normal weekly hours	Normal weekly rate
1974	40.19	44	46	34.95
1975	52.65	45	47.5	44.34
1976	62.03	45	47.5	52.24
1977	70.20	46	49	57.31
1978	76.83	46	49	62.72
1979	91.90	46	49	75.02
1980	107.51	45	47.5	90.53
1981	121.95	43	44.5	109.62

(b)

RPI	80.5	100.0	116.5	135.0	146.2	165.8	195.6	218.9
Log	1.91	2.00	2.07	2.13	2.16	2.22	2.29	2.34

Semi-logarithmic graph of the retail price index
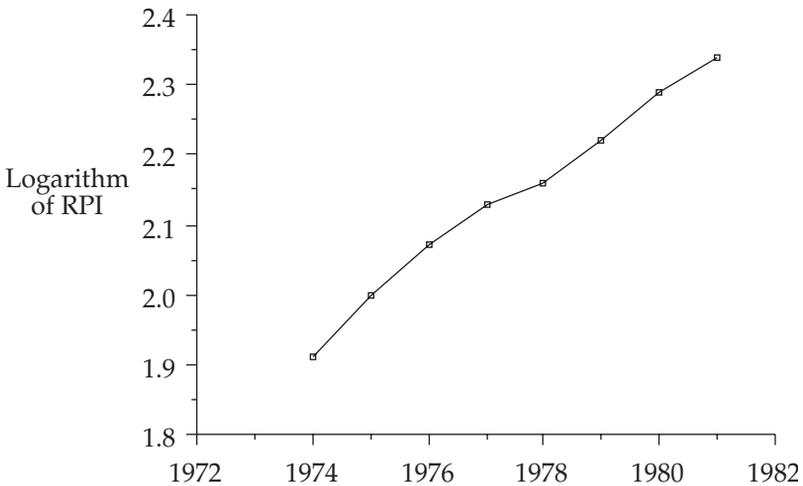
The above logarithms are plotted against the relevant year to form a semi-logarithmic graph which is shown in the figure above.

(c) Since the semi-graph in Figure 1 is an approximate straight line, this demonstrates that the rate of increase of the RPI is constant.

(d) A deflated normal weekly rate can be obtained by dividing each normal weekly rate by the value of the RPI for that year and multiplying back by 100 to bring the value back to the correct form.

e.g. deflated normal weekly rate for 1974 $= 34.95 \times \dfrac{100}{80.5} = £43.42$

Year	1974	1975	1976	1977	1978	1979	1980	1981
Deflated normal weekly rate (£)	43.42	44.34	44.84	42.45	42.90	45.25	46.28	50.08

(e) If an index of real wages is calculated, it will enable a comparison between the increase in prices and real wages to be made.

Question 4

Item (w)	Weight (I)	Index (wI)	(1) (wI)	(2) (wI)	(3) (wI)
Mining and quarrying	41	361	14801		
Manufacturing					
Food, drink and tobacco	77	106	8162	8162	8162
Chemicals	66	109	7194	7194	7194
Metal	47	72	3384	3384	3384
Engineering	298	86	25628	25628	25628
Textiles	67	70	4690	4690	4690
Other manufacturing	142	91	12922	12922	12922
Construction	182	84	15288	15288	
Gas, electricity and water	80	115	9200	9200	
	1000		101269	86468	61980

(i) (1) All industries index is given by: $\dfrac{101269}{1000} = 101.3$

(2) All industries except mining and quarrying index is: $\dfrac{86468}{1000 - 41} = 90.2$

(3) Manufacturing industries index is: $\dfrac{61980}{1000 - 41 - 182 - 80} = 88.9$

(ii) The high mining and quarrying index of 361 was severely offset by its small weight in the relatively low value of 101.3 for the overall index in (1). However, the index of only 90.2 in (2) shows the significance of mining and quarrying (particularly North Sea oil) to industrial production in the UK. The low manufacturing index of 88.9 in (3) is due to the fact that the three largest weights are assigned to relatively low indices.

Question 5

FOOD: The movement in these weights can be accounted for by the increased affluence of our society, which results in a much greater pool of disposable income left after the basic necessities (of which food is one of the most important) have been acquired. Also, since it is reasonable to assume that we are not buying less food in

1981 than we were in 1961, it means that (in relative terms) food is now cheaper. Since food has such a high weighting, the value of its index will bear the most significant effect on the RPI itself.

HOUSING: The increase in expenditure is probably due to two factors. First, a significant part of the extra disposable income is being spent on housing. Second, housing is more expensive in real terms. Changes in such things as mortgage rates and rents will now have a more significant effect on the RPI than was previously the case.

CLOTHING: The decrease in expenditure is probably due to cheap imports, since again we can only suppose that we are buying at least as much clothing in 1981 as we were in 1961.

TRANSPORT: The dramatic increase in transport costs are probably due to the increased mobility we now have as a society. We travel much further both to and from our place of occupation and also for leisure and recreation purposes. Changes in petrol prices and car tax will now have much more effect on the RPI than they did previously.

Part 6

Question 1

(a) Set P = £40,000, for convenience.

Time 0: amount owed = P.

After 1 quarter, 4% is added to amount owing, giving $P(1 + 0.04) = PR$

X is paid, leaving amount owed = $PR - X$.

After 2 quarters 4% is added to amount owing, and X is paid.

Amount now owed $= (PR - X)R - X$

$= PR^2 - XR - X$ or £$(40{,}000\,R^2 - XR - X)$

(b) After 3 quarters, amount owed $= (PR^2 - XR - X) - R - X$

$= PR^3 - XR^2 - XR - X$

And so on, until, after 80 quarters, the amount owed

$= PR^{80} - XR^{79} - XR^{78} - ... -X$

$= PR^{80} - X(R^{79} + R^{78} + ... + R + 1)$

$= PR^{80} - X\left(\dfrac{R^{80} - 1}{R - 1}\right)$ from the geometric progression formula.

Now, as the mortgage is to be paid off in this period, this amount owed must be zero, and so:

$$PR^{80} = X\left(\dfrac{R^{80} - 1}{R - 1}\right)$$

and: $\qquad P.R^{80}.\left(\dfrac{R - 1}{R^{80} - 1}\right) = X$

$0.0418P = X$ (from tables) [*]

Since $P = £40{,}000$, the quarterly repayment is £1,672.56.

(c) Using [*], if P is doubled from £40,000 to £80,000, the factor 0.0418 would be unaltered, and so the repayment figure would double to $2X$.

Question 2

(a) We are given that: $P=12000$; $i=0.15$; $n=5$. Putting the amortization payment as A, we must have that:

$$12000 = \frac{A}{1.15} + \frac{A}{1.15^2} + ... + \frac{A}{1.15^5}$$
$$= A(0.86957 + 0.75614 + 0.65752 + 0.57175 + 0.49718)$$
$$= A(3.35216)$$

Therefore, $A = \dfrac{12000}{3.35216} = 3579.79$. That is, amortization payment = £3579.79

The amortization schedule is tabulated as follows:

Year	Amount outstanding (beginning)	Interest	Payment
1983	12000.00	1800.00	3579.79
1984	10220.21	1533.03	3579.79
1985	8173.45	1226.02	3579.79
1986	5819.68	872.95	3579.79
1987	3112.84	466.93	3579.79
1988	(0.02)		

(b) Here, there are two interest rates. The investment rate, $j=0.1$ and the borrowing rate, $i=0.15$. Also, $P=12000$ and $n=5$.

The calculations for the sinking fund payment (ordinary annuity) is given in the following.

The debt will amount to $12000(1.15)5 = £24,136.29$ after 5 years. Thus, the sinking fund must mature to this amount. If A is the annual deposit into the fund, then we must have that:

$$24136.29 = A + A(1.1) + A(1.1)\,2 + A(1.1)\,3 + A(1.1)\,4$$
$$= A(1 + 1.1 + 1.21 + 1.331 + 1.4641) = A(6.1051)$$

Therefore, $A = \dfrac{24136.29}{6.1051} = £3953.46$

The Sinking Fund schedule is tabulated as follows:

Year	Debt outstanding	Interest on debt	Deposit	Amount in fund	Interest on fund
1983	12000.00	1800.00	0	0	0
1984	13800.00	2070.00	3953.46	3953.46	395.35
1985	15870.00	2380.50	3953.46	8302.27	830.23
1986	18250.50	2737.58	3953.46	13085.96	1308.60
1987	20988.08	3148.21	3953.46	18348.02	1834.80
1988	24136.29		3953.46	24136.28	

(c) Discounted cash flow table for calculation of NPV:

Year	Net cash flow	Discount Factor (10%)	Discount Factor (15%)	Present value (10%)	Present value (15%)
1983	(12000)	1.0000	1.0000	(12000.00)	(12000.00)
1984	6600	0.9091	0.8696	6000.06	5739.36
1985	6000	0.8264	0.7561	4958.40	4536.60
1986	4500	0.7513	0.6575	3380.85	2958.75
1987	(1000)	0.6830	0.5718	(683.00)	(571.80)
1988	(2600)	0.6209	0.4972	(1614.34)	(1292.72)
			NVP	41.97	(629.81)

(d) Using the formula method to determine the IRR, we have:

$I_1=10$; $N_1=41.97$; $I_2=15$; $N_2=-629.81$

and IRR $= \dfrac{N_1 I_2 - N_2 I_1}{N_1 - N_2} = \dfrac{(41.97)(15) - (-629.81)(10)}{41.97 - (-629.81)} = \dfrac{629.55 + 6298.1}{671.78}$

giving IRR = 10.3%.

The IRR gives the rate which makes NPV=0.

Question 3

(a) This can be calculated using a schedule as follows:

Year	Amount in fund	Interest	Total in fund	
1	10000	1000	11000	
2	21000	2100	23100	
3	33100	3310	36410	
4	46410	4641	51051	= value

(b) To calculate present value:

Year	Net savings	11% discount factor	Present value
1	2000	0.9009	1801.81
2	2000	0.8116	1623.20
3	2000	0.7312	1462.40
4	2000	0.6587	1317.40
			6204.80

(c) This can again be calculated using a schedule:

Year	Total amount invested	Interest (10%)	Total	
1	1000.00	100.00	1100.00	
2	1600.00	160.00	1760.00	
3	2260.00	226.00	2486.00	
4	2986.00	298.60	3284.60	
5	3784.60	378.46	4163.06	= acc sum

(d) This requires the value of an amortization annuity. Given that: $P = 20000$; $n = 20$; $i = 0.14$. If the yearly repayment is A, then:

$$20000 = \frac{A}{1.14} + \frac{A}{1.14^2} + \dots + \frac{A}{1.14^{20}}$$

$$= A\left(\frac{1}{1.14} + \frac{1}{1.14^2} + \dots + \frac{1}{1.14^{20}}\right)$$

But the terms in the bracket form a gp with $a = \frac{1}{1.14}$ and $r = \frac{1}{1.14}$.

The sum to 20 terms of this gp is given by:

$$\frac{\frac{1}{1.14}\left[1 - \left(\frac{1}{1.14}\right)^{20}\right]}{1 - \frac{1}{1.14}} = \frac{0.8772 \times 0.9272}{0.1228} = 6.6233$$

Therefore, $20000 = A(6.6233)$, giving $A = \frac{20000}{6.6233} = £3019.65$

Question 4

(a) The amount of the mortgage must sum to the present value of all the payments made. If A is the annual payment, then:

$$10000 = \frac{A}{1.12} + \frac{A}{1.12^2} + \dots + \frac{A}{1.12^5}$$

Thus: $10,000 = A(0.8929 + 0.7972 + 0.7118 + 0.6355 + 0.5674)$

$$= 3.6048A$$

Therefore, $A = \frac{10000}{3.6048} = £2774.10$

(b)

Year	Outstanding debt	Interest paid	Payment	Principal repaid
1	£10,000.00	£1,200.00	£2,774.10	£1,574.10
2	£8,425.90	£1,011.11	£2,774.10	£1,762.99
3	£6,662.91	£799.55	£2,774.10	£1,974.55
4	£4,688.37	£562.60	£2,774.10	£2,211.49
5	£2,476.87	£297.22	£2,774.10	£2,476.87

(c) After 5 years, the debt will amount to: $£10,000(1.12)^5 = £17,623.42$, which is therefore the amount that the fund must mature to. Putting A as the annual premium into the fund, gives:

$$17,623.42 = A(1.15) + A(1.15)^2 + \dots + A(1.15)^5$$

$$= A(1.15 + 1.3225 + 1.5209 + 1.7490 + 2.0114)$$

$$= 7.7538A$$

Thus, $A = £2272.89$

Answers to examination questions – part 6

(d) Year	Debt Outstanding	Interest on Debt	Deposit	Amount in Fund	Interest on Fund
1	£10,000.00	£1,200.00	£2,272.89	£2,272.89	£340.93
2	£11,200.00	£1,344.00	£2,272.89	£4,886.72	£733.01
3	£12,544.00	£1,505.28	£2,272.89	£7,892.62	£1,183.89
4	£14,049.28	£1,685.91	£2,272.89	£11,349.41	£1,702.41
5	£15,735.19	£1,888.22	£2,272.89	£15,324.71	£2,298.71
6	£17,623.42			£17,623.42	

Question 5

(a) This question refers to repayments including both capital and interest; that is, amortization. With $P = 100{,}000$ $n = 4$ and $i = 0.12$, the amortization payment, A say, must satisfy:

$$100{,}000 = \frac{A}{1.12} + \frac{A}{1.12^2} + \frac{A}{1.12^3} + \frac{A}{1.12^4}$$

$$= A(0.89286 + 0.79719 + 0.71178 + 0.63552)$$

$$= A(3.03735).$$

Therefore, $A = \dfrac{100000}{3.03735} = £32{,}923.44$.

(b) The reducing balance depreciation formula can be used, namely: $D = B.(1- i)^n$ where $D = 1000$, $B = 50000$ and $n = 5$. Here, i is to be determined.

Re-arranging gives $\dfrac{D}{B} = (1- i)^n$ or $1- i = \left(\dfrac{D}{B}\right)^{\frac{1}{n}}$

So that: $1- i = \left(\dfrac{1000}{50000}\right)^{\frac{1}{5}} = (0.02)^{0.2} = 0.457$

Therefore $i = 1 - 0.457 = 0.543 = 54.3\%$

(c) Working in time periods of one quarter (year), we have $A = P.(1+i)^n$ where: $P = 1000$, $A = 3000$

and $i = \dfrac{0.12}{4} = 0.03$ (per quarter).

Re-arranging gives: $\qquad (1+ i)^n = \dfrac{A}{P}$

and substituting: $\qquad (1.03)^n = \dfrac{3000}{1000} = 3$

Using logarithms: $\qquad n.\log(1.03) = \log(3)$

Therefore $n = \dfrac{\log(3)}{\log(1.03)} = \dfrac{0.4771}{0.01284} = 37$ (approximately).

Thus, number of years $= \dfrac{37}{4} = 9.25$.

Question 6

(a) Cash flow table:

End year	Machinery	Maintenance	Revenue	Net
0	(75000)			(75000.00)
1		(1000)	20000.00	19000.00
2		(1100)	21500.00	20400.00
3		(1210)	23112.50	21902.50
4		(1331)	24845.94	23514.94
5			26709.38	26709.38
6	1250			1250.00

(b) and (c) Discounted cash flow table:

Year	Net flow	Discount factor (10%)	Discount factor (15%)	Present value (10%)	Present value (15%)
0	(75000.00)	1.0000	1.0000	(75000.00)	(75000.00)
1	19000.00	0.9091	0.8696	17272.90	16522.40
2	20400.00	0.8264	0.7561	16858.55	15424.44
3	21902.50	0.7513	0.6575	16455.34	14400.89
4	23514.94	0.6830	0.5718	16060.70	13445.84
5	26709.38	0.6209	0.4972	16583.85	13279.90
6	1250.00	0.5645	0.4323	705.62	540.37
				8936.99	(1386.15)

(d)

Graphical estimation of IRR

Net present value (£) — Discount rate (%)

IRR estimate = 14.3%
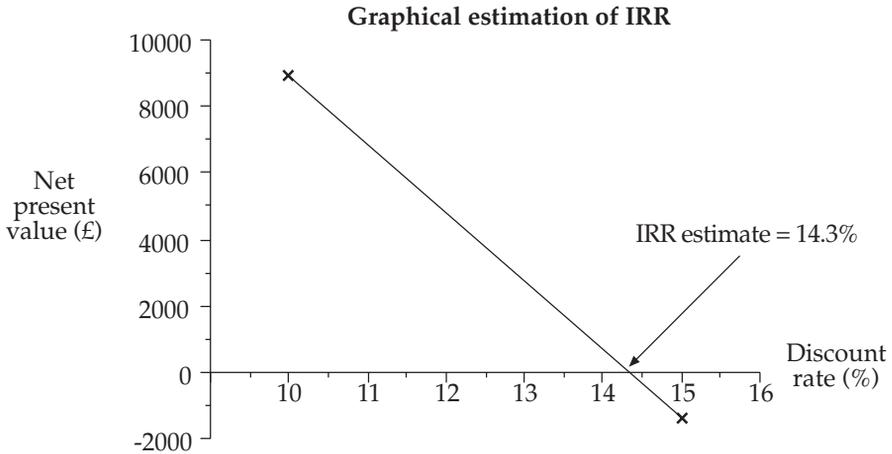
(e) The value of 14.3% for the IRR can be interpreted as the rate of return that the project earns.

Question 7

The calculations for the net present value for each one of the three decisions are tabulated below.

Discount factor at 12%	Decision (i)		Decision (ii)		Decision (iii)	
	Net cost	Present value	Net cost	Present value	Net cost	Present value
1	95000	95000	52000	52000	32000	32000
0.8929	20000	17857	27000	24107	32000	28571
0.7972	20000	15944	27000	21524	32000	25510
0.7118	20000	14236	27000	19218	32000	22777
0.6355	20000	12710	27000	17159	32000	20337
0.5674	–10000	–5674	–10000	–5674	0	0
		150073		128334		129195

Decision (ii) is the least costly and thus, on pure financial grounds, should be chosen. However, this does not take into account the usefulness of the computer over the next five years. For example, could the old machine run newly developed software?

Question 8

(a) (i) After four years (i.e. 8×6 months):

$A = P.(1+i)^n = 2750(1.035)^8 = £3621.22.$

(ii) $(1.035)^2 - 1 = 1.071 - 1 = 0.071$ or 7.1%.

(b) (i) The amortization method of debt repayment is required here. We are given: $P=37500$, $i=0.12$ and $n=5$. So, if A is the annual payment to be found, then:

$$37500 = \frac{A}{1.12} + \frac{A}{1.12^2} + \frac{A}{1.12^3} + \frac{A}{1.12^4} + \frac{A}{1.12^5}$$

$$= A(0.89286 + 0.79719 + 0.71178 + 0.63552 + 0.56743)$$

$$= A(3.60478)$$

Thus, $A = \dfrac{37500}{3.60478} = £10,402.86.$

(ii) Amortization schedule:

Year	Amount outstanding	Interest paid	Payment
1	37500.00	4500.00	10402.86
2	31597.14	3791.66	10402.86
3	24985.94	2998.31	10402.86
4	17581.39	2109.77	10402.86
5	9288.29	1114.59	10402.86
Balance	0.03		

(c) (i) Borrowing rate, $i = 12\% = 0.12$; Investment rate, $j = 8\% = 0.08$. Principal amount borrowed, $P = £37500$, $n = 5$ (years).

Notice that the payments into the fund are in advance. i.e. the payments form a due annuity.

The debt will amount to £37500(1.12)5 = £66,087.81 after 5 years. Thus the fund must mature to this amount. Putting A as the annual deposit into the fund, we must have that:

$$66087.81 = A(1.08) + A(1.08)^2 + A(1.08)^3 + A(1.08)^4 + A(1.08)^5$$
$$= A(1.08 + 1.1664 + 1.25971 + 1.36049 + 1.46933)$$
$$= A(6.33593).$$

Therefore, $A = \dfrac{66087.81}{6.33593} = £10,430.64.$

(ii) Schedule:

Year	Debt outstanding	Interest paid	Deposit	Amount in fund	Interest earned
1	37500.00	4500.00	10430.64	10430.64	834.45
2	42000.00	5040.00	10430.64	21695.73	1735.66
3	47040.00	5644.80	10430.64	33862.03	2708.96
4	52684.80	6322.18	10430.64	47001.63	3760.13
5	59006.98	7080.84	10430.64	61192.40	4895.39
6	66087.82			66087.79	

Question 9

(a)

Time (year)	Receipt (£)	Present value at start of annuity (£)
0.5	1,500	$\dfrac{1500}{1045}$
1.0	1,500	$\dfrac{1500}{1045^2}$
1.5	1,500	$\dfrac{1500}{1045^3}$
10.0	1,500	$\dfrac{1500}{1045^{20}}$

The net present value of the annuity is thus a geometric progression with $n = 20$ terms, first term 'a' = $\dfrac{1500}{1.045}$ and ratio 'r' = $\dfrac{1}{1.045}$

The sum is therefore = $\dfrac{\dfrac{1500}{1.045}\left(1 - \dfrac{1}{1.045^{20}}\right)}{1 - \dfrac{1}{1.045}}$

It is therefore worth paying up to £19,512, for the annuity.

(b) If we denote the quarterly amount by £X

after 1 quarter, value of amount paid   = X

after 2 quarters, value of amounts paid   = $X(1 + 0.025) + X$

after 3 quarters, value of amounts paid   = $X(1 + 0.025)^2 + X$

after 100 quarters:

$$X(1.025)^{99} + X(1.025)^{98} + ... + X = \frac{X(1.025^{100} - 1)}{1.025 - 1} = 432.5488X$$

This must balance the value the original loan has reached after 100 quarters at 2.5% per quarter.

By the compound interest formula, this is $50{,}000\,(1 + 0.025)^{100} = £590{,}686$

Hence: $432.5488X = 590{,}686$ giving $X = 1365.5939$

Thus the quarterly repayments are £1,365.59

(c) As in (b), after 59 months, the value of the scheme will have reached

$300\,(1.01)^{59} + 300\,(1.01)^{58} + ... + 300$

(Note that £300 is paid immediately, at time 0, in this case)

This value $= \dfrac{300(1.01^{60} - 1)}{1.01 - 1} = 24{,}500.91$

Now, adding on the interest for the final month, the final value is

$24{,}500.91 \times 1.01 = £24{,}746$ (to nearest £)

The real value of the investment is $\dfrac{24{,}746}{1.05^5} = £19{,}389$ (to nearest £)

(d) In (a), the administrative and other charges usually involved with annuities may vary in the future.

In (b), it is very common for mortgage interest rates to vary, thereby varying the quarterly instalments.

In (c), the scheme may pay more than the minimum 1% or the assumption of 5% inflation may prove accurate.

Question 10

(a) The NPVs at 10% cost of capital for the two machines are shown

Machine A						Machine B					
			Net Flow	10% Discount	Present value				Net Flow	10% Discount	Present value
	Year	Note	(£000)	factor	(£000)		Year	Note	(£000)	factor	(£000)
	0		-100	1.0000	-100.00		0		-120	1.0000	-120.00
	1	a	-60	0.9091	-54.55		1	a	-70	0.9091	-63.64
	2		40	0.8264	33.06		2		50	0.8264	41.32
	3		40	0.7513	30.05		3		50	0.7513	37.57
	4		40	0.6830	27.32		4		50	0.6830	34.15
	5		40	0.6209	24.84		5		50	0.6209	31.05
	6	b	60	0.5645	33.87		6	b	74	0.5645	41.77
			Net present value		-5.41				Net present value		2.22

Note a: Includes balance of cost of machine. Note b: Includes scrap value.

(b) From the figures shown, it is clear that machine B should be chosen since it has the higher NPV. It should be noted that machine A does not even return as much as 10% on the overall investment.

The assumptions made in the above recommendation are that the company can can afford the outlays inherent in the above structure. In the comparison above, no account has been taken of relative risks and it is clear that machine B, being the more expensive, is the riskier investment.

Machine B			Net	11%	Present
			Flow	Discount	value
	Year	Note	(£000)	factor	(£000)
	0		-120	1.0000	-120.00
	1	a	-70	0.9009	-63.06
	2		50	0.8116	40.58
	3		50	0.7312	36.56
	4		50	0.6587	32.94
	5		50	0.5935	29.67
	6	b	74	0.5346	39.56
			Net present value		-3.75

Note a: Includes balance of cost of machine. Note b: Includes scrap value.

(c) The table above shows the net flows at an 11% cost of capital.

The diagram below shows the NPVs obtained from 10 and 11% costs of capital plotted and the estimate of the IRR is seen to be approximately 10.4%.
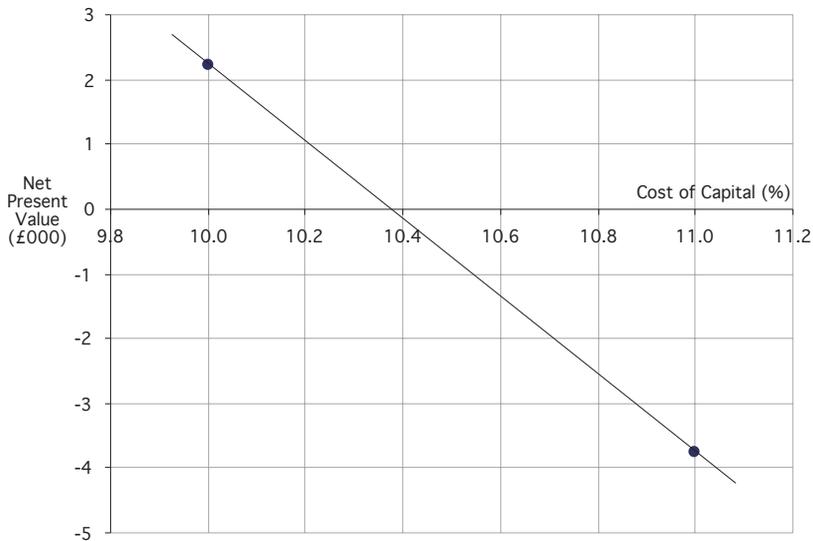
Part 7

Question 1

a), b) Materials cost £0.50 per poster, that is 0.50N (£); Labour costs £15 per hour,

and N posters take $\dfrac{N}{300} + 2$ hours to produce. Thus the labour costs are:

$$15.\left(\dfrac{N}{300} + 2\right) = \dfrac{N}{20} + 30 \ (£)$$

Administration costs are £10 per hundred posters plus £50; that is:

$$10.\dfrac{N}{100} + 50 = \dfrac{N}{10} + 50 \ (£)$$

Adding (i), (ii) and (iii), total costs are given by:

$$C = 0.50N + \dfrac{N}{20} + 30 + \dfrac{N}{10} + 50 = 0.65N + 80 \ (£)$$

Answers to examination questions – part 7

Hence producing 1,000 posters will cost: $0.65 \times 1000 + 80 = £730$

The formula for C indicates a fixed cost of £80 and a variable cost of £0.65 per poster produced.

(c) i.   If the cost is £500, then $500 = 0.65N + 80$. Hence $= \dfrac{420}{0.65} = 646$ posters

ii.   If the cost is N, then $N = 0.65N = 80$.

Hence $N = \dfrac{80}{0.35} = 229$ posters

Question 2

(a) We need to solve the equation $P = -32r^2 + 884r - 5985 = 0$

i.e. to solve $32r^2 - 884r + 5985 = 0$

Using the formula, with $a=32$, $b=-884$ and $c=5985$, gives:

$$r = \frac{884 \pm \sqrt{884^2 - 4(32)(5985)}}{2(32)} = \frac{884 \pm 124}{64}$$

That is, $r = 15.75$ or $r = 11.875$

(b) We need to tabulate and plot P for integral values of r between 11 and 16.

$-32r^2$	$884r$	-5985	P
−3872	9724	−5985	−133
−4608	10608	−5985	15
−5408	11492	−5985	99
−6272	12376	−5985	119
−7200	13260	−5985	75
−8192	14144	−5985	−33
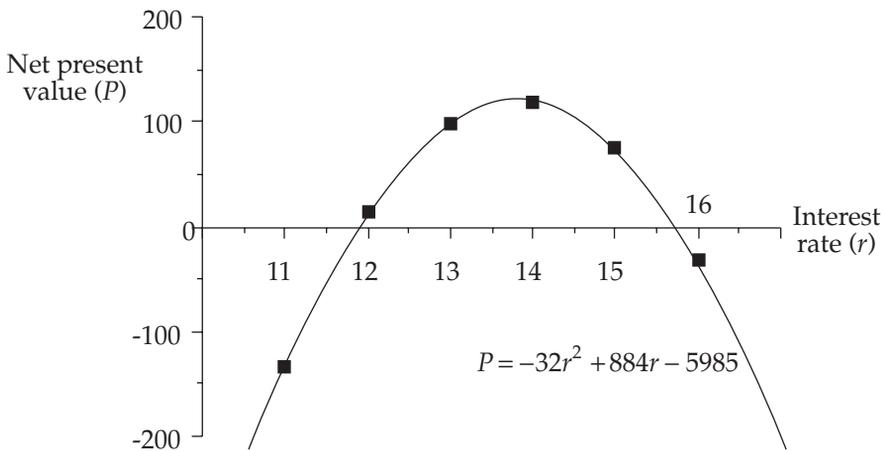
(c) With interest rates between 11.875% and 15.75%, the project yields a positive NPV and thus is worthwhile.

(d) Now, $P = -32r^2 + 884r - 5985$.

Thus: $\dfrac{dP}{dr} = -64r + 884$ and $\dfrac{dP}{dr} = 0$ when $64r = 884$. i.e. $r = 13.81$

Also, $\dfrac{d^2P}{dr^2} = -64$, and since this is negative, $r = 13.81$ signifies a max value of P.

Thus, the maximum value of r is $-32(13.81)^2 + 884(13.81) - 5985 = £120.125$.

Question 3

(a) Let n be the old number of passengers and f be the old fare.

Therefore, old revenue $= nf$.

A 30% increase in passengers gives new number of passengers as $1.3 \times n$; a 10% decrease in fare gives the new fare as $0.9 \times f$.

Thus, new revenue $=$ new number of passengers $\times$ new fare

$$= (1.3)(0.9)nf$$
$$= (1.17)nf \text{ (i.e. 0.17 increase in old revenue).}$$

Thus percentage increase in revenue is 17%

(b) New fare $= f\left(1 - \dfrac{x}{100}\right)$ and new number of passengers $= n\left(1 + \dfrac{2x}{100}\right)$

Therefore, new revenue $\quad = nf\left(1 - \dfrac{x}{100}\right)\left(1 + \dfrac{2x}{100}\right)$

Thus, the multiplier of nf $\quad = \left(1 - \dfrac{x}{100}\right)\left(1 + \dfrac{2x}{100}\right)$

$$= 1 - \dfrac{x}{100} + \dfrac{2x}{100} - \dfrac{2x^2}{10000}$$

$$= 1 + \dfrac{x}{100} - \dfrac{2x^2}{10000}$$

$$= 1 + 0.01x - 0.0002x^2.$$

(c) We need to find the value of x that maximises the multiplier:

$M = 1 + 0.01x - 0.0002x^2$.

But $\dfrac{dM}{dx} = 0.01 - 0.0004x$ and when $\dfrac{dM}{dx} = 0$, $0.0004x = 0.01$.

Thus, the value of x that maximises M is $x = \dfrac{0.01}{0.0004} = 25$.

When $x = 25$, $\text{M} \quad = 1 + (0.01)(25) - (0.0002)(25)^2$
$$= 1 + 0.25 - 0.125$$
$$= 1.125$$

Therefore, percentage increase in revenue is 12.5%.

Question 4

(a) The selling price is £15/unit. Therefore the revenue, $R = 15x$.

The costs, $C = 800 + 5x + 0.009x^2$.

Thus, profit, $P = R - C = 15x - (800 + 5x + 0.009x^2)$

i.e. $P = 10x - 800 - 0.009x^2$.

(i) We require the range of x such that $P \geq 200$.

i.e. such that $10x - 800 - 0.009x^2 \geq 200$ or $10x - 1000 - 0.009x^2 \geq 0$.

Solving $10x - 1000 - 0.009x^2 = 0$ will give the 'critical' points for x.

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Here, to use the formula, $a=-0.009$, $b=10$ and $c=-1000$, and:

$$x = \frac{-10 \pm \sqrt{10^2 - 4(-0.009)(-1000)}}{2(-0.009)} = \frac{-10 \pm \sqrt{64}}{-0.018}$$

Therefore $x=1000$ or $x=111.1$ (1D).

Now, since the graph of $y = 10x - 1000 - 0.009x^2$ is a reverse 'U' (mountain) curve, values of x between 111.1 and 1000 will give $y \geq 0$ as required.

So that, a weekly profit of at least £200 will be provided if the weekly production is between 111.1 and 1000 units.

(ii) The calculations for the graph of $P = 10x - 800 - 0.009x^2$ are tabulated:

x	50	100	200	500	800	1000	1200
$10x$	500	1000	2000	5000	8000	10000	12000
-800	-800	-800	-800	-800	-800	-800	-800
$-0.009x^2$	-22.5	-90	-360	-2250	-5760	-9000	-12960
y	-322.5	110	840	1950	1440	200	-1760

The graph is plotted below.

Company profit

$P = 10x - 800 - 0.009x^2$

Profit = £200
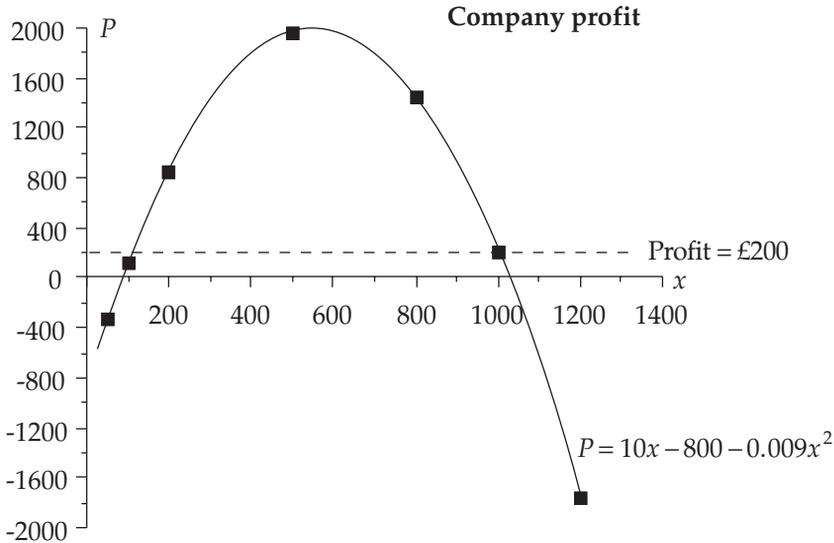
(b) Using the formula for reducing balance depreciation: $D = B(1-i)$

we have: $23500 = 32000(1-i)^6$ giving $1-i = \left(\dfrac{23500}{32000}\right)^{\frac{1}{6}}$

Hence, $1-i = 0.9498$. Therefore, $i=0.0502$. The rate of depreciation is thus 5.02%.

Question 5

(a) Total cost, $C = x^2 + 16x + 39$.

Average cost per unit $= \dfrac{C}{x} = x + 16 + \dfrac{39}{x}$

The calculations for the plot of the average cost per unit are tabulated below.

x	0	1	2	3	4	5	6	7	8
$x + 16$	16	17	18	19	20	21	22	23	24
$\frac{39}{x}$	∞	39	19.5	13	9.75	7.8	6.5	5.6	4.9
$\frac{C}{x}$	∞	56	37.5	32	29.75	28.8	28.5	28.6	28.9

and the graph is shown in the figure.

Average cost per unit
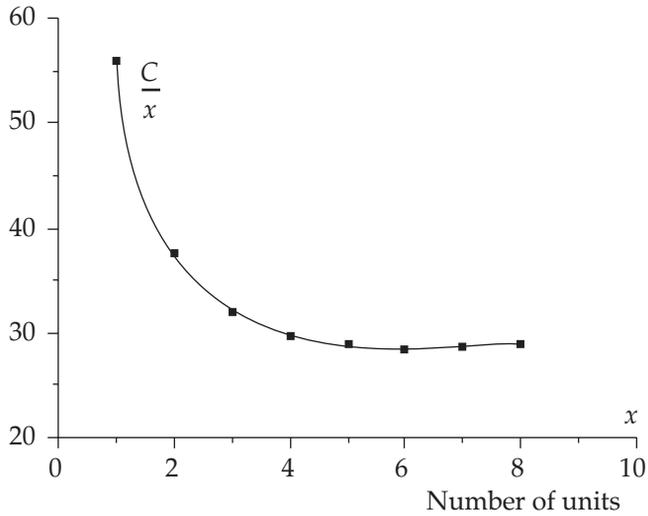
(b) (i) Demand function: $p = x^2 - 24x + 117$.

Total revenue: $R = p.x = x^3 - 24x^2 + 117x$.

(ii) R is maximised where $\dfrac{dR}{dr} = 0$ and $\dfrac{dR}{dr} = 3x^2 - 48x + 117$.

Therefore $\dfrac{dR}{dr} = 0$ where $3x^2 - 48x + 117 = 0$.

Solving the previous quadratic equation using the formula with $a=3$, $b=-48$ and $c=117$ gives:

$$x = \frac{48 \pm \sqrt{48^2 - 4(3)(117)}}{2(3)} = \frac{48 \pm \sqrt{900}}{6} = \frac{48 \pm 30}{6} \text{ i.e. } x=3 \text{ or } x=13.$$

Now, $\dfrac{d^2R}{dx^2} = 6x - 48$, and when $x=3$, $\dfrac{d^2R}{dx^2} = 6(3) - 48 = -30$.

Thus R is a maximum when $x=3$.

Price at $x=3$ is $p(3) = 3^2 - 24(3) + 117 = £54/\text{unit}$.

(iii) $\dfrac{dp}{dx} = 2x - 24$.

Therefore, elasticity of demand

$$= \left(\frac{x^2 - 24x + 117}{x}\right)\left(\frac{1}{2x - 24}\right) = \frac{x^2 - 24x + 117}{2x(x - 12)}$$

and at $x=3$ (the maximum revenue point): elasticity

$$= \frac{3^2 - 24(3) + 117}{2(3)(3 - 12)} = \frac{54}{-54} = -1$$

Question 6

(a) The total cost function is obtained by integrating the marginal cost function.

i.e. $C = \int(92 - 2x)dx = 92x - x^2 + K$ (where K is the fixed cost).

Fixed cost (K) is given as 800 (£000). Thus $C = 92x - x^2 + 800$.

(b) The total revenue function is obtained by integrating the marginal revenue function.

i.e. $R = \int(112 - 2x)dx = 112x - x^2$ (there is no constant term for revenue since $R=0$ when $x=0$)

(c) Now, profit, $P = R - C = 112x - x^2 - (92x - x^2 + 800)$. i.e. $P = 20x - 800$.

The break-even situation is where $P=0$. i.e. where $20x - 800 = 0$ or $x=40$.

(d) For maximum revenue, we need $\frac{dR}{dx} = 0$. i.e. $112 - 2x = 0$ or $x=56$.

Therefore, $R_{max} = R(\text{at } x=56) = 112(56) - 56^2 = 3136$ (£000).

For maximum costs, we need $\frac{dC}{dx} = 0$. i.e. $92 - 2x = 0$ or $x=46$.

Therefore, $C_{max} = C(\text{at } x=46) = 92(46) - 46^2 + 800 = 2916$ (£000).

(e) The calculations for the plots of the two graphs are tabulated below.

x	0	10	20	30	40	50	60
$-x^2$	0	−100	−400	−900	−1600	−2500	−3600
$92x$	0	920	1840	2760	3680	4600	5520
800	800	800	800	800	800	800	800
C	800	1620	2240	2660	2880	2900	2720
$-x^2$	0	−100	−400	−900	−1600	−2500	−3600
$112x$	0	1120	2240	3360	4480	5600	6720
R	0	1020	1840	2460	2880	3100	3120

The two graphs are plotted in the figure on the following page.

From the graphs, it is clear that, since costs are falling while revenue is still increasing between the break-even production point and the maximum production point, the profit is increasing. Hence, maximum profit is obtained at maximum production.
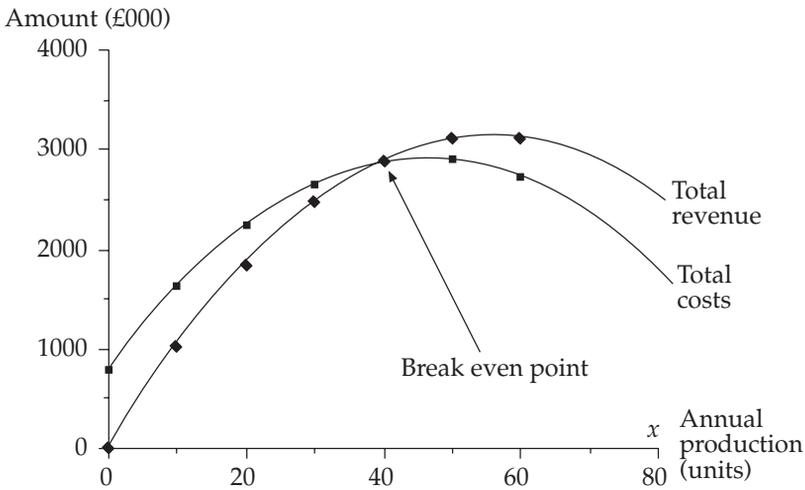
Question 7

(a) Put x as the old price of the ticket. Then the number of tickets which could be purchased previously is $\dfrac{2850}{x}$ and the number which could be purchased after the

£6 price increase is $\dfrac{2850}{x+6}$

The price increase results in a reduction of 36 in the number of tickets which can be purchased. Therefore:

$$\frac{2850}{x} = \frac{2850}{x+6} + 36$$

$$2850(x+6) = 2850x + 36x(x+6)$$

$$2850x + 17100 = 2850x + 36x^2 + 216x$$

$$0 = 36x^2 + 216x - 17100$$

$$0 = x^2 + 6x - 475$$

Thus $x = -25$ (not feasible) or $x = 19$

The percentage increase in price is therefore $\dfrac{6}{19} \times 100 = 31.58\%$

(b) Quantity sold is $\dfrac{100}{p^2}$, price (P) is £p and cost (C) is 15p per toy.

Now, profit $=$ quantity sold $\times$ (price $-$ cost)

$$= \frac{100}{p^2} \times (P - C)$$

$$= 100p^{-1} - 15p^{-2}$$

(i) $\dfrac{d\text{Profit}}{dx} = 100p^{-2} + 2 \times 15p^{-3}$

But the price at which profit is maximised is obtained by solving

$$\frac{d\text{Profit}}{dx} = 0$$

That is, $\quad 0 = (2)15p^{-3} - 100p^{-2}$

$\qquad\qquad 0 = 30p^{-1} - 100 \;$ giving $\; \dfrac{30}{p} = 100.$

Therefore, $p = 0.30$

(ii) Given that profit is maximised at a price of 30p, the maximum profit is given by $\dfrac{100}{p} - \dfrac{15}{p^2} = £166.67$

(iii) The quantity of toys sold at this level is $\dfrac{100}{0.3^2} = 1111.$

Question 8

(a) $TC = \int(x^2 - 28x + 211)dx = \dfrac{x^3}{3} - 14x^2 + 211x + c$

When $x = 0$, $TC = c$. and from the question when $x = 0$, $TC = 10$, so $c = 10$.

Therefore: $TC = \dfrac{x^3}{3} - 14x^2 + 211x + 10$

(b) $TR = (200 - 8x)x = 200x - 8x^2$

(c) Profit, $P = TR - TC = 200x - 8x^2 - \left(\dfrac{x^3}{3} - 14x^2 + 211x + 10\right) = \dfrac{x^3}{3} + 6x^2 - 11x - 10$

$$\frac{dP}{dx} = -x^2 + 12x - 11 = 0 \text{ for critical values.}$$

$-x^2 + 12x - 11 \; = 0 = x^2 - 12x + 11$

$(x - 11)(x - 1) \; = 0$

$\qquad\qquad x \; = 1 \text{ or } 11 \text{ (can also use the quadratic formula)}$

$\dfrac{d^2R}{dx^2} = -2x + 12$ and when $x=1$, $\dfrac{d^2R}{dx^2} = 10$ (minimum),

when $x=11$, $\dfrac{d^2R}{dx^2} = -10$ (maximum)

Therefore profit is maximised when output is 11.

(d) $MR = \dfrac{dTR}{dx} = 200 - 16x$. When $x=0$, $MR=200$. When $MR=0$, $x=12.5$.

When $x=0$, $MC=211$.

(e)
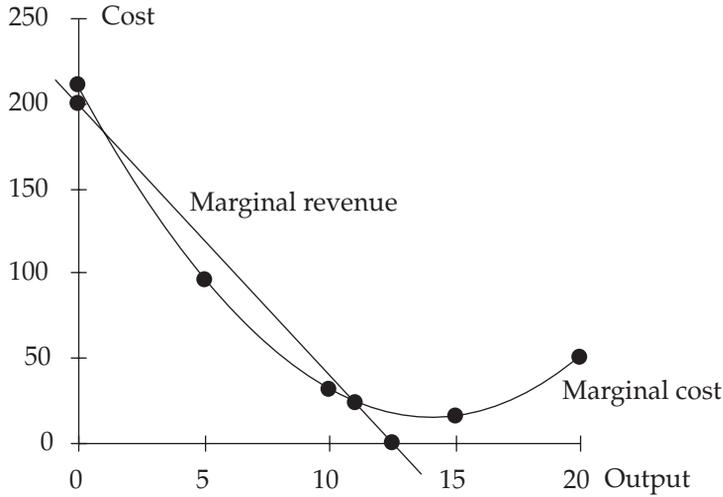
Question 9

(a) (i)

Given $C = aQ^2 - bQ + c$ and substituting for the information given, we have:

$$2900 = 100a - 10b \qquad \text{....... 1}$$

$$800 = 1600a - 40b + c \qquad \text{....... 2}$$

$$2000 = 10000a - 100b + c \qquad \text{....... 3}$$

$1 - 2:$ $\qquad 2100 = -1500a + 30b \qquad \text{....... 4}$

$3 - 2:$ $\qquad 1200 = 8400a + 60b \qquad \text{....... 5}$

$4 \times 2:$ $\qquad 4200 = -3000a + 60b \qquad \text{....... 6}$

$5 + 6:$ $\qquad 540000 = 5400a.$ Therefore: $a = 1.$

Substitute for a in 4 : $\qquad 2100 = -1500 + 30b$ Therefore: $b = 120.$

Substitute for a, b in 1 : $\qquad 2900 = 100 - 1200 + c$ Therefore $c = 4000$

Check for a,b and c in 2 : $\qquad 800 = 1600 - 4800 + 4000.$ OK!

Thus: $C = Q^2 - 120Q + 4000$

(ii) The following table shows the values calculated for Total Costs (*TC*) for *Q* in the range [0,120] and the subsequent graph of *TC* plotted. It is easily seen that the quantity that minimises Total Costs (£400) is *Q*=60.

(iii) The Revenue Function (*R*) is shown tabulated and then plotted on the graph. The profit range required is such that $R > TC$ and the graph shows this is true for *Q* between 40 and 100.

(iv) If we call the minimum price MP, then we require a separate extra Revenue Function such that $R = MP \times Q$ will satisfy the relationship that it minimises Total Costs and the dotted line that does this is shown on the graph. The gradient of this line (ie MP) can be calculated as the gradient at point A which is (approximately) $800/120 = 6.6$.

Q	TC	R
0	4000	0
20	2000	400
40	800	800
60	400	1200
80	800	1600
100	2000	2000
120	4000	2400
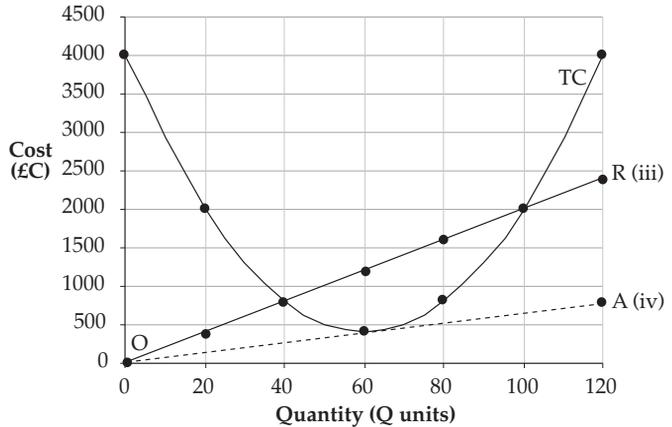
(b)   Future Total Cost $(FTC) = Q^2 - 120Q + 8000$

The Net Present Value of this $= NPV(FTC) = \dfrac{Q^2 - 120Q + 8000}{(1.1)^{10}}$

$= 0.38554 \times NPV(FTC)$ from discount tables.

(i) These values are shown tabulated below and then plotted as shown. The production quantities for which real costs are the same are seen to be in the range [14, 106]

Q	FTC	NPV(FTC)
0	8000	3084
20	6000	2313
40	4800	1851
60	4400	1696
80	4800	1851
100	6000	2313
120	8000	3084
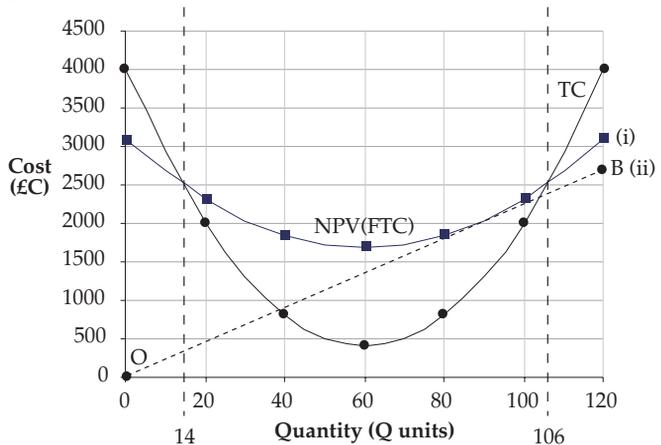
(ii) The minimum price that the product must be sold at in 10 years can be calculated by finding the gradient at point B $= \dfrac{2700}{120} = 22.5$ (approx) and then inflating this to $22.5 \times (1.1)^{10} = 58.36$.

Part 8

Question 1

Here, Pr(Female) = Pr(F) = 0.65 and Pr(Male) = Pr(M) = 0.35.

(i) Pr(all female) = Pr(FFF) = $(0.65)^3$ = 0.275 (3D)

(ii) Pr(all male) = Pr(MMM) = $(0.35)^3$ = 0.043 (3D)

(iii) Pr(at least one male)   = 1 – Pr(none male)

$$= 1 - Pr(\text{all female})$$
$$= 1 - 0.275 \text{ [from (i) above]}$$
$$= 0.725 \text{ (3D)}$$

Question 2

Pr(system operates properly)

= Pr(at least one of A or B functions and at least one of C or D functions)

= Pr(at least one of A or B functions) × Pr(at least one of C or D functions)

= [1 – Pr(both A and B fail)] × [1 – Pr(both B and C fail)]

= [1 – (0.1)(0.1)] × [1 – (0.1)(0.1)] = [1 – 0.01] × [1 – 0.01]

= [0.99] × [0.99] = 0.9801

Question 3

(a) Pr(obtaining both contracts) = Pr(obtaining A and obtaining B)

$$= Pr(\text{obtaining A}) \times Pr(\text{obtaining B})$$
$$= \frac{1}{3} \times \frac{1}{4}$$
$$= \frac{1}{12} = 0.083 \text{ (3D)}$$

(b) (i) The Venn diagram is shown in the figure on the following page.

(ii) Number of typists who can use word processors

= $n(W) = 30 = w + 6 + 3 + 9$.

Thus, $w=12$. Similarly, $n(A) = 25 = a + 5 + 3 + 6$.

So that, $a=11$. And $n(S) = 28 = s + 9 + 3 + 5$. Therefore, $s=11$.

Therefore, the total number of typists is the sum of the number of elements in the seven distinct areas in Figure 1 = 9+3+5+6+12+11+11 = 57.

(iii) The number of typists with just one skill is $a+w+s = 34$.

Question 4

(a) Two events are statistically independent if the occurrence (or not) of one in no way affects the occurrence (or not) of the other.

Word
processors

Audio

w

6

9   3

a

5

s

Shorthand
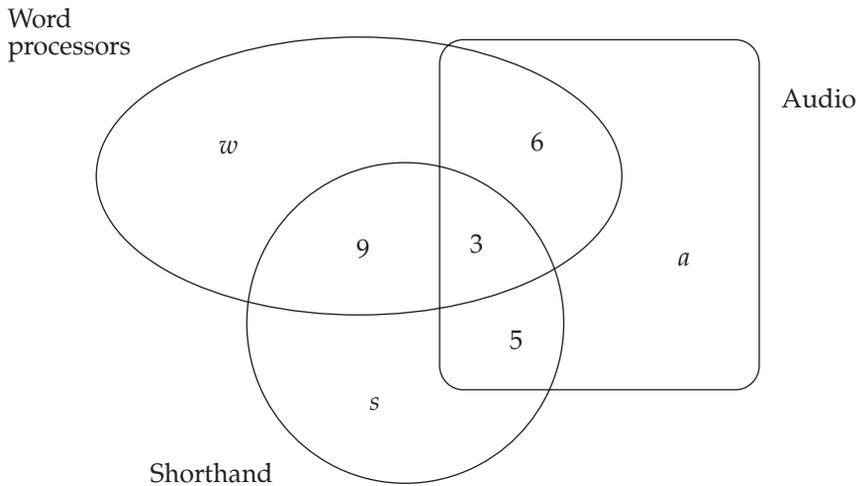
(b) There are two factors involved here, minor accidents (yes/no) and safety instructions (yes/no). Using a numeric approach for the solution of the problem means evaluating the number of men that fall into each one of the four categories defined (see table).

Total of men who had minor accidents = 1% of 10,000 = 100. Thus, 40% of 100 = 40 had safety instructions. Also, 90% of the 10,000 = 9000 had safety instructions. Using these figures, the table below can be filled as follows:

<div align="center">Minor accidents</div>

		Yes	No	Total
Safety	Yes	40	960	1000
Instructions	No	60	8940	9000
	Total	100	9900	10000

(i) Pr(No minor accidents / no safety instructions) = 0.993 (3D)

(ii) Pr(No minor accidents / safety instructions) = 0.96

(c) If there are x winning tickets out of 100, then:

$$\text{Pr(win } / \ x \text{ tickets)} = \frac{x}{100} \text{ with } x \text{ possible between 1 and 15.}$$

But $\text{Pr}(x=1) = \text{Pr}(x=2) = ... = \text{Pr}(x=15) = \dfrac{1}{15}$

Therefore, Pr(win) $= \Sigma\,[\,\text{Pr}(x).\text{Pr(win}/x)\,]$

$= \text{Pr}(x=1).\text{Pr(win}/x=1) + \text{Pr}(x=2).\text{Pr(win}/x=2) + ...$

$... + \text{Pr}(x=15).\text{Pr(win}/x=15)$

$$= \frac{1}{15} \times \frac{1}{100} + \frac{1}{15} \times \frac{2}{100} + \ ... \ + \frac{1}{15} \times \frac{15}{100}$$

$$= \frac{1}{1500}(1 + 2 + \ ... \ + 15) = \frac{120}{1500} = 0.08$$

Question 5

If advertising method A is used in the next period, the expected weekly sales can be calculated:

week sales, x units (mid-points)	probability, p	$x.p$
100	0	0
300	0.3	90
500	0.4	200
700	0.2	140
900	0.1	90
		520

Hence the expected weekly sales are 520 units.

If advertising B is used, this would increase expected weekly sales by 50%, to 780 units.

If advertising C is used, this would increase expected weekly sales by 100%, to 1,040 units.

The decision tree showing the options open to the company in the next period is:

advertising A — expected contribution = 520 × £100 = £52,000/wk

advertising B — expected contribution = 780 × £100 – £27,000 = £51,000/wk

advertising C — expected contribution = 1,040 × £100 – £50,000 = £54,000/wk
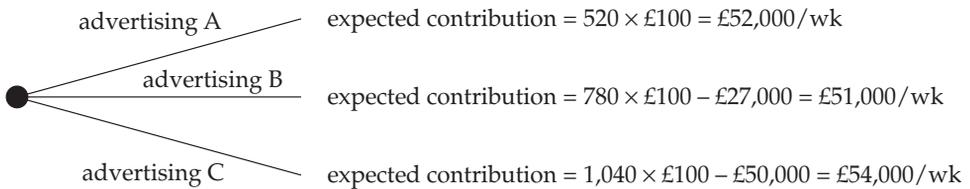
(**Note**: The contribution of £100/unit is based on using A. The figures of £27,000 and £50,000 in the above are therefore the additional advertising expenditure, above that of A.)

Thus based on expected values, the best option is to use advertising C.

The expected profit from A is a more reliable estimate as it is based on past experiance, but even this depends on the assumption that what has happened in the past will happen again during the next period. The expected profits for B and C are more speculative as they are based on estimates (possibly subjective) of the affects of different, new types of advertising on sales.

All other things being equal, the expected profits for B and C are equally unreliable, and so C would seem to be a better option than B. It is far more debatable whether C is indeed a 'better' option than A: all the expected profits are so close that only a small variation in any of the estimates could alter their relative values. For example, if C's sales are only 2% down on those expected, the company will be worse off than sticking with A. This, combined with the relatively higher reliability of the expected value for A, would lead many organisations to choose A. Only the more risk-seeking would choose C.

Question 6

(a) To find the expected (mean) sales.

Number of loaves sold (x)	0	100	200	300	400	Total
Number of days (f)	10	60	60	50	20	200
Probability (p)	0.05	0.30	0.30	0.25	0.10	

Expected number of loaves sold $= \sum$ (number of loaves × probability)

$\sum px = 0(0.05)+100(0.30)+200(0.30)+300(0.25)+400(0.10) = 205.$

(b)

PROFIT		100	200	300	400
Sales	0	–10	–20	–30	–40
	100	20	10	0	–10
	200	20	40	30	20
	300	20	40	60	50
	400	20	40	60	80

Production

(c) Expected profit = actual profit × probability. The following table shows expected profit. i.e. each profit value from the table in (b) multiplied by the probability of the respective sales level given.

EXPECTED PROFIT TABLE

Production

Sales	Value	Pr	100	200	300	400
	0	0.05	–0.5	–1	–1.5	–2
	100	0.30	6	3	0	–3
	200	0.30	6	12	9	6
	300	0.25	5	10	15	12.5
	400	0.10	2	4	6	8
Expected profit			18.5	28	28.5	21.5

(d) The optimum level of production is 300 loaves per day, since this has the highest expected profit of £28.50 per day.

Question 7

EV = expected value

At C: EV (shelve) = –4

   EV (market) = $(0.8 \times 10) + (0.2 \times 20) = 12$

   ∴ market

At D: EV (shelve) = –4

   EV (market) = $(0.5 \times 0) + (0.5 \times 5) = 2.5$

   ∴ market

At E: EV (shelve) = –4

   EV (market) = $(–4 \times 0.1) + (–3 \times 0.9) = –3.1$

   ∴ market

The 'rolled-back' tree is shown on the opposite page.

At A: EV (do not develop) = 0

   EV (develop) = $(0.3 \times 12) + (0.5 \times 2.5) + (1.2 \times – 3.1) = 4.23$

∴ The company should develop.

All of the above decisions have been based on the criterion of maximising expected values (of profits). At F and G, the decisions are clear-cut, since the choices in both cases are between one option which at least breaks even and one which loses £4m. The decision is also clear-cut at E, because one option guarantees losing £4m, while the other can lose at most £4m, but might incur a reduced loss of £3m.

At A, the decision is more debatable, as the 'develop' option could lead to a loss (at E), while 'not develop' guarantees no loss. The chances of the loss are, however, small (20%) and the potential gains are large, and so many companies would be willing to take the risk of developing.
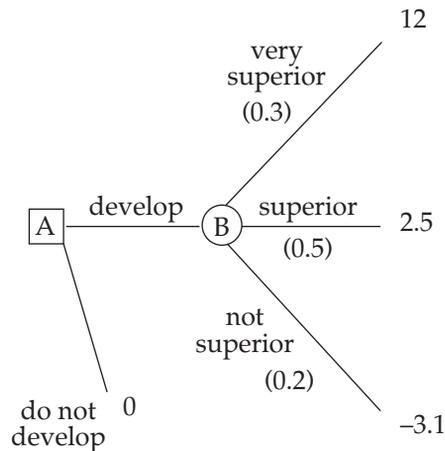
Question 8

(a) For product A

Probability of low demand $= 1 - 0.7 = 0.3$

Expected profit £ $= 0.2 \times 2 + 0.5 \times 1.5 + 0.3 \times 0.75 = 0.4 + 0.75 + 0.225 = 1.375$

For product B

Probability of medium demand $= 1 - 0.4 = 0.6$

Expected profit £ $= 0.3 \times 1.5 + 0.6 \times 1 + 0.1 \times 1.5 = 0.45 + 0.6 + 0.05 = 1.1$

The shop should display product A.

(b) i.   A random sample is one where each member of the population has an equal chance of being selected for the sample. The appropriate measures for such a sample are the mean and standard deviation.

ii. In a quota sample, all the members of the population do not have an equal chance of being selected for the sample. An interviewer is required to select a number of interviewees who may be required to have certain characteristics. As the selection is not random the appropriate statistical measures are the median and semi-interquartile range.

iii. A cluster is one where the population is divided into sub-groups which may represent geographical areas. A sample is then chosen at random within each cluster. As the sampling is random the mean and standard deviation may be used. Alternatively a cluster may be chosen at random and all the members of the cluster interviewed, again the mean and standard deviation are suitable measures of location and dispersion.

Part 9

Question 1

(a) The bar heights of the classes that have a width different to the standard (taken here as 1,000) need to be adjusted.

Lower limit	Upper limit	Act f	Bar height
0	500	20	2×20=40
500	1,000	40	2×40=80
1,000	2,000	80	80
2,000	4,000	150	0.5×150=75
4,000	5,000	60	60
5,000	6,000	30	30
6,000	7,000	20	20

The chart is shown below.
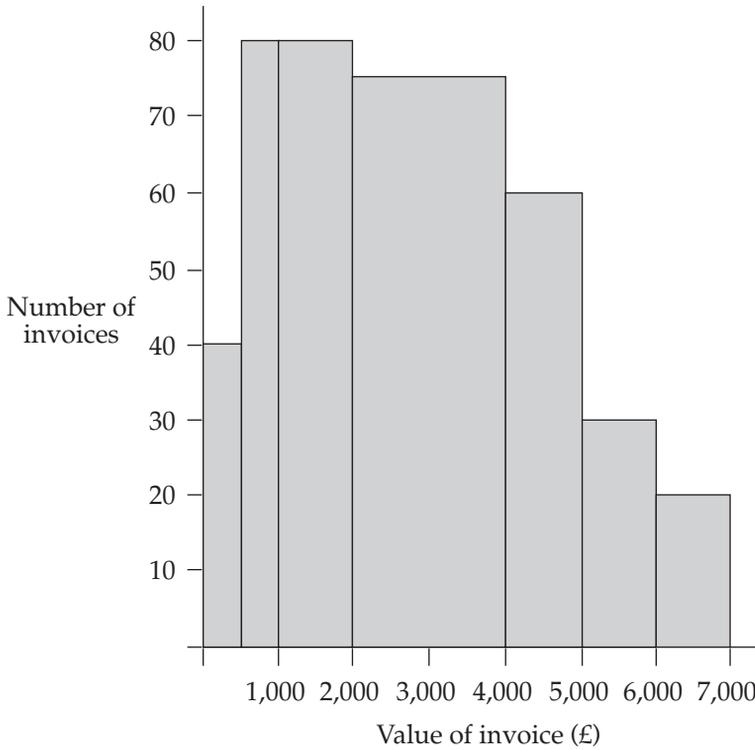
(b) Using the mid-points of each class as a representative figure

x £	f	fx
250	20	5,000
750	40	30,000
1,500	80	120,000
3,000	150	450,000
4,500	60	270,000
5,500	30	165,000
6,500	20	130,000
	400	1,170,000

The mean value is $\bar{x} = \dfrac{\sum fx}{\sum f} = \dfrac{1,170,000}{400} = £2,925.$

Note that s = standard deviation = £1,600 (given)

Thus a 90% CI is: $\bar{x} = 1.64\dfrac{s}{\sqrt{n}} = 1.64\dfrac{1600}{\sqrt{400}} = 2925 \pm 131.2 = (27938.8, 3056.2)$

(c) If 20 invoices (out of 400) contain errors then $p = \dfrac{1}{20} = 0.05$

Thus a 95% confidence interval is:

$$p \pm 1.96\sqrt{\dfrac{p(1-p)}{n}} = 0.05 \pm 1.96\sqrt{\dfrac{(0.05)(0.95)}{400}}$$

$$= 0.05 \pm 0.02$$

$$= (0.03, 0.07)$$

Question 2

(a) The lifetimes (L, say) are distributed Normally with mean, $m=100$ and standard deviation, s unknown.

(i) If 90% of the batteries last at least 40 hours, we have: $\Pr(L>40) = 0.9$.

Standardising gives $\Pr\left(Z > \dfrac{40-100}{s}\right) = 0.9$. i.e. $\Pr\left(Z > \dfrac{-60}{s}\right) = 0.9$

Now, from Standard Normal tables, a table probability of 0.9 yields $Z = 1.28$.

Hence, $\dfrac{-60}{s} = 1.28$, giving $s = 46.88$ (2D).

(ii) We require $\Pr(L<70)$.

Standardising gives: $\Pr\left(Z < \dfrac{70-100}{46.88}\right) = \Pr(Z < -0.64)$

$= 1 - 0.7389$ [from tables]

$= 0.2611$

Thus, 26% of batteries will not last 70 hours.

(b) This is a binomial situation with:

n = size of sample = 5 and p = Pr(defective) = 0.1.

(i) We require Pr(at least 3 defectives) = Pr(3 or 4 or 5 defectives)

$= \Pr(3) + \Pr(4) + \Pr(5)$

$= 5C3(0.1)3(0.9)2 + 5C4(0.1)4(0.9)1 + 5C5(0.1)5(0.9)2$

$= 10(0.1)3(0.9)2 + 5(0.1)4(0.9)1 + (0.1)5$

$= 0.0081 + 0.00045 + 0.00001$

$= 0.00856$

$= 0.01$ (2D).

(ii) The result of (i) shows that only one time in a hundred should there be 3 or more defectives out of a total of 5 calculators examined. The fact that this has happened after one sample (i.e. 1 out of 1) must throw suspicion on the original assumption that only 10% of calculators are defective. The conclusion would be that the process is not working satisfactorily.

Question 3

(a) A has a Normal distribution with $m=1000$ kgs and $s =100$ kgs; B has a Normal distribution with $m =900$ kgs and $s=50$ kgs; the breaking strength must be at least 750 kgs (given).

Thus we need to find those ropes which have the greatest probability of a breaking strength greater than 750 kgs.

$\Pr(A>750) = \Pr\left(Z > \dfrac{750-1000}{100}\right)$ [standardising] $= \Pr(Z>-2.5) = 0.9938$ [tables]

$\Pr(B>750) = \Pr\left(Z > \dfrac{750-900}{50}\right)$ [standardising] $= \Pr(Z>-3) = 0.9987$ [tables]

Thus, supplier B's ropes should be bought.

(b) This is a binomial situation with $n=50$ and $p=$Pr(defective)$=0.01$.

(i) Using the binomial distribution:

$\Pr(0 \text{ defectives}) = {}^{50}C_0(0.01)^0(0.99)^{50} = (0.99)^{50} = 0.605$ (3D).

(ii) Using the poisson distribution, the mean is calculated as the mean of the binomial.

i.e. $m = np = 50(0.01) = 0.5$. Therefore, $\Pr(0 \text{ defectives}) = e^{-0.5} = 0.607$ (3D).

Question 4

(a) We are given a Normal distribution of claims, with $m = 200$ and $s = \sqrt{2500} = 50$

   (i) $\text{Pr(claim<100)} = \text{Pr}\left(Z > \dfrac{100 - 200}{50}\right)$ [standardising] $= \text{Pr}(Z<-2) = 1 - 0.9772$

   [from tables] $= 0.0228$.

   Thus, only about 2% of claims will be under £100.

   (ii) $\text{Pr(claim<150)} = \text{Pr}\left(Z > \dfrac{150 - 200}{50}\right)$ [standardising] $= \text{Pr}(Z<-1) = 1 - 0.8413$

   [from tables] $= 0.1587$.

   Thus, about 16% of claims will be under £150.

   (iii) $\text{Pr(claim>350)} = \text{Pr}\left(Z > \dfrac{350 - 200}{50}\right)$ [standardising] $= \text{Pr}(Z>-3) = 1 - 0.9987$

   [from tables] $= 0.0013$.

   That is, only about 0.1% of claims will be over £350.

(b) This is a binomial situation with $p = \text{Pr(passenger turns up)} = 0.9$ and $n = 290$
   (number of bookings taken). Using the Normal approximation, we have:
   $m = \text{mean} = n.p = 290(0.9) = 261$
   $s = \text{standard deviation} = \sqrt{[n.p.(1-p)]} = \sqrt{[290(0.9)(0.1)]} = 5.11$ (2D)
   We need the probability that the number of passengers who turn up (P, say) will
   exceed 275 (i.e. 276 or 277 or ... etc), where it should be carefully noted that 276 for
   a binomial is the equivalent of the range 275.5 to 276.5 for a Normal distribution.

   Thus, we require: $\text{Pr(P>275.5)} = \text{Pr}\left(Z > \dfrac{275.5 - 261}{5.11}\right)$ [standardising]

   $$= \text{Pr}(Z>2.84)$$

   $$= 1 - 0.9977 \text{ [from tables]} = 0.0023.$$

Question 5

(a) We are given a poisson situation with mean, $m=2$ (demands for a coach each
   day). NOTE: Proportions and probabilities are identical concepts.
   (i) $\text{Pr(neither coach used)} = \text{Pr(0 demands)} = e^{-2} = 0.1353$.
   (ii) At least 1 demand refused means that at least 3 demands have been made
   (since there are 2 coaches available)
   Thus we require Pr(at least 3 demands)

   $$= 1 - \text{Pr(0 or 1 or 2 demands)}$$
   $$= 1 - [\text{Pr}(0) + \text{Pr}(1) + \text{Pr}(2)]$$
   $$= \left[0.1353 + 2.e^{-1} + \frac{2^2}{2}.e^{-2}\right]$$
   $$= 1 - [0.1353 + 0.2706 + 0.2706]$$
   $$= 0.324 \text{ (3D)}.$$

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    (iii) Consider coach A. If no coaches are demanded, then coach A will not be used. This will happen proportion $Pr(0) = 0.1353$ of the time. If 1 coach is demanded, there is only a 50% chance that it will be A that is not used. This will happen with probability: $0.5 \times Pr(1) = 0.5 \times 0.2706 = 0.1353$. Thus the total proportion of times that coach A will not be used is: $0.1353 + 0.1353 = 0.271$ (3D).

(b) The lengths (L, say) are given as normal with mean, $m=20.02$ and standard deviation, $s=0.05$ cm.

(i) $Pr(L \text{ will be undersize}) = Pr(L<19.9) = Pr\left(Z < \dfrac{19.1 - 20.02}{0.05}\right) = Pr(Z<-2.4)$

$$= 1 - 0.9918 \text{ [from tables]} = 0.0082.$$

i.e. 0.8% of rods will be undersize.

(ii) $Pr(L \text{ will be oversize}) = Pr(L>20.1) = Pr\left(Z < \dfrac{20.1 - 20.02}{0.05}\right) = Pr(Z>1.6)$

$$= 1 - 0.9452 \text{ [from tables]} = 0.0548.$$

Thus, approximately 5.5% of rods will be oversize.

Question 6

(a) The Normal distribution is known as the distribution of 'natural phenomena' and is the most commonly occurring continuous distribution. Heights, weights, lengths and times commonly form Normal distributions, which are characterized by their 'bell-shaped' curves.

(b) Consumption (C, say) is given as a Normal distribution with $m = 10,000$ and $s = 2000$.

(i) $Pr(C>13000) = Pr\left(Z < \dfrac{13000 - 10000}{2000}\right)$ [standardising]

$$= Pr(Z>1.5) = 1 - 0.9332 \text{ [from tables]}$$

$$= 0.0668.$$

(ii) $Pr(C<8000) = Pr\left(Z < \dfrac{8000 - 10000}{2000}\right)$ [standardising]

$$= Pr(Z<-1) = 1 - 0.8413 \text{ [from tables]}$$

$$= 0.1587.$$

(iii) To find $Pr(7500<C<14000) = 1 - Pr(C<7500) - Pr(C>14000)$.

But $Pr(C<7500) = Pr\left(Z < \dfrac{7500 - 10000}{2000}\right)$ [standardising]

$$= Pr(Z<-1.25) = 1 - 0.8944 \text{ [from tables]}$$

$$= 0.1056$$

Also $Pr(C>14000) = Pr\left(Z < \dfrac{14000 - 10000}{2000}\right)$ [standardising]

$$= Pr(Z>2) = 1 - 0.9772 \text{ [from tables]}$$

$$= 0.0228.$$

Therefore, $Pr(7500<C<14000) = 1 - 0.1056 - 0.0228 = 0.8716$.

Question 7

Putting S as the scores Normal variable, with mean 60 and standard deviation 12, we require:

$$\Pr(S > 75) = \Pr\left(Z < \frac{75 - 60}{60}\right) = \Pr(Z > 1.25) = 1 - \Pr(Z < 1.25) = 1 - 0.8944 = 0.1056$$

Question 8

(a) Mean in sample of 10 = 1.6 Therefore p(box underweight) = $1.6/10 = 0.16$

(b) We have n=6 and p(underweight)=0.16 and a binomial situation.

   (i) $p(0) = {}^{6}C_0.(0.16)^0(0.84)^6 = 0.3513$

   (ii) $p(2) = {}^{6}C_2.(0.16)^2(0.84)^4 = 15(0.056)(0.4979) = 0.1911$

   Also $p(1) = {}^{6}C_1.(0.16)^1(0.84)^5 = 0.4015$

   (iii) p(at least 3) $= 1 - p(0, 1 \text{ or } 2)$

$$= 1 - (0.3513 + 0.1911 + 0.4015)$$
$$= 0.0561$$

(c) Since n=100, we can approximate to a Normal distribution with mean = np = $100(0.16) = 16$ and sd = $\sqrt{100(0.16)(0.84)} = 3.67$

   (i) $p(<10) = p(<9.5)$ with continuity correction

Standardising 9.5 gives $z = \dfrac{9.5 - 16}{3.67} = -1.77$

    Thus: $p(<10) = p(Z<-1.77)$
$$= 1 - p(Z<1.77)$$
$$= 1 - 0.9938$$
$$= 0.01$$

   (ii) $p(>28) = p(>28.5)$ with continuity correction

Standardising 28.5 gives $z = \dfrac{28.5 - 16}{3.67} = 3.41$

    Thus: $p(>28) = p(Z>3.41)$
$$= 1 - p(Z<3.41)$$
$$= 1 - 1$$
$$= 0$$

   (iii)   p(between 16 and 24) $= p(<24) - p(<16)$
$$= p(<23.5) - p(15.5) \text{ with continuity correction}$$

Standardising 23.5 gives $z = \dfrac{23.5 - 16}{3.67} = 2.04$

Standardising 15.5 gives $z = \dfrac{15.5 - 16}{3.67} = -0.14$

p(between 16 and 24) $= p(Z<2.04) - p(Z<-0.14)$
$$= p(Z<2.04) - [1 - p(Z<0.14)]$$
$$= p(Z<2.04) + p(Z<0.14) - 1$$
$$= 0.98 + 0.54 - 1$$
$$= 0.52$$

Part 10

Question 1

a) i. A TV spot will generate 1000 extra sales at a gross profit of £10/unit, and will cost £5,000.

The contribution is thus: $1,000 \times 10 - 5,000 = £5,000$

ii. In the same way, the contribution of a newspaper advertisement is $400 \times 10 - 2,000 = £2,000$

b) Suppose the company buys x TV spots and y newspaper advertisements.

The objective is to maximise contribution, $z = 5,000x + 2,000y$ (£).

The constraints are: advertising budget:

$$5,000x + 2,000y \leq 100,000 \text{ or } 5x + 2y \leq 100 \text{ (1)}$$

maximum to be spent on each mode:

$$5,000x \leq 70,000 \qquad \text{or } x \leq 14 \qquad (2)$$
$$2,000y \leq 70,000 \qquad \text{or } y \leq 35 \qquad (3)$$

for marketing balance: $y \geq \frac{1}{2}x$ $\qquad$ or $2y \geq x$ $\qquad$ (4)

c)

No. of newspaper advertisements

No. of TV spots
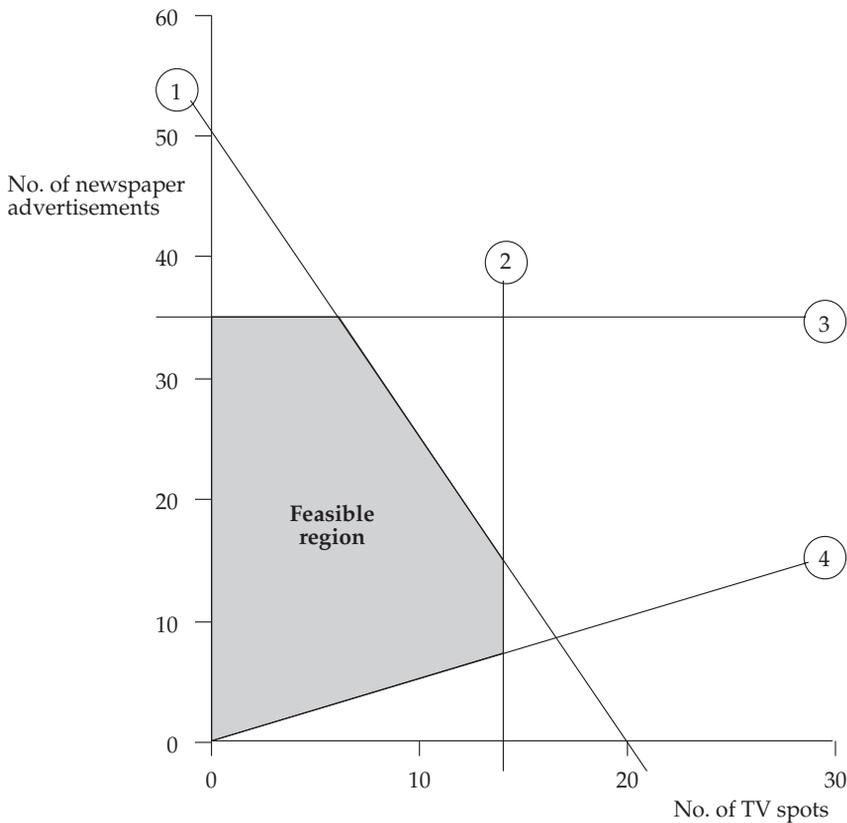
Vertex B will clearly represent an advertising mix which will generate a higher contribution than that of A. Similarly, mix C will generate a higher contribution than mix D. Only these two vertices of the feasible area are therefore considered.

Vertex	(x,y)	$z = 5,000x + 2,000y$ (£)
B	(6,35)	100,000
C	(14,15)	100,000

Thus the maximum contribution which can be generated by these promotions is £100,000. This can be achieved by any mix on the line (1) of the graph, between B and C, provided x and y are whole numbers.

These are:

x (number of TV spots)	y (number of newspaper advertisements)
6	35
8	30
10	25
12	20
14	15

It can be seen that there are *five* different ways of achieving the maximum contribution. This gives the company added flexibility, in that it can choose one from the five which gives extra benefits. For example, if the company felt that television advertising gave a high profile to all its products (not just Brand X), then it could choose the last of the combinations listed, and still maximise X's contribution.

Question 2

The purpose of this question is to test the candidates knowledge of probability and of matrix notation.

a) i. Let F represent the event of failing the test and A the event of being appointed.

Then $P(F) = 0.8$ hence $P(\text{not } F) = 1 - P(F) = 1 - 0.8 = 0.2$

From the question $P(A/\text{not } F) = 0.3$ and $P(A / F) = 0.1$

The probability of being appointed, $P(A)$, is

$P(A) = P(A/F)P(F) + P(A/\text{not } F)P(\text{not } F) = 0.1 \times 0.8 + 0.3 \times 0.2 = 0.14$

ii. $P(\text{not } A) = 1 - P(A) = 1 - 0.14 = 0.86$

$P(\text{not } A/F) = 1 - P(A/\text{not } F) = 1 - 0.3 = 0.7$

$P(\text{not } A \text{ intersection not } F) = P(\text{not } A/\text{not } F) \times P(\text{not } F) = 0.7 \times 0.2 = 0.14$

$P(\text{not } F/\text{not } A) = P(\text{not } A \text{ intersection not } F)/P(\text{not } A)$

$= \dfrac{0.14}{0.86} = \dfrac{7}{43} = 0.163$

iii. Probability of being appointed for all applicants
= probability of being selected for first interview × probability of being selected $= 0.15 \times 0.14 = 0.021$

b) i. $x(1) + x(2) + 2x(3) = 12,000$
$$x(1) + 2x(2) + 4x(3) = 22,000$$
$$2x(1) + 2x(2) + 3x(3) = 19,500$$

The above equations in matrix format are:
$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 2 & 3 \end{bmatrix} * \begin{bmatrix} x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 12,000 \\ 22,000 \\ 19,500 \end{bmatrix}$$

ii. Profit (£) = $4 \times 2,000 + 4 \times 1,000 + 2 \times 4,500 = 21,000$

iii. Profit (£) = $4 \times 3,000 + 4 \times 1,500 + 2 \times 5,000 = 28,000$
Annual market value (£) = $28,000 \times 4 \times 12 = 112,000 \times 12 = 1,344,000$

Question 3

(a) The conditions under which production is possible are:

Labour: $2X + 2Y \geq 24$ or $X + Y \geq 12$
Cost: $4000X + 12000Y \leq 120000$ or $X + 3Y \leq 30$
Space: $20X + 30Y \leq 480$ or $2X + 3Y \leq 48$

The (production) function to be maximised is $150X + 300Y$.

(b) See the figure below.
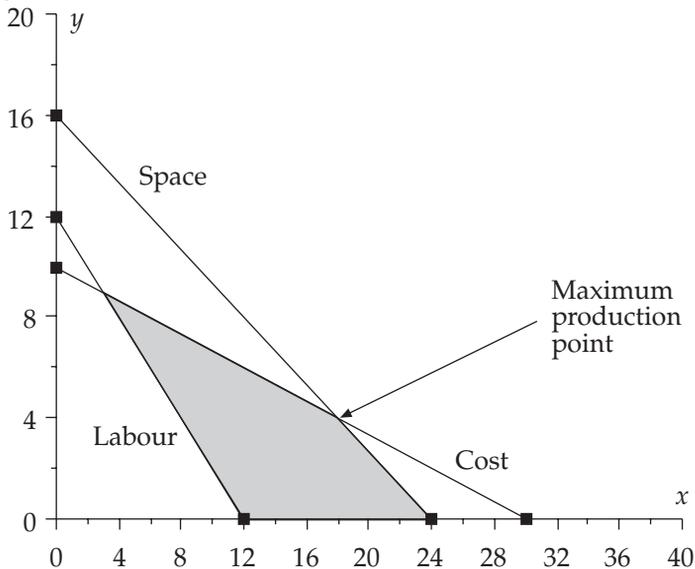
(c) From the graph, the maximum production point is $x=18$, $y=4$.
That is, 18 X presses and 4 Y presses should be bought.

(d) Maximum production is $150(18) + 300(4) = 3900$ sheets/minute.
Total cost of presses is $18(4000) + 4(12000) = £120,000$.

Question 4

(a) (i)   If $P = 192 - 28r + r^2$, then the break even point is where $P = 0$.

i.e. where $r^2 - 28r + 192 = 0$.

Solution by factorization: $(r-16)(r-12)=0$.   Therefore $r=12$ or $r=16$.

Solution by formula: (with $a=1$, $b=-28$ and $c=192$).

$$r = \frac{28 \pm \sqrt{28^2 - 4(1)(192)}}{2(1)} = \frac{28 \pm \sqrt{16}}{2} = \frac{28 + 4}{2} \text{ or } \frac{28 - 4}{2} = 16 \text{ or } 12$$

(ii) Consider a graph of $P = 192 - 28r + r^2$. It must cross the r-axis at the two points $r=12$ and $r=16$. But P is a 'U-shaped' parabola, which means that P must be negative between the values $r=12$ and $r=16$ and positive for all other values of r. Thus, the particular project in question makes a loss if the discount rate is between 12% and 16% and a profit for all other values.

(b) (i) If C is the cost matrix and D is the demand matrix, then:

$$C = \begin{array}{c} \text{Shirts} \\ \text{Shorts} \\ \text{Socks} \end{array} \begin{matrix} A & B \\ \begin{bmatrix} 5.75 & 6.25 \\ 3.99 & 4.48 \\ 1.85 & 1.97 \end{bmatrix} \end{matrix} \text{ and } D = \begin{array}{c} A \\ B \end{array} \begin{matrix} X & Y & Z \\ \begin{bmatrix} 36 & 24 & 60 \\ 48 & 72 & 0 \end{bmatrix} \end{matrix}$$

(ii) $C*D = \begin{bmatrix} 5.75 & 6.25 \\ 3.99 & 4.48 \\ 1.85 & 1.97 \end{bmatrix} * \begin{bmatrix} 36 & 24 & 60 \\ 48 & 72 & 0 \end{bmatrix}$

$$= \begin{array}{c} \text{Shirts} \\ \text{Shorts} \\ \text{Socks} \end{array} \begin{matrix} X & Y & Z \\ \begin{bmatrix} 507 & 588 & 345 \\ 358.68 & 418.32 & 239.40 \\ 161.16 & 186.24 & 111.00 \end{bmatrix} \end{matrix}$$

(c) The new demand matrix is:

$$D = \begin{array}{c} A \\ B \end{array} \begin{matrix} X & Y \\ \begin{bmatrix} 36 & 24 \\ 48 & 72 \end{bmatrix} \end{matrix}$$

and its inverse can be calculated as follows:

$$D^{-1} = \frac{1}{(36)(72) - (48)(24)} \begin{bmatrix} 72 & -24 \\ -48 & 36 \end{bmatrix} = \begin{bmatrix} 72/1440 & -24/1440 \\ -48/1440 & 36/1440 \end{bmatrix}$$

$$= \begin{bmatrix} 0.05 & -0.017 \\ -0.033 & 0.025 \end{bmatrix}$$

Question 5

a)
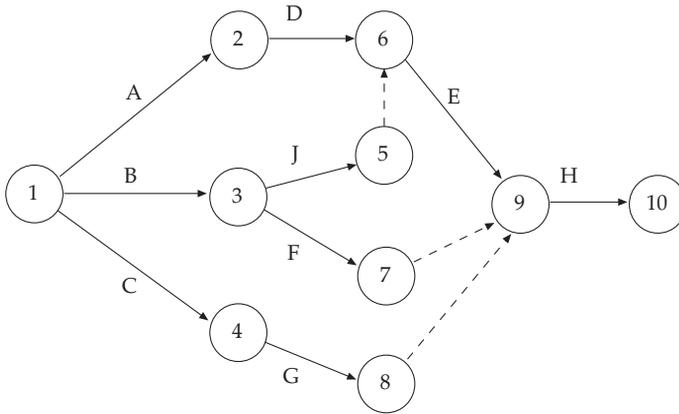
b)

Activity	Duration	Earliest start	Earliest finish	Latest start	Latest finish	Total Float
A	1–2	3	0	3	2	5	2
B	1–3	3	0	3	2	5	2
C	1–4	7	0	7	0	7	0
D	2–6	1	3	4	5	6	2
J	3–5	1	3	4	5	6	2
F	3–7	2	3	5	6	8	3
G	4–8	1	7	8	7	8	0
	5–6	0	4	4	6	6	2
E	6–9	2	4	6	6	8	2
	7–9	0	5	5	8	8	3
	8–9	0	8	8	8	8	0
H	9–10	1	8	9	8	9	0

c) The critical path is C–G–H (Alternatively 1–4–8–9–10). This path determines the minimum project completion time (9 days). All activities on the critical path must be completed without delay if the project is to be completed in 9 days.

Question 6

a) If the company buys x tables of type X and y tables of type Y, then the objective function is $x + y$, which needs to be minimised.

b) The constraints are given as follows:

Money: $40x + 30y \leq 24000$

Seating capacity: $4x + 2y \geq 1800$

Mixture of x and y: $x - y \leq 0$

c) The graphs of the constraints are shown in the figure, with the feasible region marked.

d) The optimum solution can be found by evaluating $x + y$ for each of the three vertices of the feasible region.

For the vertex at $x=300$, $y=300$, $x + y = 600$

For the vertex at $x=150$, $y=600$, $x + y = 750$

For the vertex at $x=342.9$, $y=342.9$, $x + y = 685.8$.

Thus the minimum total number of tables needed is 600, 300 of type X and 300 of type Y.
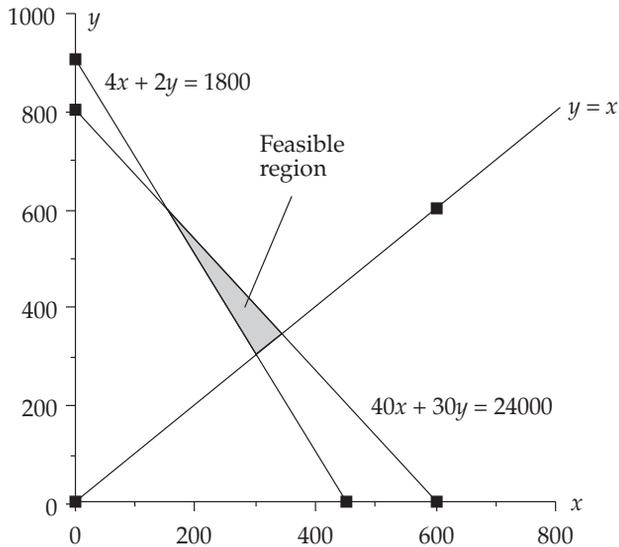
Question 7

(i) The matrix showing the pattern of retention and transfer from the first to the second month is:

$$S = \begin{array}{c} \\ BM1 \\ BM2 \\ BM3 \end{array} \begin{array}{ccc} BM1 & BM2 & BM3 \\ \begin{bmatrix} 0.70 & 0.20 & 0.10 \\ 0.25 & 0.65 & 0.10 \\ 0.05 & 0.15 & 0.80 \end{bmatrix} \end{array}$$

(ii) The product of matrix S with itself is:

$$\begin{bmatrix} 0.70 & 0.20 & 0.10 \\ 0.25 & 0.65 & 0.10 \\ 0.05 & 0.15 & 0.80 \end{bmatrix} * \begin{bmatrix} 0.70 & 0.20 & 0.10 \\ 0.25 & 0.65 & 0.10 \\ 0.05 & 0.15 & 0.80 \end{bmatrix} = \begin{bmatrix} 0.5450 & 0.2850 & 0.1700 \\ 0.3425 & 0.4875 & 0.1700 \\ 0.1125 & 0.2275 & 0.6600 \end{bmatrix}$$

(iii) The resulting matrix can be interpreted as follows.

Of the original customers who buy BM1, 54.5% will remain loyal to the brand in month 3, 28.5% will have switched to BM2 and 17% will have switched to BM3.

Of the original customers who buy BM2, 48.75% will remain loyal to the brand in month 3, 34.25% will have switched to BM1 and 17% will have switched to BM3.

Of the original customers who buy BM3, 66% will remain loyal to the brand in month 3, 11.25% will have switched to BM1 and 22.75% will have switched to BM2.

Question 8

(a) $C = D\dfrac{c_1}{q} + q\dfrac{c_2}{2};\ \ \dfrac{dC}{dq} = -D\dfrac{c_1}{q^2} + \dfrac{c_2}{2} = 0$ for critical values gives $q = \sqrt{2c_1\dfrac{D}{c_2}}$

and $\dfrac{d^2C}{dq^2} = 2D\dfrac{c_1}{q^3}$

When q is the positive square root and as all other quantities are positive then the second derivative is positive values of q. This implies that there is a minimum in cost for positive values of q.

(b) Demand rate $d = 1{,}000$; Order cost, $c_1 = 10$; Storage cost, $c_2 = 2$.

$$\mathrm{EOQ} = \sqrt{2c_1\dfrac{d}{c_2}}$$

(c)

Order size	Ordering cost £	Storage cost £	Total cost £
0	0	∞	∞
20	500	20	520
40	250	40	290
60	166.66	60	226.66
80	125	80	205
100	100	100	200
120	83.33	120	203.33
140	71.42	140	211.42
160	62.5	160	22.5

(d) Order cost, storage cost and total cost against order size. See Figure Q8(d).

(e) The model assumes that demand is regular, that stock delivery is instantaneous and that there is no chance of stockout.

Question 9

(a) Let X be the number of boxes of X produced and y be the number of boxes of Y produced.

The given restrictions (constraints) are tabulated below as linear inequalities.

Blending:	$3x + y < 900$
Baking:	$5x + 4y < 1800$
Packaging:	$x + 3y < 900$

We need to maximise the objective function, $x + 2y$.

(b) The constraints are graphed in Figure Q9(b).

The optimum solution can be read from the graph as $x=164$ and $y=245$. That is, 164 X chocolate bars and 245 Y chocolate bars should be produced in order to maximise contribution.

(c) The maximum contribution is obtained by substituting the optimal solutions into the objective function, $x + 2y$, to give $164 + 2(245) = 654$. That is, the maximum contribution is £654.

Figure Q8(d)
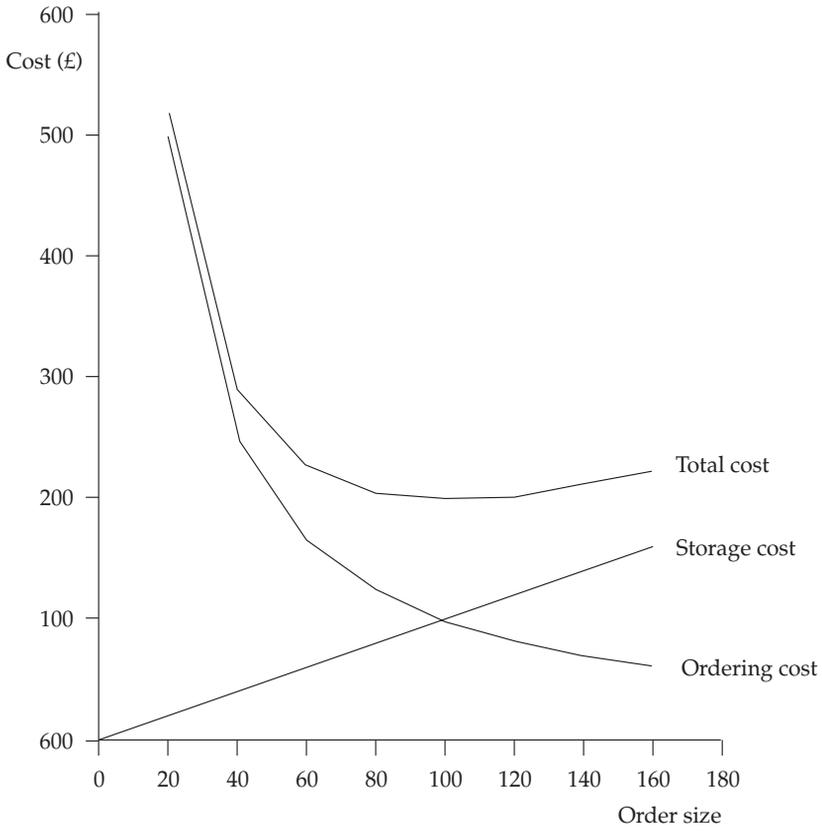
Figure Q9(b)
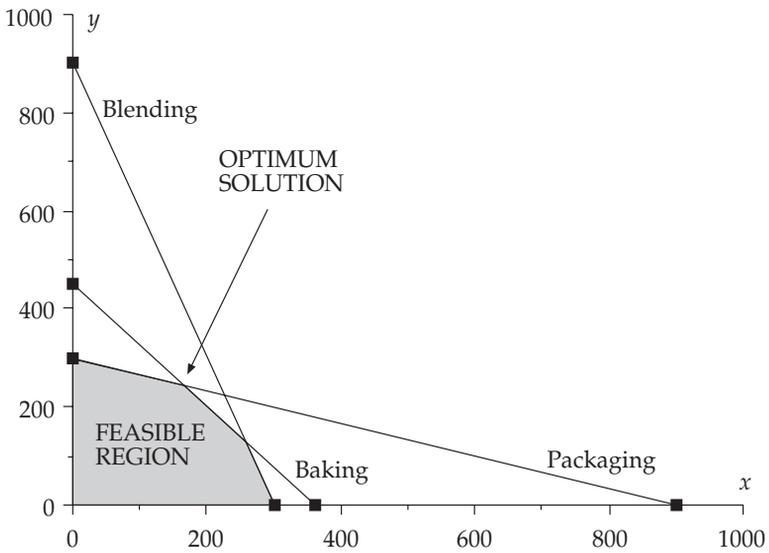
Appendices

1 Compounding and discounting tables

Range: n = 1 to 8, 1% to 12%

[C = multiplier for compounding;   D = multiplier for discounting]

		n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8
1%	C	1.0100	1.0201	1.0303	1.0406	1.0510	1.0615	1.0721	1.0829
	D	0.9901	0.9803	0.9706	0.9610	0.9515	0.9420	0.9327	0.9235
2%	C	1.0200	1.0404	1.0612	1.0824	1.1041	1.1262	1.1487	1.1717
	D	0.9804	0.9612	0.9423	0.9238	0.9057	0.8880	0.8706	0.8535
3%	C	1.0300	1.0609	1.0927	1.1255	1.1593	1.1941	1.2299	1.2668
	D	0.9709	0.9426	0.9151	0.8885	0.8626	0.8375	0.8131	0.7894
4%	C	1.0400	1.0816	1.1249	1.1699	1.2167	1.2653	1.3159	1.3686
	D	0.9615	0.9246	0.8890	0.8548	0.8219	0.7903	0.7599	0.7307
5%	C	1.0500	1.1025	1.1576	1.2155	1.2763	1.3401	1.4071	1.4775
	D	0.9524	0.9070	0.8638	0.8227	0.7835	0.7462	0.7107	0.6768
6%	C	1.0600	1.1236	1.1910	1.2625	1.3382	1.4185	1.5036	1.5938
	D	0.9434	0.8900	0.8396	0.7921	0.7473	0.7050	0.6651	0.6274
7%	C	1.0700	1.1449	1.2250	1.3108	1.4026	1.5007	1.6058	1.7182
	D	0.9346	0.8734	0.8163	0.7629	0.7130	0.6663	0.6227	0.5820
8%	C	1.0800	1.1664	1.2597	1.3605	1.4693	1.5689	1.7138	1.8509
	D	0.9259	0.8573	0.7938	0.7350	0.6806	0.6302	0.5835	0.5403
9%	C	1.0900	1.1881	1.2950	1.4116	1.5386	1.6771	1.8280	1.9926
	D	0.9174	0.8417	0.7722	0.7084	0.6499	0.5963	0.5470	0.5019
10%	C	1.1000	1.2100	1.3310	1.4641	1.6105	1.7716	1.9487	2.1436
	D	0.9091	0.8264	0.7513	0.6830	0.6209	0.5645	0.5132	0.4665
11%	C	1.1100	1.2321	1.3676	1.5181	1.6851	1.8704	2.0762	2.3045
	D	0.9009	0.8116	0.7312	0.6587	0.5935	0.5346	0.4817	0.4339
12%	C	1.1200	1.2544	1.4049	1.5735	1.7623	1.9738	2.2107	2.4760
	D	0.8929	0.7972	0.7118	0.6355	0.5674	0.5066	0.4523	0.4039

Range: n = 1 to 8, 13% to 25%

[C = multiplier for compounding;   D = multiplier for discounting]

		n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8
13%	C	1.1300	1.2769	1.4429	1.6305	1.8424	2.0820	2.3526	2.6584
	D	0.8850	0.7831	0.6931	0.6133	0.5428	0.4803	0.4251	0.3762
14%	C	1.1400	1.2996	1.4815	1.6890	1.9254	2.1950	2.5023	2.8526
	D	0.8772	0.7695	0.6750	0.5921	0.5194	0.4556	0.3996	0.3506
15%	C	1.1500	1.3225	1.5209	1.7490	2.0114	2.3131	2.6600	3.0590
	D	0.8696	0.7561	0.6575	0.5718	0.4972	0.4323	0.3759	0.3269
16%	C	1.1600	1.3456	1.5609	1.8106	2.1003	2.4364	2.8262	3.2784
	D	0.8621	0.7432	0.6407	0.5523	0.4761	0.4104	0.3538	0.3050
17%	C	1.1700	1.3689	1.6016	1.8739	2.1924	2.5652	3.0012	3.5115
	D	0.8547	0.7305	0.6244	0.5337	0.4561	0.3898	0.3332	0.2848
18%	C	1.1800	1.3924	1.6430	1.9388	2.2878	2.6996	3.1855	3.7589
	D	0.8475	0.7182	0.6086	0.5158	0.4371	0.3704	0.3139	0.2660
19%	C	1.1900	1.4161	1.6852	2.0053	2.3864	2.8398	3.3793	4.0214
	D	0.8403	0.7062	0.5934	0.4987	0.4190	0.3521	0.2959	0.2487
20%	C	1.2000	1.4400	1.7280	2.0736	2.4883	2.9860	3.5832	4.2998
	D	0.8333	0.6944	0.5787	0.4823	0.4019	0.3349	0.2791	0.2326
21%	C	1.2100	1.4641	1.7716	2.1436	2.5937	3.1384	3.7975	4.5950
	D	0.8264	0.6830	0.5645	0.4665	0.3855	0.3186	0.2633	0.2176
22%	C	1.2200	1.4884	1.8158	2.2153	2.7027	3.2973	4.0227	4.9077
	D	0.8197	0.6719	0.5507	0.4514	0.3700	0.3033	0.2486	0.2038
23%	C	1.2300	1.5129	1.8609	2.2889	2.8153	3.4628	4.2593	5.2389
	D	0.8130	0.6610	0.5374	0.4369	0.3552	0.2888	0.2348	0.1909
24%	C	1.2400	1.5376	1.9066	2.3642	2.9316	3.6352	4.5077	5.5895
	D	0.8065	0.6504	0.5245	0.4230	0.3411	0.2751	0.2218	0.1789
25%	C	1.2500	1.5625	1.9531	2.4414	3.0518	3.8147	4.7684	5.9605
	D	0.8000	0.6400	0.5120	0.4096	0.3277	0.2621	0.2097	0.1678

Range: n = 9 to 16, 1% to 12%

[C = multiplier for compounding;   D = multiplier for discounting]

		n=9	n=10	n=11	n=12	n=13	n=14	n=15	n=16
1%	C	1.0937	1.1046	1.1157	1.1268	1.1381	1.1495	1.1610	1.1726
	D	0.9143	0.9053	0.8963	0.8874	0.8787	0.8700	0.8613	0.8528
2%	C	1.1951	1.2190	1.2434	1.2682	1.2936	1.3195	1.3459	1.3728
	D	0.8368	0.8203	0.8043	0.7885	0.7730	0.7579	0.7430	0.7284
3%	C	1.3048	1.3439	1.3842	1.4258	1.4685	1.5126	1.5580	1.6047
	D	0.7664	0.7441	0.7224	0.7014	0.6810	0.6611	0.6419	0.6232
4%	C	1.4233	1.4802	1.5395	1.6010	1.6651	1.7317	1.8009	1.8730
	D	0.7026	0.6756	0.6496	0.6246	0.6006	0.5775	0.5553	0.5339
5%	C	1.5513	1.6289	1.7103	1.7959	1.8856	1.9799	2.0789	2.1829
	D	0.6446	0.6139	0.5847	0.5568	0.5303	0.5051	0.4810	0.4581
6%	C	1.6895	1.7908	1.8983	2.0122	2.1329	2.2609	2.3966	2.5404
	D	0.5919	0.5584	0.5268	0.4970	0.4688	0.4423	0.4173	0.3936
7%	C	1.8385	1.9672	2.1049	2.2522	2.4098	2.5785	2.7590	2.9522
	D	0.5439	0.5083	0.4751	0.4440	0.4150	0.3878	0.3624	0.3387
8%	C	1.9990	2.1589	2.3316	2.5182	2.7196	2.9372	3.1722	3.4259
	D	0.5002	0.4632	0.4289	0.3971	0.3677	0.3405	0.3152	0.2919
9%	C	2.1719	2.3674	2.5804	2.8127	3.0658	3.3417	3.6425	3.9703
	D	0.4604	0.4224	0.3875	0.3555	0.3262	0.2992	0.2745	0.2519
10%	C	2.3579	2.5937	2.8531	3.1384	3.4523	3.7975	4.1772	4.5950
	D	0.4241	0.3855	0.3505	0.3186	0.2897	0.2633	0.2394	0.2176
11%	C	2.5580	2.8394	3.1518	3.4985	3.8833	4.3104	4.7846	5.3109
	D	0.3909	0.3522	0.3173	0.2858	0.2575	0.2320	0.2090	0.1883
12%	C	2.7731	3.1058	3.4785	3.8960	4.3635	4.8871	5.4736	6.1304
	D	0.3606	0.3220	0.2875	0.2567	0.2292	0.2046	0.1827	0.1631

Range: n = 9 to 16, 13% to 25%

[C = multiplier for compounding;   D = multiplier for discounting]

		n=9	n=10	n=11	n=12	n=13	n=14	n=15	n=16
13%	C	3.0040	3.3946	3.8359	4.3345	4.8980	5.5348	6.2543	7.0673
	D	0.3329	0.2946	0.2607	0.2307	0.2042	0.1807	0.1599	0.1415
14%	C	3.2519	3.7072	4.2262	4.8179	5.4924	6.2613	7.1379	8.1372
	D	0.3075	0.2697	0.2366	0.2076	0.1821	0.1597	0.1401	0.1229
15%	C	3.5179	4.0456	4.6524	5.3503	6.1528	7.0757	8.1371	9.3576
	D	0.2843	0.2472	0.2149	0.1869	0.1625	0.1413	0.1229	0.1069
16%	C	3.8030	4.4114	5.1173	5.9360	6.8858	7.9875	9.2655	10.7480
	D	0.2630	0.2267	0.1954	0.1685	0.1452	0.1252	0.1079	0.0930
17%	C	4.1084	4.8068	5.6240	6.5801	7.6987	9.0075	10.5387	12.3303
	D	0.2434	0.2080	0.1778	0.1520	0.1299	0.1110	0.0949	0.0811
18%	C	4.4355	5.2338	6.1759	7.2876	8.5994	10.1472	11.9737	14.1290
	D	0.2255	0.1911	0.1619	0.1372	0.1163	0.0985	0.0835	0.0708
19%	C	4.7854	5.6947	6.7767	8.0642	9.5964	11.4198	13.5895	16.1715
	D	0.2090	0.1756	0.1476	0.1240	0.1042	0.0876	0.0736	0.0618
20%	C	5.1598	6.1917	7.4301	8.9161	10.6993	12.8392	15.4070	18.484
	D	0.1938	0.1615	0.1346	0.1122	0.0935	0.0779	0.0649	0.0541
21%	C	5.5599	6.7275	8.1403	9.8497	11.9182	14.4210	17.4494	21.1138
	D	0.1799	0.1486	0.1228	0.1015	0.0839	0.0693	0.0573	0.0471
22%	C	5.9874	7.3046	8.9117	10.8722	13.2641	16.1822	19.7423	24.0856
	D	0.1670	0.1369	0.1122	0.0920	0.0754	0.0618	0.0507	0.0415
23%	C	6.4439	7.9259	9.7489	11.9912	14.7491	18.1414	22.3140	27.4462
	D	0.1552	0.1262	0.1026	0.0834	0.0678	0.0551	0.0448	0.0364
24%	C	6.9310	8.5944	10.6571	13.2148	16.3863	20.3191	25.1956	31.2426
	D	0.1443	0.1164	0.0938	0.0757	0.0610	0.0492	0.0397	0.0320
25%	C	7.4506	9.3132	11.6415	14.5519	18.1899	22.7374	28.4217	35.5271
	D	0.1342	0.1074	0.0859	0.0687	0.0550	0.0440	0.0352	0.0281

2 Random sampling numbers

33865	04131	78302	22688	79034	01358	61724	98286	97086	21376
09356	09387	52825	93134	21731	93956	85324	68767	49490	11449
98243	37636	64825	43091	24906	13545	90172	31265	81457	93108
99052	61857	33938	86339	63531	77146	33252	81388	28302	18960
00713	24413	36920	03841	48047	04207	50930	84723	07400	81109
34819	80011	17751	03275	92511	70071	08183	72805	94618	46084
20611	34975	96712	32402	90182	94070	94711	94233	06619	34162
64972	86061	04685	53042	82685	45992	19829	45265	85589	83440
15857	73681	24790	20515	01232	25302	30785	95288	79341	54313
80276	67053	99022	36888	58643	96111	77292	03441	52856	95035
30548	51156	63914	64139	14596	35541	70324	20789	29139	66973
53530	79354	75099	89593	36449	66618	32346	37526	20084	52492
77012	18480	61852	82765	29602	10032	78925	71953	21661	95254
04304	40763	24847	07724	99223	77838	09547	47714	13302	17121
76953	39588	90708	67618	45671	19671	92674	22841	84231	59446
34479	85938	26363	12025	70315	58971	28991	35990	23542	74794
28421	16347	66638	25578	70404	67367	14730	37662	64669	16752
58160	17725	97075	99789	24304	63100	22123	83692	92997	58699
96701	73743	82979	69917	34993	36495	47023	48869	50611	61534
55600	61672	99136	73925	30250	12533	46280	03865	88049	13080
55850	38966	46303	37073	42347	36157	44357	52065	66913	06284
47089	83871	51231	32522	41543	22675	89316	38451	78694	01767
26035	86173	11115	22083	12083	43374	66542	23518	05372	33892
74920	35946	21149	70861	13235	02729	57485	23895	80607	11299
44498	00498	31354	39787	65919	61889	17690	10176	94138	95650
80045	71846	17840	23670	77769	84062	52850	20241	06073	20083
15828	95852	12124	95053	09924	91562	09419	27747	84732	81927
04100	75759	37926	70040	80884	48939	65228	60075	45056	56399
69257	48373	58911	78549	63693	43727	81058	53301	85945	54890
33915	26034	08166	59242	03881	88690	92298	48628	02698	94249
83497	62761	68609	85811	40695	08342	67386	63470	85643	68568
46466	15977	69989	90106	01432	59700	13163	56521	96687	41390
03573	87778	27696	35147	54639	20489	03688	72254	28402	98954
02046	44774	31500	30232	27434	14925	65901	34521	94104	54935
68736	12912	02579	34719	09568	21571	91111	81307	97866	76483

3 Exponential tables. Values of e^{-m}

Range: m = 0 to 2.4

m	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	1.0000	0.9900	0.9802	0.9704	0.9608	0.9512	0.9418	0.9324	0.9231	0.9139
0.1	0.9048	0.8958	0.8869	0.8781	0.8694	0.8607	0.8521	0.8437	0.8353	0.8270
0.2	0.8187	0.8106	0.8025	0.7945	0.7866	0.7788	0.7711	0.7634	0.7558	0.7483
0.3	0.7408	0.7334	0.7261	0.7189	0.7118	0.7047	0.6977	0.6907	0.6839	0.6771
0.4	0.6703	0.6637	0.6570	0.6505	0.6440	0.6376	0.6313	0.6250	0.6188	0.6126
0.5	0.6065	0.6005	0.5945	0.5886	0.5827	0.5769	0.5712	0.5655	0.5599	0.5543
0.6	0.5488	0.5434	0.5379	0.5326	0.5273	0.5220	0.5169	0.5117	0.5066	0.5016
0.7	0.4966	0.4916	0.4868	0.4819	0.4771	0.4724	0.4677	0.4630	0.4584	0.4538
0.8	0.4493	0.4449	0.4404	0.4360	0.4317	0.4274	0.4232	0.4190	0.4148	0.4107
0.9	0.4066	0.4025	0.3985	0.3946	0.3906	0.3867	0.3829	0.3791	0.3753	0.3716
1.0	0.3679	0.3642	0.3606	0.3570	0.3535	0.3499	0.3465	0.3430	0.3396	0.3362
1.1	0.3329	0.3296	0.3263	0.3230	0.3198	0.3166	0.3135	0.3104	0.3073	0.3042
1.2	0.3012	0.2982	0.2952	0.2923	0.2894	0.2865	0.2837	0.2808	0.2780	0.2753
1.3	0.2725	0.2698	0.2671	0.2645	0.2618	0.2592	0.2567	0.2541	0.2516	0.2491
1.4	0.2466	0.2441	0.2417	0.2393	0.2369	0.2346	0.2322	0.2299	0.2276	0.2254
1.5	0.2231	0.2209	0.2187	0.2165	0.2144	0.2122	0.2101	0.2080	0.2060	0.2039
1.6	0.2019	0.1999	0.1979	0.1959	0.1940	0.1920	0.1901	0.1882	0.1864	0.1845
1.7	0.1827	0.1809	0.1791	0.1773	0.1755	0.1738	0.1720	0.1703	0.1686	0.1670
1.8	0.1653	0.1637	0.1620	0.1604	0.1588	0.1572	0.1557	0.1541	0.1526	0.1511
1.9	0.1496	0.1481	0.1466	0.1451	0.1437	0.1423	0.1409	0.1395	0.1381	0.1367
2.0	0.1353	0.1340	0.1327	0.1313	0.1300	0.1287	0.1275	0.1262	0.1249	0.1237
2.1	0.1225	0.1212	0.1200	0.1188	0.1177	0.1165	0.1153	0.1142	0.1130	0.1119
2.2	0.1108	0.1097	0.1086	0.1075	0.1065	0.1054	0.1044	0.1033	0.1023	0.1013
2.3	0.1003	0.0993	0.0983	0.0973	0.0963	0.0954	0.0944	0.0935	0.0926	0.0916
2.4	0.0907	0.0898	0.0889	0.0880	0.0872	0.0863	0.0854	0.0846	0.0837	0.0829

Range: m = 2.5 to 5.0

m	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.5	0.0821	0.0813	0.0805	0.0797	0.0789	0.0781	0.0773	0.0765	0.0758	0.0750
2.6	0.0743	0.0735	0.0728	0.0721	0.0714	0.0707	0.0699	0.0693	0.0686	0.0679
2.7	0.0672	0.0665	0.0659	0.0652	0.0646	0.0639	0.0633	0.0627	0.0620	0.0614
2.8	0.0608	0.0602	0.0596	0.0590	0.0584	0.0578	0.0573	0.0567	0.0561	0.0556
2.9	0.0550	0.0545	0.0539	0.0534	0.0529	0.0523	0.0518	0.0513	0.0508	0.0503
3.0	0.0498	0.0493	0.0488	0.0483	0.0478	0.0474	0.0469	0.0464	0.0460	0.0455
3.1	0.0450	0.0446	0.0442	0.0437	0.0433	0.0429	0.0424	0.0420	0.0416	0.0412
3.2	0.0408	0.0404	0.0400	0.0396	0.0392	0.0388	0.0384	0.0380	0.0376	0.0373
3.3	0.0369	0.0365	0.0362	0.0358	0.0354	0.0351	0.0347	0.0344	0.0340	0.0337
3.4	0.0334	0.0330	0.0327	0.0324	0.0321	0.0317	0.0314	0.0311	0.0308	0.0305
3.5	0.0302	0.0299	0.0296	0.0293	0.0290	0.0287	0.0284	0.0282	0.0279	0.0276
3.6	0.0273	0.0271	0.0268	0.0265	0.0263	0.0260	0.0257	0.0255	0.0252	0.0250
3.7	0.0247	0.0245	0.0242	0.0240	0.0238	0.0235	0.0233	0.0231	0.0228	0.0226
3.8	0.0224	0.0221	0.0219	0.0217	0.0215	0.0213	0.0211	0.0209	0.0207	0.0204
3.9	0.0202	0.0200	0.0198	0.0196	0.0194	0.0193	0.0191	0.0189	0.0187	0.0185
4.0	0.0183	0.0181	0.0180	0.0178	0.0176	0.0174	0.0172	0.0171	0.0169	0.0167
4.1	0.0166	0.0164	0.0162	0.0161	0.0159	0.0158	0.0156	0.0155	0.0153	0.0151
4.2	0.0150	0.0148	0.0147	0.0146	0.0144	0.0143	0.0141	0.0140	0.0138	0.0137
4.3	0.0136	0.0134	0.0133	0.0132	0.0130	0.0129	0.0128	0.0127	0.0125	0.0124
4.4	0.0123	0.0122	0.0120	0.0119	0.0118	0.0117	0.0116	0.0114	0.0113	0.0112
4.5	0.0111	0.0110	0.0109	0.0108	0.0107	0.0106	0.0105	0.0104	0.0103	0.0102
4.6	0.0101	0.0100	0.0099	0.0098	0.0097	0.0096	0.0095	0.0094	0.0093	0.0092
4.7	0.0091	0.0090	0.0089	0.0088	0.0087	0.0087	0.0086	0.0085	0.0084	0.0083
4.8	0.0082	0.0081	0.0081	0.0080	0.0079	0.0078	0.0078	0.0077	0.0076	0.0075
4.9	0.0074	0.0074	0.0073	0.0072	0.0072	0.0071	0.0070	0.0069	0.0069	0.0068
5.0	0.0067	0.0067	0.0066	0.0065	0.0065	0.0064	0.0063	0.0063	0.0062	0.0061

4 Standard Normal distribution tables
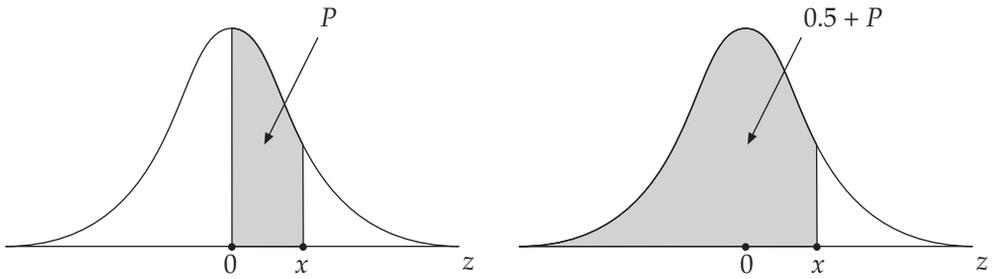
The table following gives the probability (P) that a Standard Normal variable lies between 0 and x.

This is equivalent to the shaded area in the left-hand figure.

To obtain the probability shown in the shaded area in the right-hand figure, 0.5 needs to be added to P as shown.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0754
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767

Appendices

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986

3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
0.4987	0.4990	0.4993	0.4995	0.4997	0.4998	0.4998	0.4999	0.4999	0.5000

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Note that the references below refer to chapters and sections. Thus 20:11 refers to chapter 20, section 11.

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