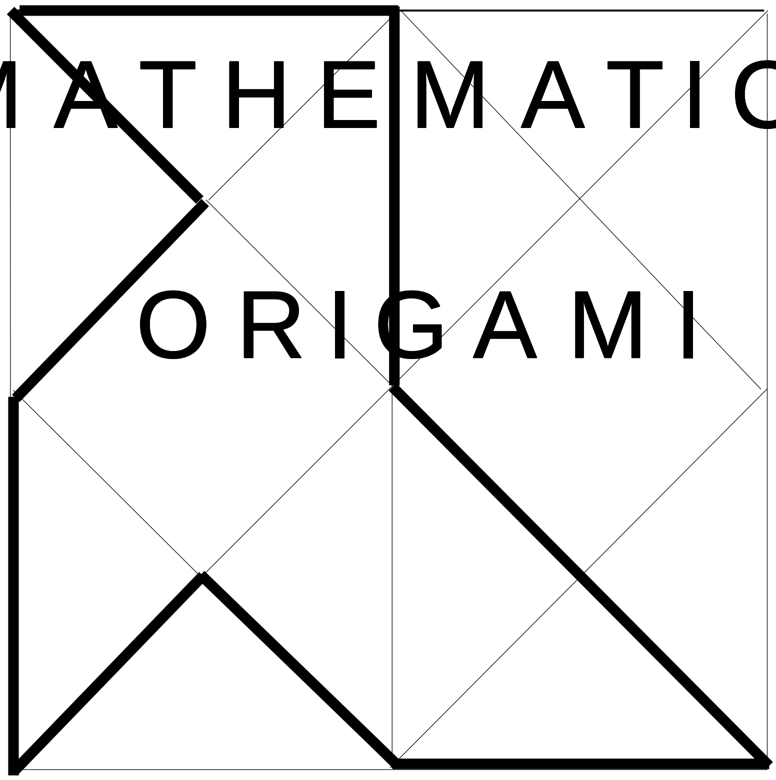
MATHEMATICS AND ORIGAMI



Jesús de la Peña Hernández

WITH COMPLIMENTS TO THE SPANISH ORIGAMI ASOCIATION

This book has been written initially by the author for the SPANISH ORIGAMI ASOCIATION.

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To my friend Jesús Voyer
because of his abnegation to correct
and enrich this book.

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SYMBOLS

Valley fold

Mountain fold

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Cut

Sink

Turn over figure

Enlargement

Bisectrix, equal angles

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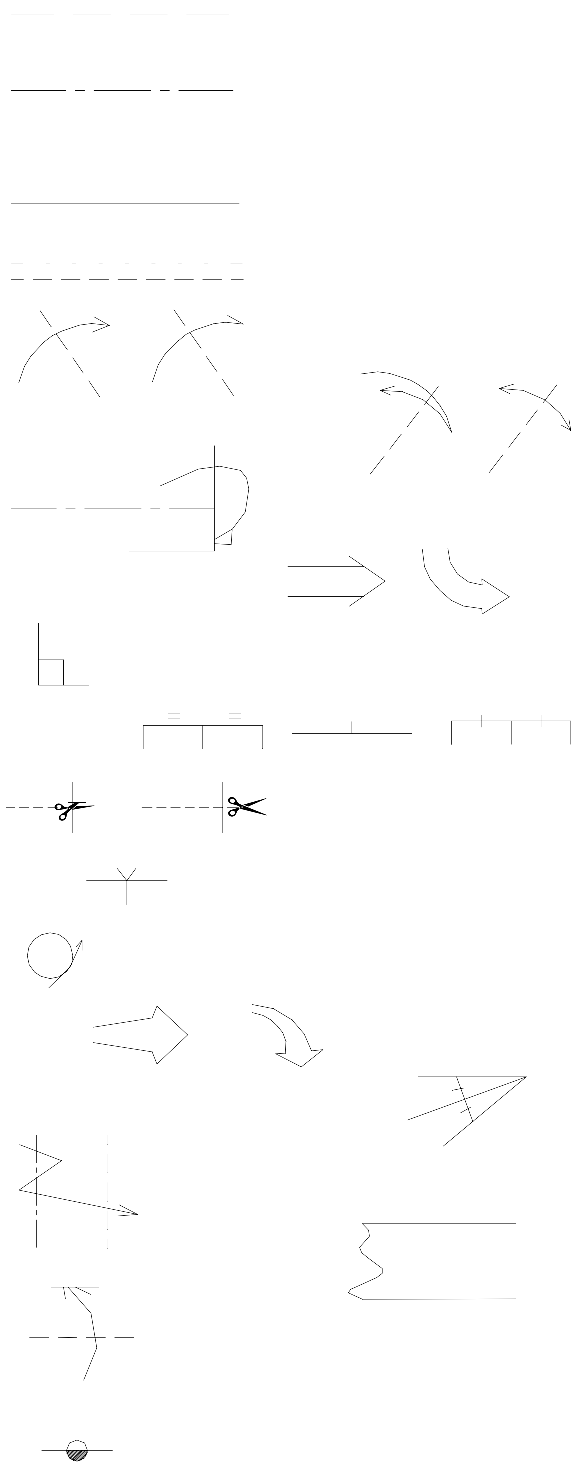

PREFACE

I must admit my difficulty to explain the title given to this book. As a matter of fact, it has been an exercise to overcome the sense of amazement exhibited by my co-speakers.

The questions:

¿Can you imagine a hatful of maths within a paper folded hat of those made for us when little children?

¿How many maths can be associated to the paper folded aeroplane of our childhood?

Not to mention our beloved *pajarita*, well known all around but difficult to draw properly by almost everybody (try it if you are sceptical).

The hereby questions have to do with persons who know at least the relationship between origami and paper folding, for most people are unaware of it. Learned persons use to mention Unamuno at this point, and that´s all. Commonly you may come across questions like this: Explain me what *papiroflexia* means (papiroflexia is the Spanish word for origami), because it sounds like the name of a disease ...

Therefore, if with origami happens what already we know, and mathematics are rather unpopular, as also is recognised, the resultant of mixing both may be at least quite risky.

Nevertheless, my consciousness of the close affinity between geometry and origami, and my fondness of geometry made me to endure an special affection towards origami.

Well before I came across origami, I had already published two treatises dealing with geometry.

The first of them, under the name of *Tubes bent in space*, was a study based on pure space geometry to solve certain problems of the automotive industry. The second one was entitled *3D measuring machines, geometric principles and practical considerations* and aimed at the computerisation of a 3D measuring mechanical outfit, through analytical geometry. It consisted basically in a great deal of combined calculus programs to enable the 3D measurements of any component at the workshop.

While digging out into the geometrical profile of origami I discovered that the art of folding paper had many other ways of relationship with the mathematics such as infinitesimal calculus, algebra, topology, projective geometry, etc. Eventually this particularity forced my decision for the final title of the book.

In spite of that, there remains an important question that should be clarified: ¿Which helps to which? ¿Origami to mathematics or viceversa?

The answer is not a simple one, for sometimes the paper folder employs mathematics not been aware of it. For example, if I take a square of paper and fold its lower side over the upper one, the result is a folding line which is the axis of symmetry that converts one side into the other; but I do not necessarily need to know that I had played with the geometrical concept of symmetry to proceed with the rest of my folding.

Origami receives more sophisticated geometrical help to design folding bases. In passing, I tackle this matter when dealing with the triangle´s incenter and its related hyperbola. But one who masters this subject is our ingenious creator Anibal Voyer. Not long ago, he lectured on that in the conference held at the Spanish Institute of Engineering under the title of *Engineering, origami and creative design*. Afterwards, and with the same materials, he wrote a comprehensive article in **PAJARITA** (No 68, October 1999; bulletin of the AEP –Spanish Association for paperfolding-), this time entitled **INTRODUCTION TO CREATION**.

Many mathematical demonstrations can be fulfilled by means of origami. Nevertheless, to be fair, both, maths and origami demonstrations should be performed in order to obviate the risk of taking for exact a folded figure which is not such. Moreover, the best will be to add some CAD (Computer Aided Design) evidences.

It should be recognised the ingeniousness that led origami to demonstrations such as the limits of convergent series, the Poncelet´s theorem on conics or problem solving of maxima and minimums. The book will deal with all this and with some other simpler things developed under the excuse of not having available neither a pencil nor a rule, a square or a set square.

In this respect, I wondered whether splitting out simple and no so simple matters. Eventually I decide not to do so. I thought the entanglement produced would be greater as compared with the advantages to be obtained.

I hope the reader will follow out each subject up to his mathematics limitations, ignoring the points to be jumped: that will not impair him to proceed.

Another question I should like to point out is this: now and again I intend to prove the lack of mathematical rigour shown in certain paperfoldings which, on the other hand are believed to enclose perfection.

Anybody can object that my strict attitude does not make any sense because the inherent imperfection of folding (not being the less important that induced by the thickness of paper) hide, by far, the supposed geometrical imperfection.

I am prepared to agree with the objectors. Not only that: I should like to render here my most sincere homage to those who had the intuition to almost reach perfection in folding geometrical figures.

But that will not weaken my purpose to discern perfect –there are many of them- and imperfect constructions, from the point of view of pure geometry. It should be added that the contrary is equally true: a faultless design, mathematically speaking, will end up in an imperfect construction by the reasons already mentioned. What matters is to know the cause of imperfection.

Finally I should like to assert my limitations. I am not an origami creator. To me, origami, as well as mathematics are a source of recreation, of personal fun.

Much of the content of this book was already widespread throughout countless publications. My main work has consisted in adding coherence and math demonstrations whenever needed. The reader will be able to judge how much in it is due to my own creativity or to my profile of recreator.

In connection with the above mentioned demonstrations I have to say that not always are shown all their steps, in some occasions because of its simplicity and in others, on the contrary, due to its complexity. When this complexity would carry us too far away of origami, I have preferred to take for granted what is already demonstrated in specialised books of mathematics.

A demonstration is made once and is not repeated anymore. So, it may happen that if the order of a certain themes has been changed as required by book editing, a subject may have its demonstration not in its first appearance. Nevertheless I hope that the intended demonstrative rigour is kept generally throughout the book

I wish to add that I had no intention whatsoever to cover all the exhaustive information available. My ruling criterion to choose the subjects was, in the first place to show a didactic or research projection, and then to keep an adequate balance between maths and paperfolding.

I mentioned before the use of CAD along this book. All its figures have been drawn with CAD. There is no special difficulty with 2D figures but that may not be the case with 3D ones: sometimes I have been forced to develop calculus programs to help the analytical CAD support.

The reader will find a bibliography at the end of the book. Since in origami not always is easy to assign the name of the author to a given work, I have made this assignment in the text only in cases when the situation was clear to me. The list of *acknowledgements* is, besides that, a kind of fuzzy bibliography.

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Hereinbefore I said that origami is performed just only with paper used at the same time as raw material and instrument. But as a matter of fact, I have taken licence to some exceptions that do not injure its foundation and, at the same time, help the practice. Those exceptions may be such as marking with a pencil by transparency, using scissors to cut or mark creases, the use of adhesive paper or glue to fix 3D figures, etc.

I never had the intention to compete with origami process creators that achieve complex geometrical figures of a great merit: now bodies out of a single paper, now figures made out of moduli, in both cases perfectly locked up: whenever the risk of loosing the balance between origami and maths, I always reverted to the licence I mentioned before.

I have to refer also that the reader will find in between some chapters (just to cover blank spaces in the text) the so-called **Interlude.** What is shown there has nothing to do with maths, but represents a series of beautiful figures that will break with the possible tiresome mathematical developments. The crease pattern and complete figure are shown together to graphic scale. I should recommend the curious reader to try to get the first without looking at it, just fixing his attention in the finished figure: it´s an exciting exercise. Many of the complete figures have been taken from, or merely inspired by Makio Araki and Toshikazu Kawasaki.

A relation of origami symbols is included to help those fond of maths but not familiar with origami practises. At the end, the only thing left is to ask for patience and comprehension to those inclined to origami rather than to maths. This is the kindest request of

THE AUTHOR

OVER THE AUTHOR

He is a member of the AEP (Spanish Origami Association) and Dr. Electromechanical Engineer (Madrid University). He is the Vice-president of the Creativity Committee and a fellow of the Committee of Terminology, both belonging to the IIE (Spanish Institute of Engineering)

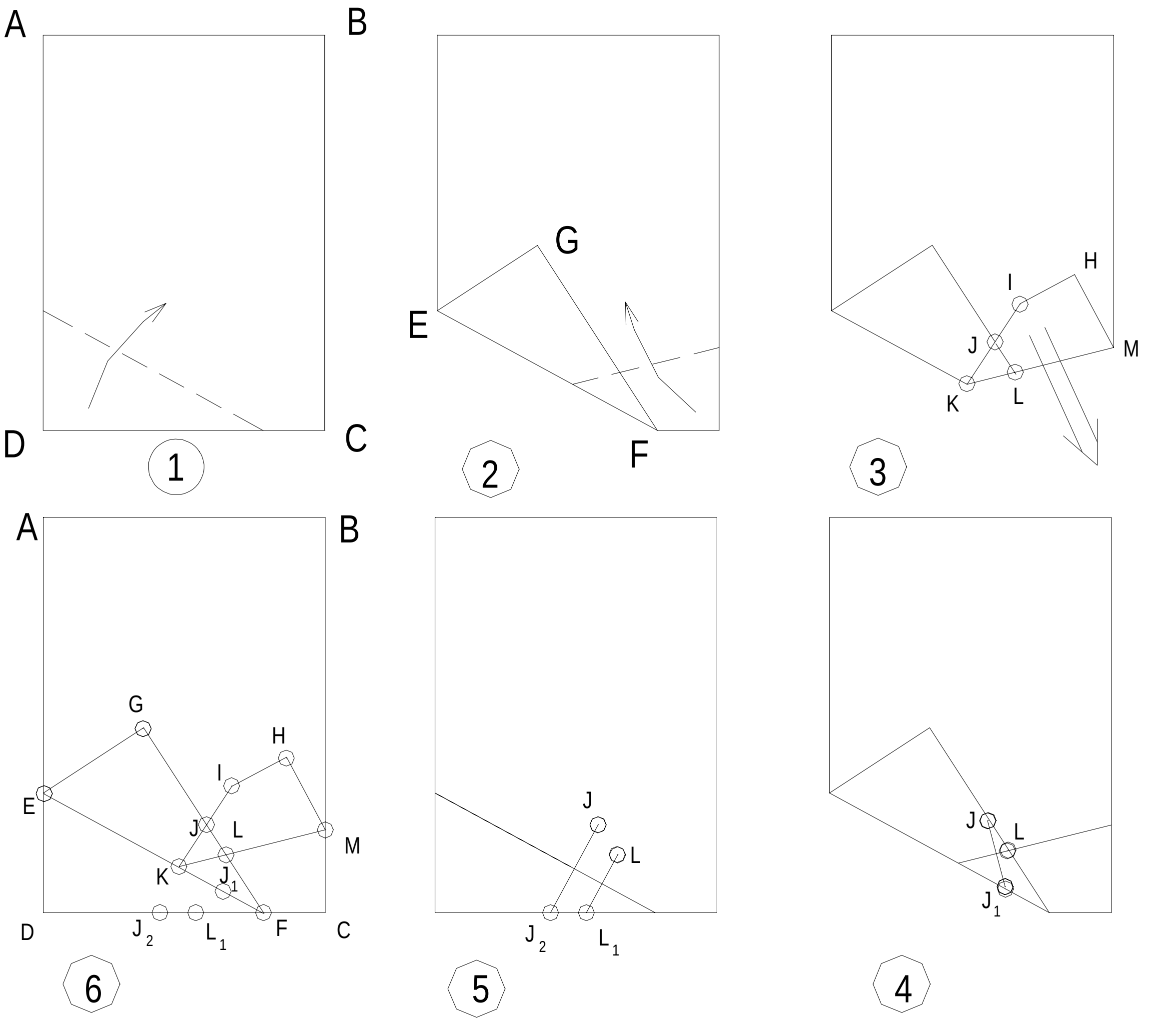
He took courses in Engineering at the Universities of Birmingham (UK) and Pittsburgh (PA). His professional activity was always developed in the fields of projects, design, development, engineering and quality.

He has been professor of Quality and Reliability at the Higher School of Industrial Engineers (Comillas University, Madrid) and has published, apart those books mentioned in the Preface, the following: "*Quality, Reliability*" ; "*Total Quality: a practical utopia*".

He has entered the field of poetry and composed these audiorama: "*Madrid on horseback*"; "*Children are like that*"; "*A museum of forged rail fences*"; "*Romanesque architecture in the city of Soria*"; "*Mingote, an anthology of gesture*". The last one, entitled "*Esopus*" is a recreation of his fables supported with animal figures made out of paperfoldings.

1. ORIGAMI RESOURCES TO DEAL WITH POINTS, STRAIGHT LINES AND SURFACES.

    1.1. OBTENTION OF POINTS.



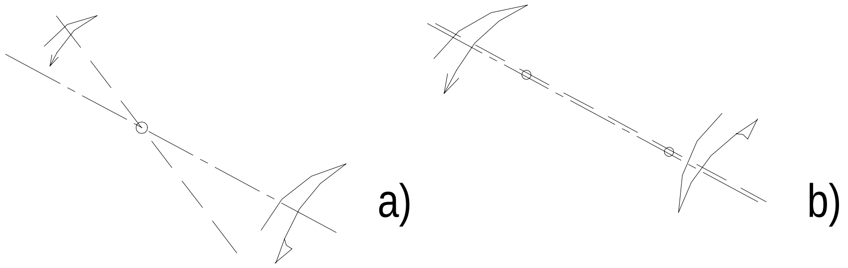
POINTS OBTAINED AS:				
ORIGI-NAL	INTERSECTION EDGE / FOLD	TRANSPORTATION PREVIOUS POINTS	INTERSECTION OF TWO FOLD LINES	
1	2 3 4 5	2 3 4 5	3 4 5	PHASES
A B C D	E J F L M	G(D) H(C) J_1(J) J_2(J) I(F) L_1(L)	K	

Example. G(D): G is obtained in Fig. 2 when revolving D around EF.

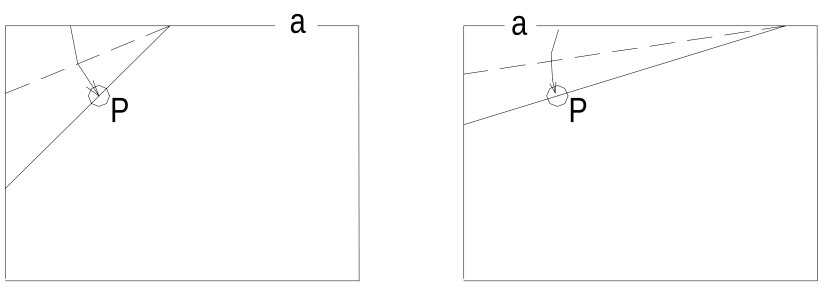
1.2. OBTENTION OF STRAIGHT LINES (valley, mountain)

1.2.1. THROUGH ONE POINT: infinity, a)

1.2.2. THROUGH TWO POINTS: just one, b)

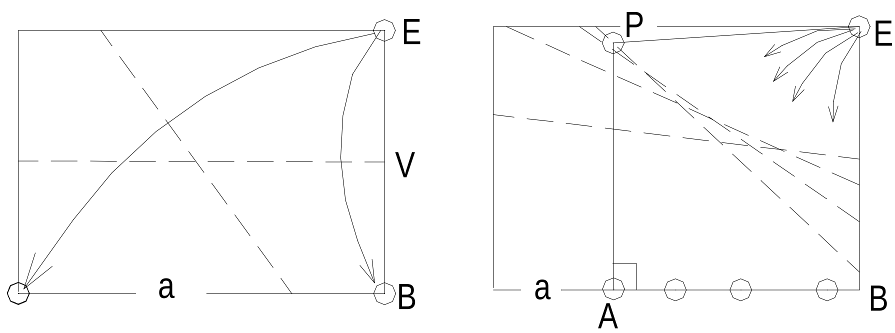


1.2.3. SYMMETRICAL TO THE EDGE a THROUGH POINT P: The folding line is the axis of symmetry. There is an infinity of solutions: a paper edge can be



folded over one inner point, in many ways.

1.2.4. FOLDING OF A PAPER CORNER E over the edge a.



extreme points intermediate points

As can be seen, there is an infinity of solutions: The envelope of all those folding lines is a parabola so defined: focus E; axis EB; directrix a; vertex V. Therefore, for point P on the parabola, we have PA = PE.

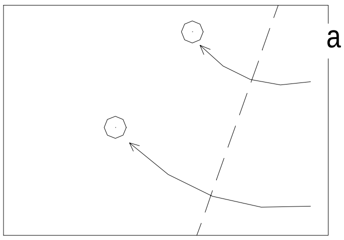
1.2.5 FOLDING OF ONE EDGE OVER AN INTERNAL POINT.
Same approach as in Point 1.2.4:

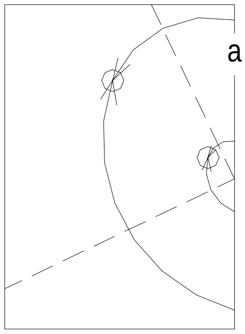
- Once the folding line is obtained, it makes no difference to fold the point over the edge or viceversa.
- Corner E can be considered as an internal point of a larger rectangle.

Therefore, the result is a series of straight lines whose envelope is a parabola.

1.2.6 FOLD A PAPER EDGE over two inner points.

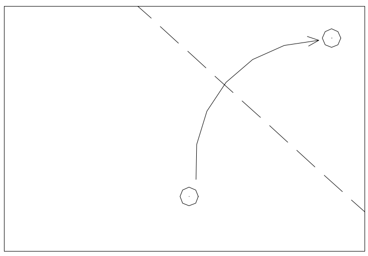
There may be a unique or a double solution; the edge and the straight line determined by those two points are symmetrical with respect to the folding line. In the second case both creases are perpendicular: bisectors of two supplementary angles.





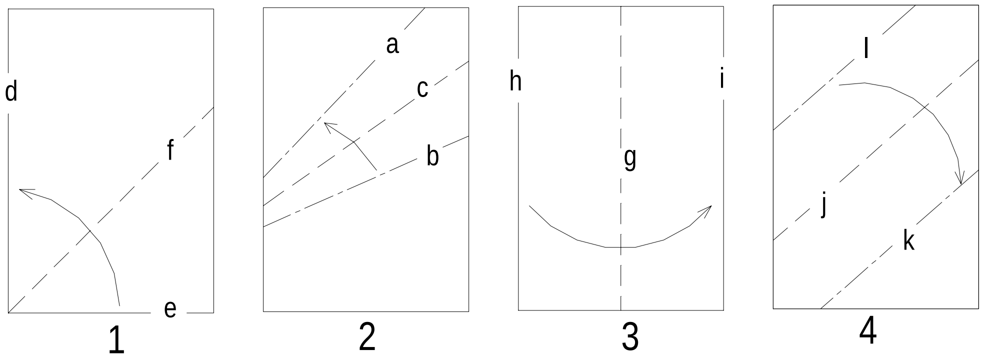
1.2.7 PERPENDICULAR BISECTOR OF A SEGMENT.

Let two points in a piece of paper. If we fold it in such a way that one of the points will lay over the other, the folding line will be the perpendicular bisector of the segment determined by both points. By so doing we can see that the points are symmetrical with respect to the folding line, which is the characteristic of a perpendicular bisector. It´s advisable to mark heavily both points in order to let them coincide by transparency.

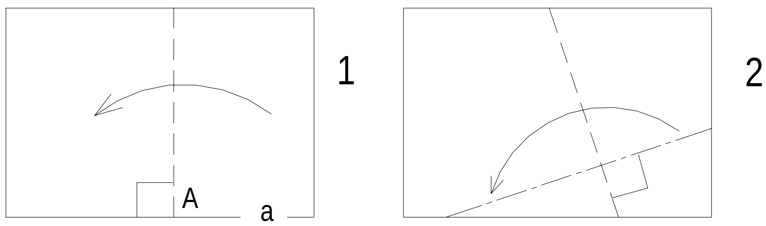


1.2.8 BISECTOR OF THE ANGLE FORMED BY TWO STRAIGHT LINES.

1- The bisector of edges ed is the folding line f.
2- c (valley) is the bisector of lines ab (mountain).
3- g (valley) is the mean parallel of edges hi.
4- j (valley) is the mean parallel of lk (mountains).

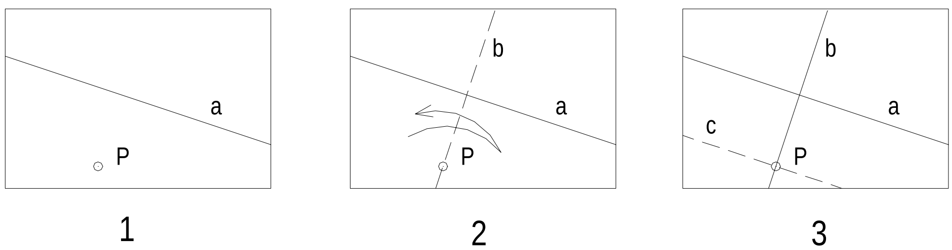


1.2.9 PERPENDICULAR LINES.



1- If edge a is folded over itself, the folding line becomes perpendicular to that edge. This can be expressed as: A → A; a → a.
2- If an existing fold (mountain) is folded over itself by means of a valley fold, both, mountain and valley folds become perpendicular.

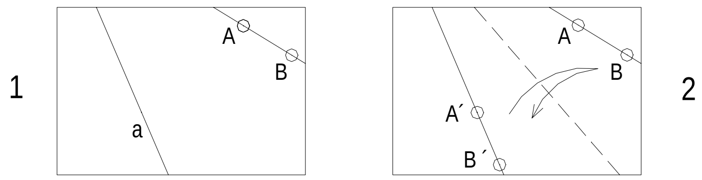
1.2.10 STRAIGHT LINE PARALLEL TO ANOTHER THROUGH ONE POINT.



1- Get the parallel to existing fold a, through point P.
2- Get b perpendicular to a through P (P → P; a → a).
3- Get through P the perpendicular to b: the answer is c, parallel to a.

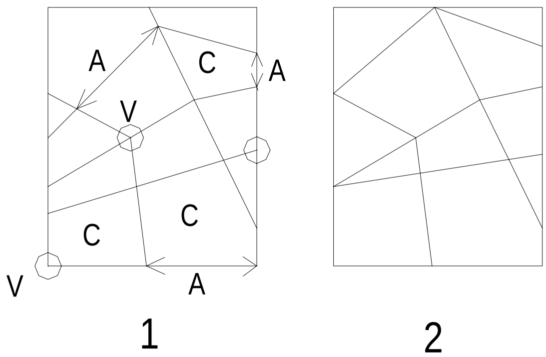
1.2.11 TRANSPORTATION OF A SEGMENT OVER A STRAIGHT LINE

1- Transport segment AB, situated on an existing fold line, over the other fold a.
2- Fold the bisector of lines a / AB to get points A´ B´ which are the answer.



1.3 RELATION WITH SURFACES

1.3.1 EULER CHARACTERISTIC APPLIED TO THE PLANE



Faces (C) + Vertices (V) = Sides (A) + 1

If both figures are obtained by folding a rectangle, the above condition can be checked:

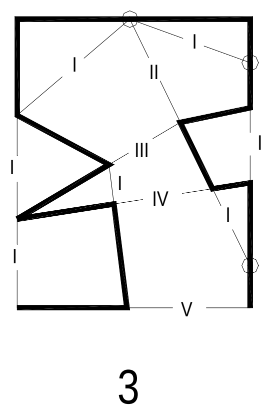
1) Faces = 12 Vertices = 20 Sides = 31 .... 12 + 20 = 31 + 1
2) Faces = 11 Vertices = 16 Sides = 26 .... 11 + 16 = 26 + 1

We shall show it from figures 2 and 3.
The heavy broken line is such that it passes just once through all the vertices. As any side has two vertices, it follows that the broken line is made out of so many sides as vertices in it, minus 1.

15 A = 16 V – 1 ; Sides in broken line = Vertices – 1 (1)

There are left 26 – 15 Sides to be determined. Let´s associate these 11 Sides left, to the amount of faces.

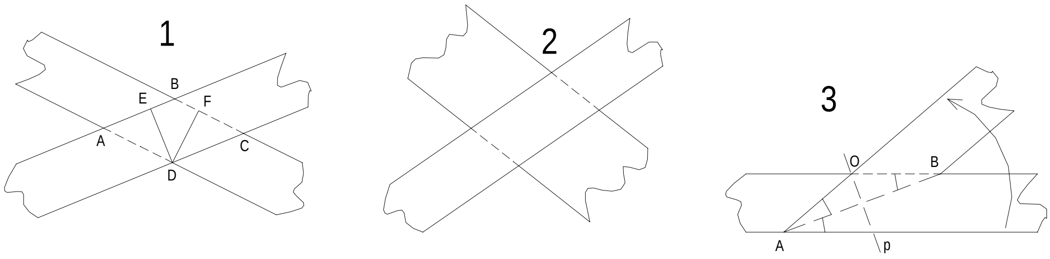
- Faces to which, from broken line, one only side is left (I): 7.
- After the former operation: faces to which one only side is left (II): 1
- Idem (III): 1
- Idem (IV): 1
- Idem (V): 1



Total of faces by left sides: 7+1+1+1+1= 11 = total amount of faces = C (2)
Adding up (1) and (2) we´ll have:
Total sides A = V – 1 + C ; C + V = A + 1

1.3.2 INTERACTION OF STRAIGHT LINES AND SURFACES.

We shall call paper strip to a rectangle such that its length is much greater than its width.

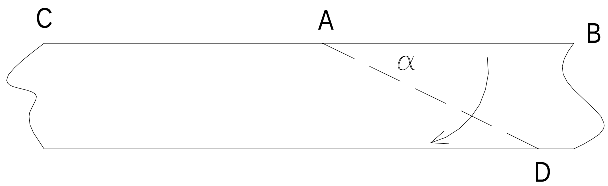


1. If two paper strips of the same width are superimposed, the common surface is a rhomb (one square if the strips lay orthogonally). That common surface is, obviously, a parallelogram: congruent acute angles in A, B, C (vertical and alternate / interior). ΔADE = ΔCDF as straight angled with ang.A = ang.C and ED = EF (same strip width). The parallelogram with two adjacent congruent sides is a rhomb or a square.

2. If both strips have different width, the common surface is a rhomboid (a rectangle in case of orthogonality): Parallel lines cut by those other parallel produce a parallelogram with not congruent adjacent sides.

3. A strip of paper folded over itself in any way, produces an isosceles triangle as common surface. If fold and paper edge form a 45º angle, the triangle is a straight angled one; it´s equilateral if that angle is 60º. By means of this one can make very useful bevel squares.
   The angles marked in A or B are equal, now because of symmetry now for being alternate interior angles; so AB is the base of an isosceles triangle. Moreover, sides OA and OB of that isosceles triangle are equal as the fold p makes evident: O → O ; B → A. Therefore, ΔOAB is isosceles.

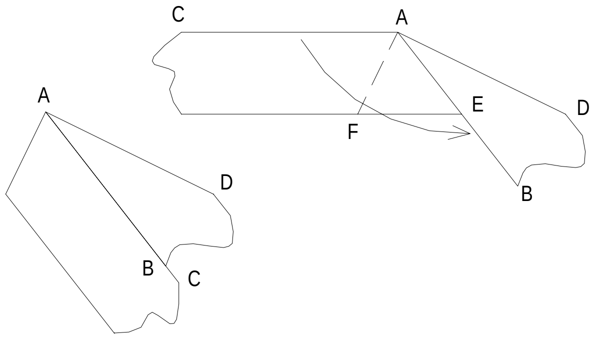
1.3.3 THE RIGHT ANGLE

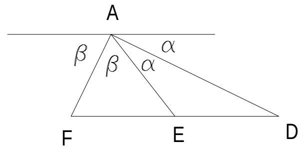


Let a strip having A in its upper edge and make the fold AD according to α.
Then produce fold AF to carry AC over AE.

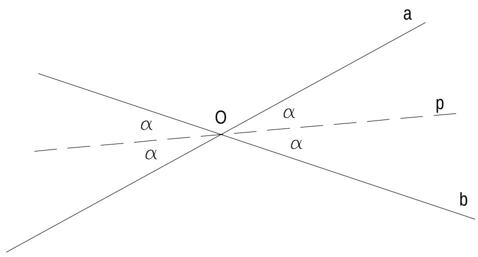
Resulting angle FAD is a right one:
Symmetry in last figure makes equal the pair of α angles and the pair of β, respectively. Straight angle in A gives:

$$\alpha+\alpha+\beta+\beta=180 \quad ; \quad \alpha+\beta=90$$



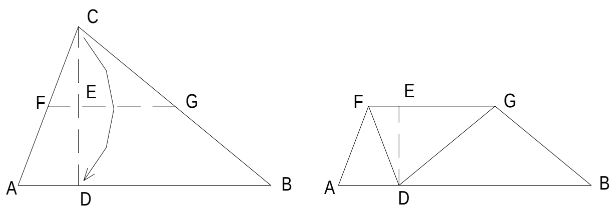


1.3.4 VERTICAL ANGLES



Let a and b, lines meeting at O.
If we produce the fold p: O $\rightarrow$ O ; b $\rightarrow$a
angles α will be equal, respectively, because of symmetry. Therefore vertical angles 2α and 2α will also be equal

1.3.5 SUM OF THE ANGLES OF A TRIANGLE



Let´s produce the following folds in ΔABC:
1- CD: C $\rightarrow$ C ; AB $\rightarrow$ AB
2- FG: C $\rightarrow$ D
As result we have:

$$ED = \frac{1}{2}CD \quad ; \quad FA = \frac{1}{2}CA \quad ; \quad GB = \frac{1}{2}CB$$

Besides, because of the symmetry, it is:

$$FC = FD \quad ; \quad GC = GD$$

which proves that ΔAFD and ΔDGB are isosceles and therefore:

Ang.FDA = Ang.FAD ; Ang.GDB = Ang.GBD

The straight angle in D can be expressed as:

180 = Ang.ADF + Ang. FDG +Ang. GDB

or its equivalent:

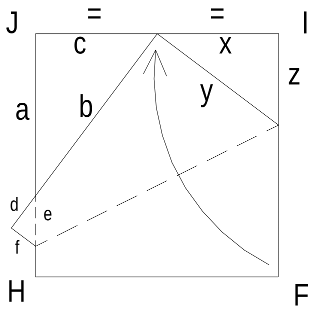
180 = Ang.CAB + Ang. ACB +Ang. CBA

This proves that the three angles in a triangle add up to 180º.

2. **HAGA**´s THEOREM

Enunciation:

Let the square FIJH whose side measures one unit. If we fold vertex F over the mid-point of JI, three right-angled triangles Δ(abc); Δ(xyz); Δ(def) are obtained in such a way that their sides keep the proportion 3,4,5. Besides, being $x = \frac{1}{2}$, it is also $a = \frac{2}{3}$



3. **HAGA**´s THEOREM EXTENSION

3.1 For any value of x, THE PERIMETER of Δ(abc) equals the sum of perimeters of Δ(xyz) and Δ(def).

Moreover, the perimeter of Δ(abc) is equal to half the perimeter of square FIJH.

3.2 **KOHJI** and **MITSUE FUSHIMI**

If $x = \frac{1}{3}$, it follows that $a = \frac{1}{2}$; and if $x = \frac{1}{4}$, it´s $a = \frac{2}{5}$

4. **HAGA**´s THEOREM DEMONSTRATION; likewise its extension will be demonstrated for any value of x.

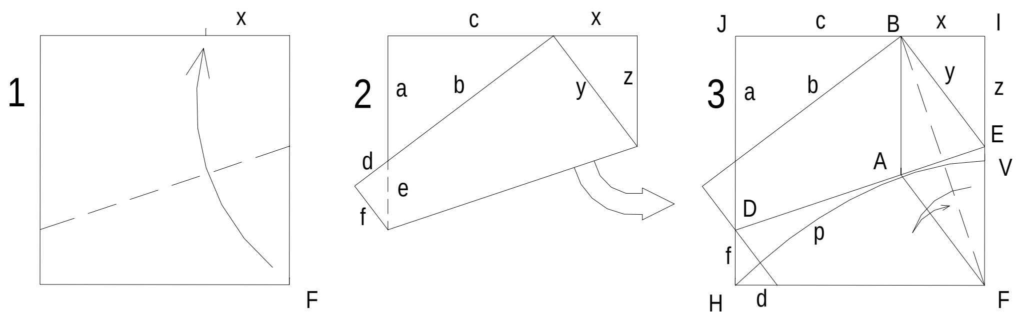


Fig. 1 is a square whose side is equal to 1. F is folded over the upper side, being x the independent variable. By so doing, fig. 2 is produced. In Fig. 3 BA is drawn perpendicular to JI through B, so BA and EF are parallel. Fold BF is perpendicular to DE because DE is the perpendicular bisector of BF. Therefore ABEF is a quadrilateral having perpendicular diagonals and opposite vertices (B and F) equidistant from the intersection point of those diagonals; moreover, opposite sides BA and EF are parallel as said hereby. Consequently this quadrilateral is a rhomb.

Under these conditions AB = AF; in other words, A is a point on the parabola p which has focus F, directrix IJ, vertex V (VI = VF) and passes through H (because HJ = HF). According to point 1.2.4, DE is the tangent to parabola p in A.

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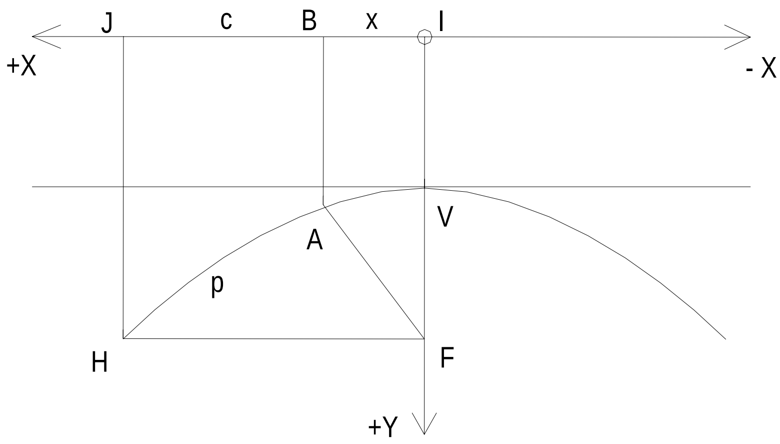


Fig. 4 shows the structure of parabola p; its equation is:

$$y = \frac{1}{2} + Kx^2 \quad ; \quad \text{IV} = \frac{1}{2}$$

As the parabola passes through H, we´ll have:

$$x = 1 \quad ; \quad y = 1 \quad ; \quad \text{replacing:}$$

$$1 = \frac{1}{2} + K \quad ; \quad K = \frac{1}{2}$$

Resultant equation of the parabola is

$$y = \frac{1}{2} + \frac{1}{2}x^2 ; \qquad y = \frac{1}{2}\left(1 + x^2\right) \qquad (1)$$

For the demonstrations we are after, now we already have the first motive: the parabola´s configuration. The second motive we need is the similarity of the three triangles we are dealing with.

Those triangles are similar because they are right angled (three vertices of the square) and each two of them have either acute angles, equal.

Both motives will allow us to define in a direct or indirect way, the nine sides as functions of the independent variable x, which at the same time is one of the sides.

$$x \qquad y = \frac{1}{2}\left(1 + x^2\right) \qquad z = \sqrt{y^2 - x^2}$$

$$a = \frac{x(1 - x)}{\sqrt{y^2 - x^2}} \qquad b = \frac{y(1 - x)}{\sqrt{y^2 - x^2}} \qquad c = 1 - x$$

$$d = 1 - b \qquad e = \frac{y(1 - b)}{x} \qquad f = 1 - a - e$$

Let´s pass to demonstrations. For x = $\frac{1}{2}$ we have:

$$x=\frac{4}{8} \qquad y=\frac{5}{8} \qquad z=\frac{3}{8}$$
$$a=\frac{4}{6} \qquad b=\frac{5}{6} \qquad c=\frac{3}{6}$$
$$d=\frac{4}{24} \qquad e=\frac{5}{24} \qquad f=\frac{3}{24}$$

Those values prove that respective sides keep the ratios 3, 4, 5.

It should be noted that the hypotenuses (the greater sides) are in proportion to 5. On the other hand it has been shown that for x = $\frac{1}{2}$ it is a = $\frac{2}{3}$

In connection with Point 3.2, it can be checked that for the two values assigned to x, we have:

$$x=\frac{1}{3} \qquad y=\frac{5}{9} \qquad a=\frac{1}{2}$$
$$x=\frac{1}{4} \qquad y=\frac{17}{32} \qquad a=\frac{2}{5}$$

As the perimeter of the square amounts to 4, the second part of Point 3.1 will be demonstrated if, once a b c expressed in function of x, the following hypothesis is verified:

2 = a + b + c , i.e., if:

$$2=\frac{x(1-x)}{\sqrt{0.25\left(1+x^2\right)^2-x^2}}+\frac{0.5\left(1+x^2\right)(1-x)}{\sqrt{0.25\left(1+x^2\right)^2-x^2}}+1-x$$

Developing denominators, we have:

$$\sqrt{0.25+0.25x^4+0.5x^2-x^2}=\sqrt{0.25\left(1+x^4-2x^2\right)}=0.5\left(1-x^2\right)$$

Therefore:

$$2=\frac{x}{0.5(1+x)}+\frac{1+x^2}{1+x}+1-x$$

0.5 (1 + x) = x + 0.5 (1 + x^2) – x 0.5 (1 + x) ,, 0 = 0 which proves the theorem

In a similar way can be proved the second part of Point 3.1.

5. COROLLARY P

Enunciation:

- Let´s assume that in a square whose side measures one unit, any of its four vertices is folded over any one of its opposite sides in such a way that the image of that vertex induces the distance x as shown in figs. 1 and 2 of Point 4.
- If x takes the form x = $\frac{1}{n}$, n being a natural number or a rational number greater than 1, a will have the value a = $\frac{2}{n+1}$ (see again fig. 2 in Point 4).

Before demonstrating the corollary, let´s see two applications of it:

- n is a natural number. If we make, e.g., x = $\frac{1}{13}$, a will be a = $\frac{2}{14} = \frac{1}{7}$. This characteristic leads to an exact procedure to divide a segment by folding, in any number of equal parts (see Point 9.16). We have to bear in mind (see Fig 2 of Point 4) that ratio a / x is biunivocal, i.e., it makes no difference to fix x to get a, than viceversa.

- n is a rational number. If the square have a side of, e.g., 600 mm and we want x = 273.5 mm, we can figure out the value of a:

x in the form $\frac{1}{n}$ ends up as: x = $\frac{1}{\frac{600}{273.5}} = \frac{1}{\frac{6000}{2735}}$, therefore a = $\frac{2}{\frac{600}{273.5}+1}$ = 0.6262163

600 x 0.6262163 = 375.72982 will be the mm measured by a.

DEMONSTRATION: For this purpose we´ll use the values of x, y, a, obtained in Point 4 as well as the simplification in the denominator of a. So we have:

$$a = \frac{x(1-x)}{\sqrt{0.25(1+x^2)^2 - x^2}} = \frac{x(1-x)}{0.5(1-x^2)} = \frac{x}{0.5(1+x)} \qquad (1)$$

If the corollary is true, it will be: $$a = \frac{2}{n+1} = \frac{\frac{1}{n}}{0.5\left(1+\frac{1}{n}\right)} \qquad (2)$$

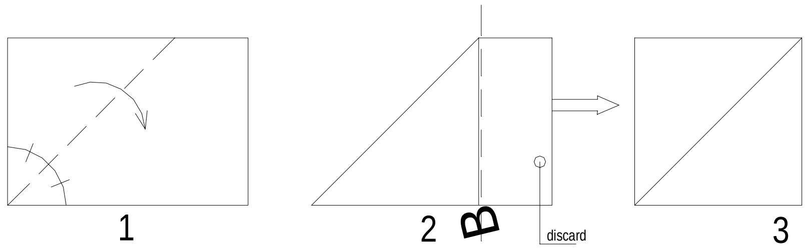
Expressions (1) and (2) are identical for $x = \frac{1}{n}$ (corollary P´s statement), therefore the corollary is fulfilled.

6. OBTENTION OF PARALLELOGRAMS

6.1 SQUARE FROM A RECTANGLE

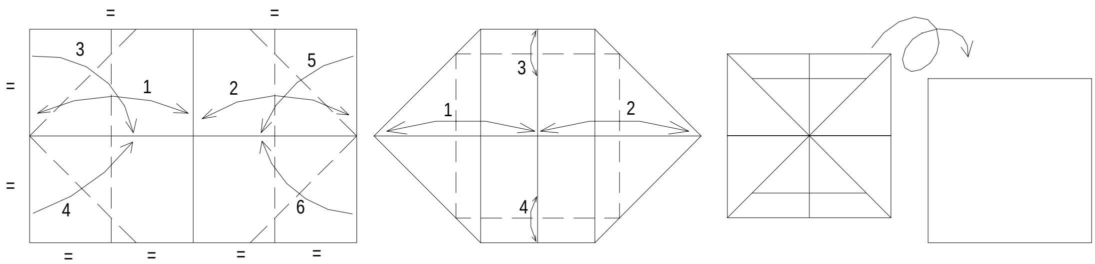
Solution 1

Rectangle with sides in any given proportion. Square side results to be equal to the smaller side of rectangle.

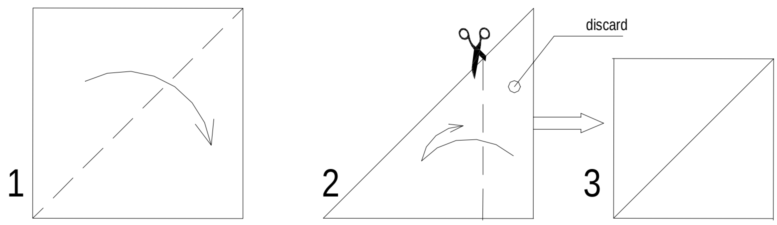


Solution 2

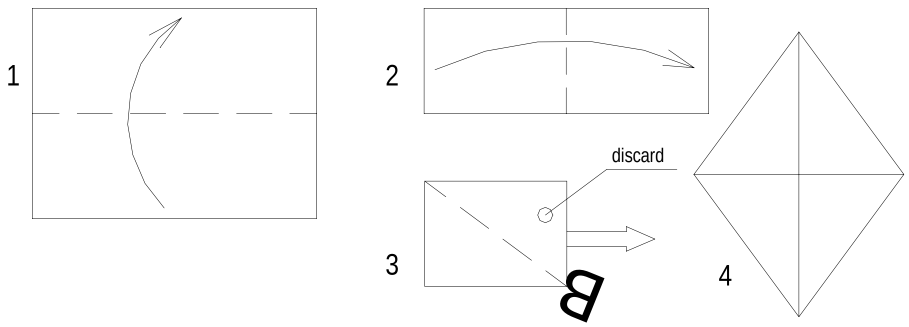
Square centered in rectangle. If the latter has b as base and a as height, the condition $a \geq \frac{b}{2}$ must be fulfilled to enable construction.



6.2 SQUARE FROM ANOTHER SQUARE.

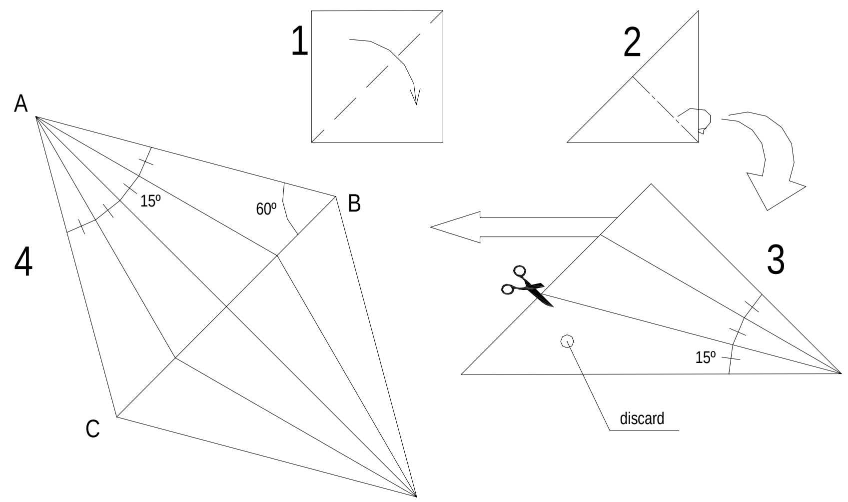


6.3 RHOMB FROM A RECTANGLE. Rhomb diagonals are equal to the sides of the rectangle.

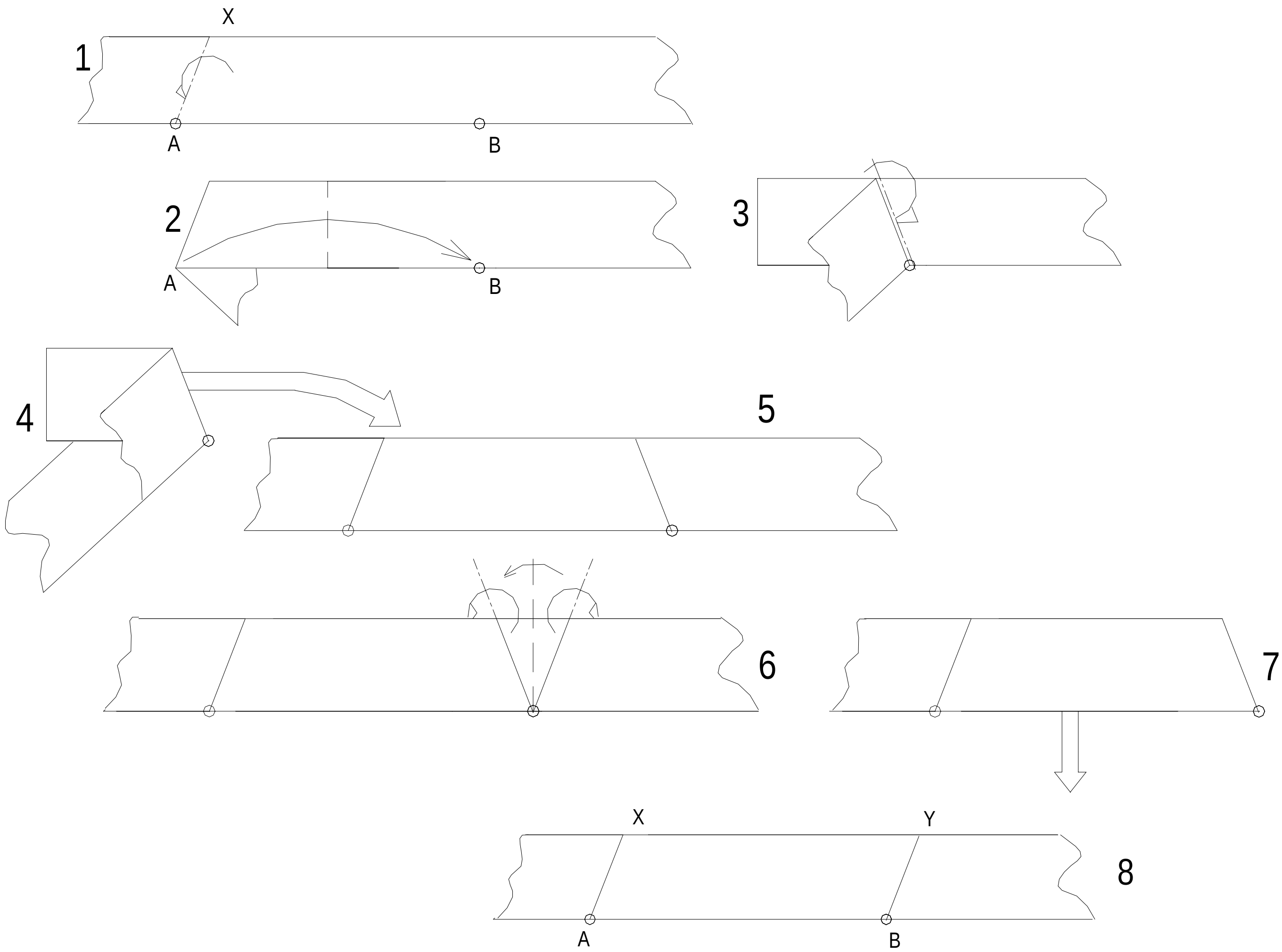


6.4 RHOMB FROM A SQUARE, the rhomb being the sum of two equilateral triangles.

Observe that Δ(ABC) is a real equilateral triangle. To trisect an angle of 45º, see Point 8.2.2.2 hereinafter.



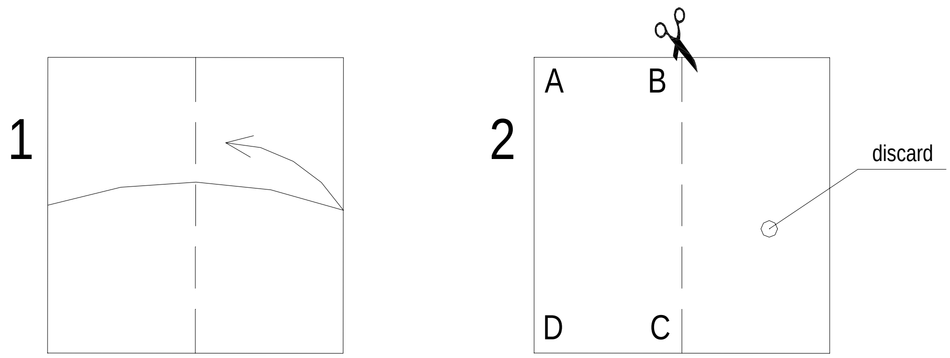
6.5 RHOMBOID FROM A PAPER STRIP.



Start up with the greater side AB and fold over small side AX with any slope.
The rhomboid AXYB is obtained. Observe how an isosceles trapezium associated to the rhomboid, is produced in step 5.

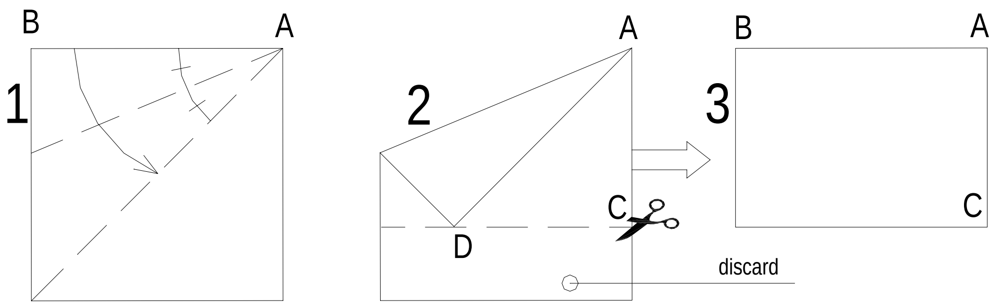
6.6 OBTENTION OF RECTANGLES WITH THEIR SIDES IN VARIOUS PROPORTIONS:

6.6.1 **2:1**, start up with a square whose side is one unit. ABCD is the rectangle obtained.

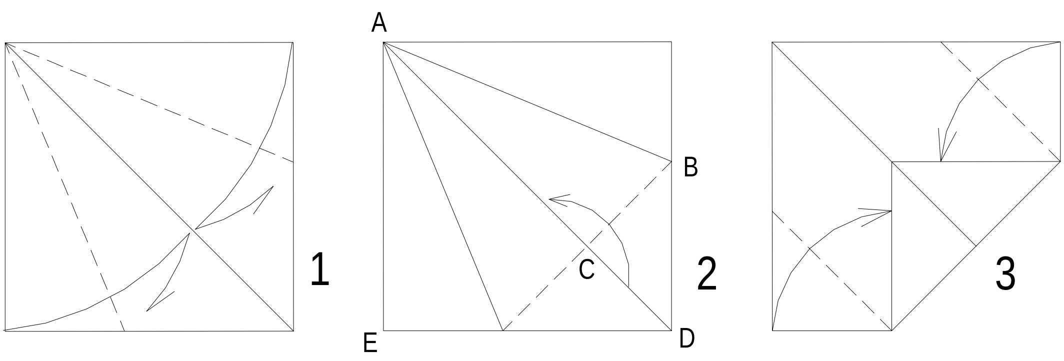


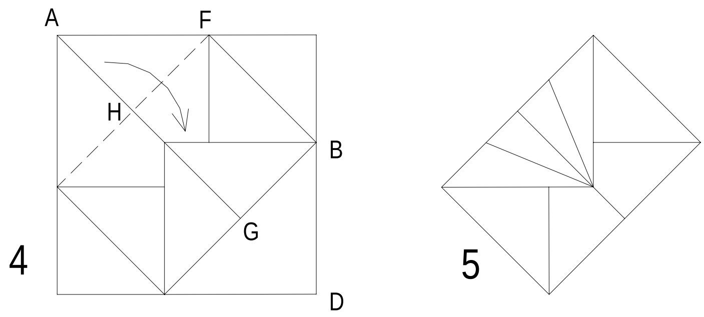
6.6.2 $\sqrt{2}:1$ (DIN A), with BA = 1. The solution is in step 3:

In ΔADC, ang DAC = 45º; AD = AB = 1; $AC = \dfrac{AD}{\sqrt{2}}$; $\dfrac{AB}{AC} = \dfrac{\sqrt{2}}{1}$



6.6.3 $\sqrt{2}:1$ (envelope version)





Let L and l be the greater and smaller side of the rectangle finally obtained in step 5.

In 2: CB = $\frac{L}{2}$; CB = CD = AD – AC ; AD = $\sqrt{2}$; AC = AE = 1. therefore

$$\frac{L}{2} = \sqrt{2} - 1$$

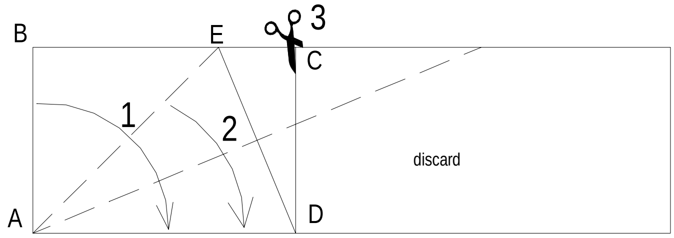
In 4: FB = l ; FB = HG = AD – 2AH ; AH = HF = $\frac{L}{2}$. So

$$l = \sqrt{2} - L$$

Then,

$$\frac{L}{I} = \frac{2\left(\sqrt{2}-1\right)}{\sqrt{2}-L} = \frac{2\left(\sqrt{2}-1\right)}{\sqrt{2}-2\left(\sqrt{2}-1\right)} = \sqrt{2}$$

6.6.4 DIN A FROM ANY RECTANGLE



IF AB = 1, then AE = AD = $\sqrt{2}$

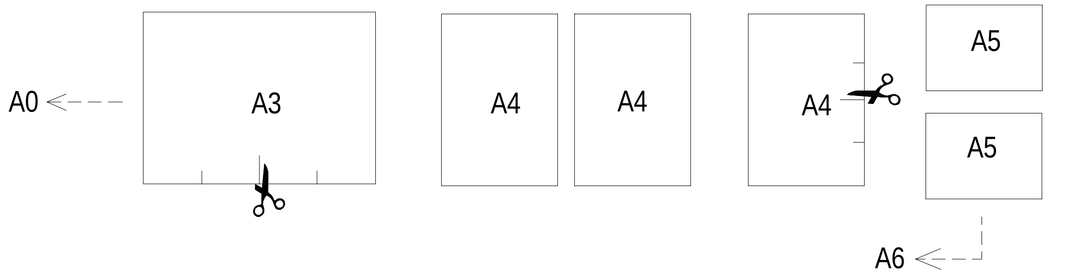
ABCD ≡ DIN A

If a DIN A4 is wanted, AB should measure 210 mm.

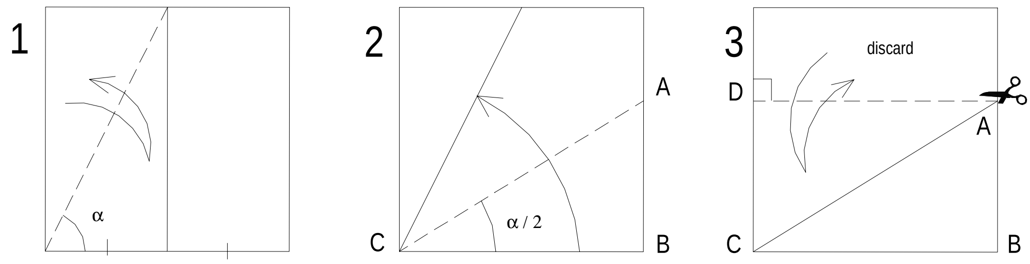
6.6.5 DIN A FROM ANOTHER DIN A

A4 ≡ 297 x 210 ≡ 210 $\sqrt{2}$ x 210

A5 ≡ $210 \times \frac{297}{2} \equiv 210 \times \frac{210\sqrt{2}}{2} \equiv 210 \times \frac{210}{\sqrt{2}}$



6.6.6 $2 : \left(\sqrt{5}-1\right) = \left(1+\sqrt{5}\right) : 2$



$$x = \mathrm{tg}\ \alpha = 1 : \frac{1}{2} = 2 \quad ; \quad \mathrm{tg}\frac{\alpha}{2} = \frac{-1}{x} \pm \frac{\sqrt{1+x^2}}{x} = \frac{-1}{2} \pm \frac{\sqrt{1+4}}{2} = \frac{\sqrt{5}-1}{2} = \frac{AB}{BC} = AB$$

(AB must be positive)

ABCD is the wanted rectangle because:

$$\frac{CB}{AB} = 1 : \frac{\sqrt{5}-1}{2} = 2 : \left(\sqrt{5}-1\right)$$

6.6.7 ARGENTIC RECTANGLE: GREATER SIDE $= 2+\sqrt{5}$; DIAGONAL = $3+\sqrt{5}$

1 Start up with rectangle ABCD having AB = 6 units and AD = 4 units
2 To get XY: X $\rightarrow$ X; AD $\rightarrow$ AD
3 In square with opposite vertices AF, get EF = $\frac{\sqrt{5}-1}{2}$ the same way as AB was obtained in Point 6.6.6
4 To get G: F$\rightarrow$ F; XY $\rightarrow$ XY; E $\rightarrow$ G
5 To get H: G $\rightarrow$ G; GZ $\rightarrow$ XY (by means of simultaneous folds GZ (mountain) and GU (valley).

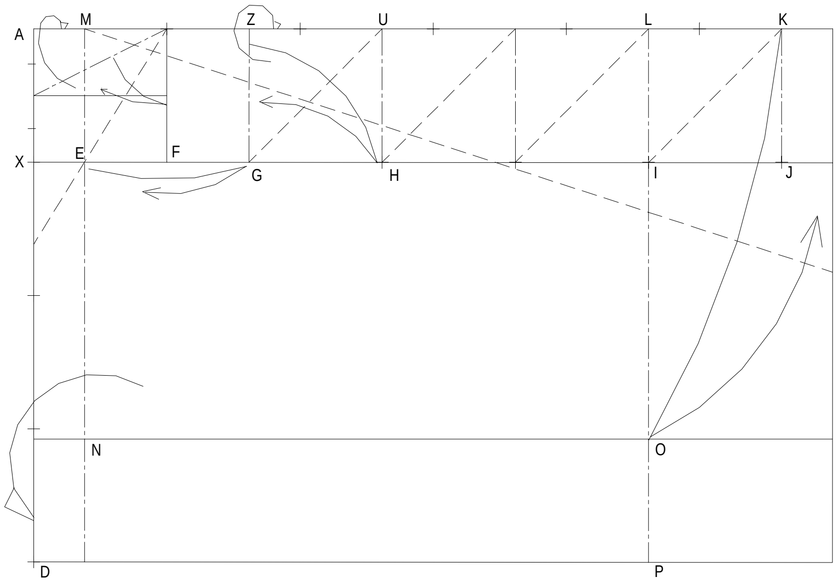
6 So we get EH = EG + 1 = 2EF + 1 = $\sqrt{5}$. With successive folds similar to those of step 5, I and J are obtained. Result is EI = $\sqrt{5}$ + 2 y EJ = $\sqrt{5}$ + 3

7 To get O: K $\rightarrow$ LP; M $\rightarrow$ M

8 LMNO is the wanted rectangle because LM = EI = 2 + $\sqrt{5}$; MO = MK = EJ = 3 + $\sqrt{5}$.
That makes: LM = 4,236068 ; MO = 5,236068 ; MN = 3,0776835

9 Discard lower and lateral correspondent strips.

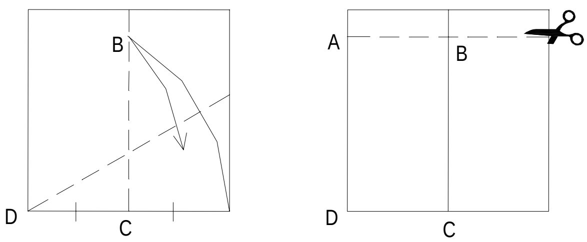
10 NOTE: the rectangle so obtained is a must to get a perfect convex regular pentagon.



6.6.8 $\sqrt{3}:1$

In ΔDBC, it is: BC = $\sqrt{1-(1/2)^2} = \dfrac{\sqrt{3}}{2}$

In rectangle ABCD: $BC = \dfrac{\sqrt{3}}{2}$; $DC = \dfrac{1}{2}$, therefore $\dfrac{BC}{DC} = \sqrt{3} : 1$



6.6.9 AURIC RECTANGLE

It is that in which the greater side is divided by the smaller one in media and extreme ratio, i.e., the small side is the geometric mean of the length of the greater side and the difference in length between both sides.

Let´s recall what is the division of a segment in media and extreme ratio.

When that division takes place, it happens that the ratio of the total segment to the great subsegment is the same as the ratio of the latter to the small subsegment. It´s shown in Fig. 1

Total segment AB is divided into the great CB and the small AC, in such a way that:

$$\frac{AB}{CB} = \frac{CB}{AC}$$

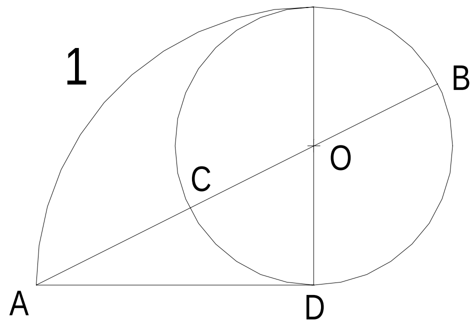
Or:

$$CB^2 = AB \times AC$$

As CB = AD, we can write:

$$AD^2 = AB \times AC$$

which is true as a consequence of the power of point A with respect to the circumference of center O and radius OD.



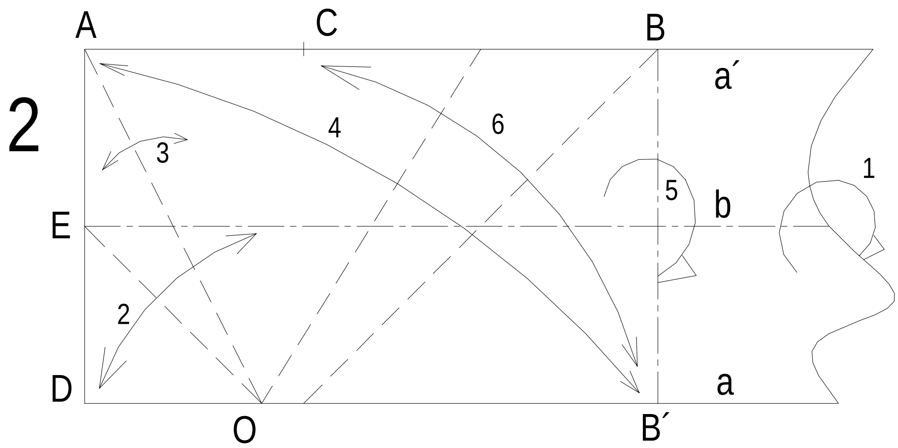


Fig. 2 shows how to get an auric rectangle through folding. Its small side is taken for the great subsegment. We´ll begin with a paper strip of adequate dimensions. Folding process and results are as follows:

1- a´ → a. We get E.
2- D → b; E → E. We get O.
3- Fold over AO.

Till now we have constructed ΔAOD which is the same in Fig. 1: To get total segment AB (larger side of the auric rectangle) OB should be added to hypotenuse AO, OB been equal to small leg OD.

4- A → a; O → O. We get B´.
5- a → a; B´ → B´. We get B: AB is the greater side of the auric rectangle.
6- B´ → a´; B → B. We get C.

The sides of the auric rectangle are AB (great) and AD (small); subsegments are AC y CB. Since Fig. 2 gets by folding the segments of Fig. 1, we´ll have:

$$\frac{AB}{CB} = \frac{CB}{AC} \quad ; \quad CB^2 = AB \times AC$$

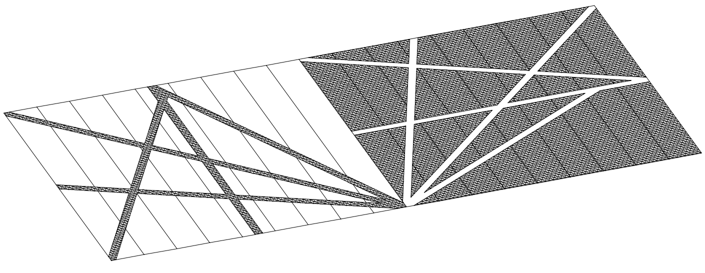
It is pertinent to quote now that rectangle ABCD in Point 6.6.6 is an auric one (S. Turrión). If that would be the case, we should have:

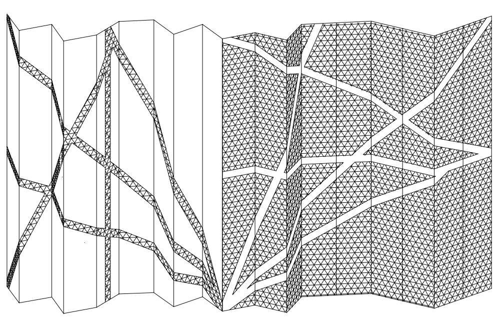
$$AB^2 = CB\,(CB - AB\,)$$

expression that is equivalent to the following identity which confirms the hypothesis:

$$\left(\frac{\sqrt{5}-1}{2}\right)^2 = 1 \times \left(1 - \frac{\sqrt{5}-1}{2}\right)$$

Interlude

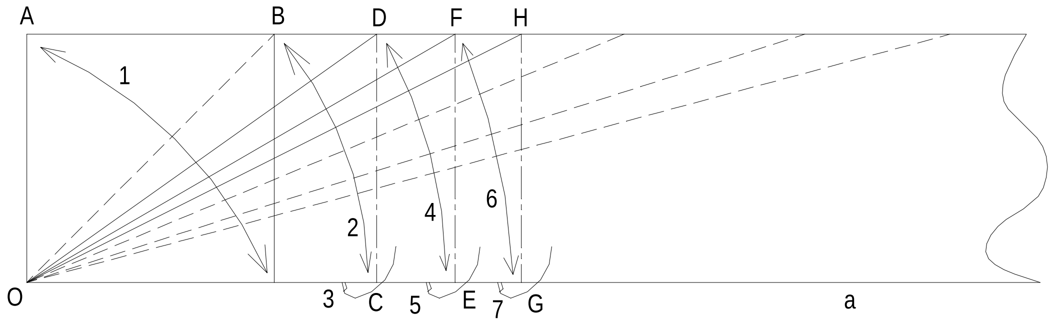




6.7 DYNAMIC RECTANGLES. SQUARE ROOT OF SUCCESSIVE NATURAL NUMBERS.

PROCEDURE 1 (Fig. 1)

1



Let´s start with a paper strip of length a and width one unit.

1- A $\rightarrow$ a; O $\rightarrow$ O. We get **OB** $= \sqrt{2}$.

2- B $\rightarrow$ a; O $\rightarrow$ O. We get C.

3- a $\rightarrow$ a; C $\rightarrow$ C. We get D: **OD** $= \sqrt{1+OC^2} = \sqrt{1+OB^2} = \sqrt{1+\left(\sqrt{2}\right)^2} = \sqrt{3}$

4- D $\rightarrow$ a; O $\rightarrow$ O. We get E.

5- a $\rightarrow$ a; E $\rightarrow$ E. We get F: **OF** $= \sqrt{1+OE^2} = \sqrt{1+OD^2} = \sqrt{1+3} = \sqrt{4}$

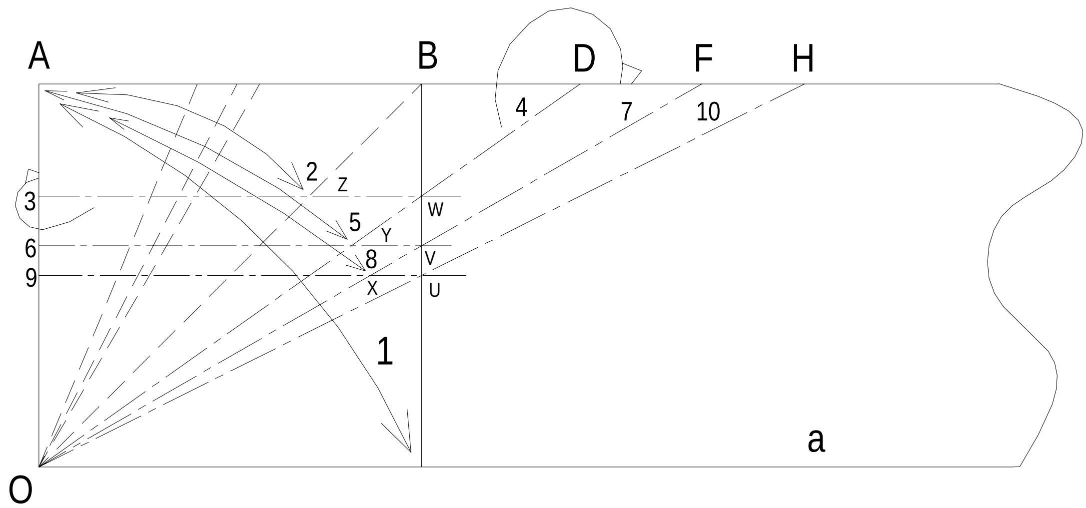
6- F $\rightarrow$ a; O $\rightarrow$ O. We get G.

7- a $\rightarrow$ a; G $\rightarrow$ G. We get H: **OH** $= \sqrt{1+OG^2} = \sqrt{1+OF^2} = \sqrt{1+4} = \sqrt{5}$

... and so on.

PROCEDURE 2 (Fig. 2)

2



In this case foldings will be:

1- A $\rightarrow$ a; O $\rightarrow$ O. Getting **OB** $= \sqrt{2}$

2- A $\rightarrow$ OB; O $\rightarrow$ O. ,, Z

3- AO $\rightarrow$ AO; Z $\rightarrow$ Z ,, W

4- OW. Getting D; **OD** $= \sqrt{3}$

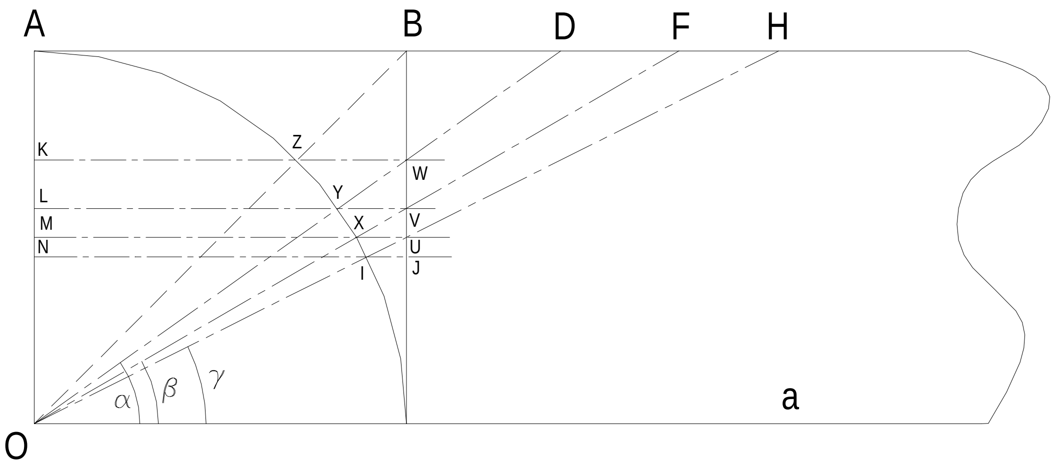
5- A $\rightarrow$ OD; O $\rightarrow$ O. Getting Y

6- AO $\rightarrow$ AO; Y $\rightarrow$ Y. ,, V

7- OV. Getting F; $\mathbf{OF} = \sqrt{4}$
8- A $\rightarrow$ OF; O $\rightarrow$ O. Getting X
9- AO $\rightarrow$ AO; X $\rightarrow$ X. ,, : U
10- OU. Getting H; $\mathbf{OH} = \sqrt{5}$

Justification of procedure 2 (Fig. 3)

3



- Let´s go back from end to beginning.
  In ΔAOH, $OH = \dfrac{OA}{\operatorname{sen}\gamma}$; $\operatorname{sen}\gamma = \sqrt{1-\cos^2\gamma}$
  In ΔION, $\cos\gamma = \dfrac{IN}{OI} = IN = 1 - IJ$
  To reach the end we should have to get the successive values BZ, ZW, WY, YV, VX, XU, UI, IJ, etc. as functions of the unit (width of the strip). Let´s see how to do it.
- As OI = OX = OY = OZ = OA = 1, we can draw a circumference (whose first quadrant is shown) with center O and radius OA.
- Power of B with respect to the circumference at O:
  $1^2 = BZ(BZ+2)$; $BZ^2 + 2BZ - 1 = 0$; $\mathbf{BZ} = \dfrac{-2 \pm \sqrt{4+4}}{2} = \sqrt{2} - 1$ (discarding the negative value of BZ).
  In the isosceles rectangled triangle BZW we have:
  $$\mathbf{WZ} = \frac{BZ}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}$$
- Power of W with respect to the same circumference:
  $$WZ(WK + KZ) = WY(WO + OY)$$
  $$\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \left(1 - WZ\right)\right) = WY\left(1 + WY + 1\right)$$
  $$\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{2}}\right) = 2WY + WY^2$$

$$WY^2 + 2WY - \frac{1}{2} = 0 \quad ; \quad \mathbf{WY} = \sqrt{\frac{3}{2}} - 1$$

- $VY = WY\cos\alpha = WY \times LY = WY(1 - VY) \quad ; \quad VY(1 + WY) = WY$

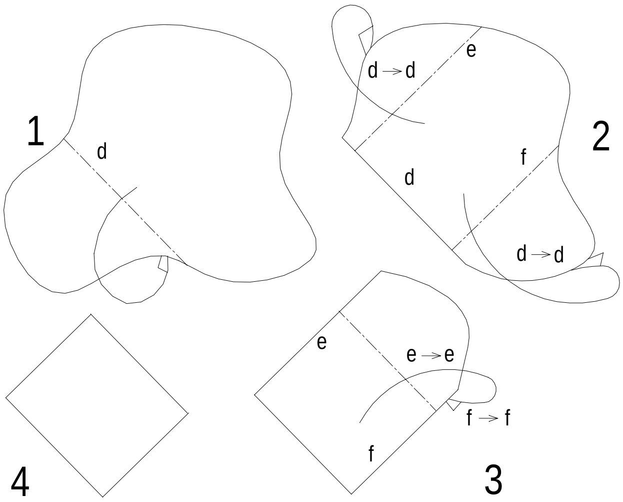
$$\mathbf{VY} = 1 - \sqrt{\frac{2}{3}}$$

- Till now we have got segments **BZ, WZ, WY, VY**. The following stage would be to figure out the power of V with respect to the already mentioned circumference to carry on getting the successive stepped segments indicated above.
- At this moment we are prepared to get OB (its straight away value is $\sqrt{2}$), and also OD:

$$\mathbf{OD} = \frac{1}{\operatorname{sen}\alpha} = \frac{1}{\sqrt{1-\cos^2\alpha}} = \frac{1}{\sqrt{1-LY^2}} = \frac{1}{\sqrt{1-(1-YV)^2}} = \frac{1}{\sqrt{1-\left(1-\left(1-\left(1-\sqrt{\frac{2}{3}}\right)\right)\right)^2}} = \sqrt{3}$$

- In a similar way we can get values of **OF** = $\sqrt{4}$, **OH** = $\sqrt{5}$, etc.

6.8 HOW TO GET A RECTANGLE FROM AN IRREGULAR PIECE OF PAPER.
Solution in step 4.



6.9 STELLATE RECTANGLE

As we know, to get stellate polygons we must play properly with their diagonals. So we only will have stellate polygons from the pentagon, forward.

Nevertheless, all polygons (triangles and quadrilaterals included) may take a stellate appearance, not a flat one, but volumelike. In this respect they leave off to be polygons.

A very simple solution is to fold the angles´ bisectors and the bisectors of the angles so obtained. By means of this you can obtain beautiful shapes evoking architectonic forms (see Figs. 2 and 3 as an example).

1

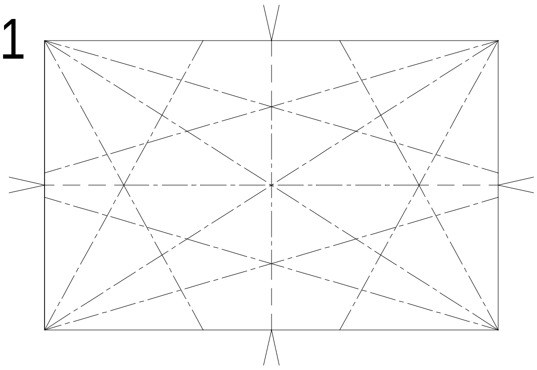
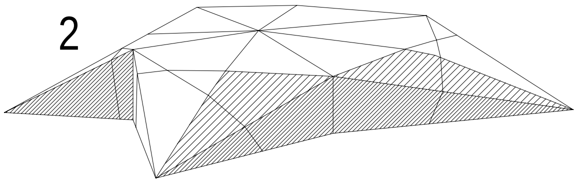
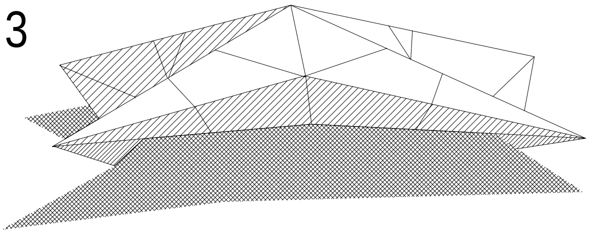


Fig. 1 brings out figs. 2 and 3 by merely changing the mountain / valley configurations

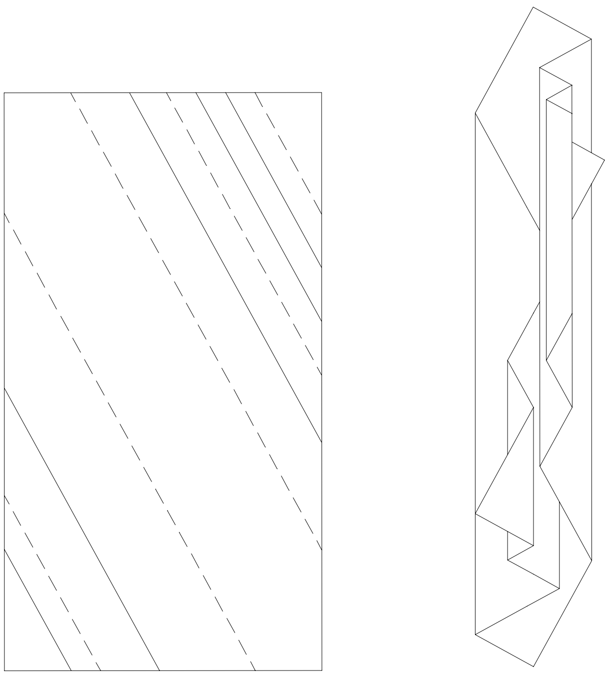
2



3

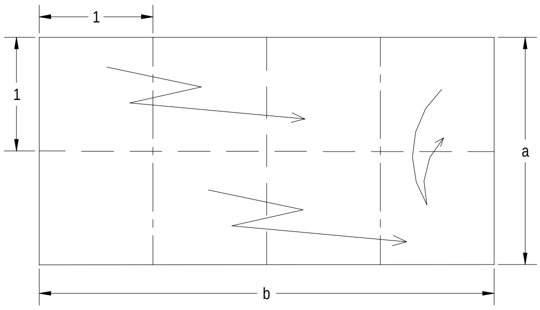


Interlude

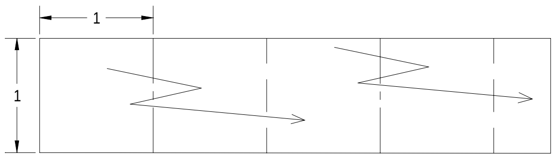


7 GEOMETRY IN THE PLANE. CARTESIAN PLANE. ALGEBRA.

7.1 AREA OF A RECTANGLE.

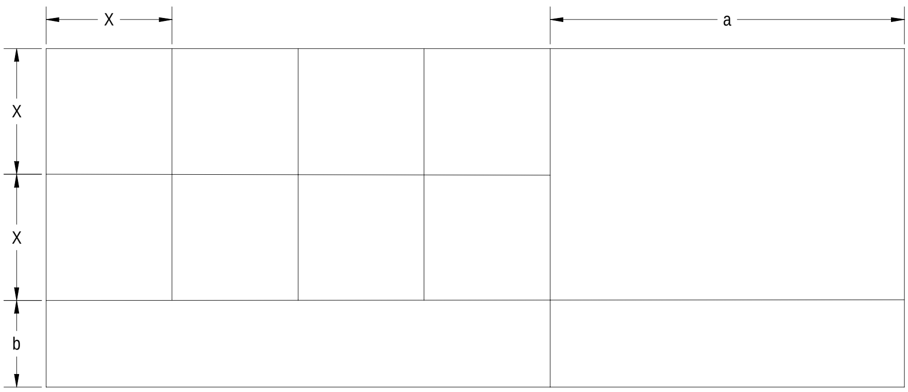


That area is expressed by the amount of unit squares contained within its surface. If the linear unit is contained exactly an integer number of times in both sides of the rectangle, it is evident that the area amounts to $b \times a = 4 \times 2 = 8$ unit squares.



If we consider the small side of the rectangle as the linear unit, the area of the small rectangle left aside is equivalent to the fraction of unit square expressing the decimal part of the big side. Therefore, in all cases the product of its base times the height defines the area of a rectangle.

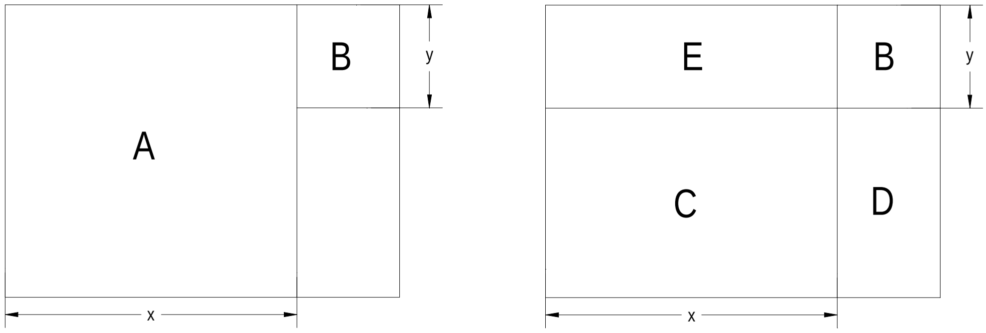
7.2.1 BINOMIAL PRODUCT



$$(4x + a) \times (2x + b) = 8x^2 + 2xa + 4xb + ab$$

Once the paper is folded to get the figure, we can see that the second member of the expression above relates to the sum of areas of the four correspondent rectangles.

7.2.2 SQUARES´ DIFERENCE.



Let squares A and B with respective sides x, y.
Algebraically we know that:

$$(x+y)(x-y)=x^2-y^2$$

The first member is equivalent to rectangle C + D
The second, to areas C + E – B
So,

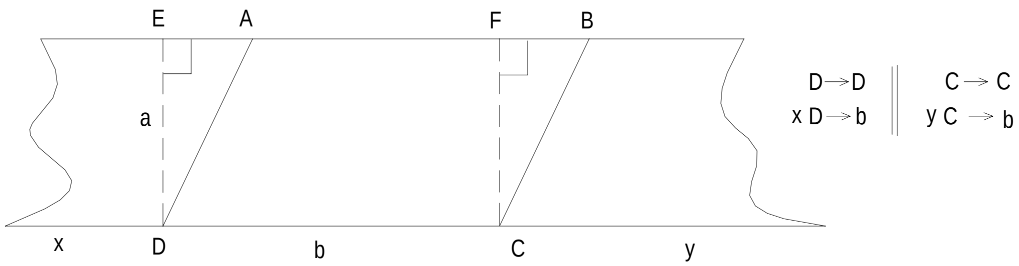
$$C + D = C + E - B \quad ; \quad D = E - B$$

Coming back to geometry from algebra, last expression is equivalent to:

$$y(x-y)=xy-y^2$$

as the latter expression is an identity, it proves that also geometrically sum times difference is equal to the difference of squares.

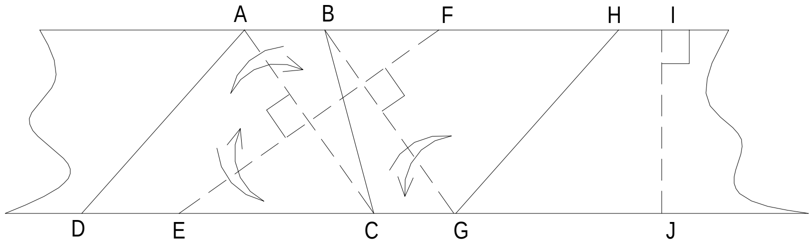
7.3 AREA OF THE OTHER PARALELOGRAMS



Let´s take the rhomboid ABCD as representative (see Point 6.5 for construction). Folding through D and C the perpendicular to AB, we get equal triangles AED and FBC. Therefore, the area of rhomboid ABCD is equal to that of rectangle EDCF, i.e., $b \times a$ (base times height).

7.4 AREA OF TRAPEZIUM ABCD

1- AC (valley fold).
2- EF perpendicular to AC (AC→ AC).
3- BG perpendicular to EF through B (B → B; EF → EF).
4- This way CG = AB is obtained.
5- Likewise, get BH = DC.
6- Thus we get the trapezium BCGH which is equal to ABCD because it has equal angles in B and C, as well as their three associated sides.



7- Therefore, trapezium ABCD has an area half of the rhomboid ADGH.

8- So: area ABCD = $\frac{1}{2}$ DG × IJ = $\frac{DC + AB}{2}$ IJ

7.5 PROBLEMS IN THE CARTESIAN PLANE

To obtain the coordinates of point P:

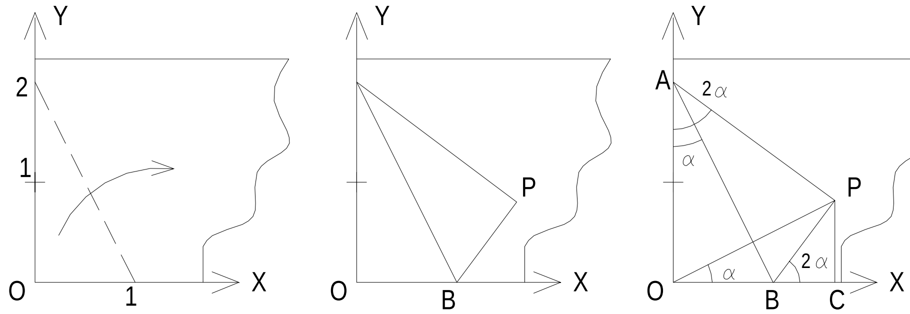
AB = $\sqrt{4+1} = \sqrt{5}$

ΔAOB is similar to ΔOPC

$\frac{y_P}{x_P} = \frac{PC}{OC} = \frac{1}{2}$; y_P = sen 2α

sen α = $\frac{1}{\sqrt{5}}$; sen 2α = 2 sen α cos α = $\frac{2}{\sqrt{5}}\sqrt{1-\frac{1}{5}} = \frac{4}{5}$

$y_P = \text{sen}\, 2\alpha = \frac{4}{5}$; $x_P = 2y_P = \frac{8}{5}$



X

To get the coordinates of point P and the area of quadrilateral OFPD .

$$\alpha = \text{ArTang}\left(\frac{2}{3}\right)$$

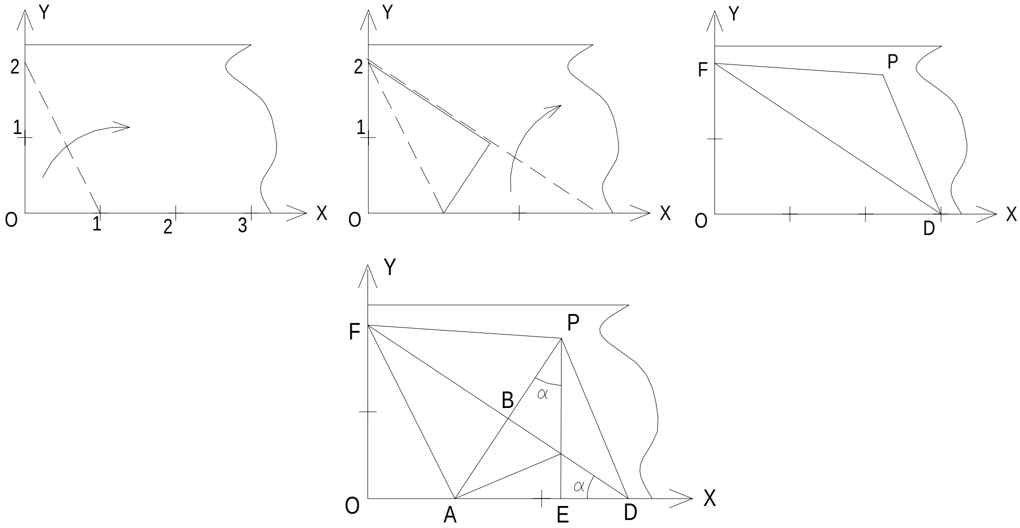
ΔABD≡AB=2senα

AP = 2 AB = 4sen α

x_P = 1+AE = 1+AP sen α = 1 +4sen^2 α = 2,23

y_P = (x_P -1) / tang α = $\frac{1{,}23}{2}\times 3 = 1{,}846$

area (OFPD) = (OFA) + (AFPD) = $1 + 2\times 2 = 5$

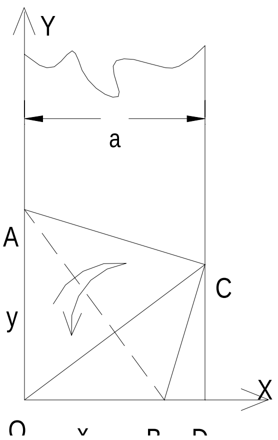


7.6 MAXIMA AND MINIMUMS

In a rectangle of width a, to fold O over the edge CD in such a way that, being B within OD, the length of fold AB will be minimum. If

$$y = OA \quad ; \quad x = OB$$

we must find an x value such that it will make minimum the expression $\sqrt{x^2 + y^2}$



ΔAOB and ΔCOD are similar:

$$\frac{y}{x} = \frac{a}{CD} \quad ; \quad \text{in } \Delta CBD: \quad CD = \sqrt{x^2 - (a-x)^2}$$

$$y^2 = \frac{a^2 x^2}{x^2 - (a-x)^2}$$ Then we seek the minimum of $\sqrt{x^2 + \frac{a^2 x^2}{2ax - a^2}}$

making

$$z = \left(x^2 + \frac{x^2}{\frac{2}{a}x - 1} \right)^{\frac{1}{2}}$$ and deriving, we have:

$$z' = \frac{1}{2}\left(x^2 + \frac{x^2}{\frac{2}{a}x - 1} \right)^{-\frac{1}{2}} \left(2x + \frac{\left(\frac{2}{a}x - 1\right) 2x - x^2 \frac{2}{a}}{\left(\frac{2}{a}x - 1\right)^2} \right)$$

The value of x that will make z’ = 0 is the minimum for AB we are trying to find. It is obvious that the maximum is ∞ (what happens when O is folded over D).

Let´s set the hypothesis that z’ = 0 when its last factor´s value is 0. As the previous factor is a square root in a denominator, we´ll have to check afterwards that this factor´s value is not 0 for the x solution.
Therefore it should be

$$2x\left(\frac{2}{a}x-1\right)^2+\left(\frac{2}{a}x-1\right)2x-x^2\frac{2}{a}=0 \quad \text{under the condition of} \quad \left(\frac{2}{a}x-1\right)^2\neq 0$$

Simplifying last equation it is easy to see that

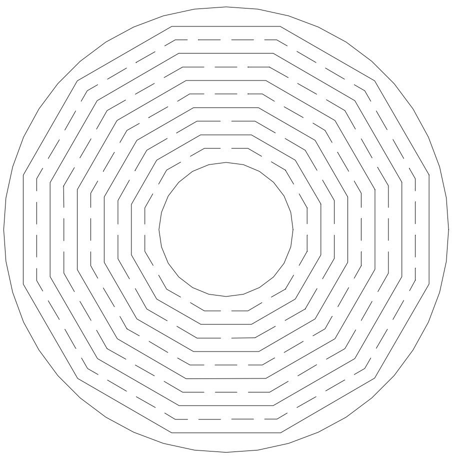
$$x=\frac{3}{4}a$$

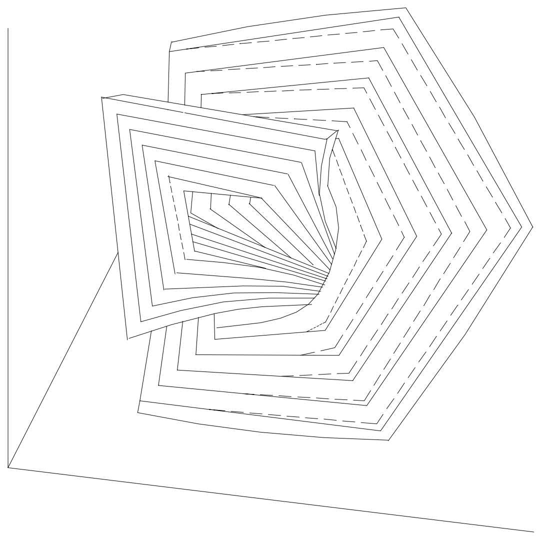
which in fact does not nullify either the expressions mentioned above.

Folding O over CD through a point B such that $x=\frac{3}{4}a$ we get the minimum segment AB.

The reader can check it by himself trying successive folds to approximate AB minimum.

Interlude





7.7 RESOLUTION OF A QUADRATIC EQUATION

Let the equation

$$x^2 + mx + n = 0 \qquad (1)$$

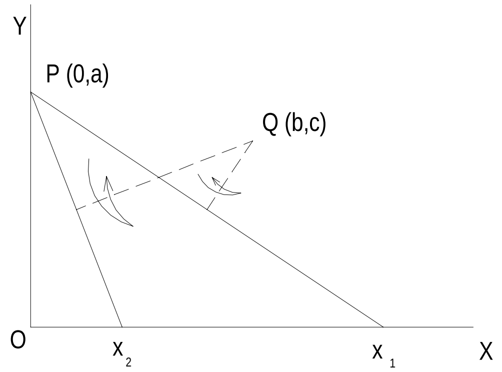
As it´s well known, any equation may be changed to have 1 as first coefficient: if this is not the case, the whole equation can be divided just by that coefficient.
Let´s define in the Cartesian plane the points

P(0,a) and Q(b,c)

And make this folding: P → OX ; Q → Q

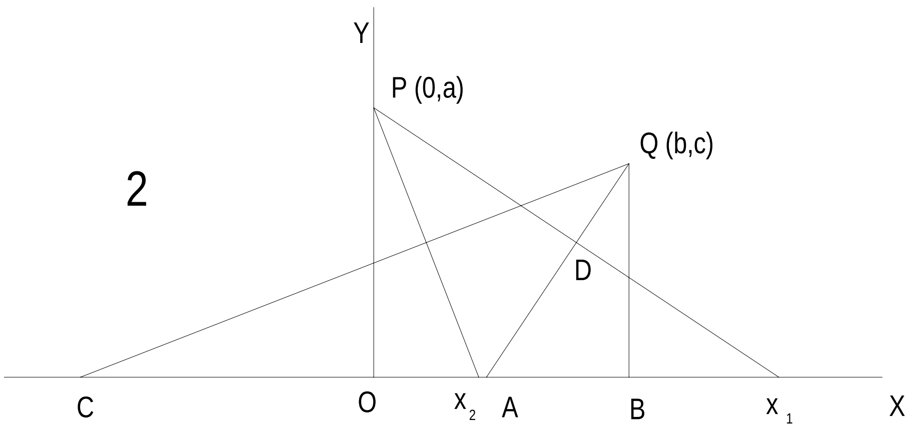
There are two ways of folding, giving respectively points x_1 and x_2. They are the two solutions of the proposed equation. Fig. 1 describes the process.

1



In both cases P has being folded over the two axes of symmetry (valley) passing through Q to give both points x. Let´s see the grounds in fig 2.

2



ΔPOX_1 and ΔADX_1 are similar, therefore:

$$\frac{Px_1}{Ox_1} = \frac{Ax_1}{Dx_1} \quad ; \quad Px_1 \times Px_1 / 2 = x_1 \times Ax_1 \qquad (2)$$

$$Ax_1 = x_1 - b + AB \qquad (3)$$

ΔPOX_1 and ΔQAB are also similar, so:

$$\frac{AB}{c} = \frac{a}{x_1} \quad ; \quad AB = \frac{ac}{x_1} \quad ; \quad \text{substituting in (3):}$$

$$Ax_1 = x_1 - b + \frac{ac}{x_1} \quad ; \quad \text{substituting in (2):}$$

$$a^2 + x_1^{\ 2} = 2x_1\left(x_1 - b + \frac{ac}{x_1}\right)$$

$$a^2 + x_1^{\ 2} = 2x_1^{\ 2} - 2bx_1 + 2ac$$

$$x_1^{\ 2} - 2bx_1 + a(2c - a) = 0$$

If former development would have been made for solution x_2 instead of x_1, the same expression is reached, therefore the quadratic equation obtained can be made general:

$$x^2 - 2bx + a(2c - a) = 0 \qquad (4)$$

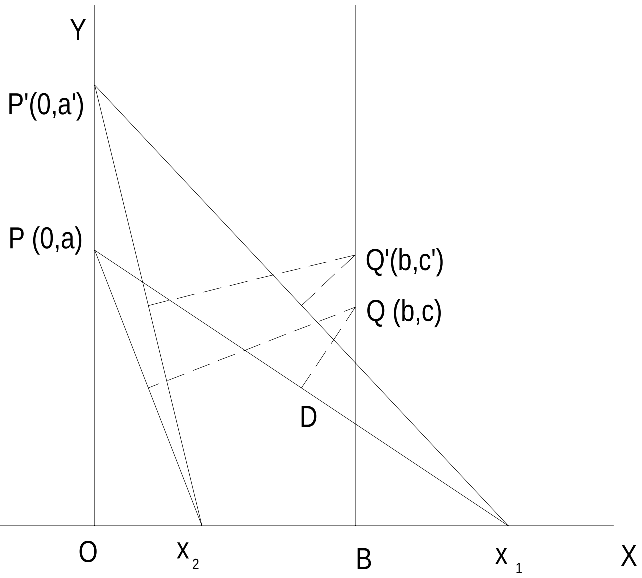
Comparing equations (1) and (4) we have:

$$m = -2b \quad ; \quad n = a(2c - a) \qquad (5)$$

At the end, what we are after is the values of a,b,c that enables us to draw fig 1. So we have 3 unknowns and 2 relations (5) that give:

$$b = -\frac{m}{2} \quad ; \quad c = \frac{1}{2}\left(\frac{n}{a} + a\right) \qquad (6)$$

To overcome this difficulty we may consider that c is a function of a, besides being also a function of n (6).



3

Fig 3 shows that fixing any arbitrary value for a, we get each time different values of c, but the same couple x. In (5) we can see that playing with values a,c we get the same value for n which is the given independent term in (1).

Summarising. To solve (1):

- Assign an adequate value to a in order to have the drawing properly covered by the paper.
- Get b,c according to (6), giving an arbitrary value to a.
- Fold to fig 1.

By means of this, any quadratic equation with real roots (positive, negative, double, etc) can be solved. We come across the exception of imaginary roots (negative discriminant of the equation). In that case it is impossible to draw fig. 2.

7.8 SQUARE ROOT OF A NUMBER

This is a particular case of Point 7.7: it´s a matter of solving a quadratic equation without lineal term and having a negative independent term:

$$x^2 - n = 0$$

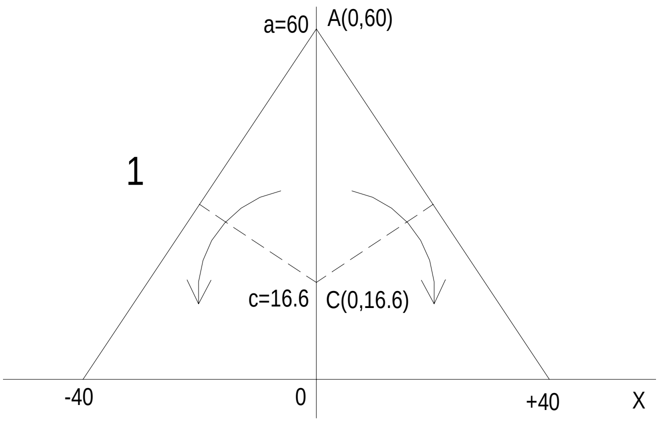
Expressions (5) in Point 7.7 take this form:

$$-2b = 0 \quad ; \quad -n = a(2c - a) \qquad (1)$$

which transform (6) to:

$$b = 0 \quad ; \quad c = \frac{1}{2}\left(-\frac{n}{a} + a\right)$$

Let´s apply this to the following example (fig 1): we wish to find out by means of folding, the square root of 1.600.



We´ll take arbitrarily a = 60

As a consequence we have $c = \frac{1}{2}\left(\frac{-1600}{60} + 60\right) = 16.6$

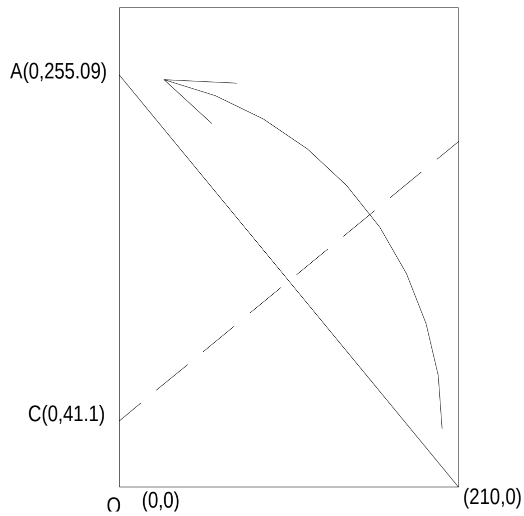
After folding A over the OX axis around C, we get as result $x = \pm 40$

7.9 SQUARE OF A NUMBER

It´s the inverse of latter exercise. Suppose we want to find out the square of 210 (the small side of a DIN A4 rectangle).

We take the lower right-hand side corner over any point A on left-hand side of paper. The ordinate of A must be greater than 210 to enable the construction of point C.

Points A and C are obtained after folding: taking their respective ordinates a,c into (1) of Point 7.8, it gives:



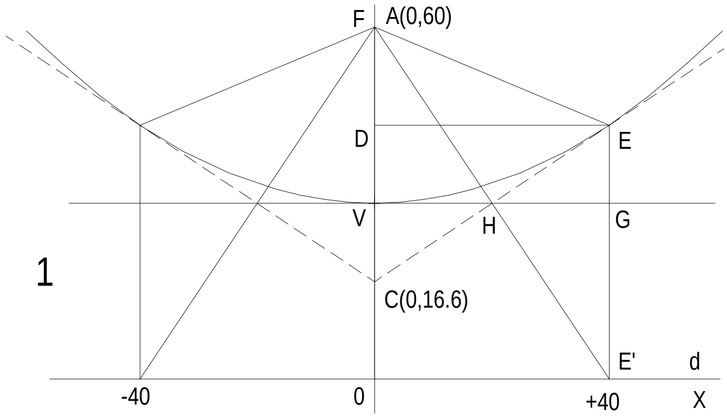
$$-n = a(2c - a) = 255.09(2\times 41.1 - 255.09) = -44102.5$$

n = 44100 (the difference is due to error in screen resolution).

7.10 PARABOLA ASSOCIATED TO THE FOLDING OF A QUADRATIC EQUATION

Point 1.2.4 explained how in folding a point over a straight line, the crease became the tangent to a parabola whose focus was the point to be moved and its directrix was the line which receives the point. That was already proved in Point 4 when demonstrating Haga´s theorem.

Now then, in fig 1 of Point 7.8 we have reproduced the same operation, so we can complete it now by drawing the parabola with focus F ≡ A, directrix d, vertex V (midpoint of AO) and tangent CE on point E whose abscissa is just 40.



The equation of this parabola is:

$$y = \frac{60}{2} + \frac{1}{2 \times 60} x^2 \quad ; \quad y = 30 + \frac{1}{120} x^2 \qquad (1)$$

This equation has the same structure as (1) in Point 4: we may observe that its first term has a length dimension (L) whereas the coefficient of x^2 has L^{-1} as dimension.

Fig 1 of present Point 7.10 makes obvious that distances from E to F and from E to d are equal because folding line EC is the symmetry axis. Eventually the result is the pair of tangents to a parabola from point C.

Fig 1 also shows:
ΔAOE' is similar to ΔHVC and therefore

$$\frac{60}{40} = \frac{\frac{40}{2}}{VC} \quad ; \quad VC = \frac{40}{3}$$

ΔVHC = ΔEHG, then:

$$EG = VC = \frac{40}{3}$$

$$y_E = EE' = E'G + GE = 30 + \frac{40}{3} = \frac{130}{3}$$

i.e. it is $E\left(40, \frac{130}{3}\right)$

Let´s make now a translation of axes from the origin to D. Equation (1) will become:

$$y = 30 + \frac{1}{120}x^2 \quad \rightarrow \quad y + \frac{130}{3} = 30 + \frac{1}{120}x^2$$

Which for y = 0 gives

$$x = \sqrt{\left(\frac{130}{3} - 30\right)120} = \pm 40$$

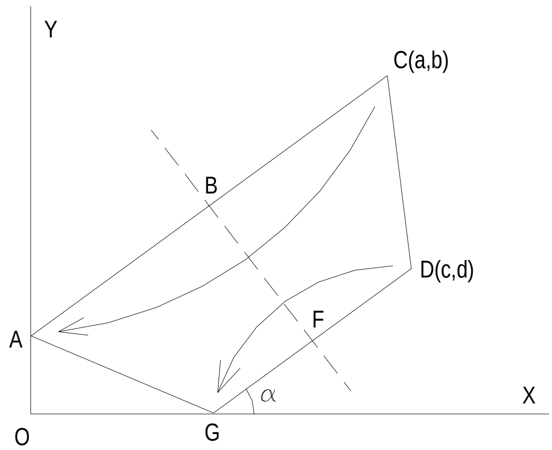
What proves that the roots of one equation coincide with the cutting points of the curve representing that equation, and the x axis.

7.11 COMPLET EQUATION OF 3rd DEGREE (J. JUSTIN)

Let it be

$$t^3 + pt^2 + qt + r = 0 \qquad (1)$$

1



In Fig 1 we fix the points C(a,b) and D(c,d), from the coefficients of (1), as we´ll see later.
Then we shall do **simultaneously** these foldings:

$$C \rightarrow OY \quad ; \quad D \rightarrow OX$$

The gradient of the normal to crease BF is the solution of (1), i.e.,

$$t = \mathrm{tg}\,(DGX) = \mathrm{tg}\,\alpha$$

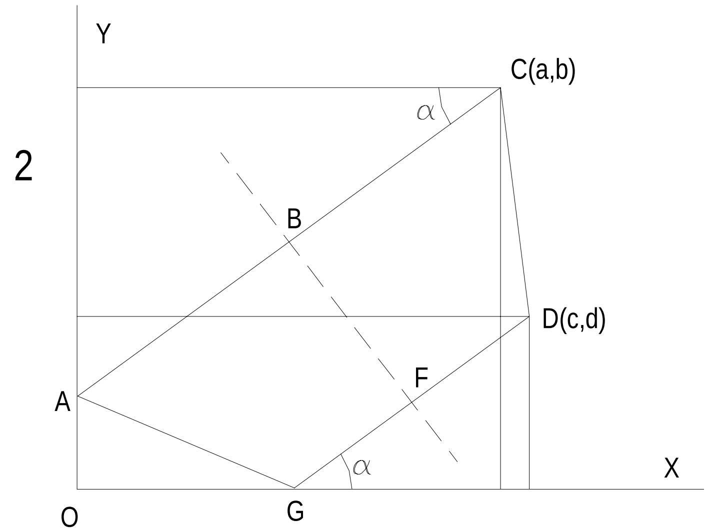
With the configuration and scale of Fig 1, there is only one solution for t: there is one only way of folding. That´s because the following conditions are fulfilled:

$$\frac{3q - p^2}{3} > 0 \quad \text{and} \quad \frac{2p^3 - 9pq + 27r}{27} < 0$$

So equation (1), in this case, has one real solution and two conjugate imaginary. Let´s discuss the solution (Fig 2):

$$\left.\begin{aligned} OG &= c - \frac{d}{\mathrm{tg}\,\alpha} \\ OA &= b - a\,\mathrm{tg}\,\alpha \end{aligned}\right\}$$

$$\left.\begin{aligned} x_F &= \frac{1}{2}(c + OG) = \frac{1}{2}\left(2c - \frac{d}{\mathrm{tg}\,\alpha}\right) \\ y_F &= \frac{1}{2}d \end{aligned}\right\} \qquad \left.\begin{aligned} x_B &= \frac{a}{2} \\ y_B &= \frac{1}{2}(b + OA) = \frac{1}{2}(2b - a\,\mathrm{tg}\,\alpha) \end{aligned}\right\}$$



$$\left.\begin{aligned} x_F &= c - \frac{d}{2t} \\ y_F &= \frac{d}{2} \end{aligned}\right\} \qquad \left.\begin{aligned} x_B &= \frac{a}{2} \\ y_B &= b - \frac{at}{2} \end{aligned}\right\}$$

On the other hand,

$$t = \frac{x_F - x_B}{y_B - y_F} \quad \text{, i.e.:}$$

$$t = \frac{c - \frac{d}{2t} - \frac{a}{2}}{b - \frac{at}{2} - \frac{d}{2}} = \frac{2ct - d - at}{2bt - at^2 - dt}$$

$$2bt^2 - at^3 - dt^2 = 2ct - d - at$$
$$at^3 + (d - 2b)t^2 + (2c - a)t - d = 0 \qquad (2)$$

Comparing equations (1) and (2), we have:

$$a = 1 \quad ; \quad d - 2b = p \quad ; \quad 2c - a = q \quad ; \quad d = -r$$

$$a = 1 \quad ; \quad b = \frac{-r - p}{2} \quad ; \quad c = \frac{q+1}{2} \quad ; \quad d = -r$$

Now we have available the four values a,b,c,d, i.e., we have the coordinates of C and D; that enables us to fold **simultaneously** these points over the axes OY and OX: the folding line that bears to t = tgα is the solution of (1). You may ease folding BF, now marking axes and points by transparency, now mountain-folding CD previously.

7.12 COMPLETE EQUATION OF 4$^{\text{th}}$ DEGREE: ITS RESOLUTION

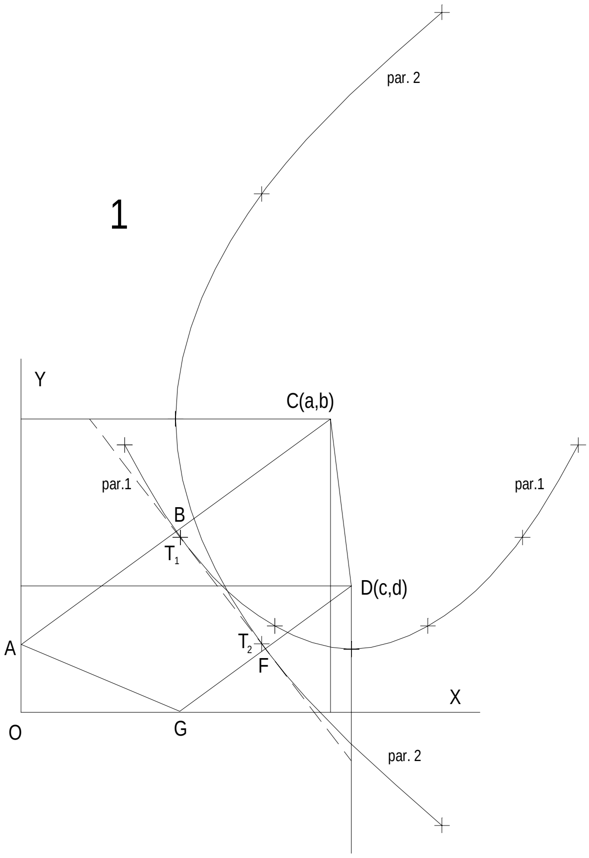
It´s not our intention to explore in detail this subject now. We´ll only say that this equation can be transformed in a complete 3$^{\text{rd}}$ degree one by a variable change and the application of Cardano´s transformation. Then Point 7.11 may be employed. The reader can refer to any treatise on equations general theory.

7.13 PARABOLAS ASSOCIATED TO THE FOLDING OF A COMPLETE EQUATION OF 3rd DEGREE

Folding operation in fig 1 (Point 7.11) leads to get BF as a common tangent to these two parabolas:

PARABOLA	FOCUS	DIRECTRIX	TANGENCY on BF
1	D	OX	T_1
2	C	OY	T_2

In fig 1 of present Point 7.13 these two parabolas are shown overlapped with fig 2 (Point 7.11)



If equation

$$t^3 + pt^2 + qt + r = 0$$

has a negative discriminant, i.e.:

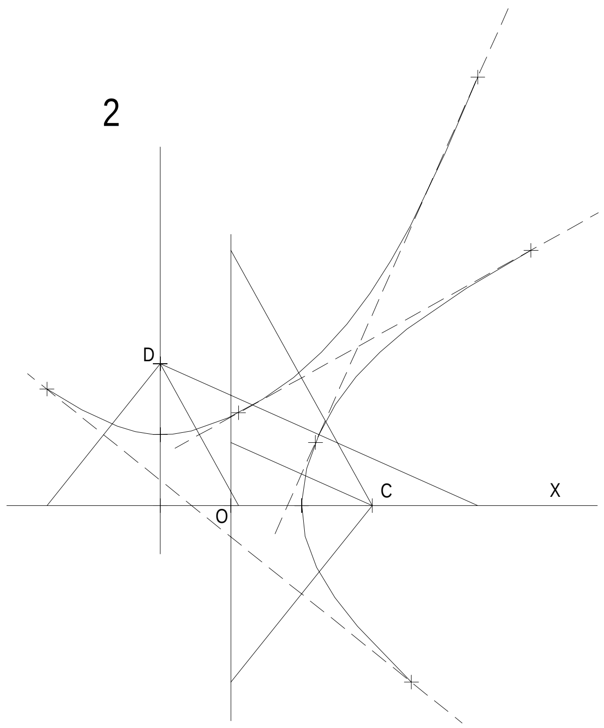
$$\frac{1}{4}\left(\frac{2p^3 - 9pq + 27r}{27}\right)^2 + \frac{1}{27}\left(\frac{3q - p^2}{3}\right)^3 < 0$$

We have three different forms of folding simultaneously points C and D over the correspondent axes.

Such is the case with the following equation that will be studied in another place:

$$t^3 + t^2 - 2t - 1 = 0$$

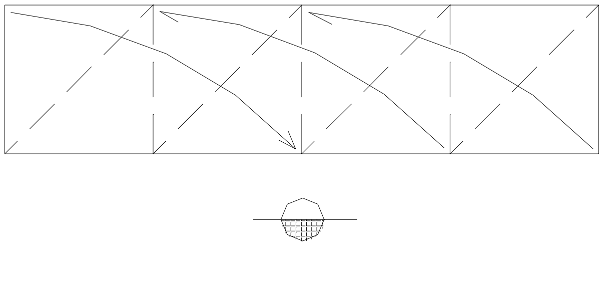
Fig 2 of present Point 7.13 shows the two parabolas and the three common tangents. These tangents are the symmetry axes in the simultaneous folds that carry focuses over directrices. Last equation has, therefore, three real roots.

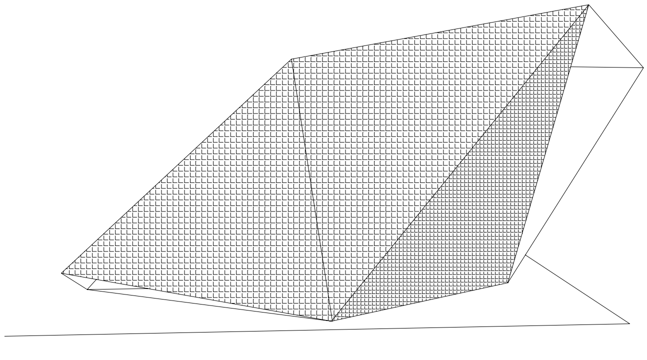


In this figure we may observe:

- Points D, C have been obtained by calculus (Point 7.11).
- From focuses D, C three radiating lines are cast, respectively, toward axes OX and OY.
- Each two of these three lines are parallel (perpendicular to the same tangent). The tangent of the angle formed by one of the rays and OX is the solution of the equation: obviously there are three of them.
- The three folding lines resulting from the simultaneous fold of D over OX and C over OY are the common tangents to these two parabolas: focus D (and directrix OX) and focus C (and directrix OY).
- The axes of those parabolas are, of course, the perpendiculars to the directrices from the focuses.

Interlude

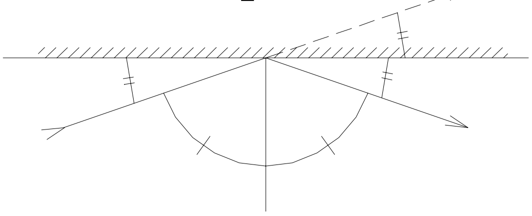


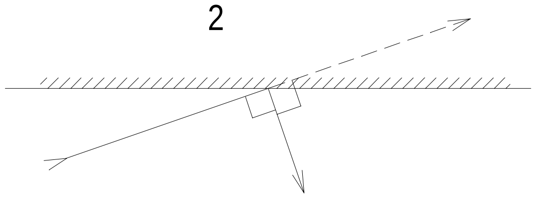


7.14 FUNDAMENT OF ORTHOGONAL BILLIARDS GAME (H. HUZITA)

This game is an ingenious recreation of conventional billiards. In this, any ball cast against the tableside is rebounded out in such a way that angles of incidence and reflexion are congruent (Fig. 1).

In orthogonal billiards, this other hypothesis is set up: once the ball hits the tableside, it is always repelled in a direction normal to incidence (Fig. 2).

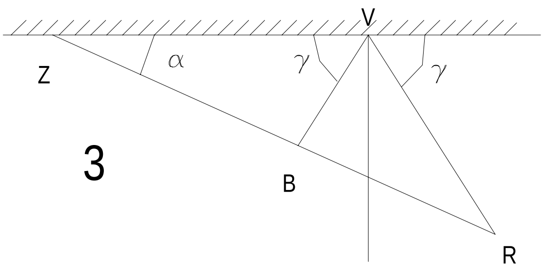




As we shall see, HH´s hypothesis is very useful to solve different problems (geometric as well as algebraic). Let´s see first, how balls behave under each of the following conditions.

Play to one tableside, only

In conventional billiards (Fig. 3), when we hit the white ball B against the tableside to reach red ball R, there is always one solution:



Data are BZ; RZ; a; γ is the unknown.

Being ZV common in ΔZVB and ΔZVR, we have:

$$\frac{ZV}{\text{sen}(\alpha+\gamma)} = \frac{ZB}{\text{sen}\,\gamma} \quad ; \quad \frac{ZV}{\text{sen}(\pi-\alpha-(\pi-\gamma))} = \frac{ZR}{\text{sen}(\pi-\gamma)}$$

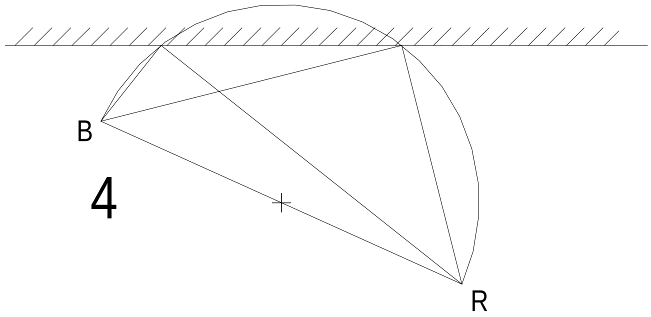
Equalising ZV:

$$ZB\frac{\text{sen}(\alpha+\gamma)}{\text{sen}\,\gamma} = ZR\frac{\text{sen}(\gamma-\alpha)}{\text{sen}\,\gamma} \quad ; \quad \text{developing:}$$

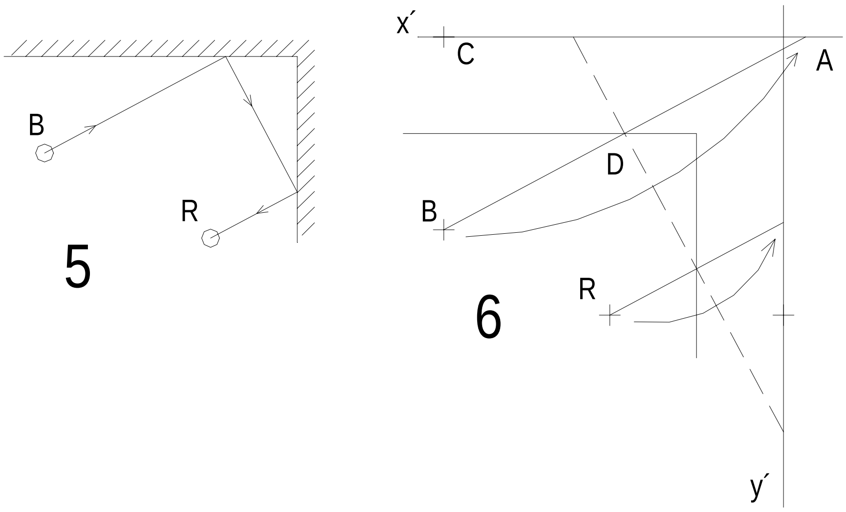
$$(ZR+ZB)\,\text{sen}\,\alpha\cos\gamma = (ZR-ZB)\,\cos\alpha\,\text{sen}\,\gamma$$

$$\text{tg}\,\gamma = \frac{2ZB+BR}{BR}\,\text{tg}\,\alpha$$

Contrarily, in orthogonal billiards there may be one, two or none solutions (Fig. 4) depending on the fact that the circumference with diameter BR will be tangent, secant or will not reach the tableside.



Play to two tablesides
In this case we shall fix our attention only in orthogonal billiards (Fig. 5)



Being placed balls B, R as indicated, we want to know the course that ball B will take to hit ball R after touching two sides of the table.
Solution is in (Fig. 6):

- To draw x´ , y´ parallel to tablesides distant from them as much as B and R do to their respective tablesides.
- Produce **simultaneous** folds B → x´ (A is got); R → y´.
- Folding line determines the intermediate stage in the way of B to R.
- As that folding line is the axis of symmetry and the horizontal tableside is the media parallel in ΔABC, right angle D sits on that horizontal tableside. The same applies to the lower triangle.

7.14.1 SQUARES AND SQUARE ROOTS (H. H.)

Let´s get the square of a (Fig. 1):

- To start with points C (-1,0) and A (0,a).
- To fold: C → y´ ; A → A.
- Folding line AB brings about B, whose abscissa is the square of a.

Justification:
ΔABC being a right-angled one, its altitude OA is the proportional media between OC and OB:

$$OA^2 = 1 \times a^2$$

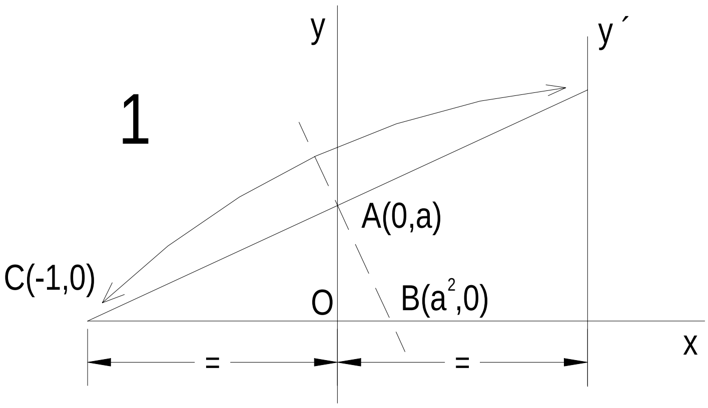
You may observe that in this case, the square (a^2) is smaller than the number (a) since the latter is smaller than 1.

Process is the inverse if we want to get the square root of b:

- Start with points C (-1, 0) and B (b,0).
- Fold: C → y´; B → B

Fold AB gives point A(0,a) such that:

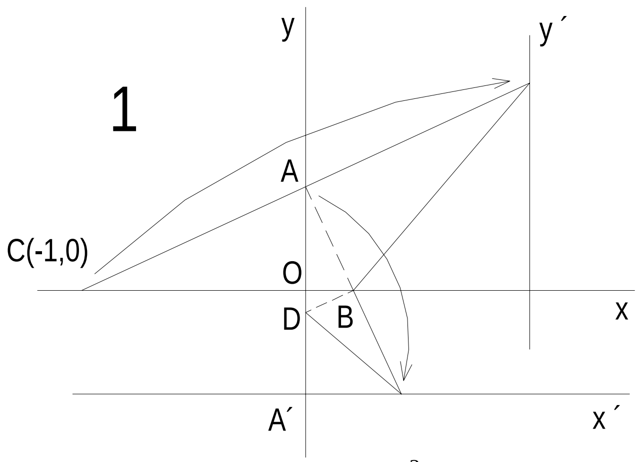
$$b \times 1 = a^2 \quad ; \quad a = \sqrt{b}$$



This process reminds orthogonal billiards, just because of orthogonality, but with the addition of this nuance: When ball C hits the tableside OY, a *virtual* reflection AB is produced outside the table.

7.14.2 CUBES AND CUBIC ROOTS (H.H.)

Iterating the former process we can find out the cube of a (Fig. 1).



- To start with points C (-1,0); A (0,a) to get B (a^2,0).
- To draw x´ such that OA´= OA.
- To fold: A → x´; B → B.
- Folding line BD produces OD = a^3.

Justification:

ΔABD is a rectangled one, and therefore:

$$OB^2 = AO \times OD \quad ; \quad \left(a^2\right)^2 = a^4 = a \times OD \quad ; \quad OD = a^3$$

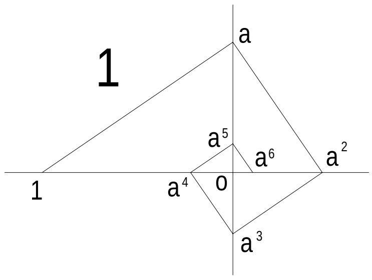
By working the opposite way as we did in former Point for the square root, we´ll reach the conclusion that $OA = \sqrt[3]{OD}$.

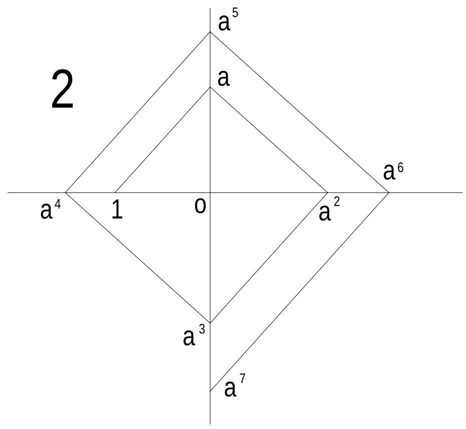
7.14.3 THE ORTHOGONAL SPIRAL OF POWERS (H.H.)

Observing the process pursued along the two previous Points, we see that successive powers of *a* can be obtained without limit, by means of folding, i.e. we can get a^n and, conversely, $a^{\frac{1}{n}}$, n being any natural number.

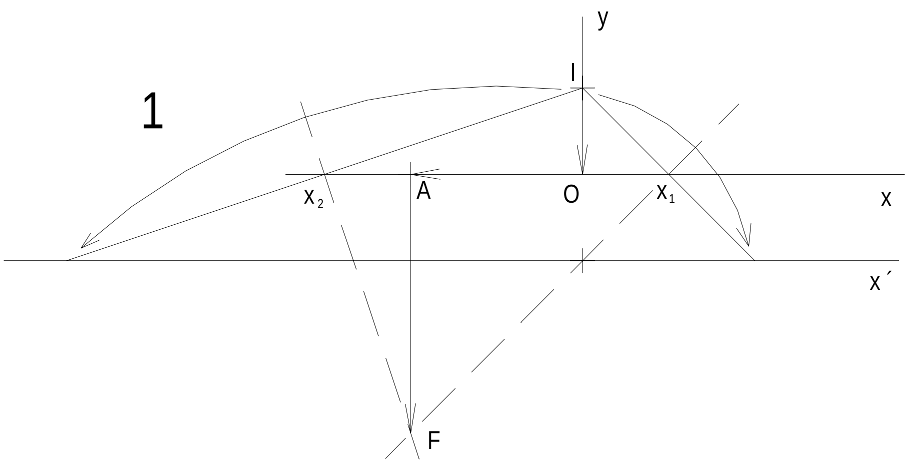
It should be noted that lines y´, x´, y´´, x´´, etc. the receivers of folding points, are parallel to the coordinate axes at a distance equal to that in between initial points and coordinate axes.

It is evident that if a <1, we have a closing *orthogonal spiral*, whereas the *spiral* opens if a >1. The values of successive powers of a are measured along the coordinate axes: even in abscissas and odd in ordinate. Figs. 1 and 2 show all that.





7.14.4 RESOLUTION OF A QUADRATIC EQUATION (H.H.)



First, let´s figure out the quadratic equation with roots $x_1 = 1$ and $x_2 = -3$

$$(x-1)\,(x+3) = 0 \quad ; \quad x^2+2x-3 = 0 \quad (1)$$

Fig. 1 shows the folding process to get its two roots:

- To set axes OX; OY.
- To draw x´ distant one unit from OX. This is because the coefficient of greater degree –the 2nd- is 1.

- Start in I (IO = 1) a series of coefficient vectors according to these criteria:
  IO = 1: coefficient of x^2.
  OA = 2: absolute value of the x coefficient; at right angle with IO; clockwise direction because from 1st to 2nd term there is no sign change.
  AF = 3: absolute value of the independent term; at right angle with OA; anticlockwise direction because passing from the 2nd to the 3rd term there is sign change.
  Finally we get **F** which is the end point of the three successive vectors.
- To fold

$$I \rightarrow x' \quad ; \quad F \rightarrow F$$

As can be seen, there are two solutions:

$$OX_1 = 1 \quad ; \quad OX_2 = -3$$

Justification:
ΔIOX_1; ΔFAX_1 as well as ΔIOX_2; FAX_2 are similar, so:

$$\frac{IO}{OX_1} = \frac{AX_1}{AF} \quad ; \quad \frac{IO}{OX_2} = \frac{AX_2}{AF}$$

To assign a value to these segments, we have to bear in mind:

- Independent variable x is to be given the correspondent sign in the Cartesian plane.
- Give to the rest of segments the absolute value they have in equation (1), because its sign was already taken into account when clockwise or anticlockwise direction was assigned: these segments (the equation coefficients) have not the dimension of the independent variable though they appear overlapped with it in the Cartesian plane.

Then it follows:

$$\frac{1}{x_1} = \frac{x_1 + 2}{3} \quad ; \quad \frac{1}{-x_2} = \frac{-x_2 - 2}{3}$$

that in both cases leads to the same result (x_1; x_2 been taking by x):

$$x^2 + 2x - 3 = 0$$

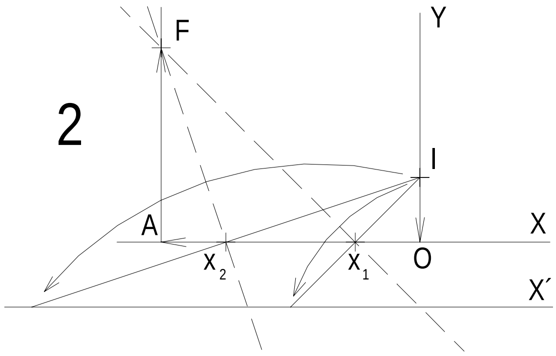
Yet we´ll see another example to settle sign attribution.
Let the quadratic equation with roots $x_1 = -1$; $x_2 = -3$

$$(x+1)(x+3) = 0 \quad ; \quad x^2 + 4x + 3 = 0 \qquad (2)$$

- Absolute values:
  IO = 1: coefficient of x^2
  OA= 4: coefficient of x
  AF= 3: independent term
- Vectors´ sequence: IOAF (clockwise all the time because in (2) there is not signs change).
- Fold: $I \rightarrow x'$; $F \rightarrow F$
- Roots come out to be: $Ox_1 = -1$; $Ox_2 = -3$

Similarity of $\Delta IOx_1 \sim \Delta FAx_1$; $\Delta IOx_2 \sim \Delta Ax_2F$ give:

$$\frac{IO}{Ox_1} = \frac{Ax_1}{AF} \quad ; \quad \frac{IO}{Ox_2} = \frac{Ax_2}{AF}$$

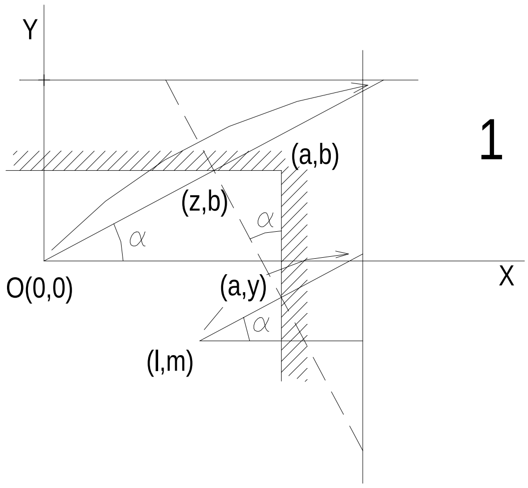


$$\frac{1}{-x_1} = \frac{4-(-x_1)}{3} \quad ; \quad \frac{1}{-x_2} = \frac{4-(-x_2)}{3}$$

In general: $x^2 + 4x + 3 = 0$

7.14.5 RESOLUTION OF THE COMPLETE EQUATION OF THIRD DEGREE (H. H)

First of all we´ll recall Fig. 6 (Point 7.14) to show how the 3rd degree equation is behind it. That figure is now completed with Fig. 1 of present Point 7.14.5



$$\left.\begin{aligned} t &= \mathrm{tg}\,\alpha \\ t &= \frac{b}{z} \\ t &= \frac{a-z}{b+y} \\ t &= \frac{m-y}{a-l} \end{aligned}\right\}$$

Let´s get a t expression just dependent of: balls coordinates (0,0) and (l,m) ; a angle (whose tangent is t); the configuration of billiards table (a,b):

$$bt + ty = a - \frac{b}{t} \quad ; \quad \frac{at - b - bt^2}{t^2} = m - at + lt$$

$$(l-a)t^3 + (m+b)t^2 - at + b = 0$$

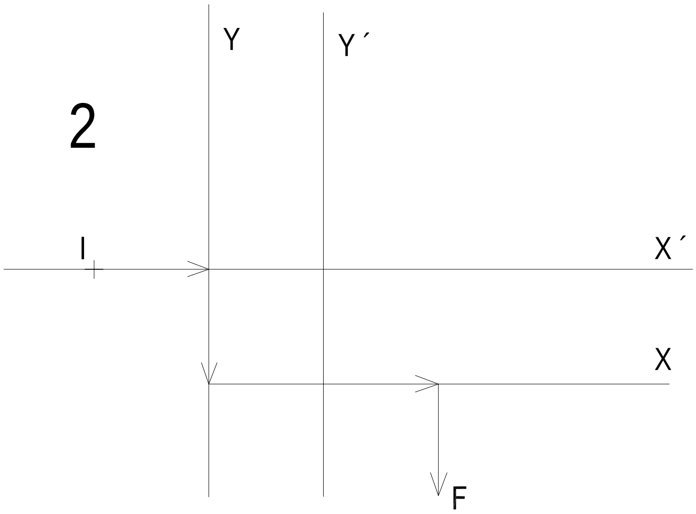
This means that the orientation given to the ball in O in order to hit the other one placed at (l,m) after rebounding orthogonally on both tablesides, is the only real root of the equation just obtained. And that is so because the equation has a positive discriminant, according to drawing scale.

It is important to insist that lines which receive points (0,0) and (l,m) along the folding operation, are parallel to their respective tablesides, and distant from them as much as the balls are distant from said tablesides.

Let´s figure out the three real roots of equation

$$t^3 + t^2 - 2t - 1 = 0$$

that was already solved with a different method in Point 7.13 (Fig. 2). The vector sequence will be: 1;1;-2;-1, as shown in present Fig. 2.



First vector starting at I *will hit side y* in such a way that it asks for line y´. In final folding operation y´ will receive point I (see also 10.3.1, Heptagon).

On the other hand, last vector ending at F *comes rebounded off side x* asking, therefore, for line x ´ to receive point F during folding operation. So the **simultaneous folding** will be:

$$I \rightarrow y\,' \quad ; \quad F \rightarrow x\,'$$

What happens though, is that this folding can be performed in three different ways as shown in

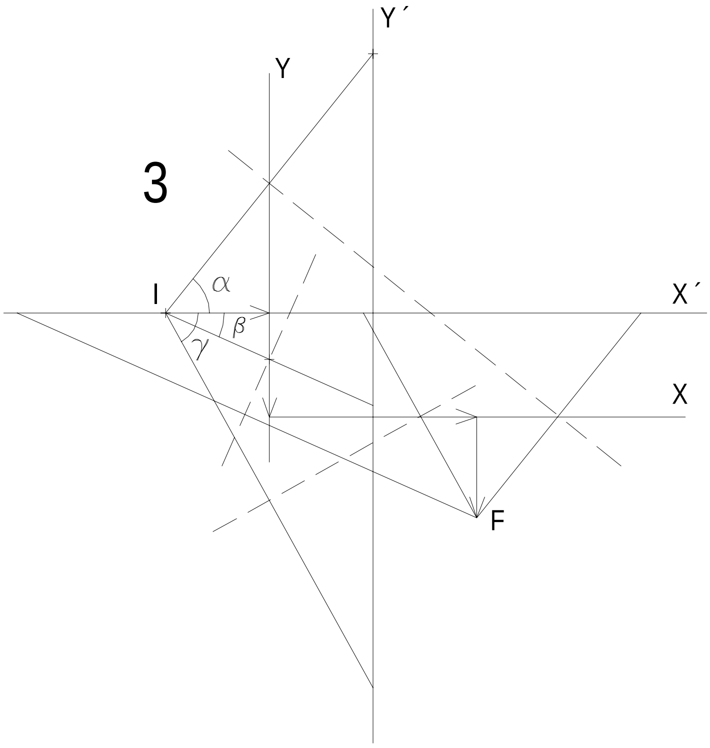


Fig. 3. In it, dashed lines are, as usual, folding lines, and angles a , b , g lead to the solutions of the equation.

You may notice that if we introduce in Fig. 3 the transformation

$$x\,' \rightarrow OX \quad ; \quad y\,' \rightarrow OY$$

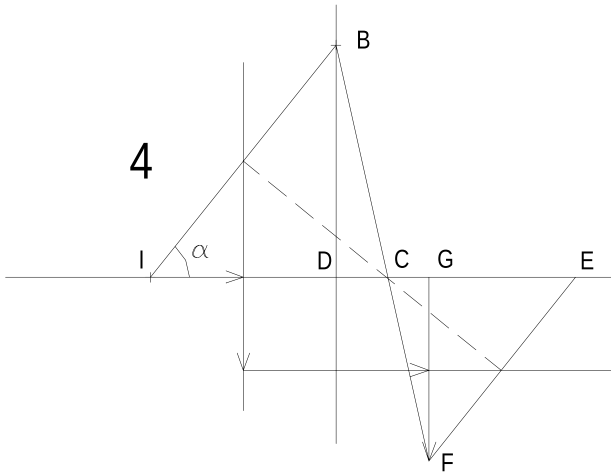
and rotate it 180º around O, Fig. 2 of Point 7.13 is obtained. Both figures give the same results for t:

$$\alpha = 51.2721^{\circ} \qquad \beta = -23.9909^{\circ} \qquad \gamma = -60.9719$$
$$t_1 = \text{tg}\,\alpha = 1.2469 \qquad t_2 = \text{tg}\,\beta = -0.445\,0 \qquad t_3 = \text{tg}\,\gamma = -1.8019$$

Any of them satisfy the equation

$$t^3 + t^2 - 2t - 1 = 0$$

Fig. 4 justifies the association of Fig. 3 to the 3rd degree equation:



ΔIBD; ΔEGF are similar:

$$\frac{BD}{ID} = \frac{FG}{GE} \quad ; \quad \frac{BD}{2} = \frac{2}{ED-1} \qquad (1)$$

ΔIBC; ΔCEF are also similar:

$$\frac{EC}{GF} = \frac{IC}{BD} \quad ; \quad \frac{ED-CD}{2} = \frac{2+CD}{BD} \qquad (2)$$

Equalising $\frac{BD}{2}$ in (1) and (2):

$$\frac{2}{ED-1} = \frac{2+CD}{ED-CD} \quad ; \quad CD = \frac{2}{ED+1} \qquad (3)$$

In ΔIBC we also have (t = tg α):

$$BD = ID\,\text{tg}\,\alpha = 2t \quad ; \quad \text{Ang.BCI} = 180 - 2\alpha \quad ; \quad BD = DC\,\text{tg}(180 - 2\alpha)$$

Then:

$$2t = -DC\,\text{tg}\,2\alpha = \frac{2DC\,\text{tg}\,\alpha}{\text{tg}^2\,\alpha - 1} \quad ; \quad t^2 - 1 = DC \qquad (4)$$

In ΔEGF, Ang.FEG = α, so:

$$t = \frac{GF}{GE} \quad ; \quad t = \frac{2}{ED-1} \qquad (5)$$

Expressions (3), (4), (5) form a t parametric system that will allow us to obtain the 3rd degree equation we are after:

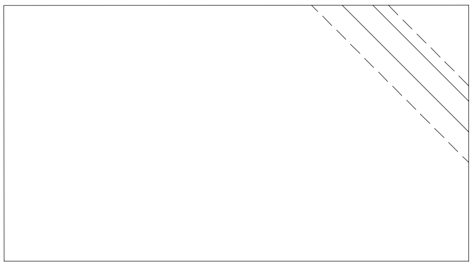
$$\left.\begin{array}{ll} t^2-1=\dfrac{2}{ED+1} \quad ; & ED+1=\dfrac{2}{t^2-1} \\ & ED-1=\dfrac{2}{t} \end{array}\right\}$$

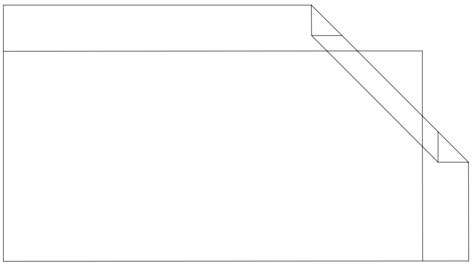
$$2=\frac{2}{t^2-1}-\frac{2}{t} \quad ; \quad (t^2-1)t=t-(t^2-1)$$

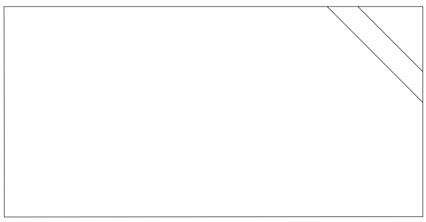
$$(t^2-1)(t+1)=t \quad ; \quad t^3-t+t^2-1=t$$

$$t^3+t^2-2t-1=0$$

Interlude



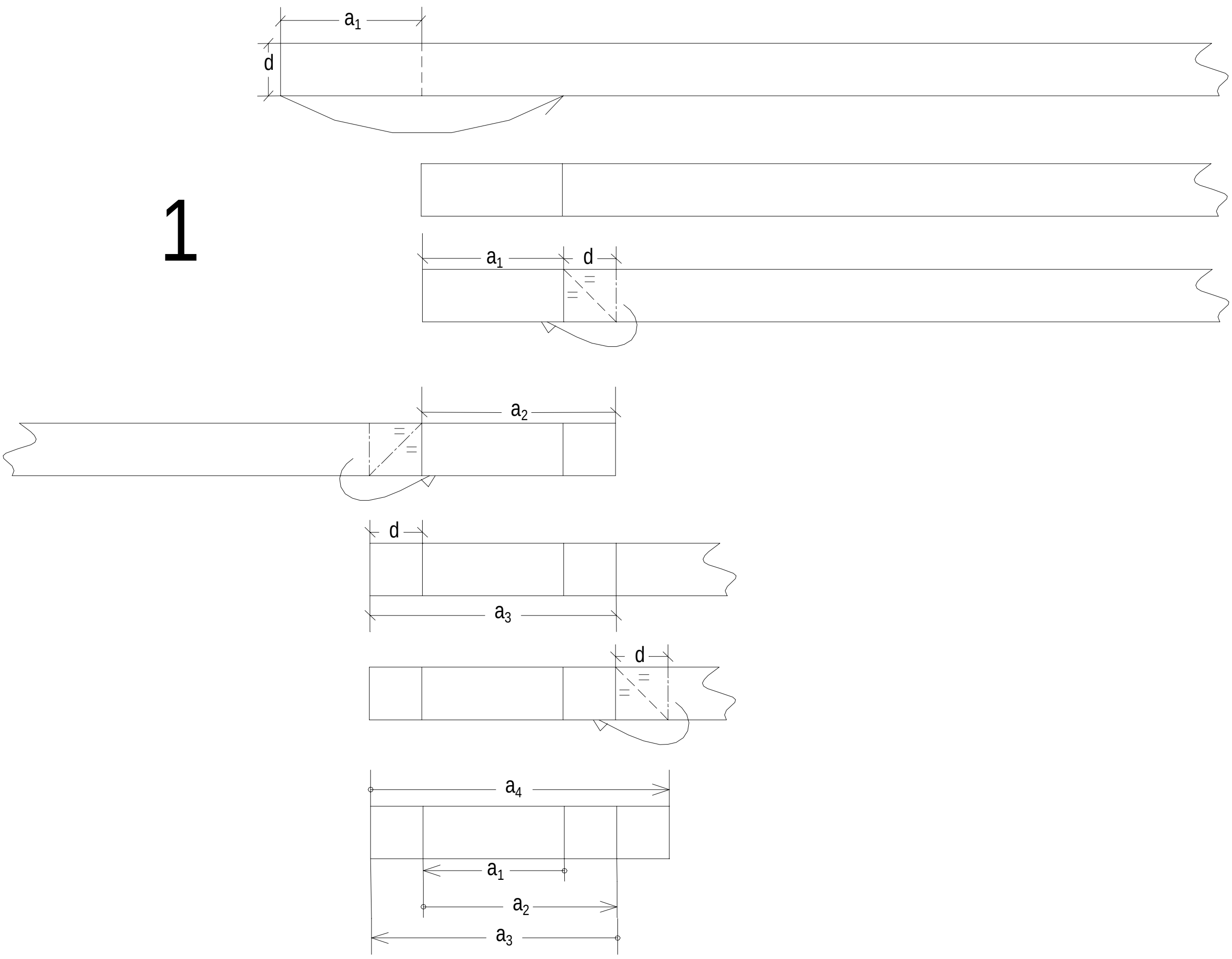




7.15 PROGRESSIONS

7.15.1 ARITHMETIC PROGRESSIONS

They are formed by a succession of quantities such that any of them is equal to the immediate preceding one plus another constant quantity called ratio d. The progression is an increasing one if d > 0 and is decreasing if d < 0.Let´s see an example of the former type.
To build it up by folding we begin with a paper strip of width d and adequate length (see first picture of Fig.1). From its left end we take the progression´s first term a_1. The obtention of a



square (side d) shown in the 3rd picture for the first time, is the key to get successive terms a_2, a_3 and a_4. It is obvious that the only limit to the number of terms is the strip length.

Last picture of Fig 1 shows how the terms of the progression do grow: it looks like a flattened bellows. Just by counting and looking at that picture, the most important properties of arithmetic progressions can be checked.

Last terms´ value:

$$a_n = a_1 + (n-1)d \quad ; \quad a_4 = a_1 + 3d$$

Continuous equidistance between three consecutive terms:

$$a_{i+1} - a_i = a_i - a_{i-1} \quad ; \quad a_4 - a_3 = a_3 - a_2$$

One term as the arithmetic media of its preceding and following terms:

$$a_i = \frac{a_{i-1} + a_{i+1}}{2} \quad ; \quad a_3 = \frac{a_2 + a_4}{2} = \frac{a_1 + d + a_1 + 3d}{2} = \frac{2a_1 + 4d}{2} = a_1 + 2d = a_3$$

Relation between term n and one of its precedent p, being i the number of terms in between the two:

$$a_n = a_p + d(i+1) \quad ; \quad a_4 = a_2 + 2d$$

The sum of two equidistant terms from the extremes, is equal to the sum of these extremes:

$$a_1 + a_n = a_{1+i} + a_{n-i} \quad ; \quad a_1 + a_4 = a_2 + a_3 \quad (i = 1)$$

Sum of all of the terms of an arithmetic progression:

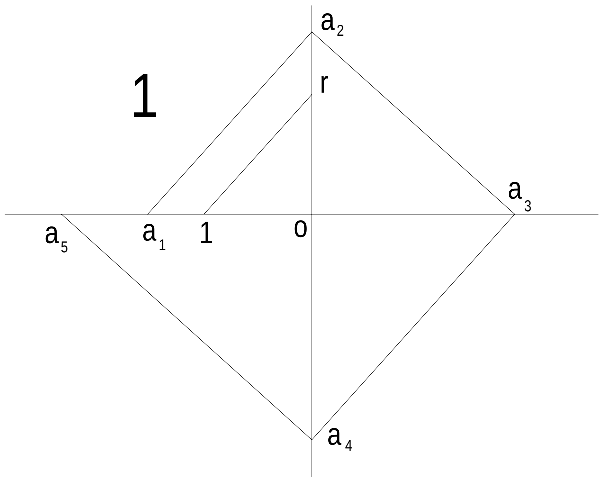
$$S = \frac{a_1 + a_n}{2} n \quad ; \quad S = \frac{a_1 + a_4}{2} 4 = 2a_1 + 2a_4 = a_1 + a_1 + 2(a_1 + 3d) = 4a_1 + 6d$$

7.15.2 GEOMETRIC PROGRESSIONS

They are those in which each term is equal to its immediate precedent multiplied by a constant r called ratio of the progression. Let´s see first, one of the increasing type (r > 1). Fig. 2 in point 7.14.3 is one example. In it, the first term is $a_1 = 1$ and the ratio is a > 1.

If we wish that the first term be $a_1 \neq 1$ (keeping the same ratio r = a) we would have to build an orthogonal spiral parallel to the former one beginning with a_1. By so doing we get Fig. 1 of present point 7.15.2. In it, the value of each term is measured from O to the correspondent a_i. Through the similarity of the triangles shown we can also see that:

1



$$\frac{a_5}{a_4} = \frac{r}{1} \quad ; \quad \frac{a_2}{a_1} = \frac{r}{1}$$

Here we have some properties that can be observed in Fig. 1:

Last term as a function of the first one and the ratio:

$$a_n = a_1 r^{n-1} \quad ; \quad a_5 = a_4 r = a_3 r \times r = a_2 r \times r \times r = a_1 r \times r \times r \times r = a_1 r^4$$

Sum of the n from the first terms of a geometric progression as a function of the first, the last and the ratio:

$S = \dfrac{a_n r - a_1}{r - 1}$; measuring on the figure, at a graphic scale, we have: (1)

$$a_5 = 2.2458 \quad ; \; a_4 = 2.0215 \quad ; \; a_3 = 1.8196 \quad ; \; a_2 = 1.6378 \quad ; \; a_1 = 1.4742$$

$$r = 1.0751 \quad ; \quad 01 = 0.9677$$

The value of r to be taken to the sum formula is 1.0751 / 0.9677 = 1.111

so, it will be

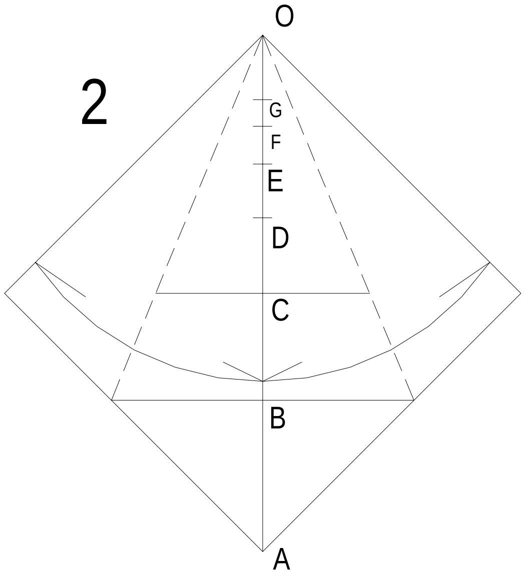
$$S = \frac{2.2458 \times 1.111 - 1.4742}{1.111 - 1} = 9.2$$

on the other hand, if we add up the 5 terms of the progression we also have:

$$\sum_{i=1}^{i=5} a_i = 9.2$$

Now we shall construct, by folding, a decreasing geometric progression (r < 1). See Fig. 2.

We can get straight away points B and C (C is the center of the square): Process to get succes-



sive points is as follows:

Fold	Is got
O $\rightarrow$ B	D
O $\rightarrow$ C	E
O $\rightarrow$ D	F
O $\rightarrow$ E	G
....	

Now we are going to see that segments AB, BC, CD, DE, EF, ... are the terms of a decreasing geometric progression.

If we assume that side of the square is one unit, then:

$$OA = \sqrt{2} \quad ; \quad OB = 1 \quad ; \quad BD = \frac{1}{2} \quad ; \quad DF = \frac{1}{4} \quad ....$$

therefore, the respective segments will measure:

$$AB = \sqrt{2} - 1 \qquad BC = \frac{\sqrt{2}}{2} - AB = 1 - \frac{\sqrt{2}}{2}$$

$$CD = \sqrt{2} - \frac{1}{2} - BC - AB = \frac{\sqrt{2}}{2} - \frac{1}{2} \qquad DE = \frac{\sqrt{2}}{4} - CD = \frac{1}{2} - \frac{\sqrt{2}}{4}$$

$$EF = \frac{1}{4} - DE = \frac{\sqrt{2}}{4} - \frac{1}{4} \qquad ....$$

We may observe that odd segments succession is a geometric progression with ratio 1 / 2. Same happens with even segments.

Let´s calculate the ratio of any two consecutive terms:

$$\frac{BC}{AB}=\frac{1-\frac{\sqrt{2}}{2}}{\sqrt{2}-1}=\frac{\sqrt{2}}{2}$$

$$\frac{CD}{BC}=\frac{AB}{2BC}=\frac{1}{2}\frac{2}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

$$\frac{DE}{CD}=\frac{BC}{2CD}=\frac{\sqrt{2}}{2}$$

$$\frac{EF}{DE}=\frac{CD}{2DE}=\frac{\sqrt{2}}{2}$$

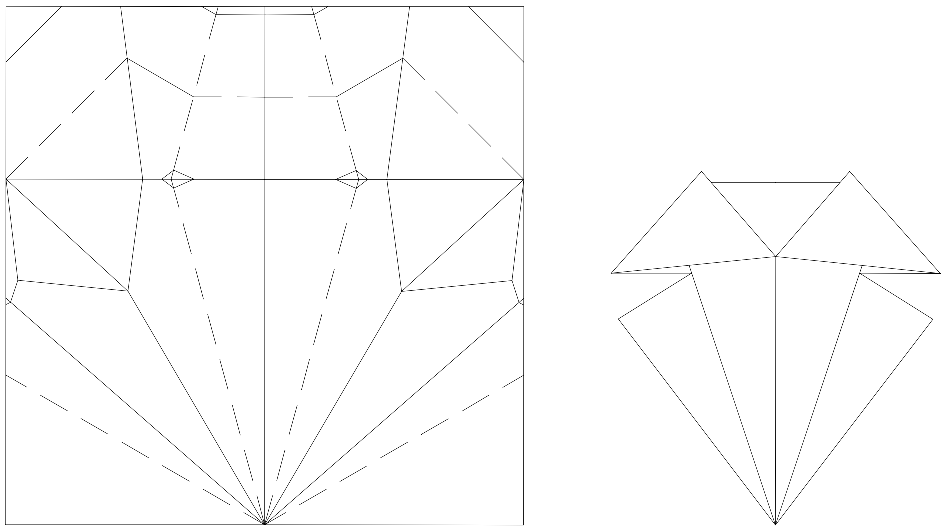
Therefore, $r=\frac{\sqrt{2}}{2}$

Let´s figure out the sum of <u>all</u> the terms of this progression. Applying formula (1) and bearing in mind that $a_n \to 0$, we have:

$$S=\frac{-a_1}{r-1}=\frac{-\left(\sqrt{2}-1\right)}{\frac{\sqrt{2}}{2}-1}=\sqrt{2}$$

Which shows that such sum has equal value than the square`s diagonal, as it should be.

Interlude



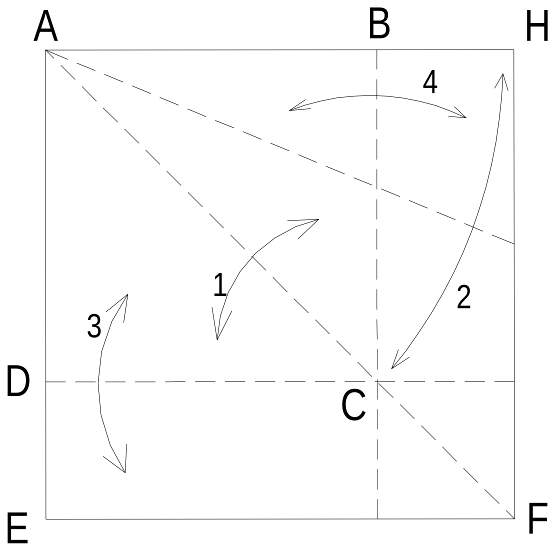
8 SQUARES / TRIANGLES / VARIOUS

8.1 SQUARE WITH HALF THE AREA OF ANOTHER ONE.

8.1.1 FOLDING SOLUTIONS

Solution 1

Produce sequentially the four folds as follows:



1- Fold AF.
2- H → AF; A → A. To get C.
3- EA → EA; C → C. To get CD.
4- HA → HA; C → C. To get CB.

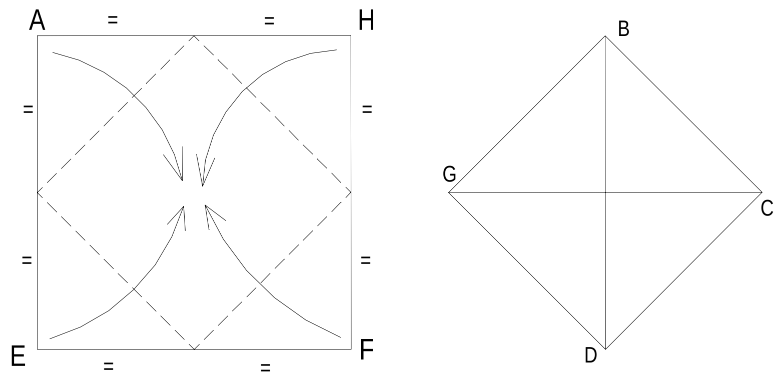
Square ABCD has half the area of AHFE:

area AHFE = AH^2 ; area ABCD = AB^2

$$AB = \frac{AC}{\sqrt{2}} = \frac{AH}{\sqrt{2}}$$

area ABCD = $\dfrac{AH^2}{2} = \dfrac{1}{2}$ area AEFH

Solution 2

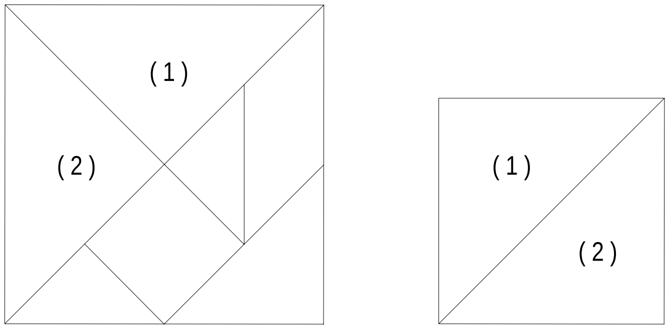


Looking at the folds it´s evident that square BCDG has half the area of square AEFH.

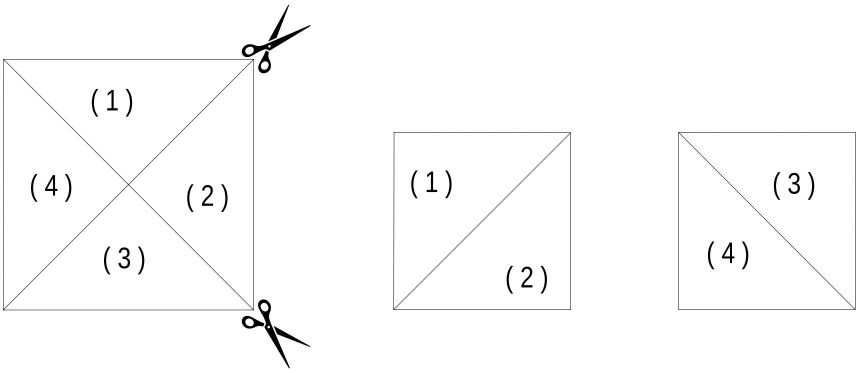
8.1.2 SOLUTIONS BY MEANS OF CUTS

Solution 1 (Tangran)

To build the main square using the 7 *tangran* figures (five right-angled isosceles triangles, one square and one rhomboid). $\Delta(1)$ and (2) are one half of the great triangle, and therefore make up the square solution (to the right).

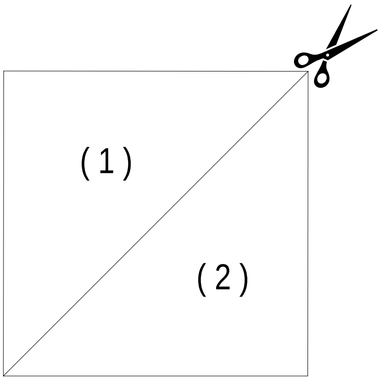


Solution 2 (very simple)



8.2 OBTENTION OF TRIANGLES

8.2.1 ISOSCELES RIGHT ANGLED TRIANGLES FROM A SQUARE



Triangles (1) and (2) are the solution.

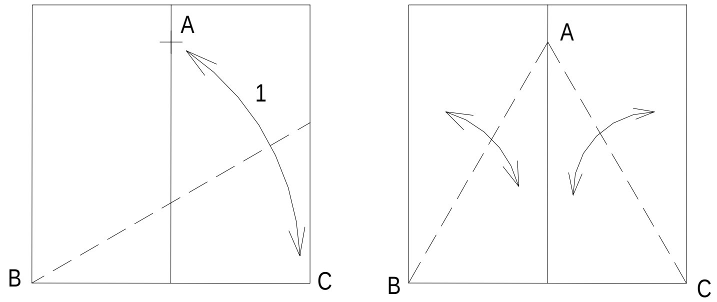
8.2.2 EQUILATERAL TRIANGLE FROM A SQUARE.

8.2.2.1 EQUAL SIDES OF TRIANGLE AND SQUARE.

Triangle ABC is the solution because AB = BC
If DC = 1

$$AE^2 + \left(\frac{1}{2}\right)^2 = 1$$

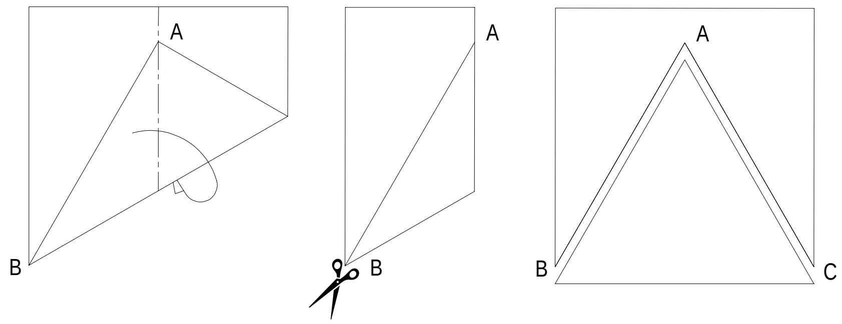
$$AE = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$$



Fold 1 obtains point A:

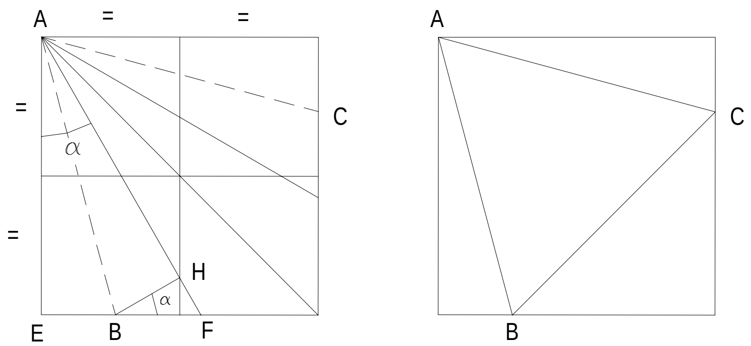
C $\rightarrow$ perpendicular bisector of BC; B $\rightarrow$ B

A version of the latter case consists in cutting the triangle out of the square:



8.2.2.2 MAXIMUM EQUILATERAL TRIANGLE (four solutions)

Solution 1



To produce the crease AB by taking corner E over the perpendicular bisector of the horizontal sides of square: we get B and H. Do the same simmetrically to diagonal through A: we get C. Triangle ABC is the solution.

$$AE = AH = 1$$

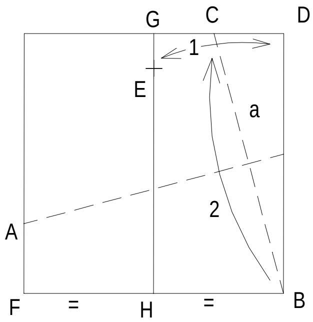
$$\operatorname{sen}\alpha = \frac{\frac{1}{2}}{AH} = \frac{1}{2} \quad ; \quad \alpha = 30^{\circ} \quad ; \quad \text{ang. BAC} = 60^{\circ}$$

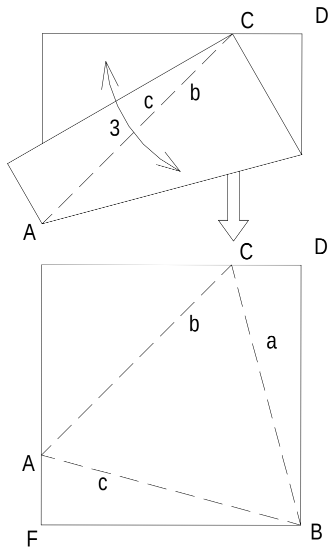
$$AB = \frac{1}{\cos 15} = 1.0352762 > AE = 1$$

Solution 2

Folds:

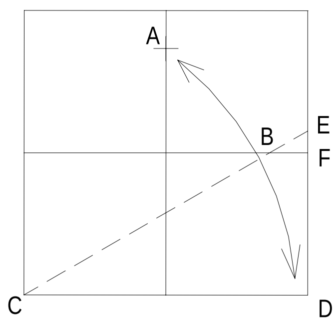
1- D → GH; B → B. We get a and E.
2- B → GD; a → a. We get C and A.
3- Fold AC. We get simultaneously b and c.

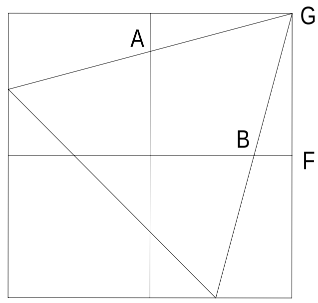




DB = EB = FB : ΔBEF, equilateral.
Then ang. EBF = 60º and CBD = 15º.
In the last figure, obtained after folding the former, and because of the symmetry, we have:
Ang. ABF = CBD =15º; ang. CBA = 90 – 30 = 60º. Therefore ΔABC is isosceles with an angle of 60º: it is equilateral.

Solution 3





CD = 1 ; ΔACD, equilateral ; ang. ACD = 60º ; ang. ECD = 30º

$$DE = \mathrm{tg}\,30 \quad ; \quad \frac{BF}{EF} = \frac{1}{\mathrm{tg}\,30} \quad ; \quad EF = ED - \frac{1}{2} = \mathrm{tg}\,30 - \frac{1}{2}$$

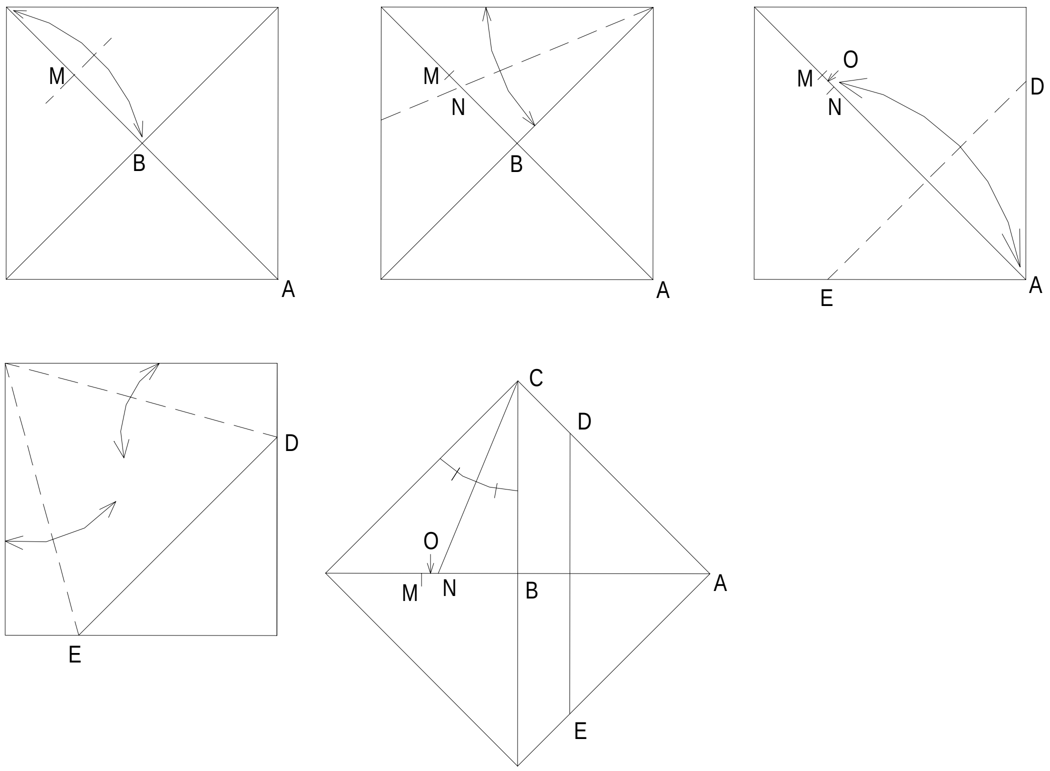
$$BF = \frac{EF}{\mathrm{tg}\,30} = 1 - \frac{1}{2\,\mathrm{tg}\,30} \quad ; \quad angBGF = Arc\,\mathrm{tg}\frac{BF}{FG} = Arc\,\mathrm{tg}\left(2 - \frac{1}{\mathrm{tg}\,30}\right) = 15^{\circ}$$

Solution 4

This solution is not perfect, though very ingenious and close to perfection.

It is a matter of folding M and N, and then, over mid-point O. A perfect solution calls for a point different from O: an undetermined point between O and M.

$$AC = 1 \quad ; \quad CB = AB = \frac{\sqrt{2}}{2} \quad ; \quad angNCB = 22.5^{\circ} \quad ; \quad NB = \frac{\sqrt{2}}{2}\,\mathrm{tg}\,22.5$$

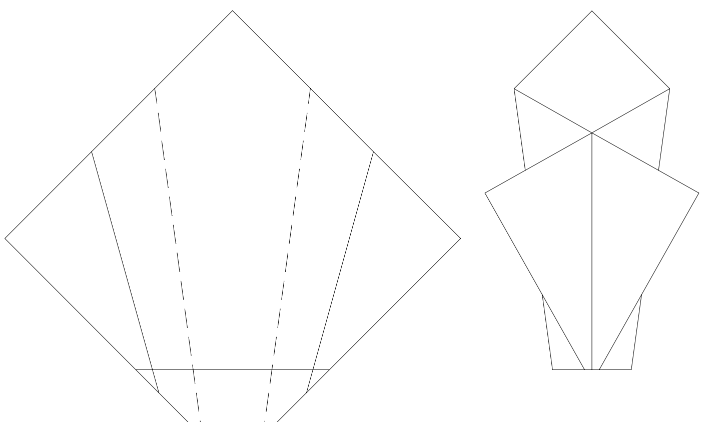


If we take O as the mid-point of MN, we´ll have:

$$OA = \frac{1}{2}\left(AM + AN\right) = \frac{1}{2}\left(\frac{3}{4}\sqrt{2} + NB + \frac{\sqrt{2}}{2}\right) = \frac{1}{2}\left(\frac{3}{4}\sqrt{2} + \frac{\sqrt{2}}{2}\,\mathrm{tg}\,22.5 + \frac{\sqrt{2}}{2}\right) = DE =$$

= 1,0303301 ≠ 1,0352762 (which is the side of a perfect maximum equilateral triangle according to solution 1)

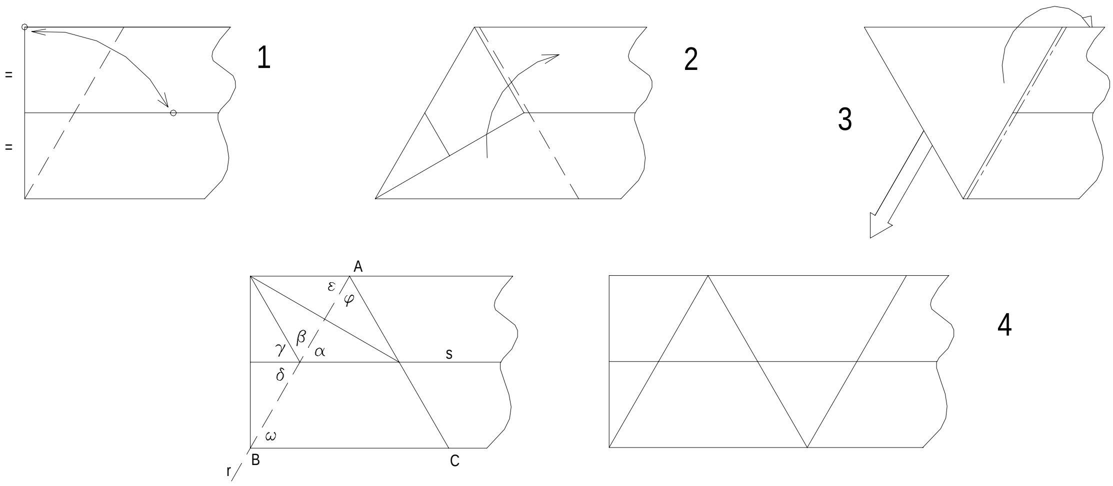
Interlude



8.2.3 EQUILATERAL TRIANGLE FROM A RECTANGLE (three solutions)

Solution 1

The rectangle is a paper strip, and the solution is very useful in hexaflexagons construction.



Justification:

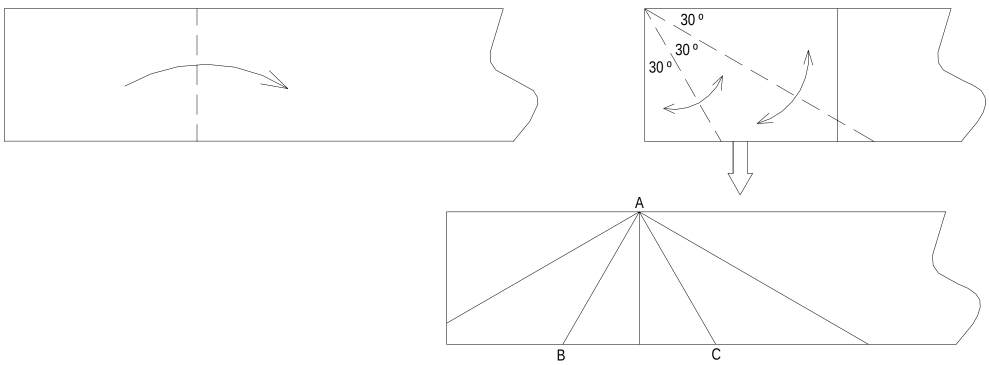
$a = b$ (simmetry r) ; $g = d$ (simmetry s) ; $a = d$ (vertical angles). Then:

$$a = b = g = \frac{180}{3} = 60 \quad ; \quad e = f \text{ (simmetry r)} \quad ; \quad w = a = e. \text{ So:}$$

$$f = w = 60^{o} \text{ and } \Delta ABC \text{, equilateral triangle.}$$

It is impossible to develop a similar construction from a square because AB is greater than the side of the supposed square.

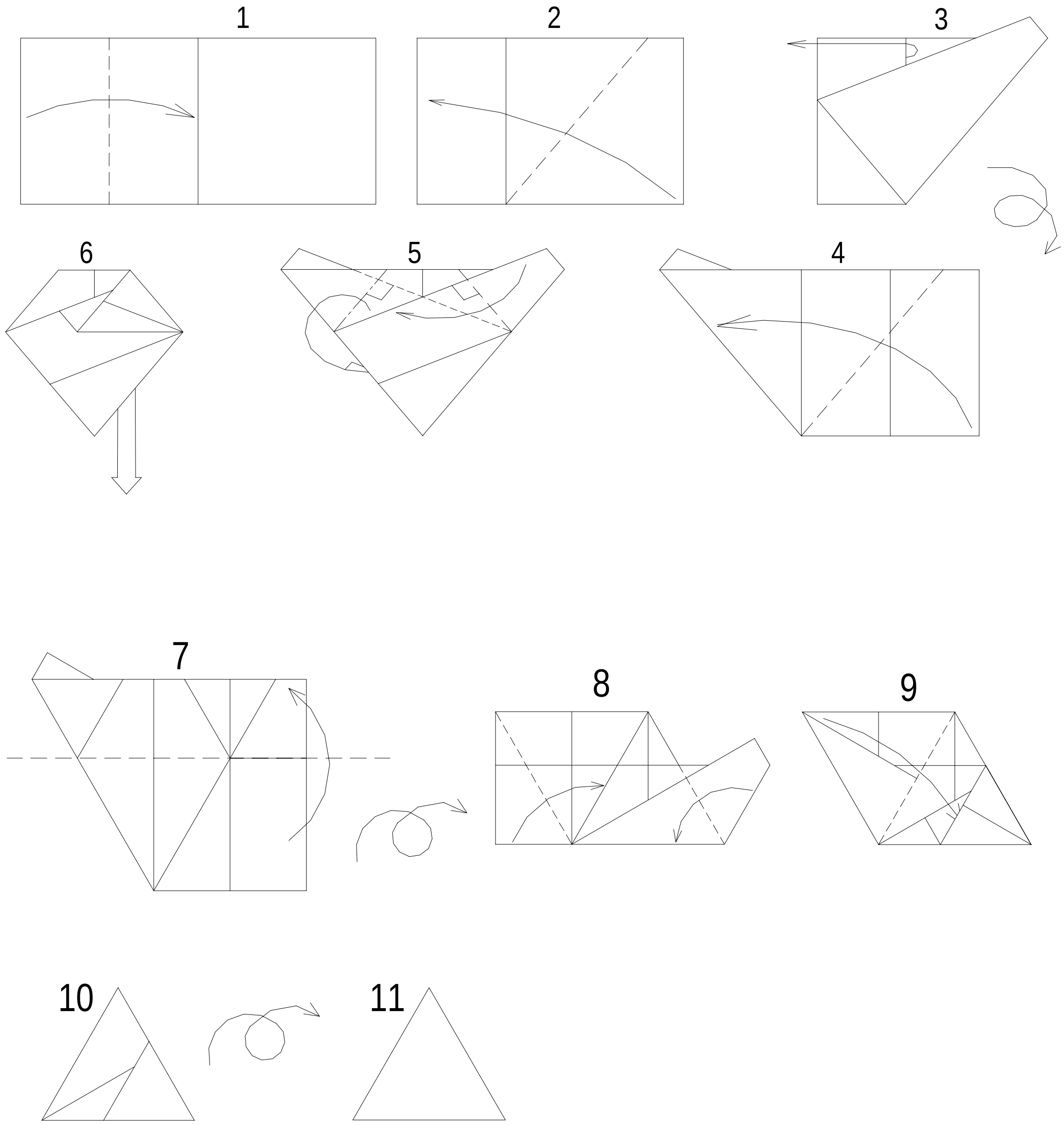
Solution 2



We also begin with a paper strip. The equilateral ΔABC is the solution. The reader may recall that in point 8.2.2.2, *Solution 1,* the right angle of a square was trisected.

Solution 3

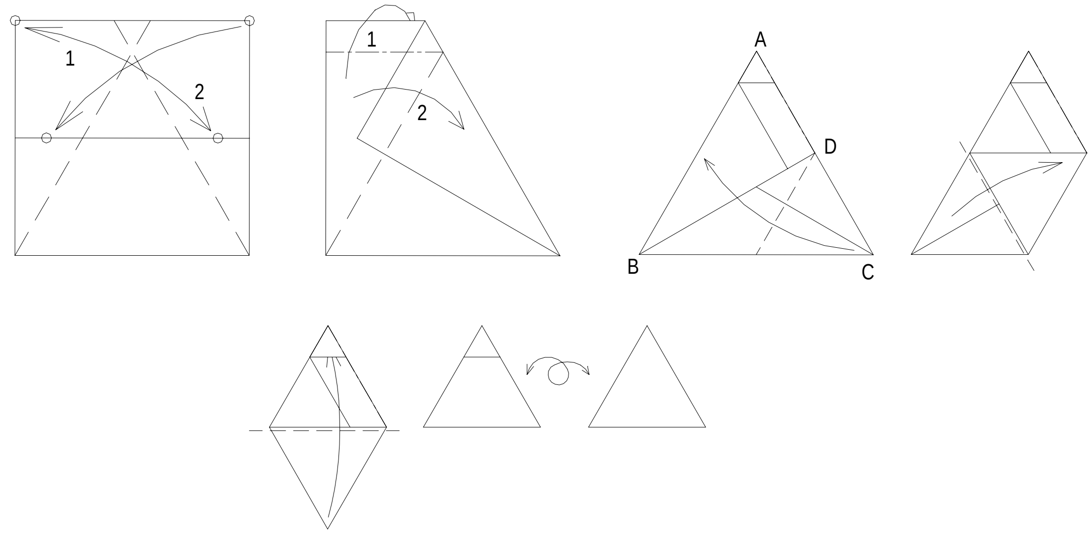
Former *Solution 1* made use of the whole width of the paper strip to allow the construction of a successive and indefinite quantity of equilateral triangles. On the contrary, present *Solution 3* implies the following proportion between a (height) and b (base): $a \geq \frac{\sqrt{3}}{4}b$. If $a > \frac{\sqrt{3}}{4}b$, an equilateral triangle of the *envelope* type (point 8.2.4) is obtained.



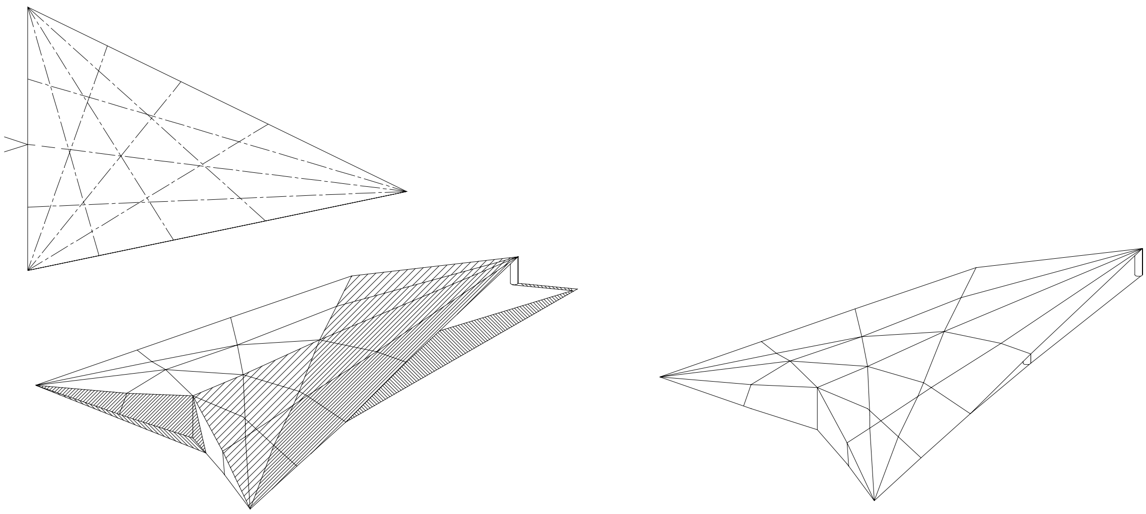
8.2.4 EQUILATERAL TRIANGLE OF THE *ENVELOPE* TYPE

It is evident that the resulting triangle (last in the process) has a side half of that in the starting square. The reader may observe that details of folding are not specified any more; e.g. we have obviated the description of folding in Fig. 3 as:

$$C \rightarrow AB;\ D \rightarrow D$$



8.2.5 STELLATE TRIANGLE

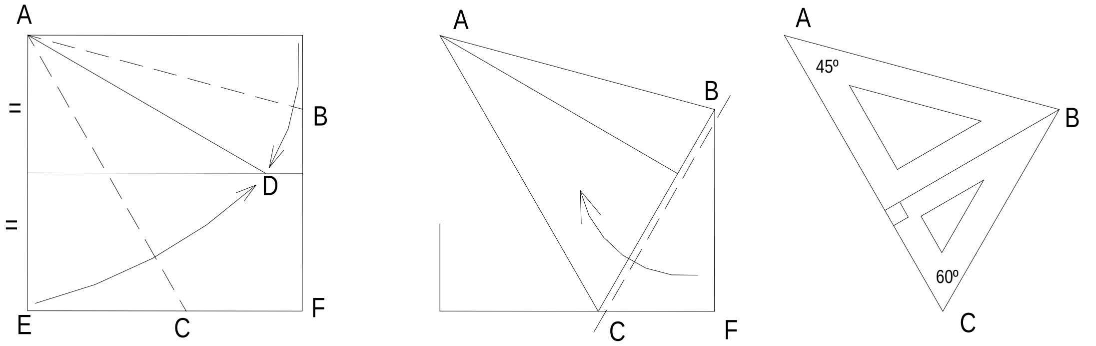


Now is the occasion to recall point 6.9 (stellate rectangle) in which the triangle was also mentioned. Now, lower left figure represents a lit *arquitectonic* stellate triangle; the figure at its right is the same, but unlit.

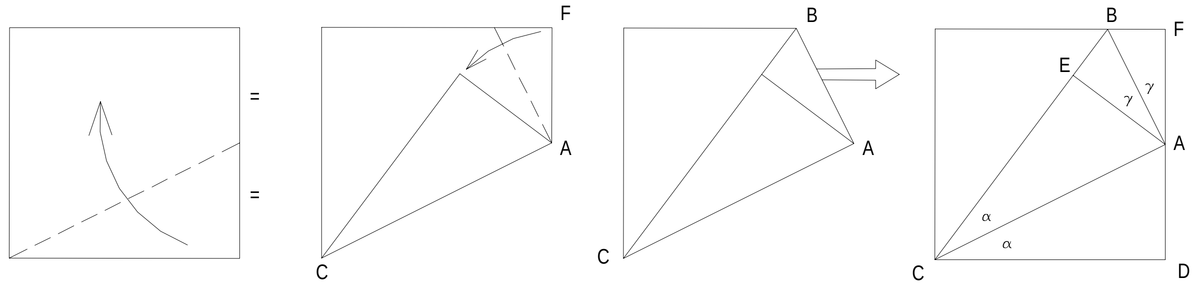
8.2.6 SQUARE / SET SQUARE

ang BAC = 45º (complementary angles bisector)
ang DAE = 60º (equilateral triangle)

ang EAC = 30º ; ang ACE = 60º ; ang ACF = 120º
ang BCF = 60º ; ang BCA= 60º



8.2.7 ANOTHER CURIOSITY



Right-angled triangle ABC is the result of the two successive folds.
To prove it we ought to see first that points B,E,C are in line. Ang BEC is a straight one as it is the sum of angles in F and D, pertaining to the square. Besides, AE = AD = AF. Therefore EA is a unique line perpendicular to BC.

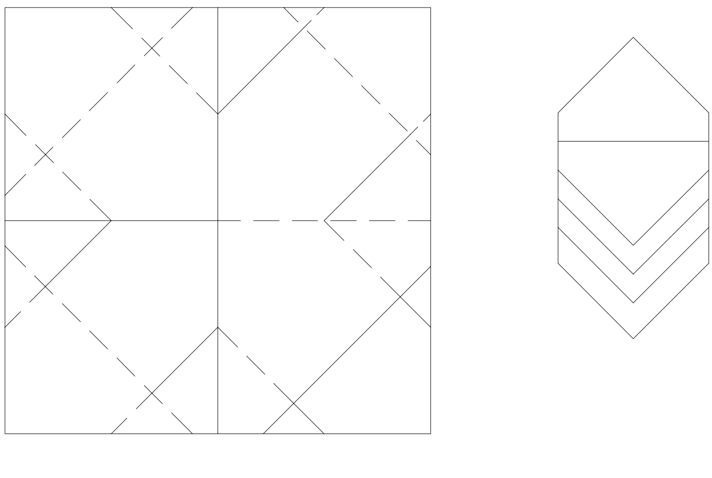
On the other hand, straight angle in A gives:

$$180 = (90 - \alpha)2 + 2\gamma = 180 - 2\alpha + 2\gamma \quad ; \quad \alpha = \gamma$$

At the end we have that ang EAC $= 90 - \alpha = 90 - \gamma$

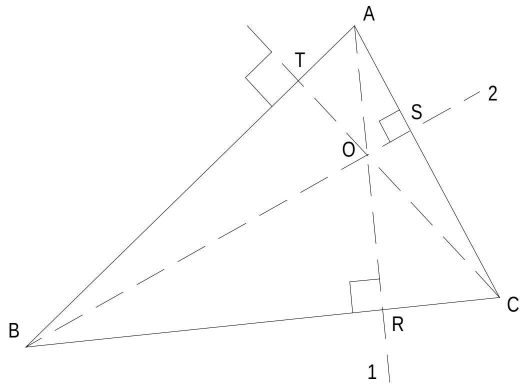
ang BAC = ang EAC $+ \gamma = 90$

Interlude



8.2.8 SINGULAR POINTS IN TRIANGLES
8.2.8.1 ORTHOCENTER

It´s the intersection point of the three altitudes of a triangle. Let´s make these two folds:
1- A $\rightarrow$ A ; BC $\rightarrow$ BC
2- B $\rightarrow$ B ; AC $\rightarrow$ AC



This way, we get intersection point O: it will be the orthocenter if fold OC becomes perpendicular to AB (starting hypothesis).

Demonstration 1

The six angles in O add up to 360º, and they are congruent (taken by pairs) as vertical angles. Moreover, Ang.B = Ang.ROC once their sides are perpendicular (if the hypothesis is fulfilled); Same applies to Angs. A and C.
Therefore it´ll be: 360 = 2A + 2B + 2C ; 180 = A + B + C
which is true because it expresses the sum of the angles of the ΔABC.

Demonstration 2

ΔACR and ΔBCS are similar (right-angled with Ang. C in common)
ΔCBT and ΔABR ,, (if the hypothesis is fulfilled)
ΔBAS and ΔCAT ,, ,, ,,
From all it is born that:

$$\frac{AC}{BC} = \frac{AR}{BS} \quad ; \quad \frac{CB}{AB} = \frac{CT}{AR} \quad ; \quad \frac{BA}{CA} = \frac{BS}{CT}$$

or its equivalent:

$$AC \times BS = BC \times AR \quad ; \quad CB \times AR = AB \times CT \quad ; \quad BA \times CT = BS \times CA$$

There will be orthocenter if the former three equalities hold true. And they do, because each one of them is equivalent to twice the area of ΔABC.

8.2.8.2 CIRCUMCENTER

Is the center of the circumference passing through the three vertices of a triangle: it coincides with the intersection of the three perpendicular bisectors of its sides.
Let these folds:

A $\rightarrow$ B (produces the perpendicular bisector c)
C $\rightarrow$ A (idem b)
B $\rightarrow$ C (idem a)

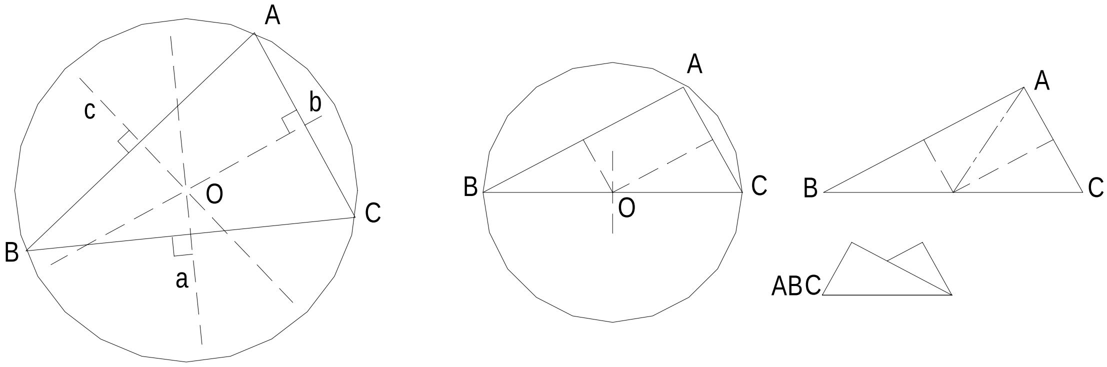
If O is the intersection of a and b, symmetry gives:

OB = OC ; OC = OA

hence:

OB = OA (fold c passes through O)

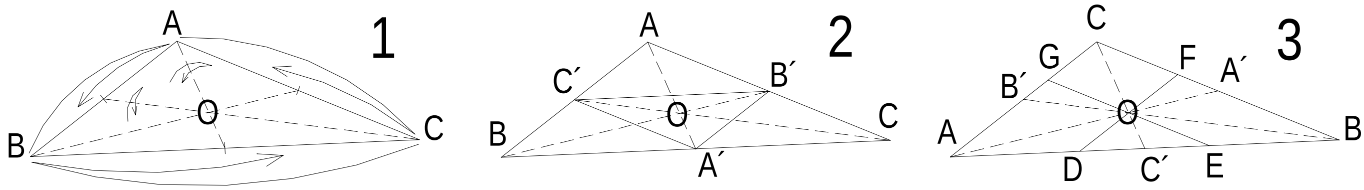
That is, O is equidistant from A, B, C; in consequence O is the center of the unique circumference passing through the three vertices of the triangle.



If ΔABC is a right-angled one, its circumcenter is placed at the mid-point of its hypotenuse. Distances form the circumcenter to the mid-points of legs are equal to half of those legs (Thales´ theorem). Last figure shows how the triangle can be flattened.

8.2.8.3 BARICENTER

Is the intersection point of the three medians of the sides of a triangle.



That´s the reason why it is also its center of gravity: The c.o.g. must be on each one of the medians and, being unique, it has to be placed over the intersection of the three. It is easy to see that the median divides a triangle in another two of equal area (equal bases and same height), hence, of equal weight (in Greek, βαρος = weighty).

To get O (fig. 1), fold mid-point of each side, then mid-point / opposite vertex.

Fig. 2 demonstrates an important property of baricenter. Uniting side mid-points we produce ΔA´B´C´ which is similar to ΔABC (Thales´ theorem). Once demonstrated the existence of baricenter O of great ΔABC, O is also the baricenter of small ΔA´B´C´. Because of the similarity of those triangles, it is: $OC' = \frac{1}{2} OC$ which proves that the baricenter is distant from a vertex twice as much as from the mid-point of the opposite side.

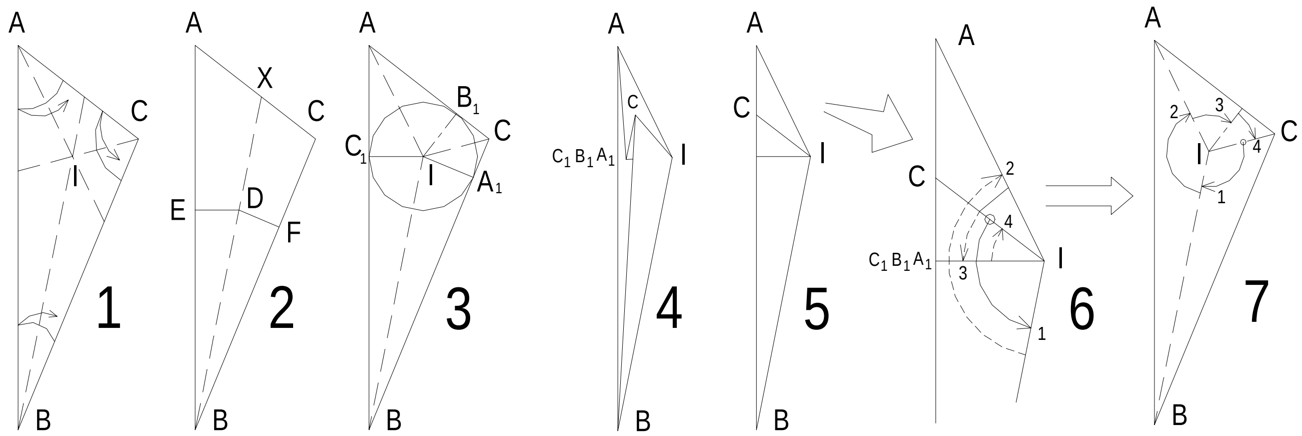
This remarkable property serves to divide a segment in three equal parts. Though this matter has its natural place when dealing with division in equal parts, I prefer to develop it now to profit of its background.

Let segment AB (fig. 3). Mark any point C and get the baricenter O of ΔABC. Fold respective parallels to BC and AC through O: O is the mid-point of DF and GE. Being similar ΔABC and BDF, if B´ is the mid-point of AC in the former, O will be also the mid-point of DF in the latter. Same demonstration applies to GE. On the other hand, ΔABC and AGE, are also similar: D is the homologous of C´, hence D is the mid-point of AE. A similar reasoning leads to E as the midpoint of DB. Therefore

$$AD = DE = EB = \frac{1}{3} AB$$

8.2.8.4 INCENTER

Is the intersection of the three bisectors of a triangle.



1- Fold the three bisectors to get I.
2- Equal distances DE and DF from a point D of a bisector BX, to corresponding sides. (ΔDEB and ΔDBF are equal: right angled with equal angles in B and common side BD).
3- Hence, equal distances from I to the three sides. One of the most important geometric properties of origami is that radii IA_1 IB_1 IC_1 I can be folded from the incenter to the respective sides.
4- FUSHIMI´S THEOREM OF INCENTER: Any triangle folded through its incenter up to their vertices and along a radius such as IB_1 of fig. 3, becomes a flattened figure.
5- There it is the resultant figure.
6- Is an enlargement of 5 to justify, together with 7, the flattening process. Angles 1, 2, 3, 4 are taken in the order they have been produced, so their sum will add up to zero, for the end falls over the beginning:

$$\text{Angs. } 1 - 2 + 3 - 4 = 0$$
$$\text{Angs. } 1 + 3 = 2 + 4$$

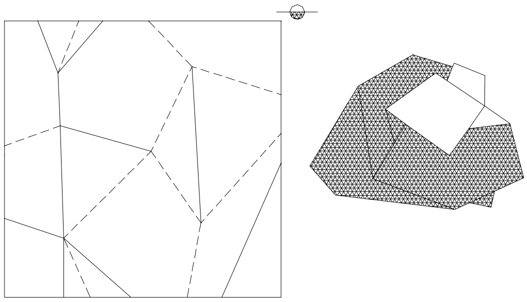
7- Unfolding fig. 6, it is:

$$\text{Angs. } 1 + 2 + 3 + 4 = 360^\circ$$
$$\text{Angs. } (1 + 3) + (2 + 4) = 360^\circ$$
$$\text{Angs. } 1 + 3 = 180^\circ \quad ; \quad \text{Angs. } 2 + 4 = 180^\circ$$

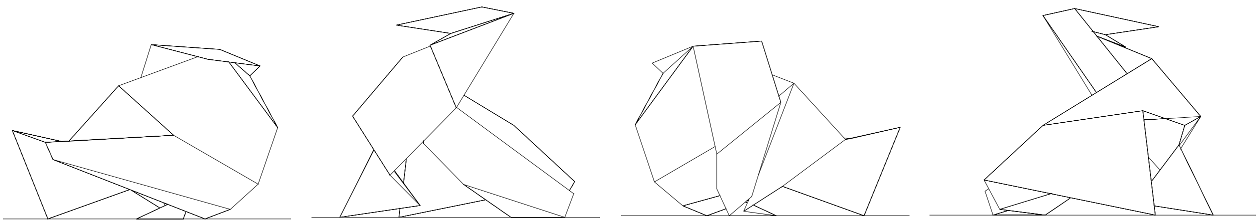
8.2.8.5 RUMPLED AND FLATTENED ORIGAMI

The first thing to ask is if this matter has anything at all to do with triangle singular points. The answer is yes: it is based upon Fushimi´s theorem.

FLATTENIG CONDITION OF A FOLDED FIGURE: To fold flat a figure around a node it is a necessary condition (but not sufficient) that the angles having their vertices on the node and their sides being the corresponding fold lines, are supplementary taken in alternate order. The other condition is that, when rumpling or flattening is produced, the paper will not interfere within itself.



The square at left (white obverse and obscure reverse) was rumpled fortuitously in the hand; before total flattening, creases were oriented to became straight lined folds; only then we obtained the flat figure (to right).



The four figures shown together are solid views (to the four cardinal points) of rumpled paper held in the hand before flattening (the drawing does not keep the same scale for all of them).

It is curious to confirm now the coincidence with Kasahara Kunihiko´s conviction regarding to the fact that hazard, and imagination together can force creativity: other wise, look at the birdlike figures that rumpled folding has provided without any intention at all.

Now, let´s undo the way: if we unfold the flattened figure, the square will show up with all the nodes and folds (mountain and valley). Then we can measure the angles around the nodes to check that they are supplementary (taken in alternate order), according to Fushimi´s theorem.

This theorem was demonstrated for 4 concurrent angles but it does not exist any limitation: the square we are dealing with has one node of 6 vertices. Obviously, to keep angles alternation, the condition is that its number must be even. If it happens to be an odd number, one of the folds will be useless. Such is the case with some bass-relieves in certain complicated tessellations.

The other necessary condition, i.e. paper not interfering within itself, was enforced while straightening folds. Of course, it was a matter of simplification, for any crease can be transformed in a broken line, but I wanted to avoid undesirable complications.

Last condition that rather is a consequence of the others, may be enunciated (theorem 4 of J. Justin): The difference between mountain and valley folds emerging from a node in a flattened construction equals ±2. It can be checked in any of the six nodes within the mentioned square.

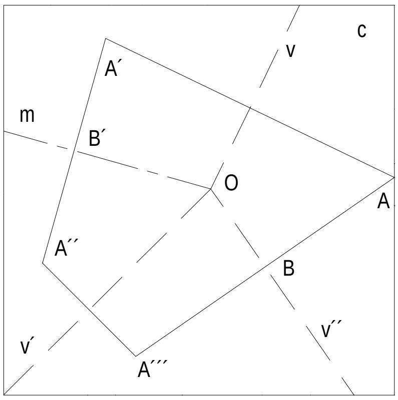
To the four flattening conditions [(even amount of concurrent angles, supplementary alternate angles, paper impenetrability and ±2 (mountain – valleys)], is to be added a fifth one: the compatibility of the four.

To fold flat around O, we have these possibilities:

m	v	v´´	v´	
— — —	—·—	—·—	—·—	flattenable
—·—	— — —	—·—	—·—	not flattenable
—·—	—·—	— — —	—·—	not flattenable
—·—	—·—	—·—	— — —	flattenable
—·—	— — —	— — —	— — —	flattenable

Fig. 1 is an enlargement of central node O in the square at left, with its four folds extended up to the sides of the square c. In said Fig.1 point A was marked to see what will happen to it with the different folds: v transforms A to A´; m transforms A´ to A´´; v´, A´´ to A´´´, and finally, v´´ gives back A´´´ to A.

1



This means that the product of several symmetries with respect to some axles (v, m, v´, v´´) concurrent in O and making a perigon, is equivalent to a rotation of 360º around O (in this case, concerning point A).

Let´s fix now our attention in the transformation of A to A´´. The conclusion is this: The product of two symmetries with respect to axes (v; m) is a rotation of an angle double of that formed by the two axes. This is because Ang. AOA´´ = AOA´ + AÓA´´ = 2Ang. vm.

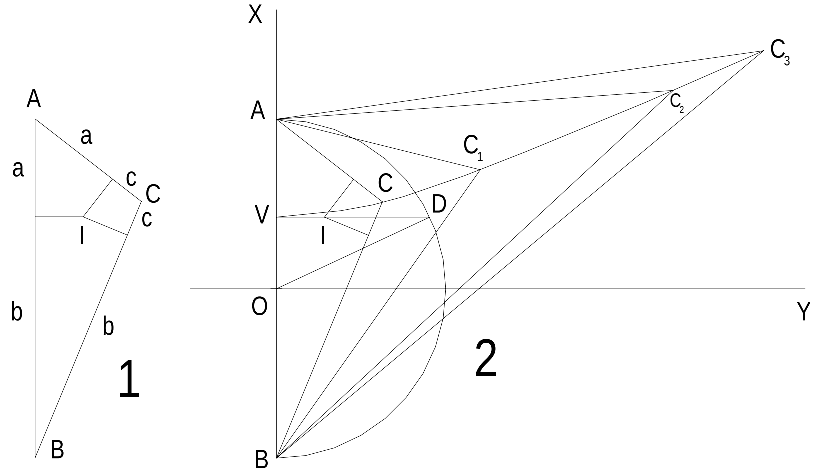
As soon as we begin to fold c, O is not coplanar any more with the four points A, but still is the center of the sphere containing them. That sphere is the same all the time: center O and radius OA. The 4 points A, are <u>always coplanar</u> but situated in different attitudes according to folding progression.

When c is folded flat, points A, A´, A´´ y A´´´ are coincident. <u>In any previous position</u> lines AA´ and A´´A´´´ meet forming a plane whose traces in c are ABA´´´ and A´B´A´´.

8.2.8.6 INCENTER AND HYPERBOLA

From the incenter theorem, Toshiyuki Meguro got at the necessary development to create folding bases to allow the construction of different figures. It is not our intention now to go deep into that development (see comprehensive article by Aníbal Voyer in Nº 68 of "PA-

JARITA", the bulletín of the AEP entitled *Introduction to creation*). On the contrary, we are going to recall some of its geometric grounds.



Each side of the triangle in fig. 1 has been divided in two parts by projecting incenter I over those sides.

Thus:

AB = a + b

BC = b + c

CA = c + a   By subtracting the two last expressions:

BC – CA = b – a

so:

$$\left.\begin{aligned} b + a &= AB \\ b - a &= BC - CA \end{aligned}\right\} \quad (1)$$

To fold a certain figure, say a quadruped, a previous task is to get an adequate folding base. There exist many of them as "pre-designed", but sometimes we will need to build one of our own according to specific requirements.

A folding base is, at the end, a triangulation of the starting paper (usually a square), by means of mountain and valley folds.

The requirements mentioned above may refer to figure tips, keeping due proportions: tail, legs, snout, ears, horns, etc.

Thus, what we really know is the pair of segments a, b in side AB (fig. 1) rather than the two other sides CA, CB; a / b must keep the tips proportion already mentioned.

Therefore, in ∆ABC we only know a, b and the expressions of system (1), i.e., side AB and the fact that vertex C is situated on the hyperbola with focuses A and B, because the distance difference from C to B and A is a constant with the known value of b – a (fig. 2).

Of course, the hyperbolas´s branch will pass through V (its vertex) for VB - VA = b – a. Its center O, is the mid-point of AB.

To draw the hyperbola in the Cartesian plane OXY we´ll start up with its equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (2)$$

such as ($a \neq$ a; $b \neq$ b): a = VO   ; b = VD   ; c = AO = OD

so it´ll be $c^2 = a^2 + b^2$

The only thing left is to assign values to x in (2) to get the corresponding values of y

$$y = \frac{1}{a}\sqrt{b^2x^2 - a^2b^2}$$

This way we have got points on the hyperbola such as C_1 C_2 C_3, besides C and V.

The design of a folding base consists of finding wanted point C as the intersection of a branch of one hyperbola (fig. 2), and another branch obtained similar way.

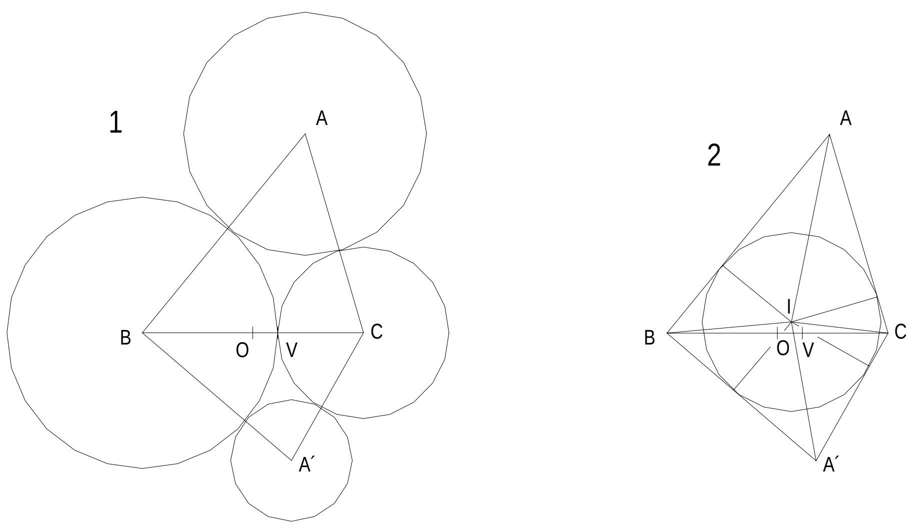
We must add that VD, perpendicular to AB at V is the locus of the incenters of triangles with vertex C_i on the hyperbola. That is evident in fig. 1, but is also true for any other triangle because AB is fixed.

8.2.8.7 FLATTENING OF A QUADRILATERAL

Point 8.2.8.4 showed how to flatten a triangle through its incenter. Now, the association of incenter and hyperbola (point 8.2.8.6) will allow us to fold flat a quadrilateral under these conditions:

> two of its opposite vertices must be on the same branch of a hyperbola whose focuses are the other two vertices of said quadrilateral.

Let focuses be the vertices B and C (Fig. 1), O the center of the hyperbola and OV = 2*a*



(*a* being the parameter of that hyperbola).

To obtain the other two vertices A, A´ of the quadrilateral, let´s draw the circumferences with centers C, B, and radii CV and BV, respectively. Then, draw arbitrary cicumferences simultaneously tangent to the other two circles: their centers A and A´ will seat on the hyperbola and therefore are the other pair of wanted vertices of the quadrilateral.

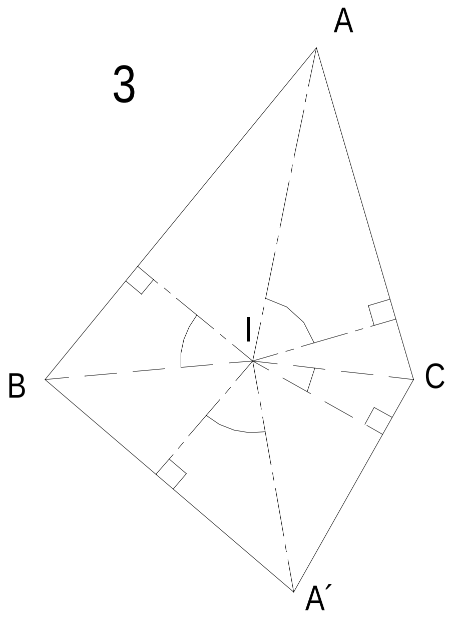
It is so because for any of them (e.g., A), it is:

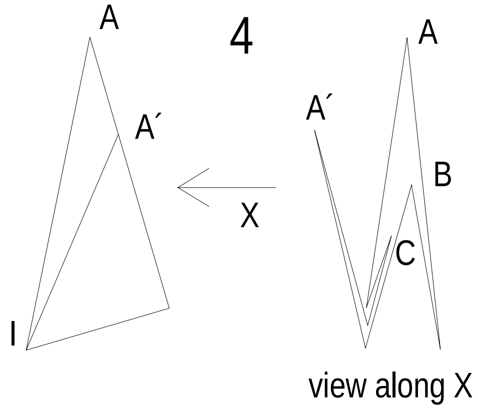
AB – AC = VB – VC = 2VO for $OV = BV - OB = BV - \frac{(BV + VC)}{2} = \frac{BV - VC}{2}$

The quadrilateral ABA´C so constructed, shows in Fig. 2 the four bisectors of its angles: they are concurrent in I, the quadrilateral incenter.

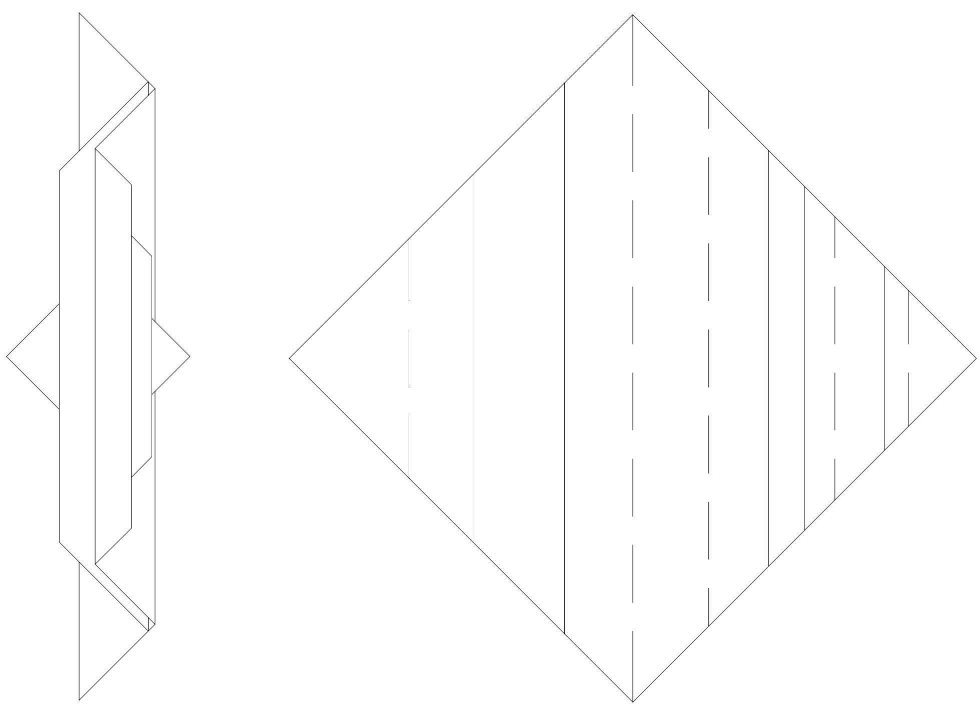
Points A y A´ are not only points of the hyperbola: they also are the tangency points on it, of lines AI and A´ I, for they are the bisectors of radius vectors AB, AC; A´C, A´ B. On the other hand, in point 13.3.4 (Poncelet´s theorem) will be demonstrated that IC is the bisector of angle A´CA.

Thus, mountain and valley folds in Fig. 3 are what we need to get a flat-folded figure. We may observe that the four marked angles are supplementary, for the double of their sum adds up to 360º. Fig. 4 shows the flattened quadrilateral with a view of its internal folds.





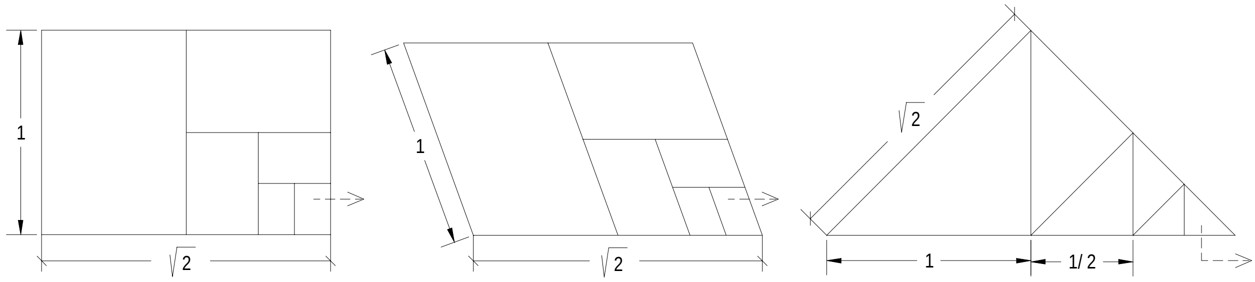
Interlude



8.3 VARIOUS

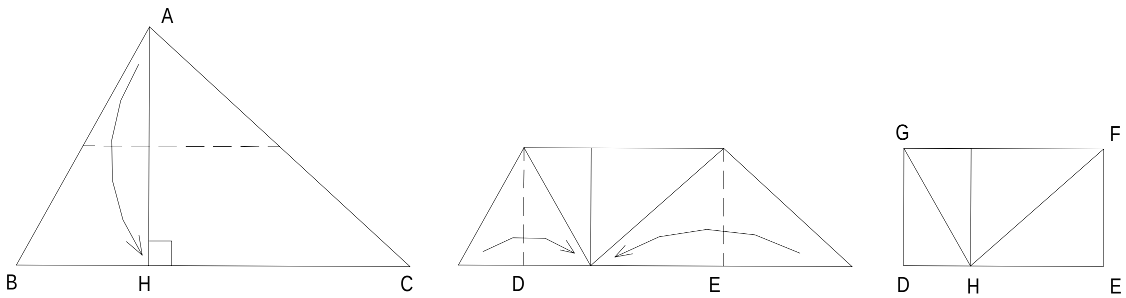
8.3.1 HOMOTOMIC FIGURES

Din A | Pseudo Din A | right-angled isosceles Δ



If we divide successively by 2 their respective areas, the original proportion of figures is kept. The ratio of similarity is $\sqrt{2}$ and therefore, the ratio of areas is 2. Of course, these divisions may be made with conventional instruments, but also by folding.

8.3.2 AREA OF A TRIANGLE



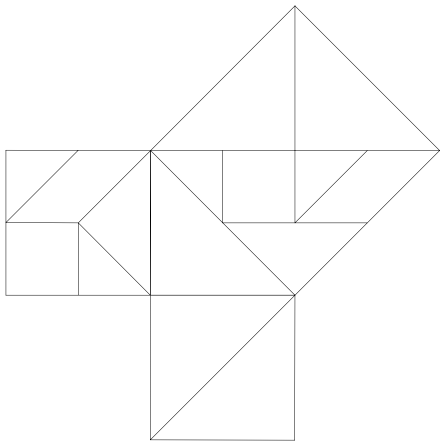
Two areas like DEFG are equivalent to ABC, so:

$$\text{area ABC} = 2\times DEFG = 2DE\times EF = 2\times\frac{1}{2}BC\times\frac{1}{2}AH = \frac{1}{2}BC\times AH$$

8.3.3 PYTHAGOREAN THEOREM

1 Isosceles right triangle.

The Tangram structure was already seen in *solution 1* of Point 8.1.2. Let´s take now two equal tangram sets, one for the hypotenuse and the other to share by the two legs; it is evident that the square built over the hypotenuse has equal area than the sum of the squares over the two legs. It is also clear that this explanation is valid only for an isosceles right triangle.



2 Escalen right triangle

Tangramlike demonstration of Fig.1 is correct when $b=\frac{4}{3}c$ ($a=\frac{5}{3}c$). It shows that the width of the central strip in square on a has equal area than the square on c.

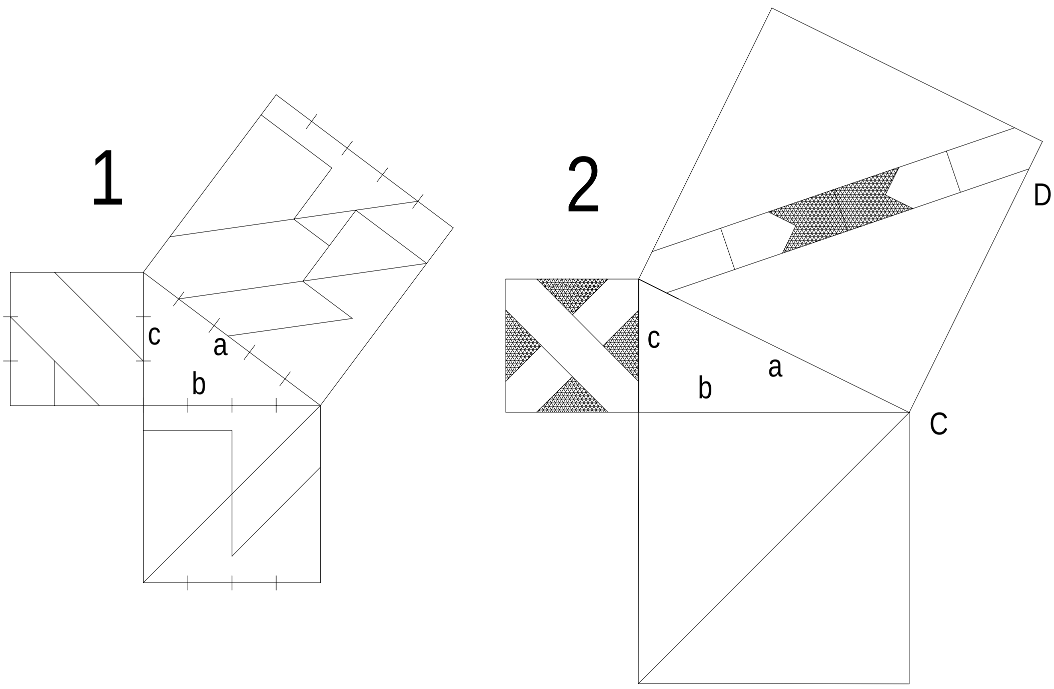


Fig. 2 (any right triangle) shows the difficulty to divide the square on c in pieces in such a manner to cover the central strip of square on a. Been CD = b, we can get the strip´s width h:

$$c^2 = 2\frac{a\sqrt{2}+b\sqrt{2}}{2}\frac{h}{2} \quad ; \quad h=\frac{\sqrt{2}\times c^2}{a+b}$$

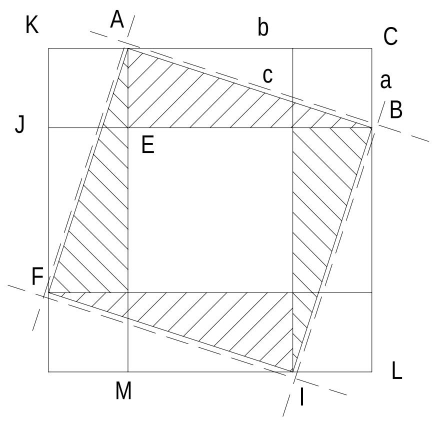
On the other hand, the value of h in the strip over a is:

$$h=\sqrt{2}(a-b)$$

Equalising both values of h we get what we know as the Pythagorean enunciation: $c^2 = a^2 - b^2$

To tangram the shaded areas of square on c to fit the other shaded area is theoretically feasible but hard to do in practice.

3 Any scalene right triangle ABC



From ∆abc we´ll construct the exterior square with side CK = a+b. Then we fold that square to get the interior square ABIF. If the Pythagorean theorem is fulfilled for ∆ABC, it follows that the area of square ABIF will be equal to the sum of AKJE and EBLM.

The square of side c is made up by 4 triangles like ABC plus the central square with side b + a – 2a = b - a.

At the same time, the square of side b is made up by said central square, two ∆ABC and a rectangle with sides a, b – a.

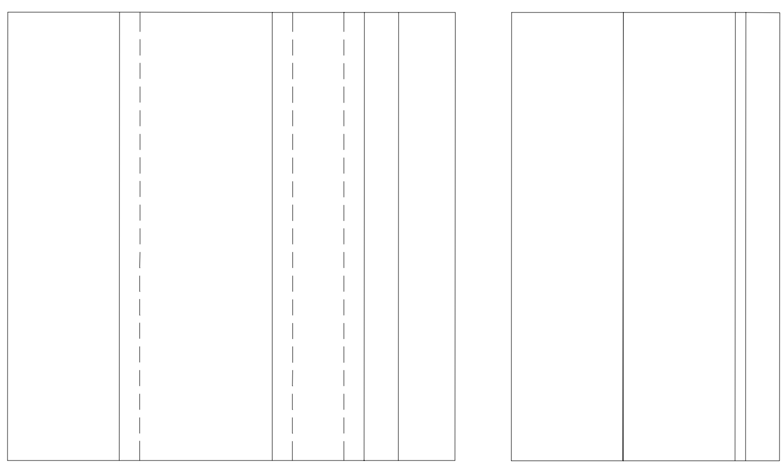
Therefore, the difference between squares with sides c and b is:

$$4\Delta ABC - 2\Delta ABC - a\,(b - a)$$

$$2\frac{ab}{2} - a(b-a) = ab - ab + a^2 = a^2$$

i.e. the difference between the square built on the hypotenuse and that constructed on the great leg, is the square on the small leg: this is also a way of enunciating the Pythagorean theorem.

Interlude



8.3.4 PYTHAGOREAN UNITS

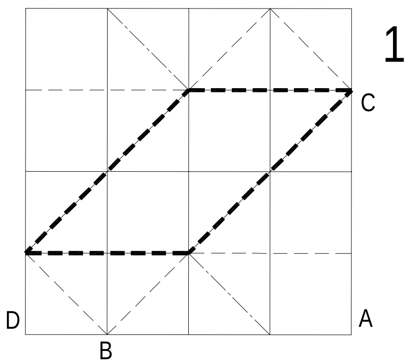
Demonstrations made in point 8.3.3 for the Pythagorean theorem are adequate for cutting and folding. Now we are going to see Toshie Takahama´s demonstration by means of a rhomboidal unit. In turn, that unit is combined with some others identical to it to form the partial surfaces that eventually will integrate the three squares associated to the right triangle.

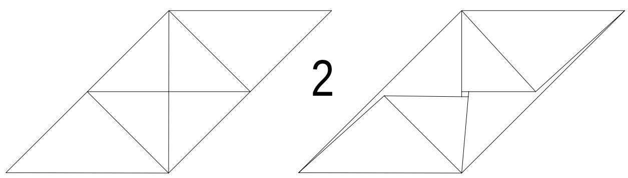
In the present case, folding only is used. It is a rigorously geometric construction and has an added pedagogic value, for 36 rhomboidal units being required, it allow a very interesting teamwork.

These observations must be made:

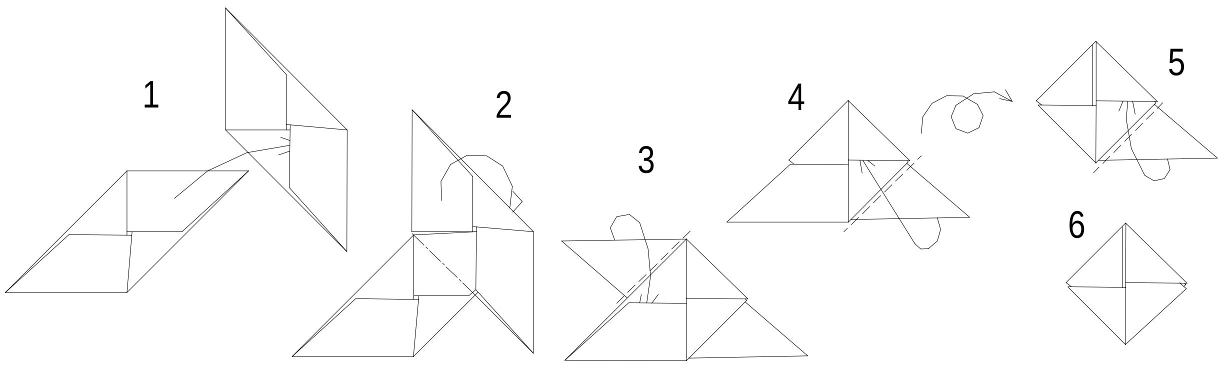
- Demonstration is valid only for right triangles with their legs in the ratio $1 : \frac{1}{2\sqrt{2}}$
- The pieces formed with the rhomboidal unit have a *flattened-consistent* structure with *flattvolumelike* shape. That´s why, at the end, the initial geometric rigor is lost, but that does not lessen its interest neither geometric nor pedagogic.

Figs.1 and 2 shows the process to get the rhomboidal unit. From them on, it follows the construction of the five pieces A, B, C, D, E.

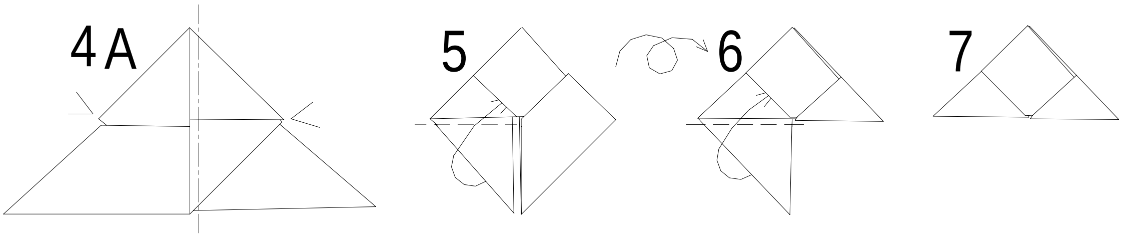




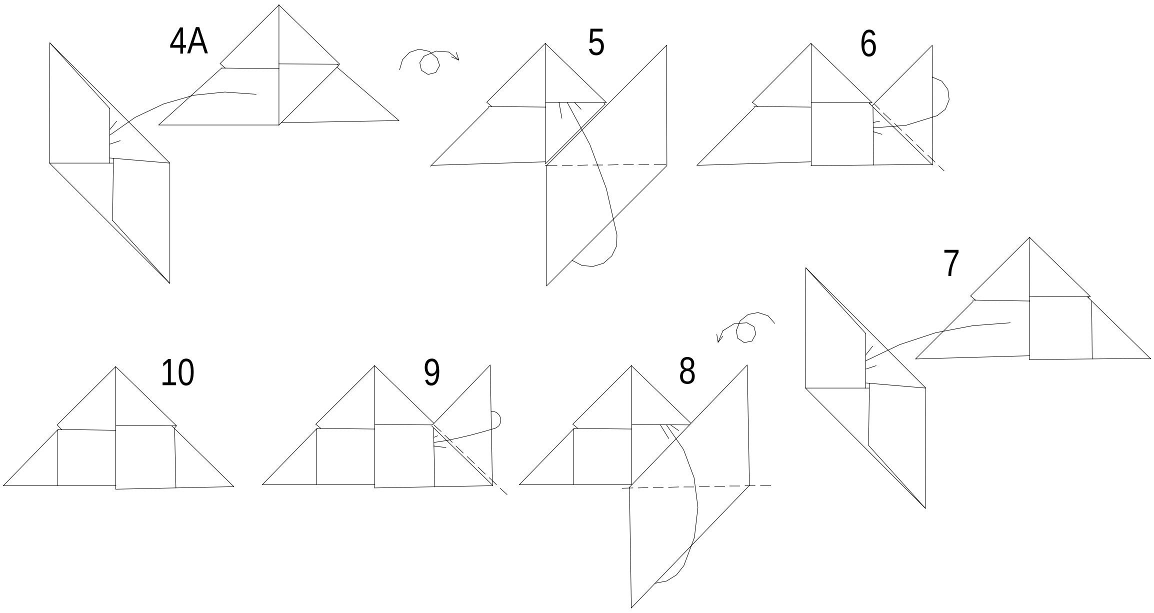
Piece A (two units)



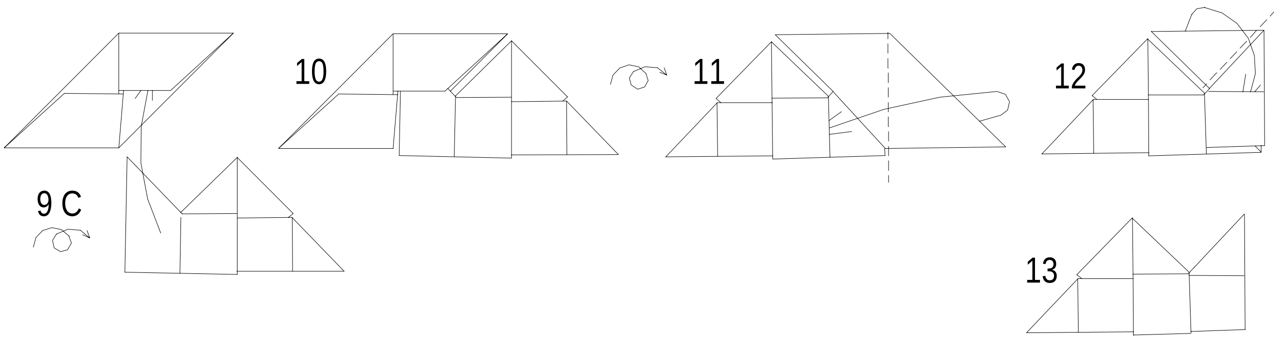
Piece B (two units)



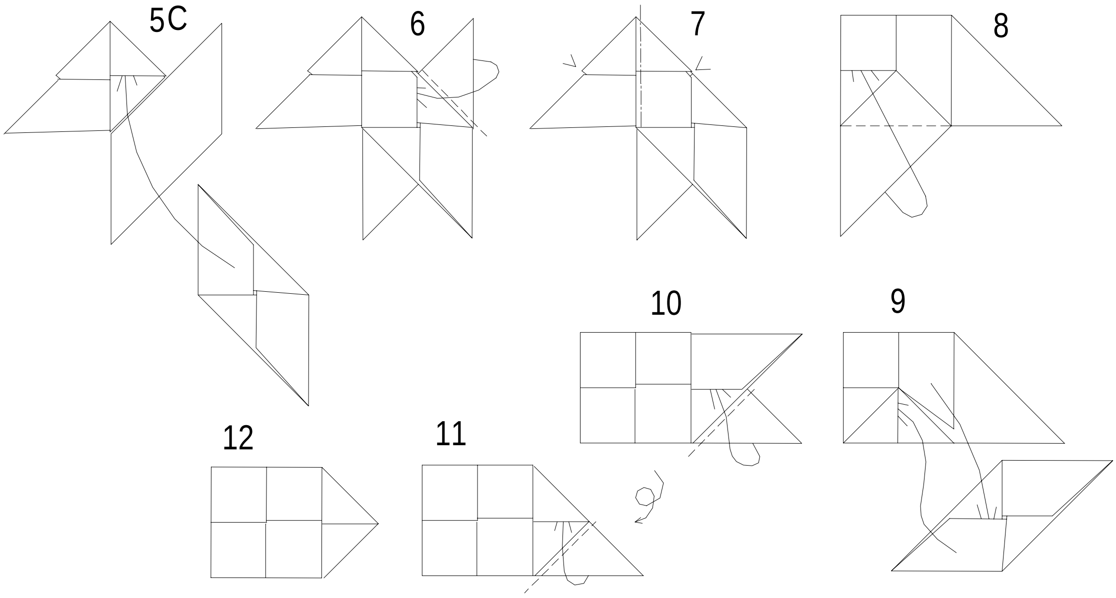
Piece C (four units)



Piece D (five units)



Piece E (five units)



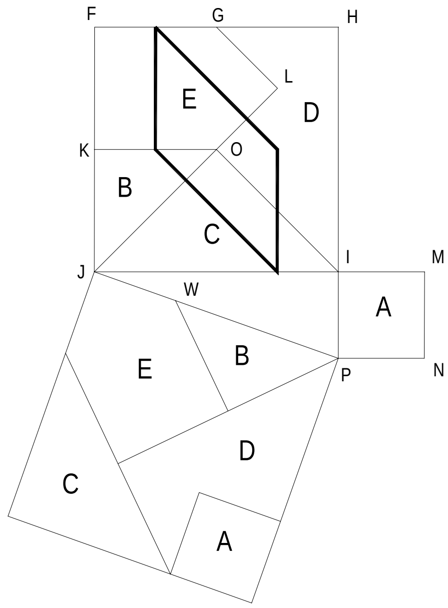
Therefore, the five pieces obtained are:

The square A6; the isosceles right triangles B7 and C10; D13 and E12.
In the following figure also appear those pieces to justify the Pythagorean theorem.

It can be seen in it that the square over the hypotenuse has the same area than the sum of those built over the legs

It may be noted also that the square over the great leg (one unit side) is equal to the initial Fig.1. The rhomboid of the basic unit and the four figures B, C, D, E are shown now overlapped.

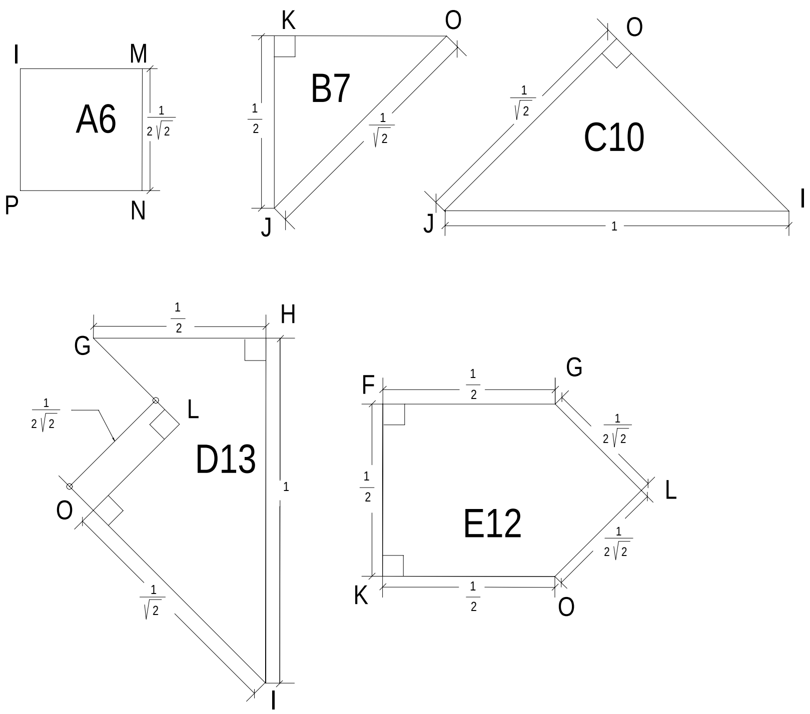
These, besides piece A are shown after with their dimensions.



Algebraically we can also see:

$$JP^2 = JI^2 + IP^2 \quad ; \quad (JW + WP)^2 = JI \ \ + IP$$

$$\left(\frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 = 1 + \left(\frac{1}{2\sqrt{2}}\right)^2 \quad ; \quad 1 + \frac{1}{8} = 1 + \frac{1}{8}$$



Remarks:

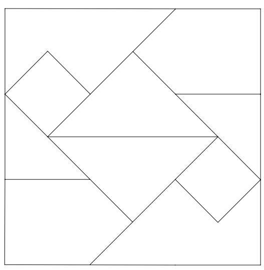
- Piece C10 is not the triangle ABC in Fig. 1. In this, BC ≠ AD (being AD= 1). There is a small difference:

$$CB = \frac{3}{4}\sqrt{2} = 1.06 \neq 1$$

That´s why in the square over the great leg, the right angle of C10 does not lie on neither of the vertices of the square. On the contrary, that happens on the square over the hypotenuse.

- It can be observed also that the sides of the squares are divided:
  - ☐ in halves (great leg).
  - ☐ in thirds (hypotenuse).
- The Pythagorean theorem approach can be presented also as a game with these alternatives:
  - ☐ Make up two different squares with the five basic pieces. The two squares over both legs are the solution.
  - ☐ Form one only square with the five pieces. The solution is the square over the hypotenuse.
  - ☐ With 2 sets of said five pieces (total of ten), build up one only square. Fig. 3 is the solution. In it you can also see 4 squares.

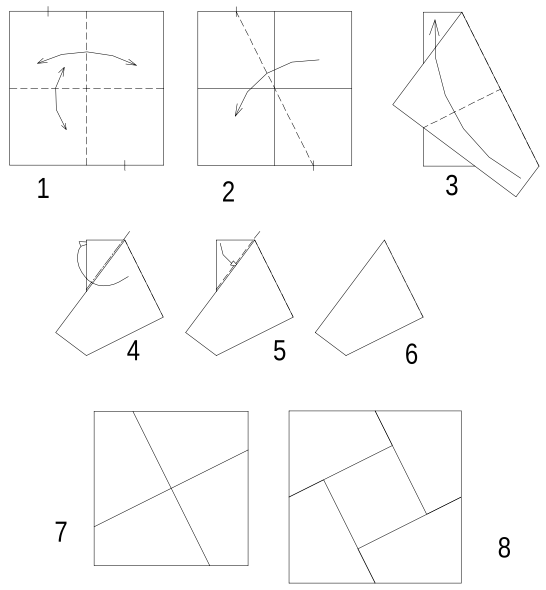
3



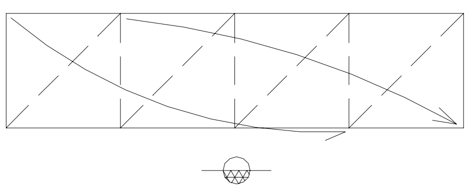
8.3.5 UNIT SQUARES (JEAN JOHNSON)

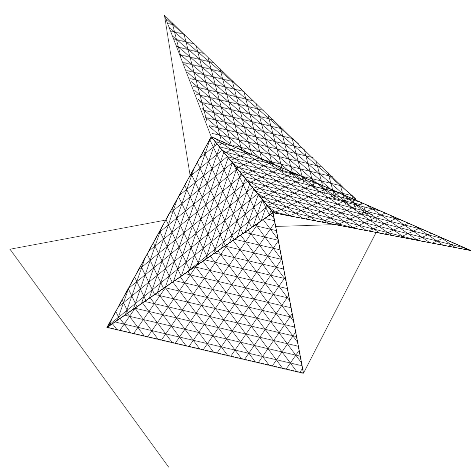
Begin with a square folded as indicated in figs. 1 to 6. An irregular quadrilateral with two straight angles is obtained eventually in fig. 6.
Ask for four units like the former: then build up squares using the four pieces in each.

Without any restrain to the process, we can play with the four pieces by: making the four pieces adjacent, overlapping or enveloping each other, turning upside down some of them, etc. The most straightforward process leads to three squares: one to fig. 7 and two (interior and exterior) to fig. 8.



Interlude

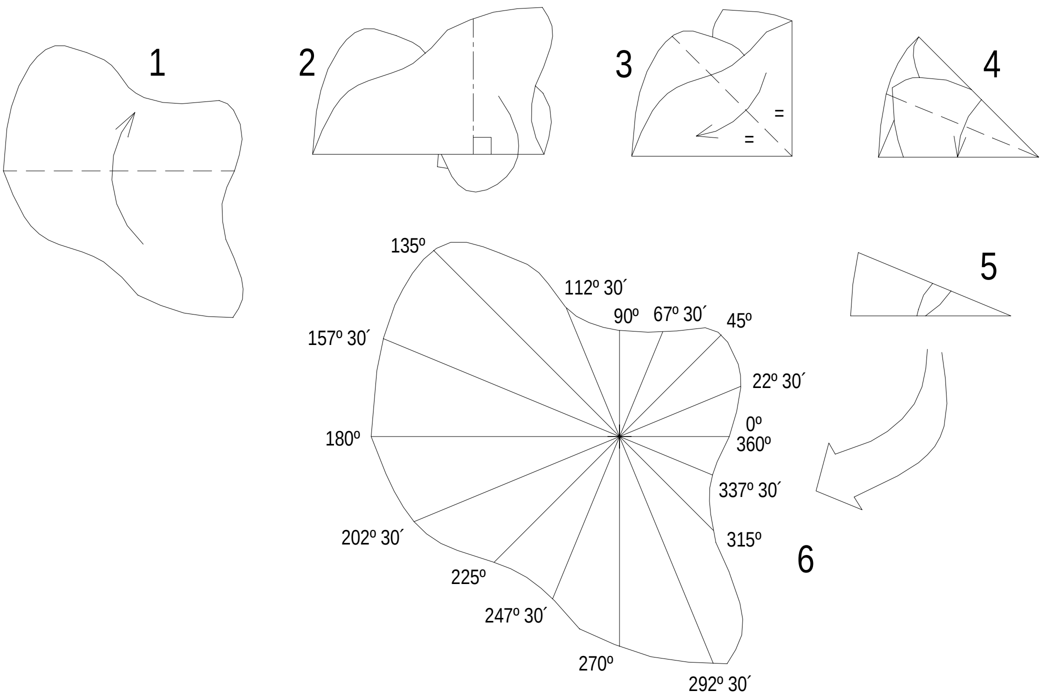




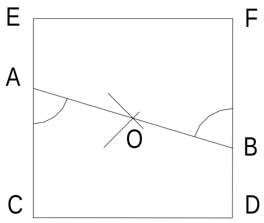
9 DIVISION IN EQUAL PARTS

9.1 EVEN PARTS OF A PERIGON

Begin with an irregular piece of paper to divide the 360º of the plane in 2^n equal parts:
n = 1, straight angle.
n = 2, right angles (perpendicular rays)
n = 3, 45º angles.
n = 4, 16 angles measuring 22º 30´ each.



9.2 A SQUARE IN TWO PARTS OF EQUAL AREA



Any segment AB passing through the center O fulfils the requirement.

AB determines two right trapeziums having all their angles, respectively congruent (ang A = ang B as alternate interior), sides CD = EF, and common side AB; hence, both areas are equal.

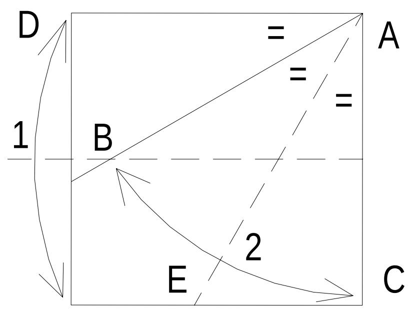
9.3 THE RIGHT ANGLE OF A SQUARE IN THREE EQUAL PARTS

The angle A is divided in three equal parts for:

ΔABC is equilateral (B lies on the perpendicular bisector of AC and AB = AC)

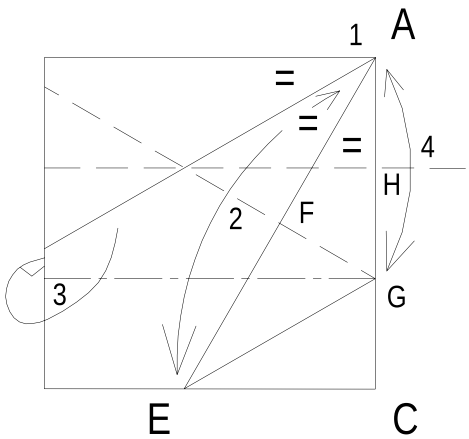
hence ang BAC = 60º ; ang DAB = 30º

ang BAE = ang EAC (symmetry) = 30º



9.4 A SQUARE IN THREE EQUAL PARTS (five exact solutions)

Solution 1



1- Divide ang A in three equal parts to Point 9.3. We get point E.

2- Fold A over E. Valleyfold FG is the symmetry axle.

- ΔAFG = ΔEFG because of symmetry.

- Being ang FAG = 30º, the three angles in G (HGF, FGE, EGC) measure 60º.

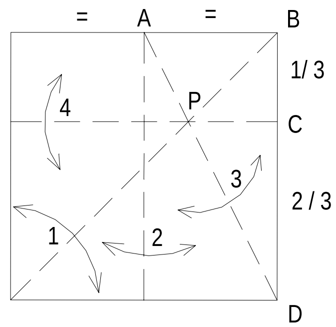
- ΔEGC is “half an equilateral triangle”, hence

$$GC = \frac{1}{2}EG = \frac{1}{2}AG$$

3- AC $\rightarrow$ AC; G $\rightarrow$ G

4- G $\rightarrow$ A: $AH = HG = GC = \frac{1}{3}AC$

Solution 2



Let a square with side one unit.
ΔABD and PCD are similar.

As $AB = \frac{1}{2}BD$, it will be $PC = CB = \frac{1}{2}CD$

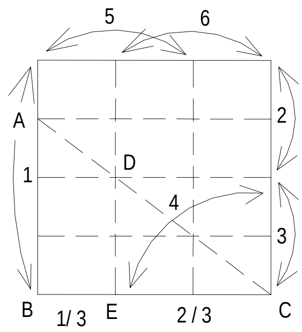
As $BC + CD = 1$, it is $BC = \frac{1}{3}$ and $CD = \frac{2}{3}$

Solution 3

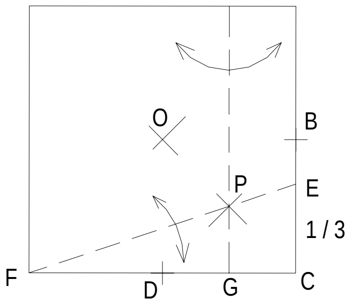
Square with side one unit.
ΔABC and ΔDEC, are similar.

$AB = \frac{3}{4}BC$

$$DE = \frac{1}{2} = \frac{3}{4}EC$$
$$EC = \frac{2}{3} \quad ; \quad BE = \frac{1}{3}$$



Solution 4



By folding, find the centers O (big square) and P (square OBCD).

Triangles FEC and FPG are similar.

As $FG = \frac{3}{4}FC$, it will be:

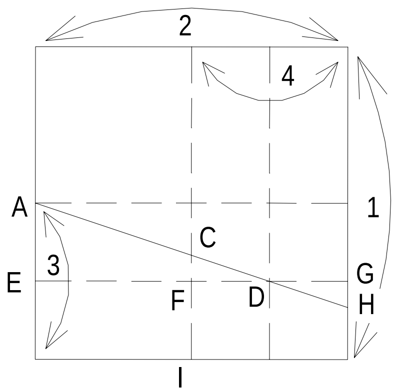
$PG = GC = \frac{1}{4} = \frac{3}{4}EC$, therefore $EC = \frac{1}{3}$

Solution 5

Begin with a square of side one unit.
Triangles CFD and DGH, are congruent.
Triangles ADE and DGH, are similar.
As $AE = \frac{1}{3}ED$, it will be $GH = \frac{1}{3}DG = \frac{1}{3}\times\frac{1}{4} = \frac{1}{12}$

$$CI = CF + FI = GH + \frac{1}{4} = \frac{1}{12} + \frac{1}{4} = \frac{4}{12} = \frac{1}{3}$$

Therefore it´s enough to fold the square horizontally over C and then to fold its upper side also over C.



9.5 A SQUARE IN THREE EQUAL PARTS (two approximate solutions)

Solution 1

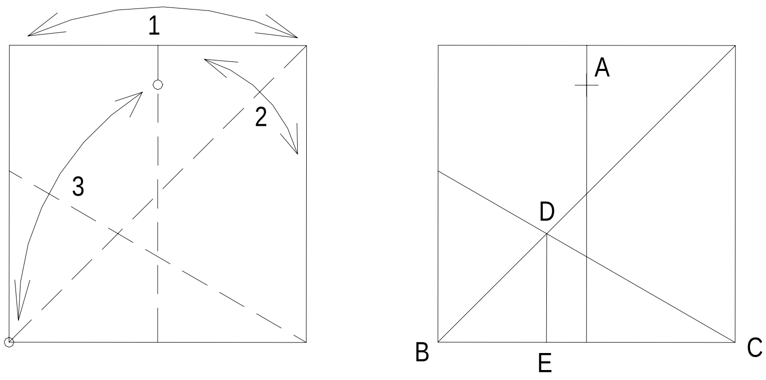
Side BC of square = 1
ΔABC, equilateral; ang DCE = 30º ; ang DBC = 45º ; ang BDC = 105º

$$\frac{DB}{\operatorname{sen}30} = \frac{BC}{\operatorname{sen}105} \quad ; \quad DB = \frac{\operatorname{sen}30}{\operatorname{sen}105} = 0.517638$$

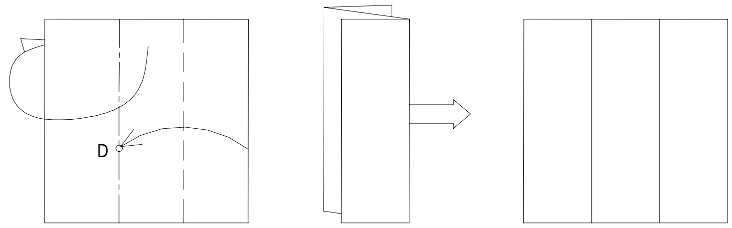
$$BE = DB\cos 45 = 0.3660254 \neq 0.333...$$

Practical folding process is:

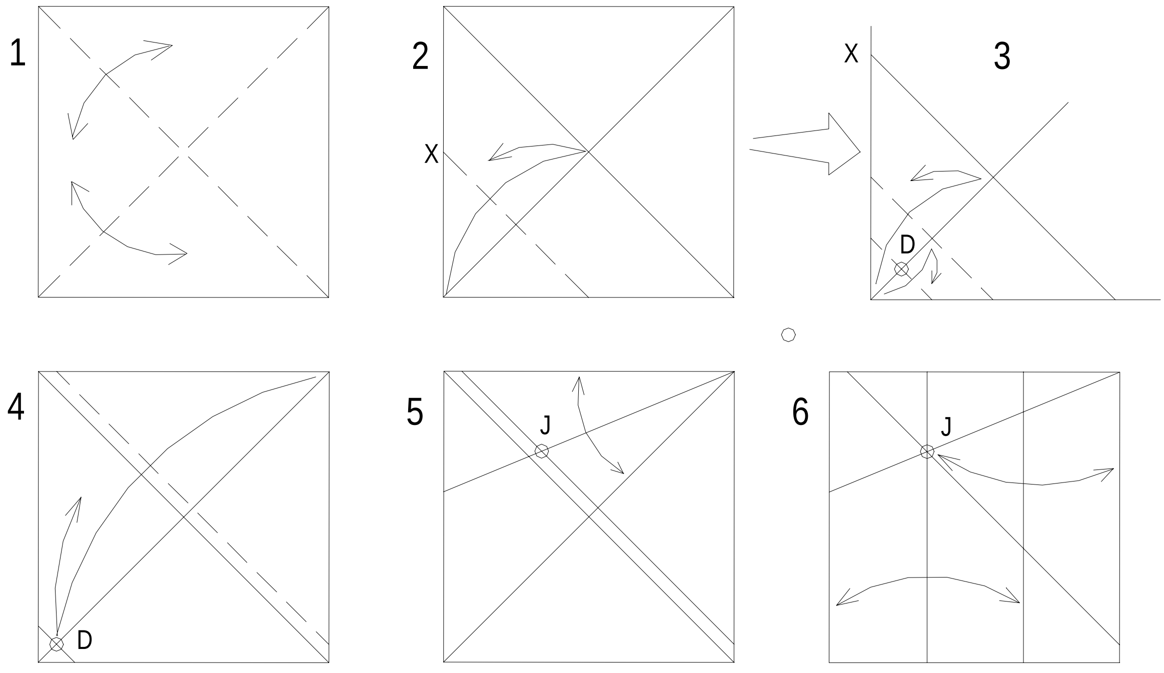
- To get point D.

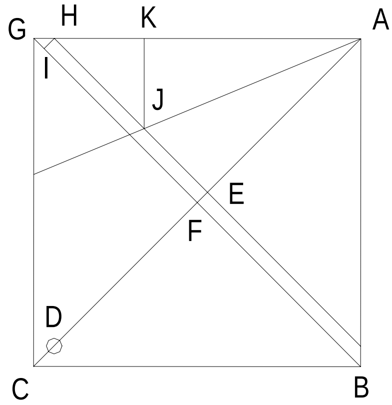


- Using D as a reference, mountain and valley fold (not flattening), to obtain a bellows configuration. Then flatten after coincidence of sides and folds.
- Unfold to see the square equally divided.



Solution2





$$AB = 1 \quad ; \quad AC = \sqrt{2} \quad ; \quad CD = \frac{\sqrt{2}}{16}$$

$$EF = AF - AE = \frac{\sqrt{2}}{2} - ED = \frac{\sqrt{2}}{2} - (EF + FD) = \frac{\sqrt{2}}{2} -$$

$$-(EF + FC - CD) = \frac{\sqrt{2}}{2} - \left(EF + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{16} \right)$$

$$EF = \frac{\sqrt{2}}{16} - EF \quad ; \quad EF = \frac{\sqrt{2}}{32}$$

$$HI = EF \quad ; \quad GH = \sqrt{2}HI = \frac{\sqrt{2}\sqrt{2}}{32} = \frac{1}{16}$$

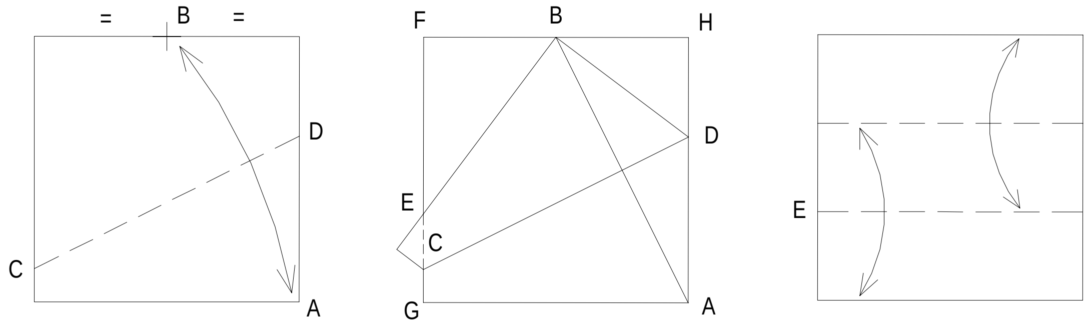
In ΔHAJ: ang H = 45º ; ang A = $\frac{45}{2}$ = 22.5º ; ang J = 180 – 45 – 22.5 = 112.5º

$$\frac{1 - \frac{1}{16}}{\operatorname{sen} 112.5} = \frac{HJ}{\operatorname{sen} 22.5} \quad ; \quad \text{HJ} = 0.3883252$$

$$GK = GH + HK = \frac{1}{16} + HJ \cos 45 = 0.3370873$$

inexact result: $\frac{1}{3} = 0.33333.....$

9.6 A SQUARE IN THREE EQUAL PARTS (Haga´s theorem)



Beginning with a square with side one unit, the result is: $EG = \frac{1}{3}$; $EF = \frac{2}{3}$

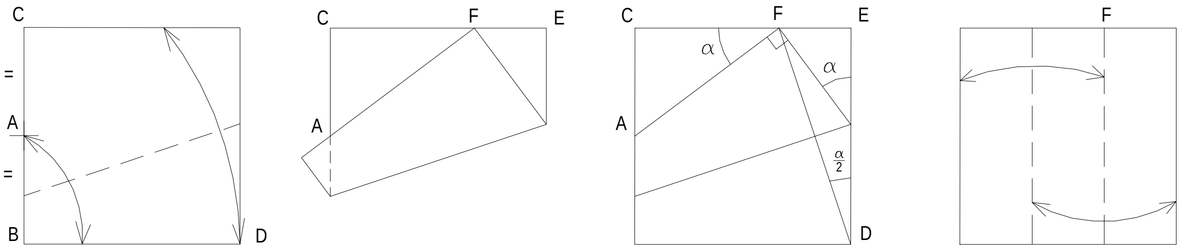
Justification:

$FH = 1$; $angABH = Arc\,\mathrm{tg}\,2$; $angDBA = angBAD = Arc\,\mathrm{tg}\,\frac{1}{2}$

$angHBD = Arc\,\mathrm{tg}\,2 - Arc\,\mathrm{tg}\,\frac{1}{2}$; $angFEB = angHBD$ (perpendicular sides)

in ΔFEB: $\mathrm{tg}\,FEB = \frac{1}{2 \times FE}$; $FE = \frac{1}{2\,\mathrm{tg}\left(Arc\,\mathrm{tg}\,2 - Arc\,\mathrm{tg}\,\frac{1}{2} \right)} = \frac{2}{3}$; $EG = 1 - \frac{2}{3} = \frac{1}{3}$

9.7 A SQUARE IN THREE EQUAL PARTS (Corollary P)



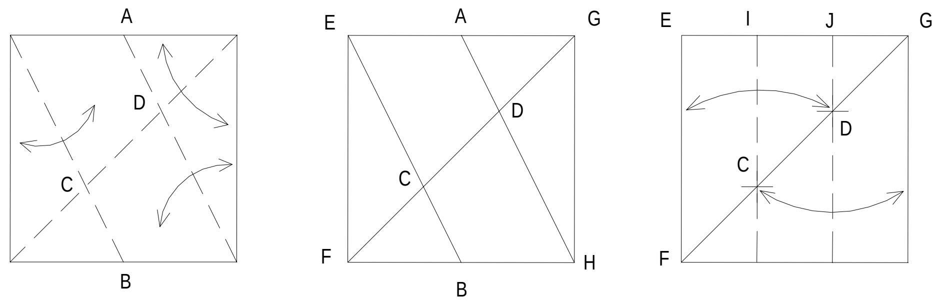
This is a consequence of the fact that a / x is biunivocal (see Point 5).
The first thing is to fold BD in such a way that:

- lower side BD lies on A, mid-point of BC.
- simultaneously, D must lie on the upper side of the square.

If the square has one unit side, the result is that $FE = \frac{1}{3}$. Justification:

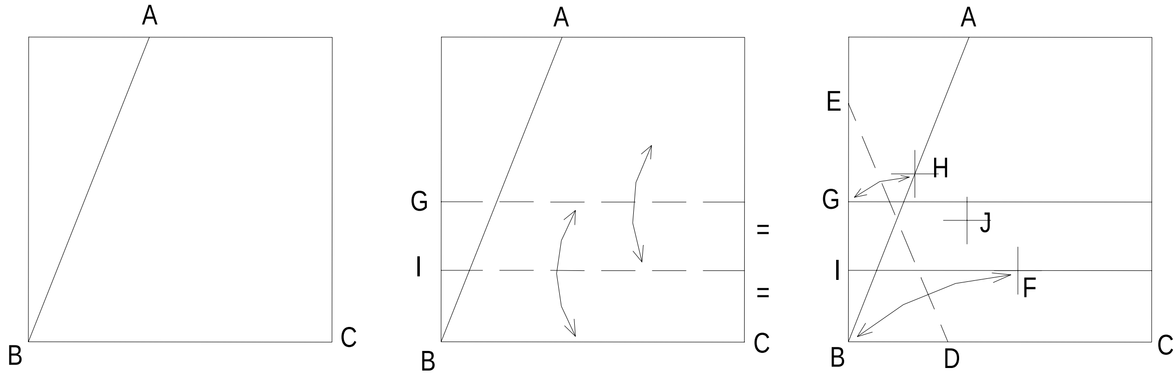
- The two angles α are congruent because their sides are perpendicular.
- In ΔACF: $\operatorname{tg}\alpha = \frac{1}{2CF}$; In ΔFED: $\operatorname{tg}\frac{\alpha}{2} = FE$
- $\operatorname{tg}\alpha = \dfrac{2\operatorname{tg}\frac{\alpha}{2}}{1-\operatorname{tg}^2\frac{\alpha}{2}}$; $\frac{1}{2CF} = \frac{2FE}{1-FE^2}$; $\frac{1}{2(1-FE)} = \frac{2FE}{(1-FE)(1+FE)}$
- $1 + FE = 4FE$; $FE = \frac{1}{3}$

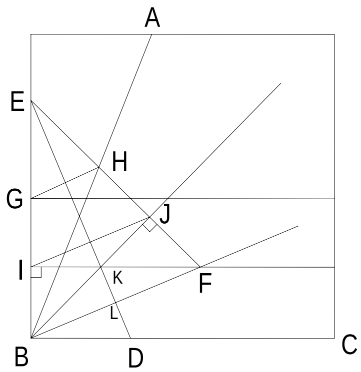
9.8 A SQUARE BY TRISECTING ITS DIAGONALS



The solution is also applicable to a rectangle.
- To get A and B, the midpoints of respective sides.
- C and D trisect the diagonal in equal parts:
  - Triangles EFB and AGH are congruent (equal and parallel legs). So EB and AH are parallel.
  - In the pencil of rays FG-FH, FB = BH. Hence FC = CD (Thales theorem).
  - In the pencil of rays GE-GF, GA = AE. Hence CD = DG (Thales theorem)
  - That is: FC = CD = DG = $\frac{1}{3}$ FG
  - Obtained C and D, folding of last square determines EI = IJ = JG (Thales theorem applied to rays GE-GF)

9.9 TRISECTION OF ANY ANGLE ABC





Process will be as follows:

1- To get two straight lines parallel to a side (e.g. BC) under the condition to be equidistant as the figure shows. Thus we get points G and I.

2- Produce valley fold ED such that B will lie on F (over lower parallel) and, **simultaneously**, G on H (over ray BA). As a consequence of the symmetry (axle ED), I will lie on the new point J.

3- Rays BJ and BF trisect Ang ABC. Justification:

ΔBKL = ΔKLF (symmetry) ; BL = LF (symmetry)

ΔBLD = ΔKLF (one side and angle, congruent)

so ΔBKL = ΔBLD, that is: ang JBF = ang FBC

Moreover: as GI = IB , HJ = JF (symmetry)

Said symmetry with respect to ED produces the right angles marked in I y J.

Then ΔHBJ = ΔFBJ, that is: ang ABJ = ang JBF

therefore: ang ABJ = ang JBF = ang FBC = $\frac{1}{3}$ ang ABC

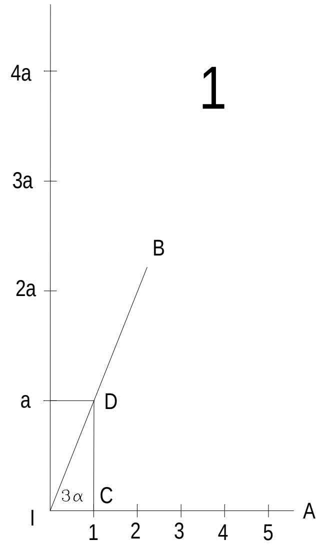
We have trisected angle ABC, associated to its corresponding square. Of course, any angle can be related to a square.

The reader may have observed that we came across the solution by simultaneously folding under two different conditions. Something similar was shown when resolving the third degree equation (Points 7.11; 7.14; 7.14.5). Now it is pertinent to say the following:

It is impossible to achieve that simultaneous coincidence by means of rule, square / set square and compass, nor even by CAD. The computer can help in a try and cut process, but that´s all. **On the contrary, folding is the unique way to integrate trying, intuition and precision.**

9.10 TRISECTION OF ANY ANGLE (H. HUZITA)

Let AIB = 3α the angle to trisect.



As $\mathrm{tg}\,2\alpha = \dfrac{2\,\mathrm{tg}\,\alpha}{1-\mathrm{tg}^2\alpha}$, it is:

$$\mathrm{tg}\,3\alpha = \mathrm{tg}(2\alpha+\alpha) = \frac{3\,\mathrm{tg}\,\alpha - \mathrm{tg}^3\alpha}{1-3\,\mathrm{tg}^2\alpha}$$

Making $\mathrm{tg}\,\alpha = t$, we´ll have:

$\mathrm{tg}\,3\alpha = \dfrac{3t-t^3}{1-3t^2}$ which leads to the complete equation of third degree:

$$t^3 - 3\,\mathrm{tg}\,3\alpha \times t^2 - 3t + \mathrm{tg}\,3\alpha = 0 \qquad (1)$$

Making $a = \mathrm{tg}\,3\alpha$, (1) takes the form:

$$t^3 - 3at^2 - 3t + a = 0 \qquad (2)$$

In Point 7.14.5 this equation was solved:

$$t^3 + t^2 - 2t - 1 = 0 \qquad (3)$$

Similarities and dissimilarities between (2) *and* (3):

- Both are complete equations of third degree.
- With negative discriminants (ignore in (2) the values of $3\alpha = n\dfrac{\pi}{2}$, n being a natural number; these exceptions will be studied below).
- Every coefficient of (3) can be expressed in the same unit.
- (2), on the contrary, requires two different types of unit to express its coefficients: the unit as such, like in (3), for the monomial terms of third and first degree, and the unit a for the terms of second degree and independent.
- Fortunately, coefficients using as unit 1 and a, alternate in (2): so, all the horizontal vectors will be measured with a type of unit and the vertical ones, with the other.

Before proceeding, let´s solve (2) for 3α equal to 90º and 180º respectively.

Being a = tg 90º = ∞, let divide (2) by a:

$$-3t^2 + 1 = 0 \quad ; \quad t = \sqrt{\frac{1}{3}} \quad ; \quad \alpha = Arc\,\mathrm{tg}\sqrt{\frac{1}{3}} = 30º$$

As a = tg 180º = 0, (2) becomes:

$$t^3 - 3t = 0 \quad ; \quad t^2 - 3 = 0 \quad ; \quad t = \sqrt{3} \quad ; \quad \alpha = Arc\,\mathrm{tg}\sqrt{3} = 60º$$

It´s evident that these particular solutions are direct: $\dfrac{90}{3} = 30 \quad ; \quad \dfrac{180}{3} = 60$

In Point 8.2.2.2 we already studied the obtention of angles of 30º and 60º by mere folding.

Let´s go on with fig. 1. If we make IC = 1, we´ll have:

DC = tg 3α = a

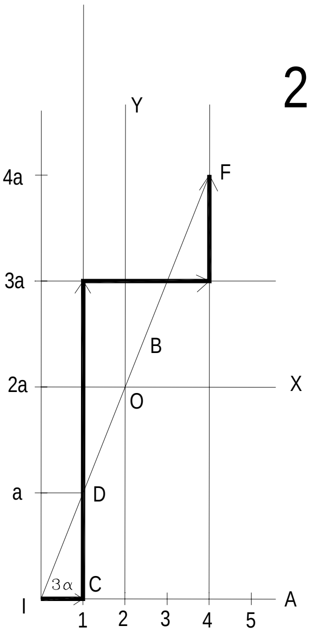
Therefore (fig. 2) we are able to draw the co-ordinate plane with origin at I (initial point), abscissas measured in conventional units and ordinates measured with a as unit.

Looking at Fig.2 we have:

- First vector will be [I → (1,0)] (which gives the axle 2,Y)
- The second vector will be [(1,0) → (1,3a)]
- The third is [(1,3a) → (4,3a)]

- And the forth vector: [(4,3a) → (4,4a)] (which gives the axle 2a,X)

2

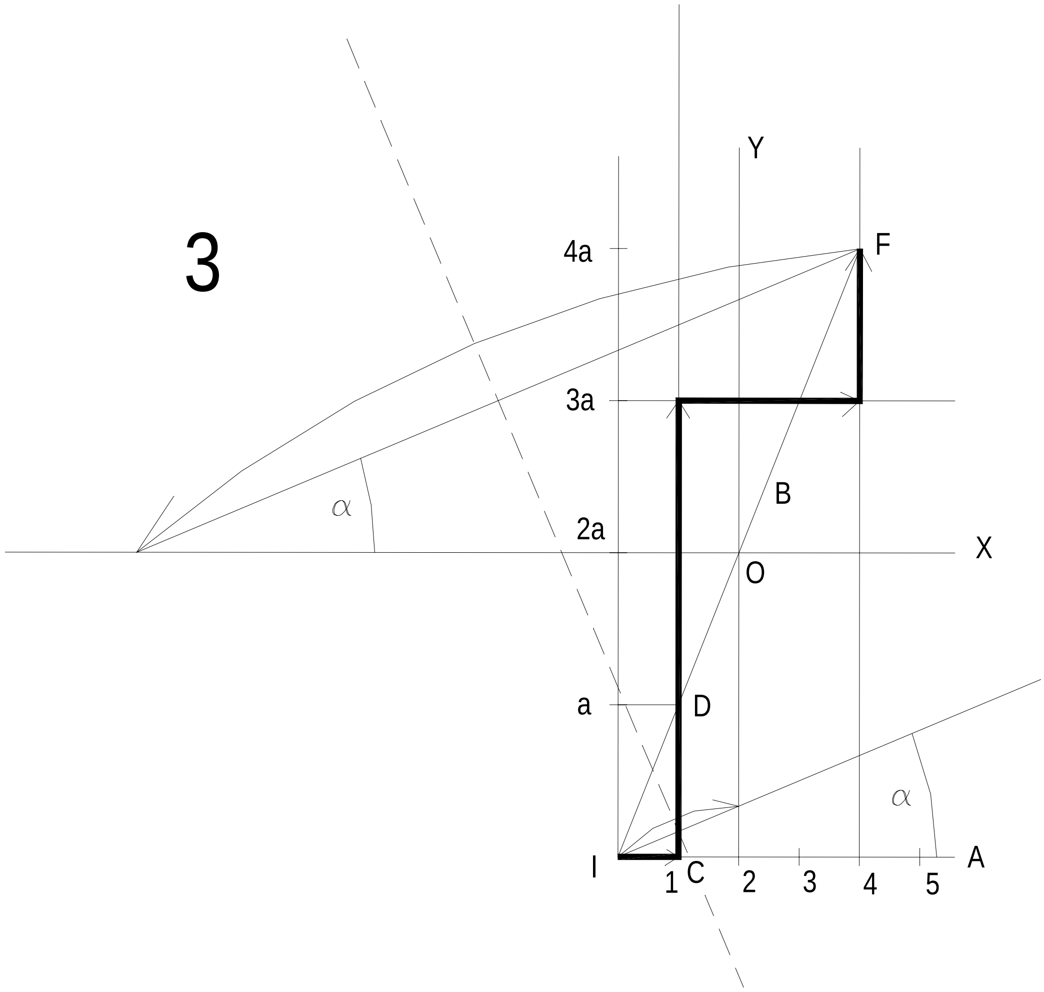


The only thing left now is to fold **simultaneously** Initial point I over OY and Final point F over OX.

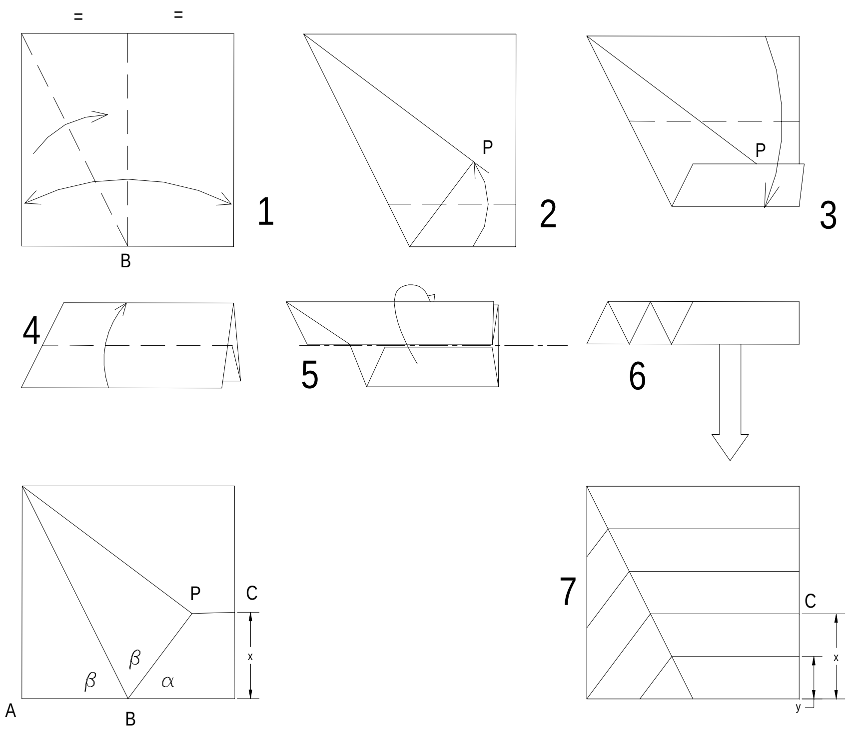
There are three solutions for t, of which, only one is acceptable. Other gives a negative angle and the third gives an angle greater than the given one.

Fig. 3 shows the final solution to be completed with a pleat fold of the angle α within 3α.

3



9.11 A SQUARE IN FIVE EQUAL PARTS



Let´s begin with a square of one unit side:

$$x = PB\,\mathrm{sen}\,\alpha = AB\,\mathrm{sen}\,\alpha = \frac{1}{2}\mathrm{sen}\,\alpha$$

$$\alpha = 180 - 2\beta = 180 - 2\,\mathrm{arctg}\,2$$

$$x = \frac{1}{2}\mathrm{sen}(2\,\mathrm{arctg}\,2) = 0.4 \quad ; \quad y = \frac{1}{2}x = 0.2 = \frac{1}{5} \quad \text{(exact solution)}$$

9.12 OTHER INEXACT FORM OF DIVISION

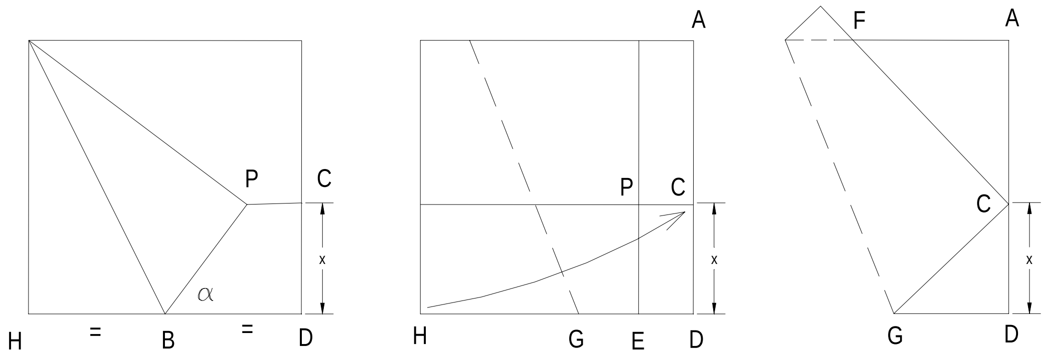
- Let C at a distance x from D, obtained according to Point 9.11.
- Fold H over C to produce G and F.
- In Point 9.11 it is $x = CD = 0.4 = \frac{2}{5} = \frac{1}{\frac{5}{2}}$; recalling Corollary P, we shall have

$$AF = \frac{2}{\frac{5}{2}+1} = \frac{2}{\frac{7}{2}} = \frac{4}{7}$$

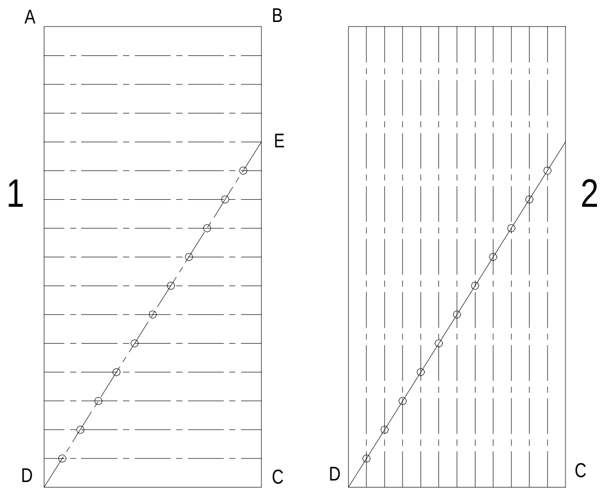
- Being similar ΔAFC and ΔDGC, it is:

$$\frac{AF}{AC} = \frac{CD}{GD} \quad ; \quad \frac{\frac{4}{7}}{1-\frac{2}{5}} = \frac{\frac{2}{5}}{GD} \quad ; \quad GD = \frac{2\times7\times3}{5\times4\times5} = \frac{21}{50}$$

- Now then, there are publications which affirm that $GD = \frac{3}{7}$ without any clarification whether that is exact or approximate.
- We have just seen that, though with a great approximation, it does not exist accuracy: GD measures $\frac{21}{50}$ which differs from $\frac{3}{7} = \frac{21}{49}$



9.13 THALES´ THEOREM: DIVISION OF A RECTANGLE IN n EQUAL PARTS



In Solution 3 (Point 9.4) we saw in advance the present application. Sidney French, in B.O.S monograph *GEOMETRICAL DIVISION* also develops this question which now we are going to generalise.

It is well known that to divide in n equal parts by folding, is immediate for n equal to the successive powers of 2: 4, 8, 16, 32 ...

For the other even values of n, e.g. 12, the operation is not immediate because each forth has to be divided in 3 parts; for any odd value of n we need specific solutions. Somebody could argue that for a division in 12 parts, we have already learned to divide by 3: that´s true but complicate. This is the reason why we shall develop now a general procedure.

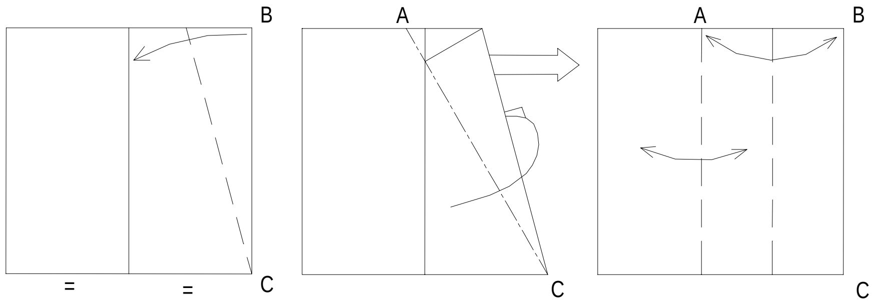
Let´s divide vertically in n = 12 parts the rectangle ABCD. We look for the power of 2 nearest to and greater than n. We find 16. Then we divide the rectangle horizontally in 16 parts (fg. 1) producing only the indispensable folds. Counting 12 parts from C we get E.

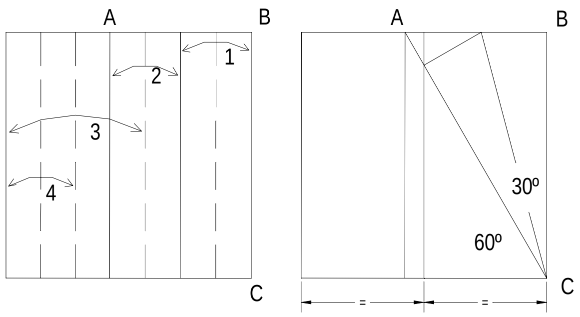
Thales theorem transfers the equality of segments in CE, to DE.

Finally we fold the rectangle (fig. 2) by the perpendiculars to DC through points marked in DE. It´s easy to see that the process is good for any value of n.

9.14 DIVISION OF A SQUARE IN 7 EQUAL PARTS (two approximate solutions)

Solution 1





For BC = 1, in ΔABC:

$$AB = tg30 = 0.5773502$$

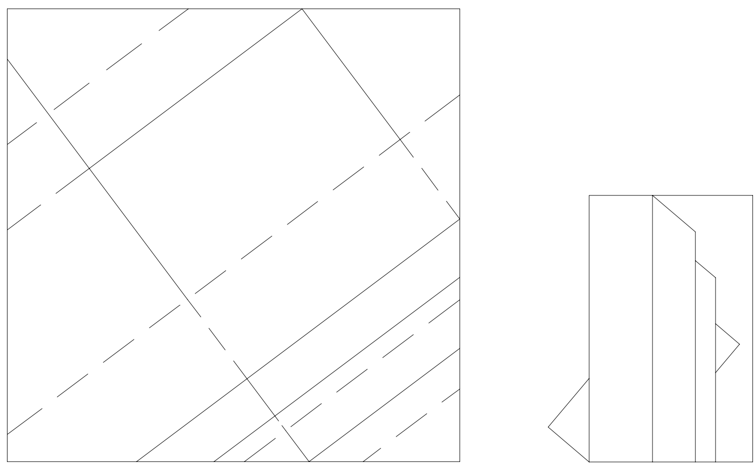
$$\frac{AB}{4} = 0.1443375$$

$$\text{as } \frac{1}{7} = 0.1428571$$

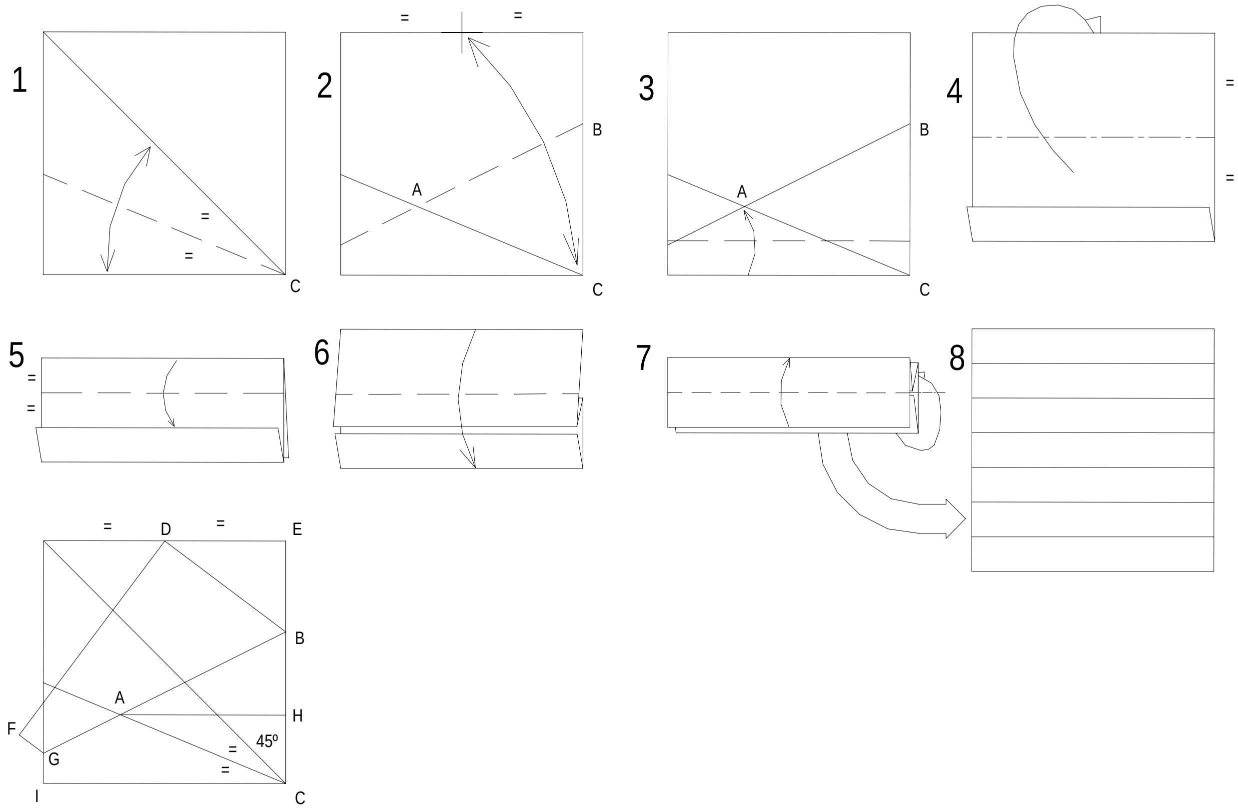
it can be seen that the result is very near accuracy, but inexact.

Besides, the seven partitions obtained are not equal: there are three sets of values. From smallest to greatest, the smallest part is the third from left, then the first and second from left and finally, the four parts at right.

Interlude



Solution 2



Beginning as usual, with a square of side 1, let´s see how much precision we get: it is supposed that $\frac{HC}{2}$ be equal to $\frac{1}{7}$.

Comparing the last figure of present *Solution 2* with Fig 2 in Point 4 (demonstration of Haga´s theorem), we have:

$$DE = x \quad ; \quad EB = z \quad ; \quad FG = GI = f$$

Now it is DE = x = 0.5; in Point 4 we had for this value of x:

$$z = 0.375 = EB \quad ; \quad f = 0.125 = FG = GI$$

From these data we can study ΔABC.

$$\text{ang ABC} = \text{arc tg } \frac{IC}{EC - EB - GI} = \frac{1}{1 - z - f} = \text{arc tg} \frac{1}{1 - 0.375 - 0.125}$$

$$\text{ang ABC} = \text{arc tg } 2 = 63.434949^{o}$$

$$\text{ang BAC} = 180 - \text{ang ABC} - \left(45 + \frac{45}{2}\right) = 49.065051^{o}$$

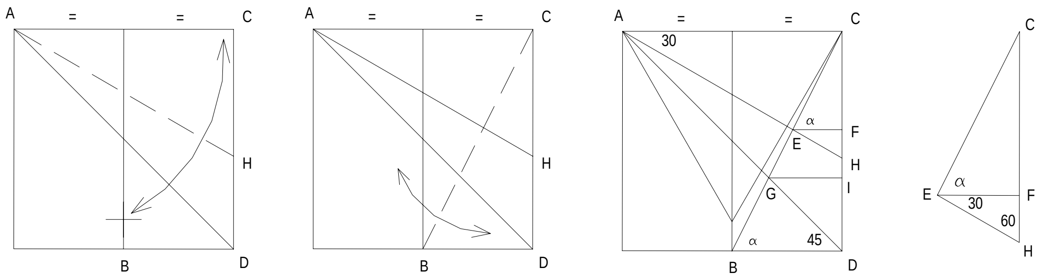
$$\frac{1 - z}{\text{sen}\, BAC} = \frac{AC}{\text{sen}\, 63.434949} \quad ; \quad AC = (1 - 0.375)\frac{\text{sen}\, 63.434949}{\text{sen}\, 49.065051} = 0.7399749$$

$$CH = AC \cos\left(45 \times \frac{3}{2}\right) = 0.2831761 \quad ; \quad \frac{CH}{2} = 0.141588$$

The reader can see the difference between the last value and $\frac{1}{7} = 0.1428571$

9.15 A SQUARE IN 9 EQUAL PARTS (two solutions)

An approximate solution



It is based in the assumption that CB is divided in segments CE, EG and GB keeping proportions 4,2,3 respectively, that is, adding up to 9:

$$\frac{CE}{4}=\frac{EG}{2}=\frac{GB}{3}=\frac{CB}{9}$$

Of this, the only certain is that $GB=\frac{1}{2}CG$:

ΔACG and GBD are similar; the sides in the small triangle are half the size of their homologous in the big one ($BD=\frac{1}{2}AC$). Therefore it will be also $GB=\frac{1}{2}GC$.

This means that folding CG in halves, CB will be divided into three parts equal to GB.

Hence, of the four ratios made equal above, the only exact is the last proportion, which, for one unit side, ends up to be:

$$\frac{GB}{3}=\frac{CB}{9}=0.124226$$

We´ll see later that

$$\frac{CE}{4}=0.1252249 \quad \text{and} \quad \frac{EG}{2}=0.122228$$

which makes evident the inexactitude.

Justification:

α = arc tang 2 = 63.434949º ; α + 30 = 93.434949º ; α + 45 = 108.43495º

CH = AC tang 30 = tang 30 = 0.5773502

In ΔCEH:

$$\frac{CE}{\operatorname{sen}60}=\frac{CH}{\operatorname{sen}(\alpha+30)} \quad ; \quad CE=CH\frac{\operatorname{sen}60}{\operatorname{sen}(\alpha+30)}=0.5008998$$

In ΔGBD:

$$\frac{GB}{\operatorname{sen}45}=\frac{BD}{\operatorname{sen}(180-\alpha-45)} \quad ; \quad GB=\frac{1}{2}\times\frac{\operatorname{sen}45}{\operatorname{sen}(\alpha+45)}=0.372678$$

On the other hand,

$$CB=\sqrt{1+\frac{1}{4}}=\frac{\sqrt{5}}{2}=1.118034$$

EG = CB – CE – GB = 0.2444561

The supposed equalities $\frac{CE}{4}=\frac{EG}{2}=\frac{GB}{3}=\frac{CB}{9}$ have these real values:

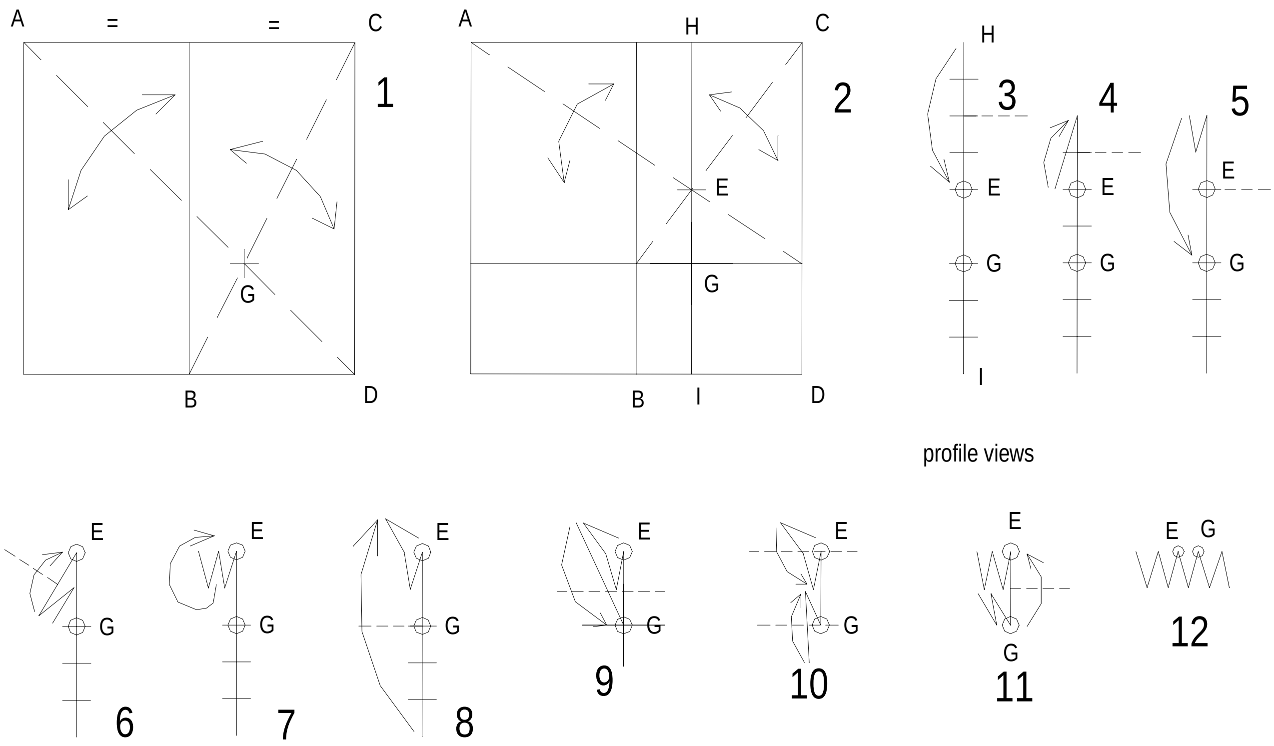
$$0.1252249 \neq 0.122228 \neq 0.124226 = 0.124226$$

Of course, to fold horizontally the given square through points E, G, etc. is equivalent to fold through F, I, etc. (Thales theorem).

We are not going to show the folding process that, in fact, is identical to the exact solution. The only difference between the two solutions is the position of point E.

Exact solution

It keeps the correct point G of former solution, and produces a new point E which will be also correct if we have applied the same process that gave G. In this case, vertical HI supersedes diagonal BC.



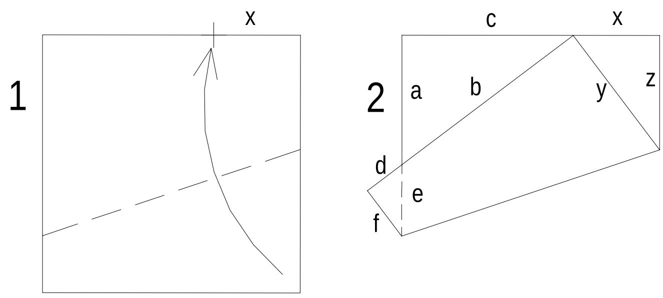
9.16 DIVISION IN n PARTS AFTER COROLLARY P.

The process is absolutely general, but for a better understanding we shall apply it to a particular case, e.g. the division of a segment in 37 equal parts. Apparently the operation may seem complicate once we are dealing with a high prime number, but we´ll see that it is not so. The first thing to do is to construct a square whose side is the given segment. We´ll suppose the problem solved and recall the biunivocal ratio a / x in Corollary P (see Point 5). That is, for $x = \frac{1}{n}$, it is $a = \frac{2}{n+1}$ (see figs. 1 and 2).

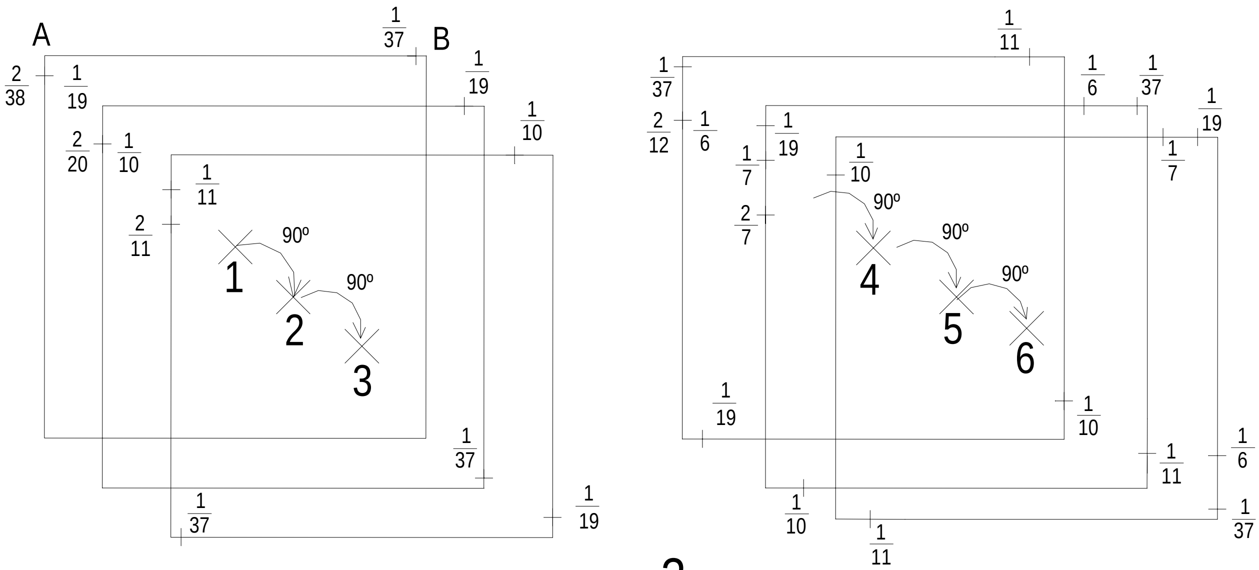
Square nº 1, side AB (fig.3), has been produced.

1. We take for granted the final solution in that square nº 1: it is the point distant $x = \frac{1}{37} AB$, from B.

2. Then, the square is folded like in figs 1 and 2 (these operations are not shown in fig 3) to get $a = \frac{2}{38} = \frac{1}{19}$



3. Copy square 1 to position 2 rotating it 90º clockwise. In that rotated square 2 appears $x = \frac{1}{19}$. You may note that in each square, former divisions are kept ($\frac{1}{37}$ in this case).



3

4. Fold square 2 to get $a = \frac{2}{20} = \frac{1}{10}$

5. Copy and rotate 2 to position 3 in which appears $x = \frac{1}{10}$

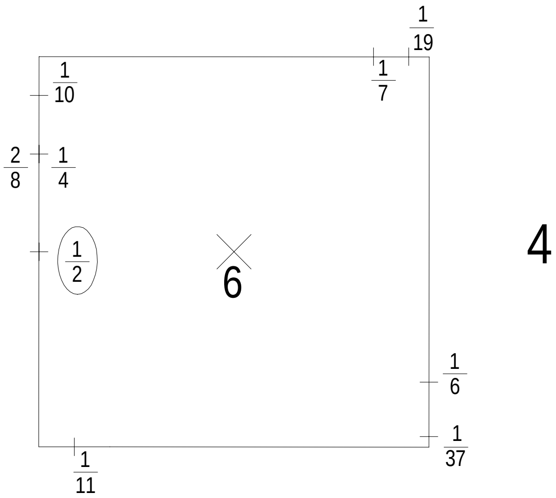
6. Fold 3 to get $a = \frac{2}{11}$; folding $\frac{2}{11}$ in halves, we get $\frac{1}{11}$

7. Copy and rotate 3 to get 4, in which $x = \frac{1}{11}$ appears.

8. Fold 4 to get $a = \frac{2}{12} = \frac{1}{6}$

9. Copy and rotate 4 to get 5 showing $x = \frac{1}{6}$

10. Fold 5 to get $a=\frac{2}{7}$ which in turn produces $\frac{1}{7}$



11. Copy and rotate 5 to get 6 which shows $x=\frac{1}{7}$

12. That square 6 is set apart in fig 4, displaying $x=\frac{1}{7}$; by folding, we get $a=\frac{2}{8}=\frac{1}{4}$

13. Finally, folding over $\frac{1}{4}$, we get $\frac{1}{2}$ (within the ellipse).

Till now we have gone the reverse way: we began with $\frac{1}{37}$ to arrive to $\frac{1}{2}$. Therefore **the direct way** will be the reverse of the reverse, beginning with $\frac{1}{2}$:

1 Get $\frac{1}{2}$, $\frac{1}{4}=\frac{2}{8}$ on the left side of 6.

2 Fold 6 in such a way that its right lower vertex lies on upper side **while** the lower side lies on $a=\frac{2}{8}$. So we get $x=\frac{1}{7}$.

3. Copy 6 in 5 rotating 90º anticlockwise.

4. In 5, fold $\frac{1}{7}$ to get $a=\frac{2}{7}$ on left side.

5. Fold square 5 in such a manner that its right lower vertex lies on upper side **while** lower side lies on $a=\frac{2}{7}$. So we get $x=\frac{1}{6}$.

6. Copy 5 in 4 rotating it 90º anticlockwise.

7. In 4 appears $a = \frac{1}{6} = \frac{2}{12}$. In said square 4 fold its right lower vertex to lie on the upper side, **while** its lower side lies on $\frac{2}{12}$. This way we get $x = \frac{1}{11}$.
8. Let´s recapitulate the x values obtained till now:

Square nº	x
6	$\frac{1}{7}$
5	$\frac{1}{6}$
4	$\frac{1}{11}$

9. With a similar procedure we would get:

Square nº	x
3	$\frac{1}{10}$
2	$\frac{1}{19}$
1	$\frac{1}{37}$

10. The whole process is contained in the following table that exhibits the successive a / x values:

a	x
$\frac{1}{2}$	
$\downarrow$	
$\frac{1}{4} = \frac{2}{8} \Rightarrow$	$\frac{1}{7}$
$\frac{1}{7}$	
$\downarrow$	
$\frac{2}{7} \Rightarrow$	$\frac{1}{6}$
$\frac{1}{6} = \frac{2}{12} \Rightarrow$	$\frac{1}{11}$
$\frac{1}{11}$	
$\downarrow$	
$\frac{2}{11} \Rightarrow$	$\frac{1}{10}$

REMARK.- One could ask: If the end of the reverse way is always $\frac{1}{2}$, which is the difference, if any, from one process to another to arrive in the direct way from that $\frac{1}{2}$ to the possible different fractions given as solution?

The answer is in the different possible combinations to get a fraction double or half of one already obtained.

You may compare the direct way shown to reach $\frac{1}{37}$ with the other we need to get $\frac{1}{27}$: the latter is shown in the following table at right.

$$\frac{1}{10} = \frac{2}{20} \Longrightarrow \frac{1}{19}$$

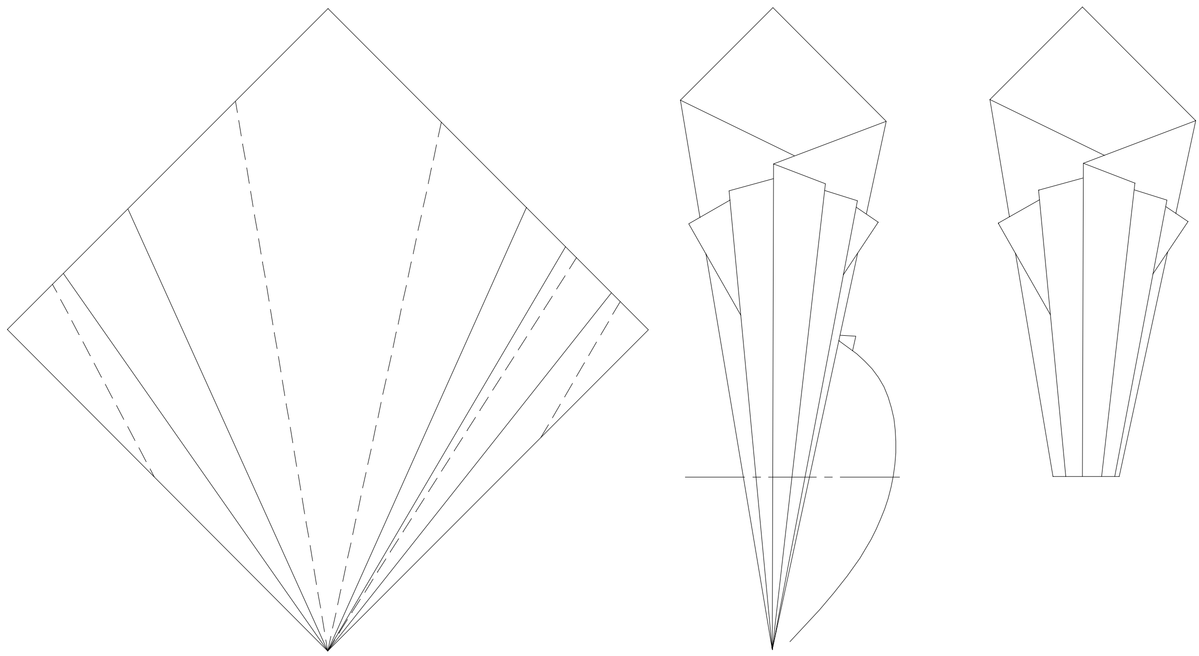
$$\frac{1}{19} = \frac{2}{38} \Longrightarrow \frac{1}{37}$$

All this proves the need to do first the reverse way to get the configuration of intermediate steps.

Of course, it is not necessary to make a precise reverse way, nor is indispensable to fold: it suffices to perform the table a / x just shown. On the contrary, folding is required in the direct way (the reverse of reverse).

a		x
$\frac{1}{2}$		
$\downarrow$		
$\frac{1}{4}$		
$\downarrow$		
$\frac{1}{8} = \frac{2}{16}$	$\Longrightarrow$	$\frac{1}{15}$
$\frac{1}{15} \to \frac{2}{15}$	$\Longrightarrow$	$\frac{1}{14}$
$\frac{1}{14} = \frac{2}{28}$	$\Longrightarrow$	$\frac{1}{27}$

Interlude



9.17 DIVISION OF A PAPER STRIP (Fujimoto´s method)

It deals with a system of successive approximations, which allows dividing the starting object in 3, 5, 7, etc equal parts through a quick, precise and practical convergence.

For the sake of simplicity we shall show only two examples: the division in 3 and 5 parts respectively, by folding a strip of paper.

Something similar could be done to divide an angle if we start up with a wider paper surface. In the case of a paper strip the little side of the strip is taken to lie each time on the previous fold. If we deal with angles, it is a ray of the angle what is revolved over its vertex to lie on the previous fold.

If a paper strip is folded from end to end, we get its two halves. Repeating the operation over one of these halves, the $\frac{1}{4}$ strip segment will appear. Continuing the same way, we´ll see how the segments $\frac{1}{8}$; $\frac{1}{16}$, etc. show up. That is, fractions such as:

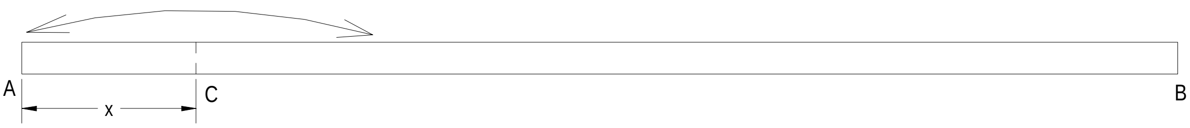
$$\frac{1}{2};\ \frac{1}{4};\ \frac{1}{8};\ \frac{1}{16};\ \frac{1}{32};\ \dots;\ \frac{1}{2^n}$$

n being the number of folds produced.

Fujimoto´s method is an application of the $\frac{1}{2^n}$ procedure, to the division in an odd number of parts. To divide in three equal parts we´ll make n = 1, i.e. the folds will take place each time in simple halves. On the contrary, to divide in 5 equal parts we´ll make n = 2: each fold will be made double each time.

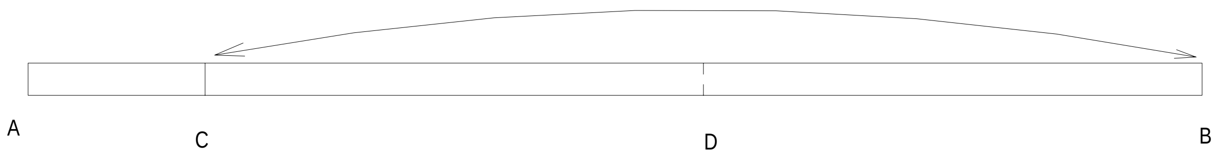
9.17.1 DIVISION IN THREE EQUAL PARTS

Before commencing, I should like to remind the reader about the division of a segment in three equal parts such as was treated in Point 8.2.8.3

1


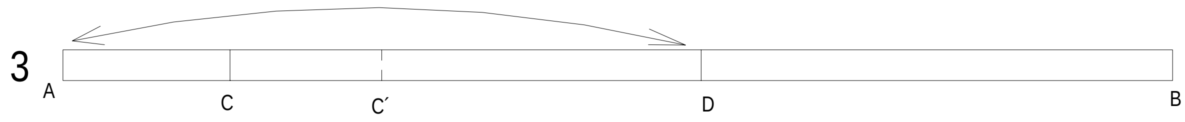
Let AB = 1; first fold is produced at C distant x from A; x may be of any measure though in practice, and in order to get a quicker convergence it should be as near as possible to 1 / 3. Nevertheless this is not an indispensable condition for the method by itself is highly convergent. In fact, fig 1, displays x strikingly smaller than 1 / 3 just to show how well the method works. Therefore it is:

$$AC = x \quad ; \quad AB = 1 \quad ; \quad CB = 1 - x$$

2


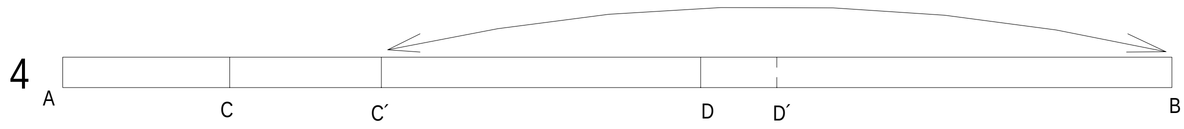
Fold B over C to get D in such a way that:

$$BD = CD = \frac{CB}{2} = \frac{1-x}{2}$$



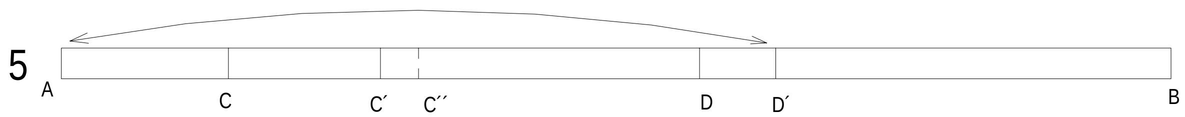
Fold A over D to get C´:

$$AC' = C'D = \frac{AD}{2} = \frac{1-BD}{2} = \frac{1-\frac{(1-x)}{2}}{2} = \frac{1}{2} - \frac{(1-x)}{4} = \frac{1}{2} - \frac{1}{4} + \frac{x}{4}$$



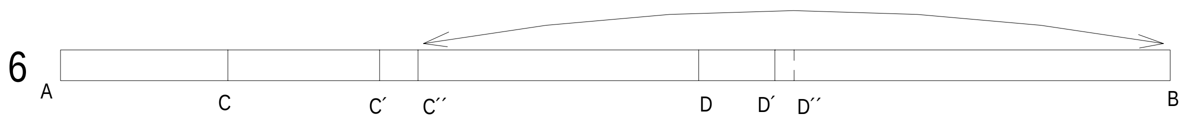
Fold B over C´ to get D´:

$$BD' = C'D' = \frac{BC'}{2} = \frac{1-AC'}{2} = \frac{1-\left(\frac{1}{2} - \frac{1}{4} + \frac{x}{4}\right)}{2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{x}{8}$$



Fold A over D´ to get C´´:

$$AC'' = C''D' = \frac{AD'}{2} = \frac{1-BD'}{2} = \frac{1-\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{x}{8}\right)}{2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{x}{16}$$



Fold B over C´´ to get D´´:

$$BD'' = C''D'' = \frac{BC''}{2} = \frac{1-AC''}{2} = \frac{1-\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{x}{16}\right)}{2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{x}{32}$$

and so on. The following step (7^{th} fold) will give:

$$AC''' = \frac{1}{2^1} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} - \frac{1}{2^6} + \frac{x}{2^6} - \ldots.. = \mathrm{Lim}\ x$$

$$\frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \frac{1}{2^6} - \frac{1}{2^7} + \frac{x}{2^7} - \ldots.. = \frac{1}{2}\mathrm{Lim}\ x$$

It may be noted that the limit of x is made up of the algebraic sum of certain powers of 2, like 2^{-j} (for j values from 1 to n).

Adding up the former two expressions, we´ll have:

$$\left(1 + \frac{1}{2}\right)\mathrm{Lim}\ x = \frac{1}{2} + \frac{x}{2^6} - \frac{1}{2^7} + \frac{x}{2^7}$$

$$\frac{3}{2}\mathrm{Lim}\ x = \frac{1}{2} - \frac{1}{2^7} + \left(\frac{1}{2^6} + \frac{1}{2^7}\right)x = \frac{1}{2} - \frac{1}{2^7} + \frac{3}{2^7}x$$

Being n the number of folds, and generalising, we have:

$$\frac{3}{2}\ \mathrm{Lim}\, x = \frac{1}{2} - \frac{1}{2^n} + \frac{3}{2^n}x$$

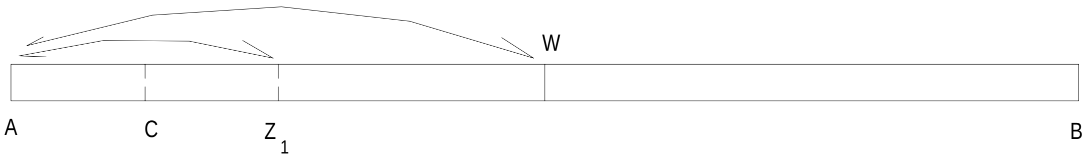
$$\mathrm{Lim}\ x = \frac{2}{3}\times\frac{1}{2} = \frac{1}{3}$$

$$n \to \infty$$

as wanted.

9.17.2 DIVISION IN 5 EQUAL PARTS

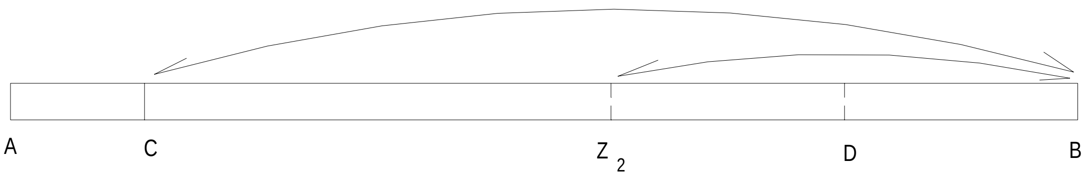
1



Exaggerating the position of starting point W, we have placed it at midpoint of AB. From A we fold, first up to W and then, up to Z_1. So we get C:

AB = 1 ; AC = x

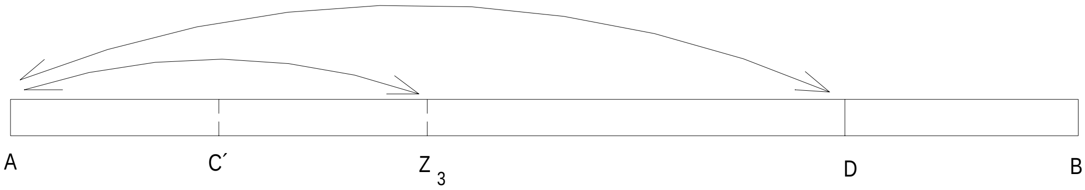
2



Fold B successively over C (getting Z_2), and over Z_2 getting D. It will be:

$$BD = \frac{1-x}{4}$$

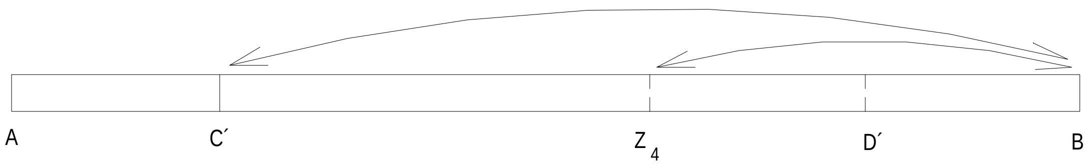
3



Similarly we get AC´:

$$AC' = \frac{1-BD}{4} = \frac{1}{4}\left(1-\frac{1}{4}(1-x)\right) = \frac{1}{4} - \frac{1}{16} + \frac{1}{16}x$$

4



Here BD´ is obtained; its value is:

$$BD' = \frac{1-AC'}{4} = \frac{1}{4}\left(1-\left(\frac{1}{4} - \frac{1}{16} + \frac{1}{16}x\right)\right) = \frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \frac{1}{64}x$$

The two following folds from A give:

$$AC'' = \frac{1-BD'}{4} = \frac{1}{4}\left(1-\left(\frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \frac{1}{64}x\right)\right)$$

$$AC''= \frac{1}{4}-\frac{1}{16}+\frac{1}{64}-\frac{1}{256}+\frac{1}{256}x$$

The latter expression gives us the formation law of AC ^{j} that is just the wanted limit of x:

$$\left.\begin{aligned} AC^{j} &= \frac{1}{2^2}-\frac{1}{2^4}+\frac{1}{2^6}-\frac{1}{2^8}+\frac{1}{2^8}x \\ \frac{1}{2^2}AC^{j} &= \qquad \frac{1}{2^4}-\frac{1}{2^6}+\frac{1}{2^8}-\frac{1}{2^{10}}+\frac{1}{2^{10}}x \end{aligned}\right\} \text{summing up}$$

$$\left(1+\frac{1}{2^2}\right)AC^{j} = \frac{1}{2^2}+\frac{1}{2^8}x-\frac{1}{2^{10}}+\frac{1}{2^{10}}x$$

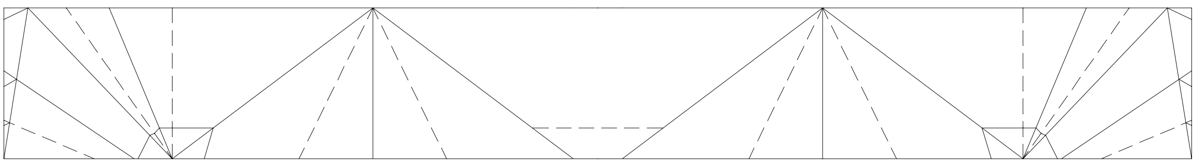
When n folds have been produced ($n \to \infty$), the three last terms of the 2^{d} member tend to zero. Thus:

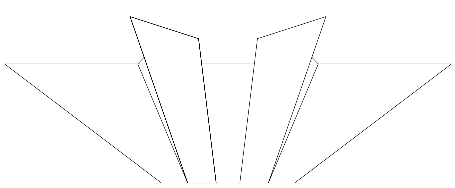
$$\left(1+\frac{1}{2^2}\right)\lim_{n \to \infty} x = \frac{1}{2^2} \qquad ; \qquad \text{Lim}\, x = \frac{4}{4\times 5} = \frac{1}{5}$$

The two preceding examples do not serve to generalise the method. In fact, Fujimoto designed a second method that is also rather complicate. The reader can realise that folding possibilities are infinite in practice: to alternate the starting from A or B; to repeat more times from one extremity than from the other; to play with simple, double, triple, etc. folds, and so on.

Binary numeration solves all difficulties associated to the division in equal parts.

Interlude





9.18 DIVISION OF A PAPER STRIP BY MEANS OF BINOMIAL NUMERATION

As we know, in binary numeration the only numbers that exist are 0 and 1. Besides, to convert an integer of the decimal system to the binary system, we must divide the integer and successive quotients by 2 till the moment a quotient 0 is reached (and, therefore a remainder of 1).

The resultant binary number is made up of those remainders: the first one (always 1, at left) is the last obtained and, successively towards the right, the others up to the first produced. Example:

13	2				$13 = 1101_{2)}$
1	6	2			
	0	3	2		
		1	1	2	
			1	0	

On the other hand, to convert a decimal fraction to binary system it´s better to follow the procedure shown in this example:

To convert into binary the decimal 43.42.

The first thing is to have available the successive powers of 2 (positive as well as negative).

$2^0 = 1$		$2^5 = 32$	$2^{-5} = 0.03125$
$2^1 = 2$	$2^{-1} = 0.5$	$2^6 = 64$	$2^{-6} = 0.015625$
$2^2 = 4$	$2^{-2} = 0.25$	$2^7 = 128$	$2^{-7} = 0.0078125$
$2^3 = 8$	$2^{-3} = 0.125$	$2^8 = 256$	$2^{-8} = 0.00390625$
$2^4 = 16$	$2^{-4} = 0.0625$	$2^9 = 512$	$2^{-9} = 0.001953125$

Then subtract the greatest possible power of 2, to the given number.

	43,42	
	- 32	
(2^5)	11,42	→ 1
	- 8	
(2^3)	3,42	→ 101
	- 2	
(2^1)	1,42	→ 10101
	- 1	
(2^0)	0,42	→ 101011,
	- 0,25	
(2^{-2})	0,17	→ 101011,01
	- 0,125	
(2^{-3})	0,045	→ 101011,011
	- 0,03125	
(2^{-5})	0,01375	→ 101011,01101

....

Put down 2^5 to the left of that result; this is the power taken into account; besides, one 1 at right (the first digit that always appears in any binary expression). Continue likewise. At the right of 3,42 repeat the 1 above and add **one** zero (**one** power of 2 has been dropped – 2^4 -); also add the 1 corresponding to the line we are dealing with.

It can be seen that the process would continue till the difference zero would be found, what not always happens.

With the bases above, let´s divide a paper strip in 7 equal parts. First of all we have to convert to binary the fraction $\frac{1}{7}$.

$$\frac{1}{7} = 0{,}1428571$$

2^{0}
2^{-1}
2^{-2}
2^{-3}

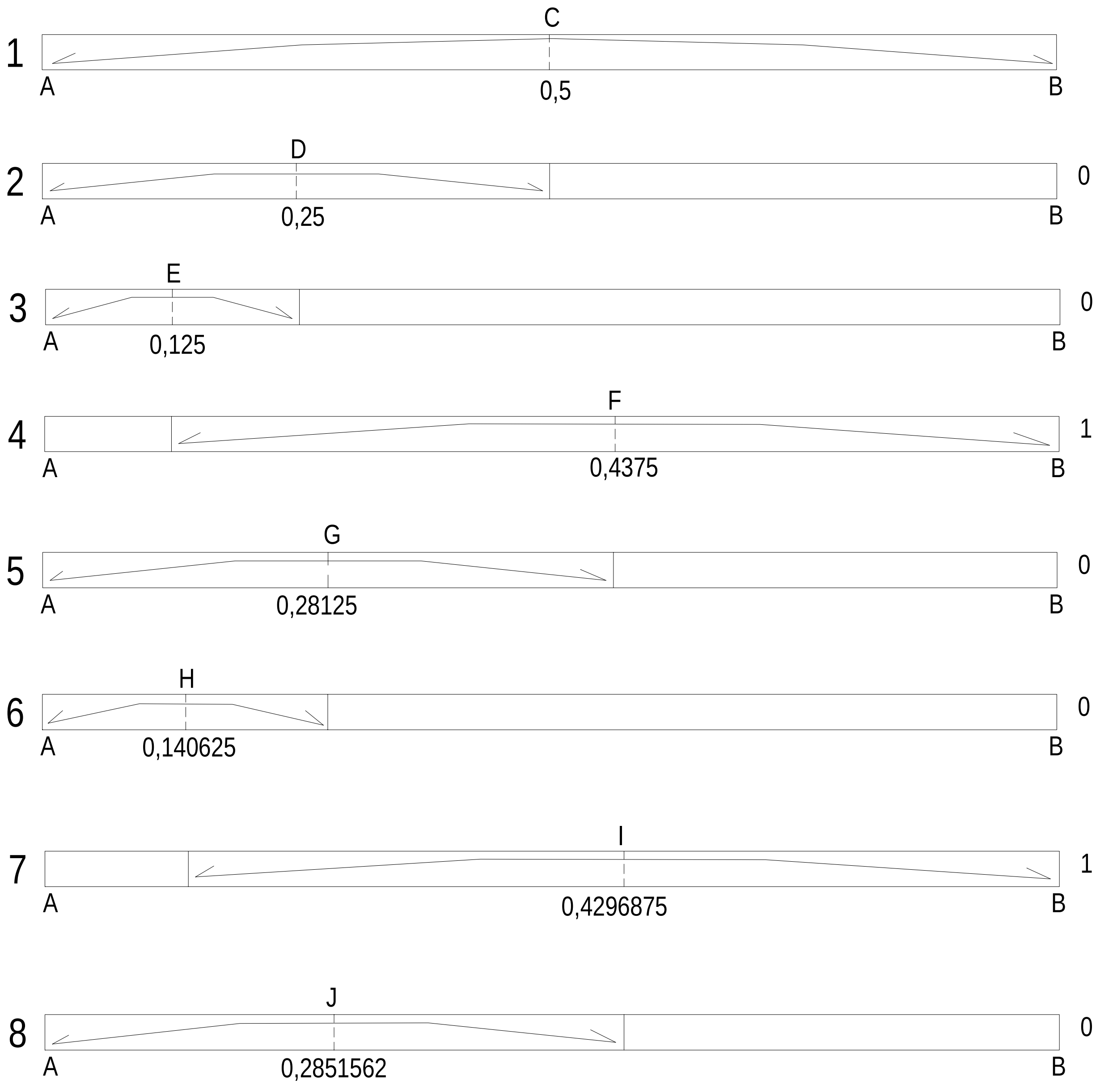
	0,1428571	
	- 0,125	
(2^{-3})	0,0178571	0,001
	- 0,015625	
(2^{-6})	0,0022321	0,001001
	- 0,001953125	
(2^{-9})	0,000278975	$0,001001\overset{\frown}{001} \equiv \dfrac{1}{7}$
		....

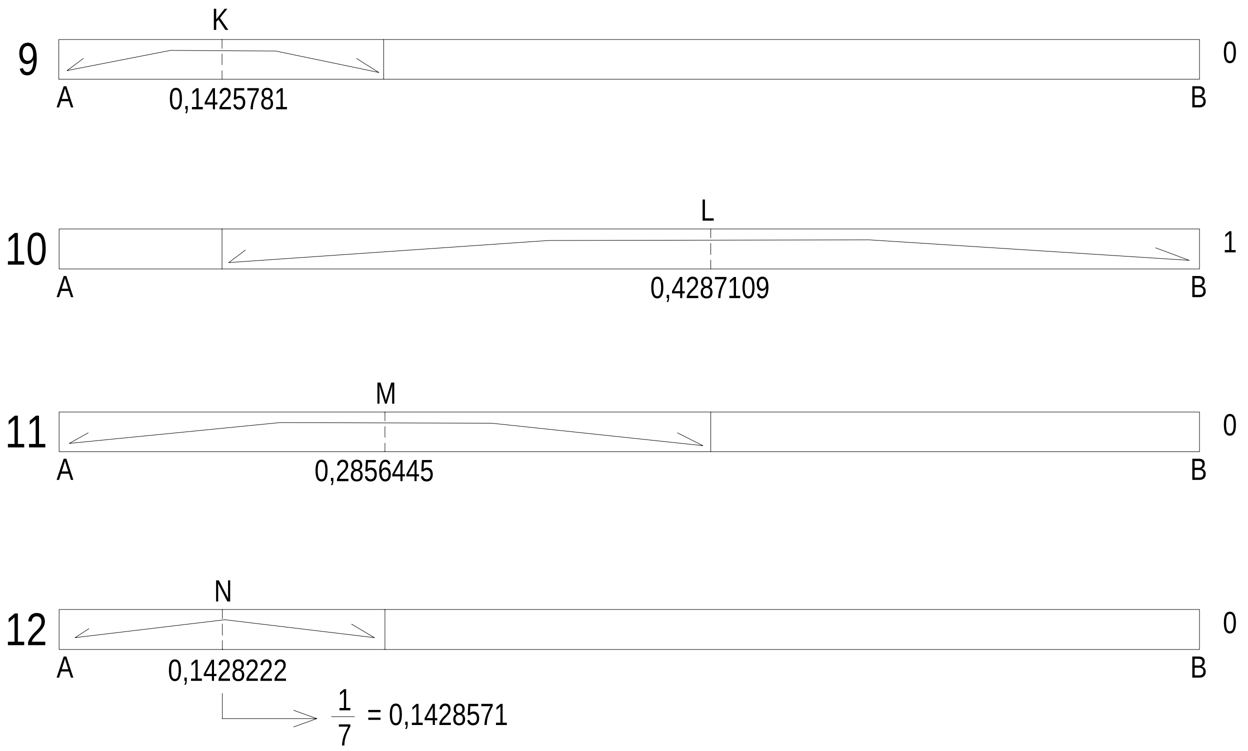
The succession of zeros and ones defines the folding sequence that was not so clear in Fujimoto´s method. That is gathered in the figures to follow.

The procedure is this: fold the end **A** over the last fold when there is a correspondence with a **0**, and end **B**, also over the last fold produced, if there is correspondence with an **1**.

1. Fold the strip by half to get $AC = \frac{1}{2} = 0{,}5$
2. and successive: assign zeros and ones as seen.
   A over C to get $AD = \frac{1}{4} = 0{,}25$
3. A ,, D ,, $AE = \frac{1}{8} = 0{,}125$
4. B ,, E ,, $BF = \frac{1}{2}(1 - AE) = \frac{1}{2} - \frac{1}{16} = 0{,}4375$
5. A ,, F ,, $AG = \frac{1}{2}(1 - BF) = \frac{1}{2} - \frac{1}{4} + \frac{1}{32} = 0{,}28125$
6. A ,, G ,, $AH = \frac{1}{2}AG = \frac{1}{4} - \frac{1}{8} + \frac{1}{64} = 0{,}140625$
7. B ,, H ,, $BI = \frac{1}{2}(1 - AH) = \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{1}{128} = 0{,}4296875$
8. A ,, I ,, $AJ = \frac{1}{2}(1 - BI) = \frac{1}{2} - \frac{1}{4} + \frac{1}{16} - \frac{1}{32} + \frac{1}{256} = 0{,}2851562$
9. A ,, J ,, $AK = \frac{1}{2}AJ = \frac{1}{4} - \frac{1}{8} + \frac{1}{32} - \frac{1}{64} + \frac{1}{512} = 0{,}1425781$
10. B ,, K ,, $BL = \frac{1}{2}(1 - AK) = \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{1}{64} + \frac{1}{128} - \frac{1}{1024} = 0{,}4287109$
11. A ,, L ,, $AM = \frac{1}{2}(1 - BL) = \frac{1}{2} - \frac{1}{4} + \frac{1}{16} - \frac{1}{32} + \frac{1}{128} - \frac{1}{256} + \frac{1}{2048} = 0{,}2856445$
12. A ,, M ,, $AN = \frac{1}{2}AM = 2^{-2} - 2^{-3} + 2^{-5} - 2^{-6} + 2^{-8} - 2^{-9} + 2^{-12} = 0{,}1428222$

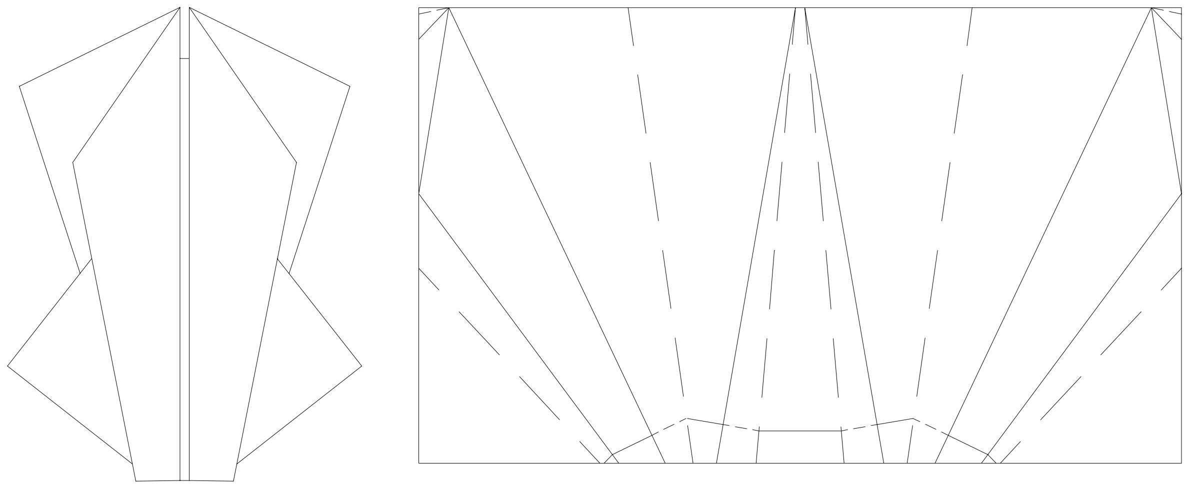
That is, values AE, AH, AK, AN ... tend to $\frac{1}{7}$, which is also the algebraic sum of terms like 2^{-j} (see Point 9.17.1).
If we want to go on with the approximation process, it will be useful to note that the binary expression of $\frac{1}{7}$ has a period of 001 which simplifies that process.





In the case of producing 15 folds, we would have approached $\frac{1}{7}$ up to the value of 0,1428527.

Interlude



10 REGULAR CONVEX POLYGONS WITH MORE THAN 4 SIDES

Our attention will focus on folded constructions leaving aside the classic that use rule and compass.

It is impossible to present the whole variety of origami solutions because of lack of space: as far as pentagon only is concerned, well overt 10 different solutions may be registered.

What we shall do is to discuss various solutions because not all of them are perfect under the point of view of geometry. Origami can deal with most of them though, even with the added handicap of folding difficulties and the accumulation of overlapped paper.

Nevertheless we are convinced that the capacity to digest errors inherent to origami should not impair to distinguish exact from approximate solutions; besides, some of these are more imperfect than others.

10.1 PENTAGON

10.1.1 FROM ONE ARGENTIC RECTANGLE

Before continuing I shall indulge myself in a semantic digression. In British origami literature, the DIN A rectangle is called *silver rectangle*: of course I have nothing to object (see D. Brill´s *BRILLIANT ORIGAMI*).

Since DIN A rectangles are well enough defined as such, I do not think they need of any added qualification despite of their $\sqrt{2}$ singularity.

On the contrary, I do fill it necessary to name *argentic rectangle* that special one whose diagonal and small side form an angle half of that of a regular convex pentagon. I like to put together, semantically speaking, those singular rectangles: *argentic* and *auric*.

To keep alive two synonyms like *silver* (taken as adjective) and *argentic* may lead us to confusion though we are aware that synonyms may have important differentiating nuances, as the present case manifests.

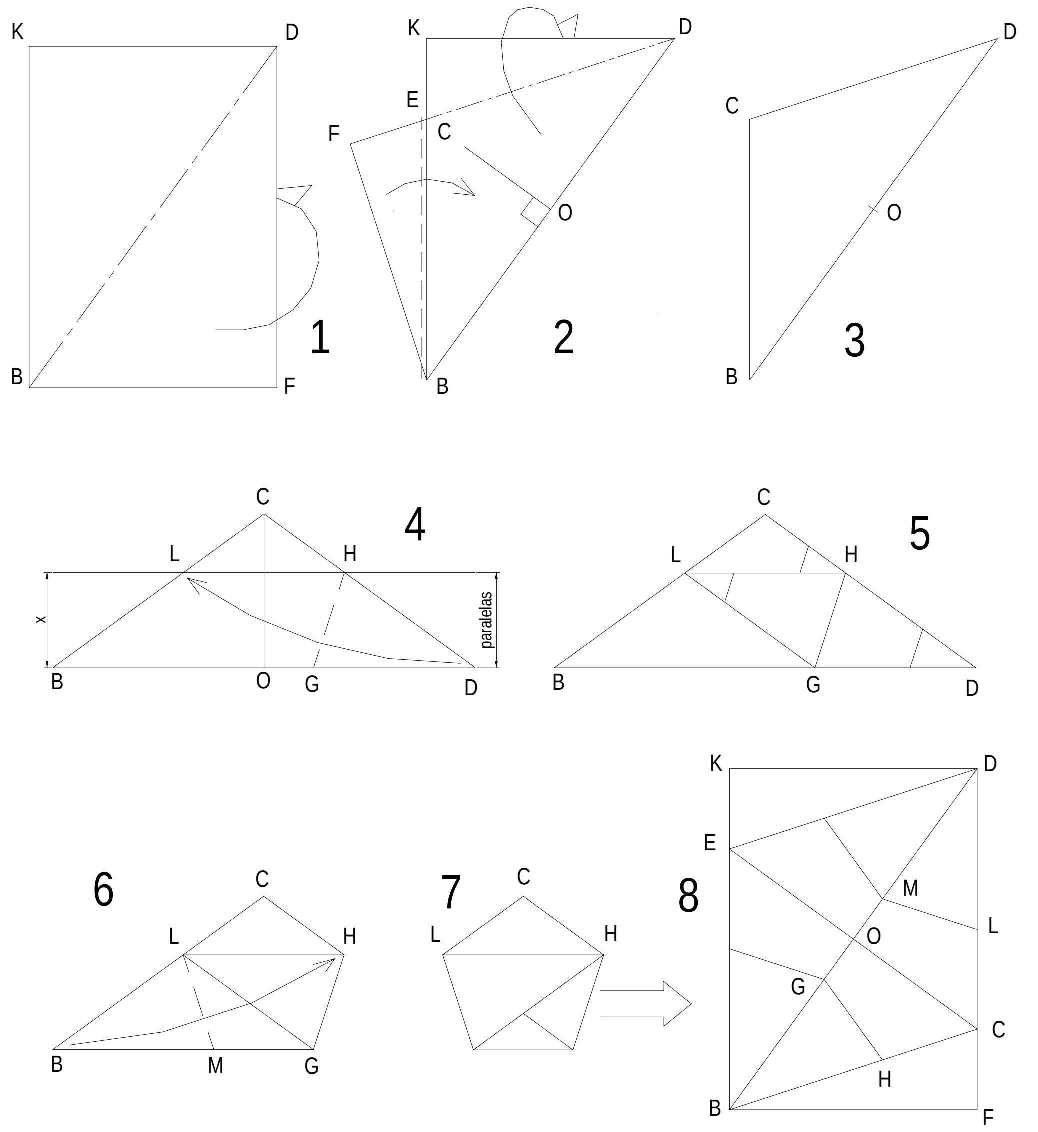
Former discussion would be useless not to be for the fact that in the above-mentioned publication, both *silver* and what I call *argentic* rectangles are mixed up. In it, pentagonal and pentagonododecahedric properties are associated to the first one, what is incorrect, but do correspond exclusively to the second. This is so, though the inexactitude is admited by the author of said publication. Let´s go, anyway, to look for the solution we are interested in.

It is a perfect geometric solution produced from an *argentic* rectangle whose sides are proportional to $\sqrt{5+2\sqrt{5}}$ (small side) and $2+\sqrt{5}$ (large side). See its construction in Point 6.6.7.

To give an idea of those proportions, we may say that they are equivalent to 210 (width of DIN A4) $\times$ 289,04021.

REMARKS TO FOLDING PROCESS.

1- BF = 1, therefore:
   FD = 1.3763819 ; BD = 1.7013016 ; Ang. DBF = g = 54º ; Ang. BDF = p = 36º
2- Ang. DCB = 2g = 2 $\times$ 54 = 108º (angle of convex regular pentagon).
3- Isosceles Δ DCB is produced: Ang. CDB = Ang. CBD = Ang. BDF = p
4- Fold D to lie on CB being HL parallel to BD. Folding operation has to be made by the rule of thumb (it´s easy to get a good result). Its analytical development is as follows:



$$CO = \frac{BD \operatorname{tg} B}{2} \qquad ; \qquad LH = HD = LB$$

$$\left.\begin{array}{l} \operatorname{sen} B = \dfrac{x}{HD} \\[3em] \operatorname{tg} B = \dfrac{CO - x}{\dfrac{LH}{2}} \end{array}\right\} \cos B = \frac{x}{2(CO - x)}$$

$$x = 2CO\cos B - 2x\cos B \qquad ; \qquad x = \frac{2CO\cos B}{1 + 2\cos B}$$

5- ΔLHG = ΔDHG by symmetry

Ang. CHG = Ang. CHL + Ang. LHG = $B + \dfrac{180 - B}{2} = 90 + \dfrac{B}{2} = 108^{o}$

6- Ang. CLM = Ang. CHG because of the symmetry with former step.
Likewise, in pentagon CLMGH, Angs. in M and G are congruent, also by symmetry.
7- A pentagon has been produced whose angles in L, C and H are congruent and have a measure of 108º; the other two angles are congruent to each other. Then, the 5 angles measure 108º: we have come across a convex regular pentagon that will have congruent its five sides.
8- Unfolding 7 we get 8 which shows the resultant pentagon CLMGH (and its symmetric).

AN INTERESTING VERIFICATION

The rhomb DHLG was formed in Fig. 5: it is made up by the two isosceles triangles with base HG. We can see in it that GD is the pentagon diagonal and GH is one side. Recalling the x value obtained at step 4, we´ll have:

$$\left.\begin{array}{l} \dfrac{x}{HD} = \text{sen}\, B \\[2ex] \dfrac{\frac{GH}{2}}{HD} = \text{sen}\,\dfrac{B}{2} \end{array}\right\} \qquad \left.\begin{array}{l} HD = \dfrac{x}{\text{sen}\, B} \\[2ex] GH = 2\,\text{sen}\,\dfrac{B}{2}\,\dfrac{x}{\text{sen}\, B} \end{array}\right\}$$

Substituting values from steps 1 and 4, we have:

HD = 0,6498393 ; GH = 0,4016228

These values satisfy the expression

$$GH^2 = (HD - GH)HD$$

This way it is verified that in a convex regular pentagon the diagonal is divided by the side in mean and extreme ratio.

10.1.2 FROM A DIN A4 RECTANGLE

Once developed the former Point, we can see now the inexactitude of present solution. For that purpose it´s enough to look at figs. 1 and 2 of Point 10.1.1.

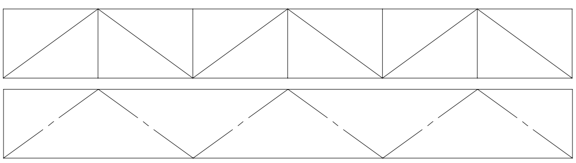
In case rectangle 1 was a DIN A4, we´ll have:

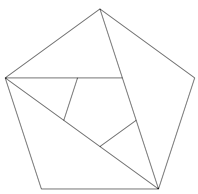
$$g = Arc\,\text{tg}\,\frac{\sqrt{2}}{1} = 54{,}73561º$$

and in fig. 2:

Ang. DCB = 2g = 109,47122 ≠ 108º

10.1.3 FROM A PAPER STRIP MADE OUT OF ARGENTIC RECTANGLES

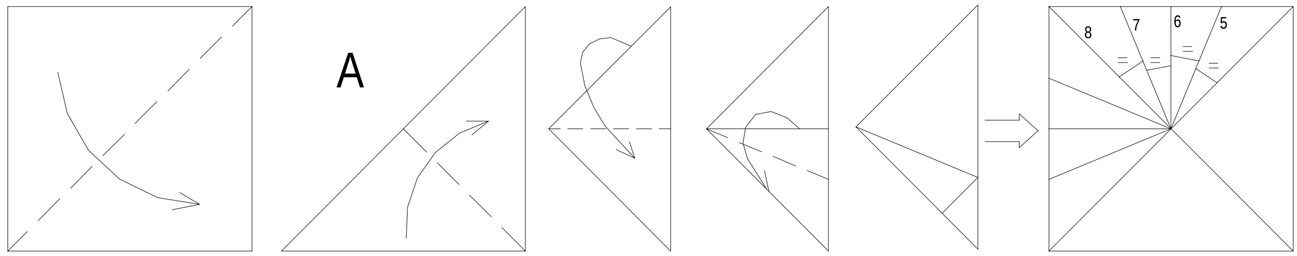
1



2

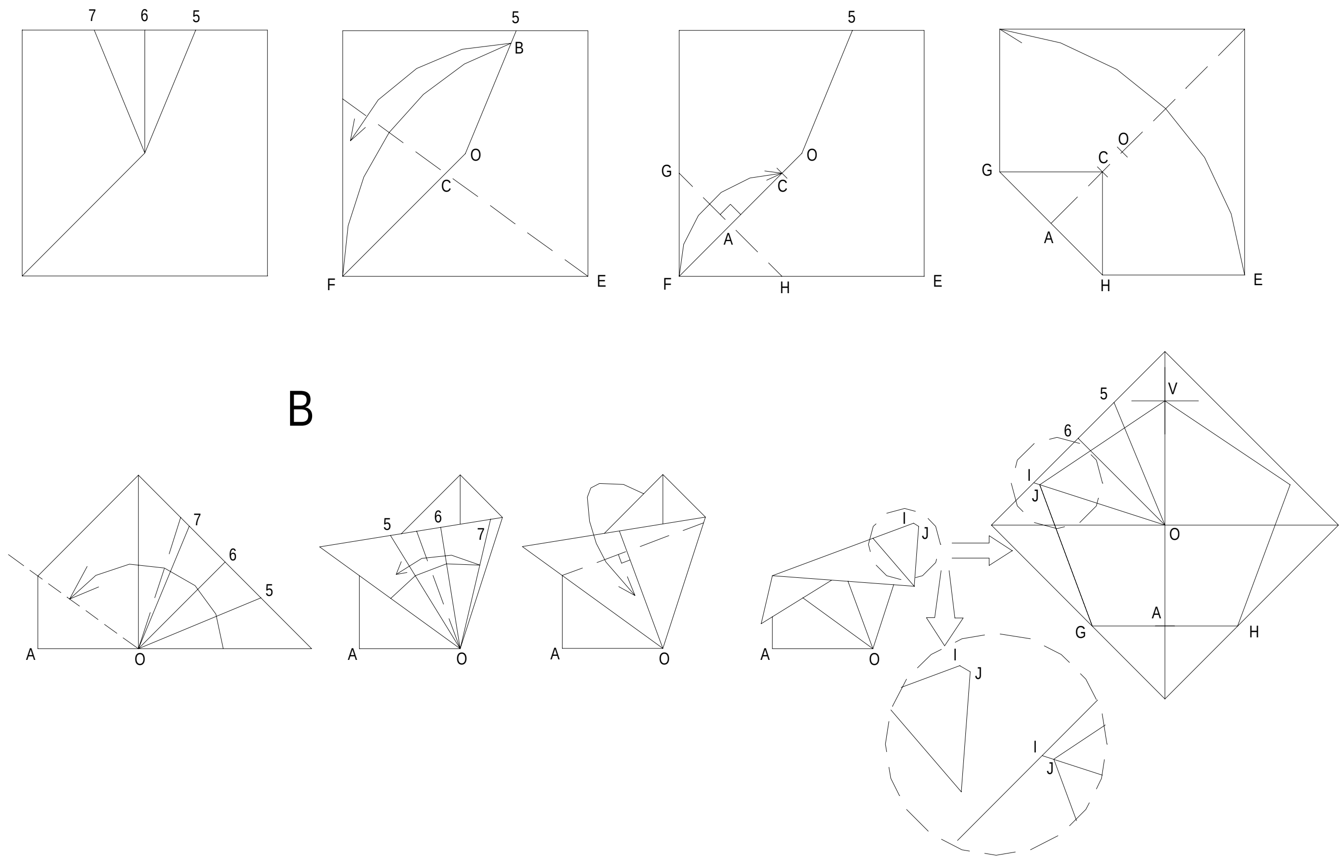
Produce six adjacent argentic rectangles after fig.1. Zigzag fold its diagonals, eliminate the small sides, mountain fold the diagonals and finally glue the end triangle equivalent to fig. 3 in Point 10.1.1.

We obtain the “perforated” pentagon of fig.2 whose side is congruent with the diagonal of the argentic rectangle we began with.

10.1.4 PENTAGON WITH A PREVIOUS FOLDING



Process shown in figs. A is common to pentagon, hexagon, heptagon and octagon.



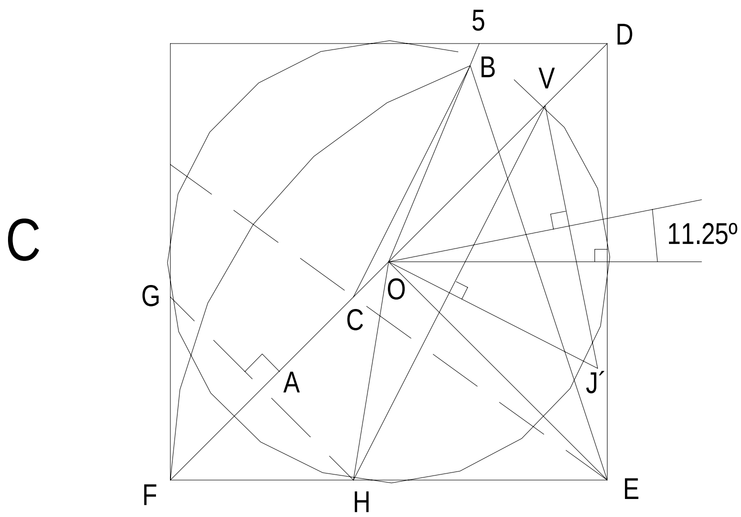
The process comprising the ten figs. B is specific for the pentagon and starts from last fig. A. This process B leads to an imperfect pentagon.

Fig. C contains all the elements from B which are necessary to easily evaluate that imperfection. Since that evaluation is rather cumbersome and in order not to bore the reader, we shall present just the final results. Taking GH as base of the pentagon, we have:

Angle in V, opposite to the base: 112,5º

Adjacent angles to the base, congruent to each other and measuring 110,395º

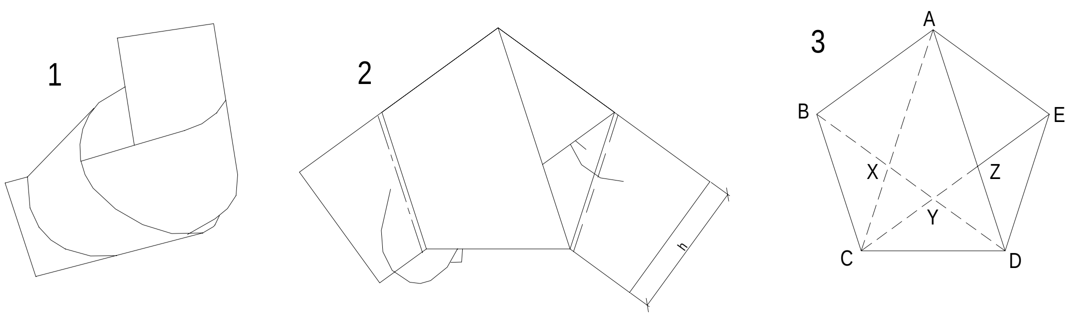
The other two angles are also congruent to each other, but with a value of 103,355º.



Taking base GH as reference, the other four sides are congruent to each other and have a length of 103,52 % of the base.

10.1.5 KNOT TYPE PENTAGON

Start with a paper strip long at least six times its width h. The pentagon is obtained just producing a knot centred on the strip (1): as it was a string, but adequately adjusting the folds. Do not flatten till the vertices are properly fastened. The result is a perfect convex regular pentagon as we are going to see (2).



First, note in (3), the three quadrilaterals

AXDE ; ABYE ; ABCZ

The three of them are parallelograms: They are determined by overlapping of two portions of the strip, obviously, of the same width. If we recall Point1.3.2 fig.1, we´ll see that those parallelograms have congruent opposite sides and also congruent adjacent sides.

Therefore, if ABCZ has an area S, it is:

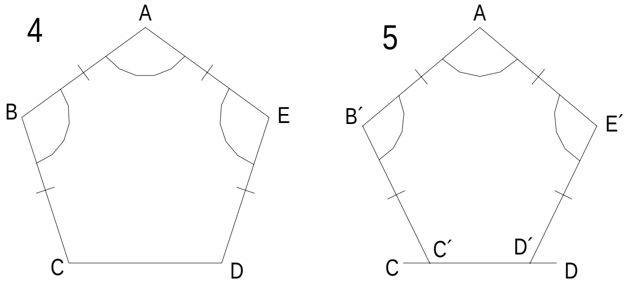
$$S = BC \times h = AB \times h \quad \text{that is} \quad BC = AB$$

Consequently, the three parallelograms are congruent rhombs with all their respective sides and angles also congruent.

As these three rhombs display some common sides, we are able to mark in pentagon (4) sides and angles congruency:

$$CB = BA = AE = ED = l \quad ; \quad \text{Ang. B} = \text{Ang. A} = \text{Ang. E}$$

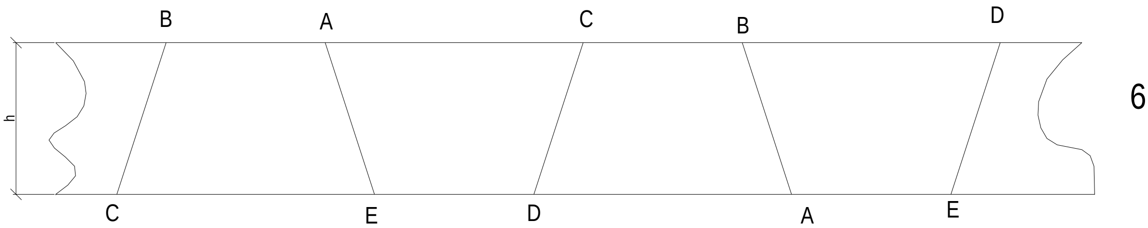
From the above we can´t jump to the conclusion that the pentagon is a regular one: we know of equalities but not of values. Last equalities are also in pentagon (5) that is not regular.



Let´s note now in (3) the four trapeziums

ABCE ; ABDE ; ACDE ; ABCD

Incidentally, when fig. 3 is unfolded, those four trapeziums appear as shown in fig. 6.



Trapeziums ABCE and ABDE are isosceles for both have:

- Their parallel bases distant h from each other.
- Big angles, congruent.
- Skew sides and small base, congruent.

Former conditions lead to trapeziums congruency. Then their diagonals should be congruent too:

$$AC = AD$$

Let´s have a look now to the trapeziums ACDE and ABCD that in turn have equal:

- The altitude h as the distance between their parallel bases.
- Their great bases AD = AC
- Their small bases BC = ED
- The small base congruent with a skew side: BC = AB = DE = AE
- The great angle formed by the small base and one skew side: Ang.B = Ang. E

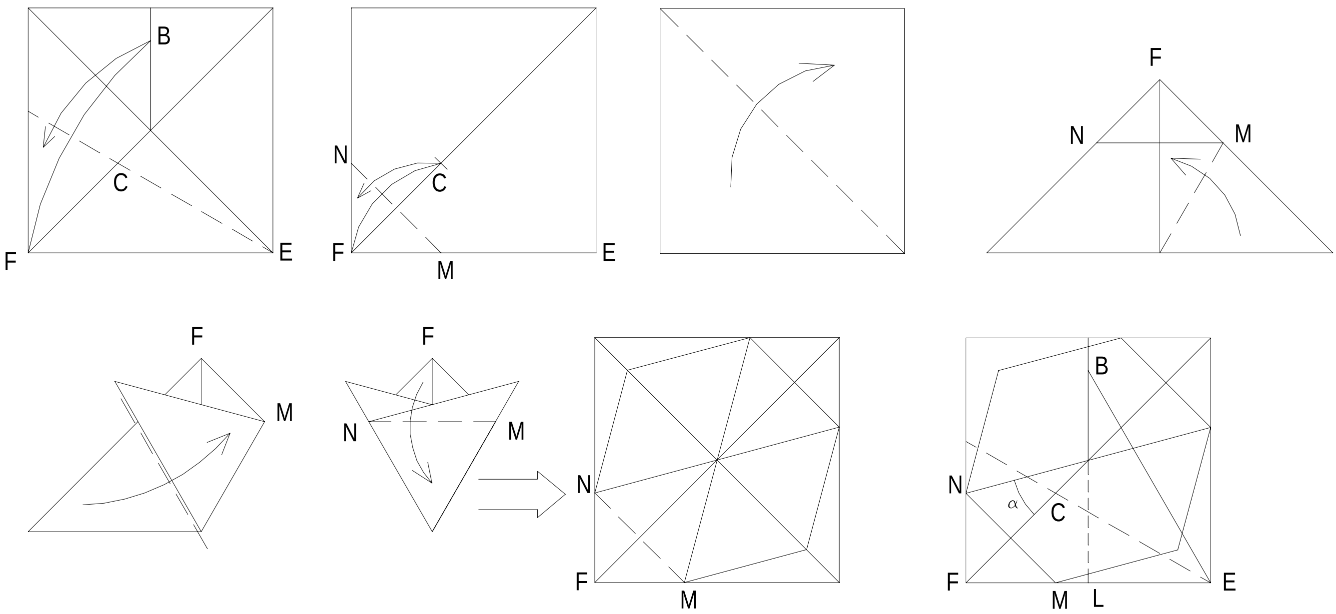
Consequently these trapeziums are congruent with each other, and also with the other pair of former trapeziums.

Therefore Ang. C = Ang. D, and congruent with the other four angles of the pentagon. As CD = l, pentagon ABCDE is a regular one for all its sides and angles are, respectively, congruent.

10.2 HEXAGON

10.2.1 WITH A PREVIOUS FOLDING

The folding process that follows leads to a perfect convex regular hexagon that is also the largest obtainable from the starting square.



In the last figure we can see the perfection of the hexagon produced (l = MN).

In ΔEBL:

$$\text{Ang. BEF} = Arc\cos\frac{l}{2l} = Arc\cos\frac{1}{2} = 60^{\circ} \quad ; \quad \text{Ang. CEF} = 30^{\circ}$$

In ΔCFE:

$$\frac{CF}{\operatorname{sen} 30} = \frac{l}{\operatorname{sen}(180-45-30)} \quad ; \quad CF = NM = \frac{l}{2\operatorname{sen} 75} = \text{side of hexagon}$$

$$\alpha = Arc\,\mathrm{tg}\frac{\frac{CF}{2}}{\left(\frac{l\sqrt{2}}{2} - \frac{CF}{2}\right)} = Arc\,\mathrm{tg}\frac{CF}{l\sqrt{2} - CF}$$

$$\alpha = Arc\,\mathrm{tg}\frac{l}{2\operatorname{sen} 75\left(l\sqrt{2} - \frac{l}{2\operatorname{sen} 75}\right)} = Arc\,\mathrm{tg}\frac{1}{2\sqrt{2}\operatorname{sen} 75 - 1} = 30^{\circ}$$

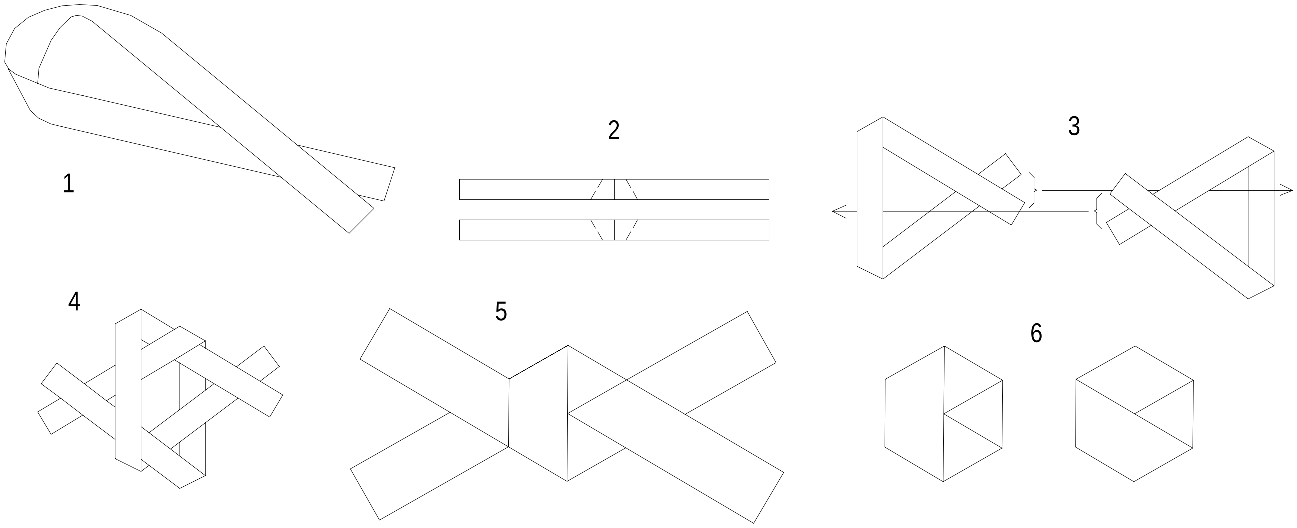
From what was shown in corner F and because of folding symmetries, we can deduce the perfection for the rest of the hexagon.

10.2.2 KNOT TYPE HEXAGON

Begin with two paper strips long 10 to 15 times its width. Produce two knots like in fig.1; one of them will be as shown; the other will be alike but set upside down.

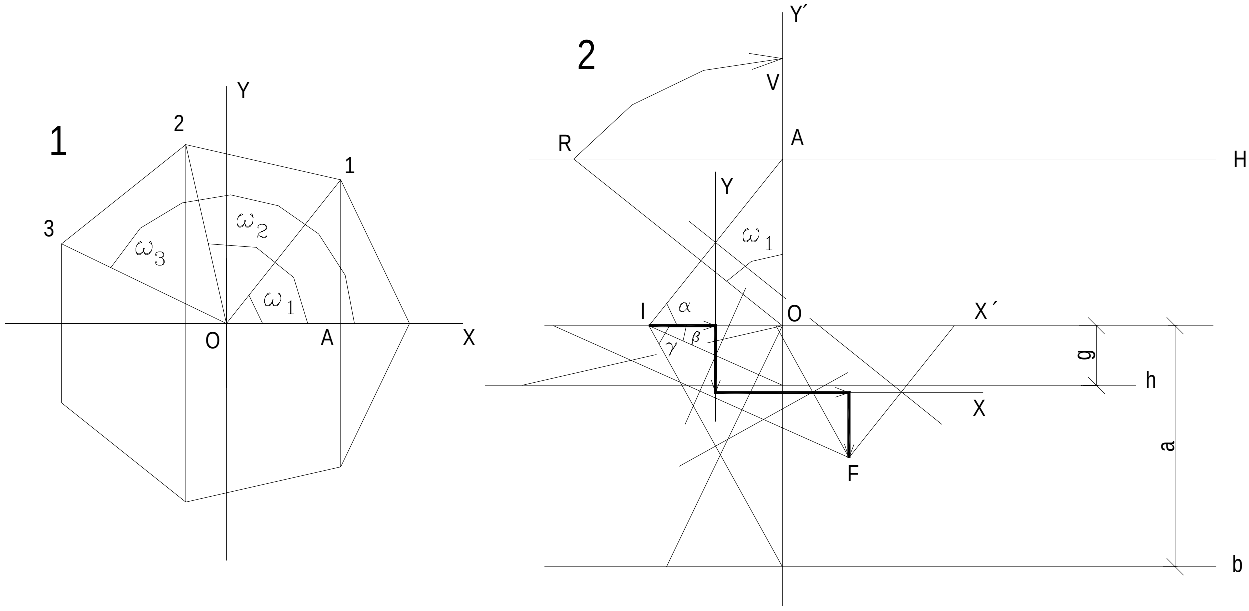
Figs. 2, 3 and 4 show flattened folds just to ease the drawing; actually, the strips are played loose as in fig 1. Only in fig. 5 the strips should be pulled tight and flattened (see precautions in Point 10.1.5).

If the length of the strip is great, the process is facilitated; the remaining paper (fig.5) can be hidden within the polygon itself. This way we produce fig. 6 that is shown in its obverse and reverse sides.



10.3 HEPTAGON

10.3.1 H. HUZITA´S SOLUTION



Let the regular heptagon of fig. 1 with radius one unit and central angle ω ($\omega_1 = \omega$; $\omega_2 = 2\omega$; $\omega_3 = 3\omega$).

Being a close polygonal line, the sum of the abscissas of its seven vertices will add up to zero:

$$1 + 2\cos\omega + 2\cos 2\omega + 2\cos 3\omega = 0 \qquad (1)$$

Reminding that:

$$\left.\begin{aligned} \cos 2\omega &= \cos^2\omega - \operatorname{sen}^2\omega = 2\cos^2\omega - 1 \\ \cos 3\omega &= 4\cos^3\omega - 3\cos\omega \end{aligned}\right\}$$

and making the variable change

$$t = 2\cos\omega \quad ; \quad \cos\omega = \frac{t}{2} \qquad (2)$$

(1) is transformed to:

$$t^3 + t^2 - 2t - 1 = 0 \qquad (3)$$

Equation (3) was already "fold solved" in Point 7.14.5.

It is clear that the coefficients of (3) are:

1; p = 1; q = -2; r = -1

what leads to a discriminant

$$\Delta = \frac{1}{4}\left[\frac{2p^3 - 9pq + 27r}{27}\right]^2 + \frac{1}{27}\left[\frac{3q - p^2}{3}\right]^3$$

that is negative, i.e., to have three real solutions for (3). In consequence, there are three different forms of simultaneous folding of point I (initial) and F (final) on axles X´ Y´ (fig 2).

According to (3), the vectors sequence in fig. 2 is:

I; 1; 1; -2; -1; F

Now we should explain how to configurate the direction of those vectors (fig. 2): if, from one coefficient to the next there is no sign change, the advancing vector turns to the right; if that change exists, it turns to the left.

Then (3) has three solutions in t, which correspond with other three solutions for cos ω in (1); besides, t_1 t_2 t_3 are, respectively, the tangents of angles α, β, γ in fig.2:

$t_1 = \text{tg}\,\alpha \rightarrow 2\cos\omega_1$

$t_2 = \text{tg}\,\beta \rightarrow 2\cos\omega_2$

$t_3 = \text{tg}\,\gamma \rightarrow 2\cos\omega_3$

Angles ω_1 ω_2 ω_3 correspond to the heptagon vertices having distinct abscissas.

Relating (2) with fig. 2, it is

$$\cos\omega_1 = \frac{t_1}{2} = \frac{\text{tg}\,\alpha}{2} = \frac{AO}{2\times 2} = \frac{AO}{4} \qquad (4)$$

If the pair of heptagon upper vertices lie on AH, when we trace $\omega_1 = \frac{360}{7}$, OR becomes the radius of said heptagon, and it´ll be:

$$\cos\omega_1 = \frac{OA}{OR} \qquad (5)$$

The consequence of equalising (4) and (5) is:

r = heptagon radius = OR = 4

Bringing r from O along OY´ we obtain vertex V.

Now we can note this:

- OR measures 4 units; in fig. 2, first vector from I towards vertical Y, is taken as one unit.
- Angles α and ω_1 have a different though very close mesure. This is unimportant, anyway: $\omega_1 = \frac{360}{7} = 51{,}428571$ and $\alpha = \text{arctg}\left[2\cos\frac{360}{7}\right] = 51{,}272558$.

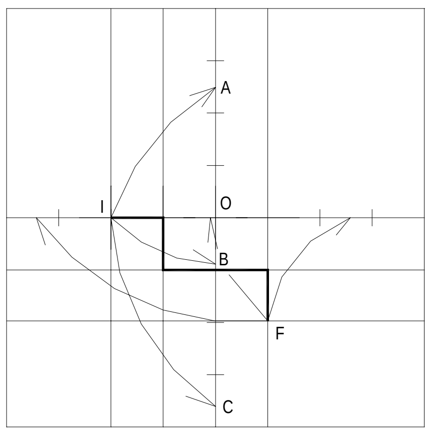
The above only means that radius OR is not parallel to its associated fold, although its construction is perfect.

Something similar can be said about lines h and b which determine the other vertices of the heptagon:

$$\left.\begin{aligned} & t_2 = \operatorname{tg}\beta = \frac{g}{2} = 2\cos 2\omega_1 \\ & \cos[\pi - 2\omega_1] = \frac{-g}{r} \\ & \cos 2\omega_1 = \frac{g}{r} \end{aligned}\right\} \quad \frac{g}{2} = \frac{2g}{r} \quad ; \quad r = 4$$

$$\left.\begin{aligned} & t_3 = \operatorname{tg}\gamma = \frac{a}{2} = 2\cos 3\omega_1 \\ & \cos[\pi - 3\omega_1] = \frac{-a}{r} \\ & \cos 3\omega_1 = \frac{a}{r} \end{aligned}\right\} \quad \frac{a}{2} = \frac{2a}{r} \quad ; \quad r = 4$$

That is, in the case we are discussing, we always reach 4 for the value of the heptagon radius.



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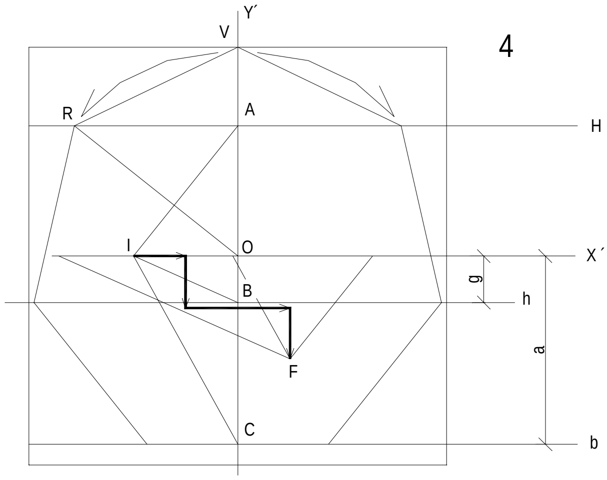


4

Therefore, the heptagon folding process will be as follows:

<u>Figure 3</u>

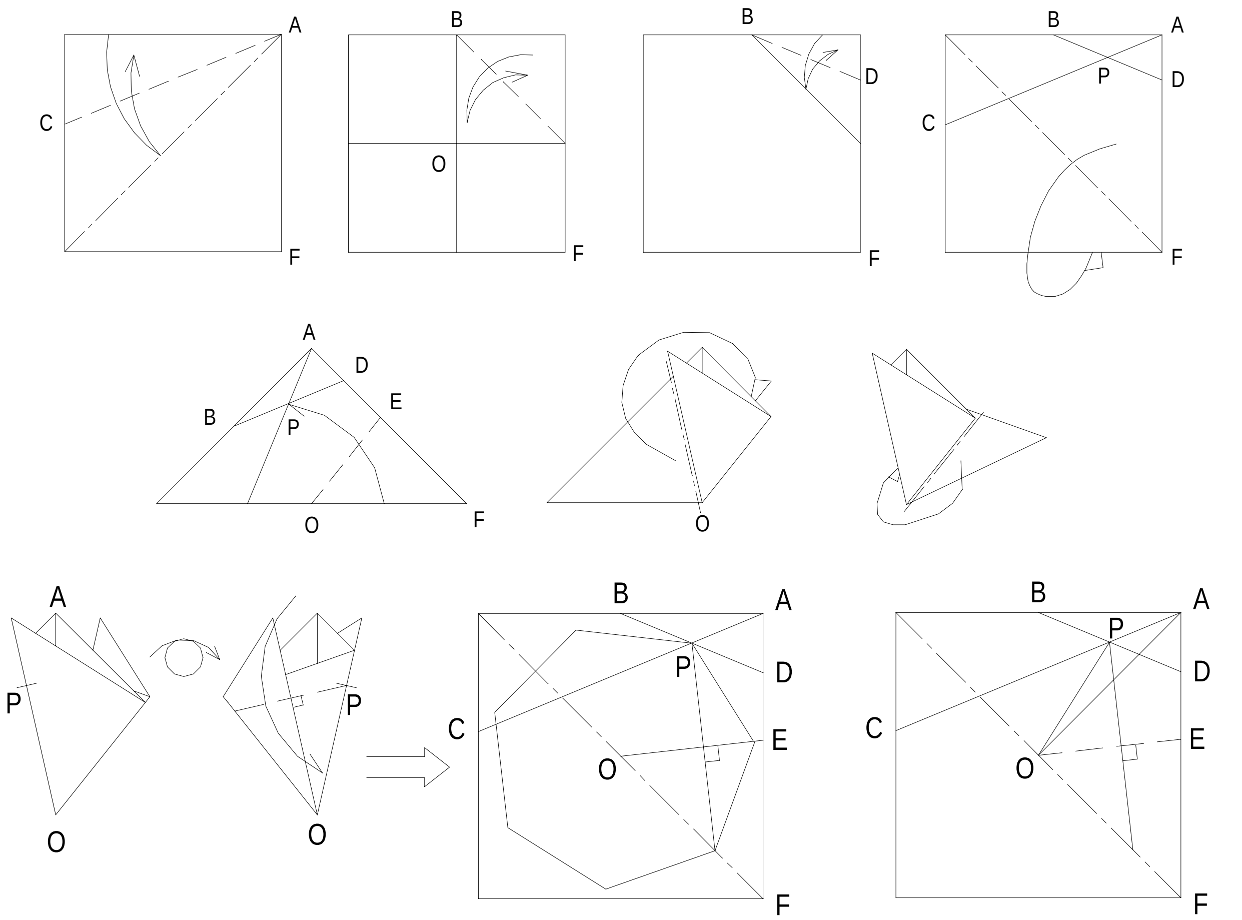
- Begin with a square of paper.
- Set on it the required quadrille centred at O, to get points I (-2,0) and F (1,-2).
- **Simultaneously** fold over the respective axles: I on the ordinates and F on the abscissas. And that, in the three possible ways.
- Thus, key points A, B, C are obtained.

<u>Figure 4</u>

- Get the folds: AH (through A); h (through B); b (through C).
- Fold: V → AH; O → O to get vertex R and its symmetric.
- Same: R → h; O → O to get both vertices on h.
- Fold around O the latter couple of vertices, to lie on b.

10.3.2 HEPTAGON: A QUASI-PERFECT SOLUTION

After the analysis of five different solutions, the one that is presented now is, no doubt, the best.



The last but one figure shows the obtained heptagon; from the last one we can figure out the value of the angles of said heptagon (the side of the given square equals 1).

Isosceles ΔBAP has Ang. B = 22,5º

$$PA = \frac{1}{4\cos 22{,}5} = 0{,}270598$$

In ΔAPO:

PA = 0,270598 ; Ang. A = 22,5º ; $AO = \dfrac{1}{\sqrt{2}}$

$$\left.\begin{array}{l} \dfrac{PA}{\operatorname{sen} O} = \dfrac{AO}{\operatorname{sen} P} \\[2ex] \dfrac{PO}{\operatorname{sen} A} = \dfrac{PA}{\operatorname{sen} O} \\[2ex] A + O + P = 180 \end{array}\right\}$$

Solving the system:

Ang. POA = 12,764389º

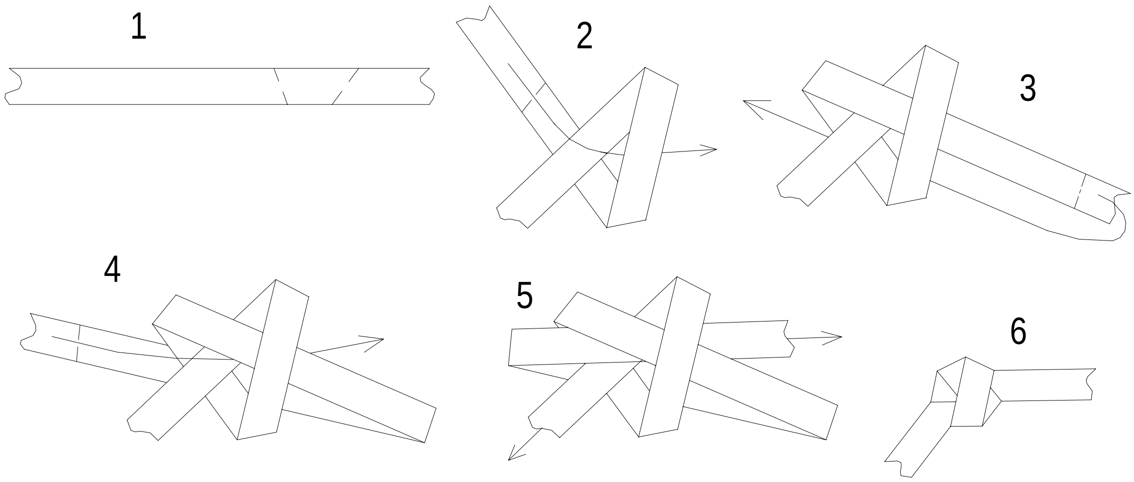
Heptagon resultant angle:

$$\text{Ang. EOF} = \text{Ang. EOP} = \frac{1}{2}(90 + POA) = 51{,}38219^{\text{o}}$$

Regular heptagon central angle: $\frac{360}{7} = 51{,}428571^{\text{o}}$

10.3.3 HEPTAGON: KNOT TYPE SOLUTION

Like all of this kind, it is a perfect one.



Recall the remark in Point 10.2.2 regarding docility and flattening of paper strip.

10.4 OCTAGON

It becomes a perfect geometrical construction as we´ll see immediately. Again, square side AC is one unit.

In ΔAOB:

$$\text{tg } A = \frac{OB}{OA} = \frac{OC - CB}{OA} = \frac{\frac{1}{\sqrt{2}} - ED}{\frac{1}{\sqrt{2}}} = 1 - ED\sqrt{2} \qquad (1)$$

and in ΔABD:

$$\text{tg } A = \frac{BD}{DA} = \frac{\frac{ED}{\sqrt{2}}}{1 - \frac{ED}{\sqrt{2}}} = \frac{ED}{\sqrt{2} - ED} \qquad (2)$$

Equalising (1) and (2), and making ED = k (octagon side), it is:

$$1 - k\sqrt{2} = \frac{k}{\sqrt{2} - k}$$

which leads to the following quadratic equation in k:

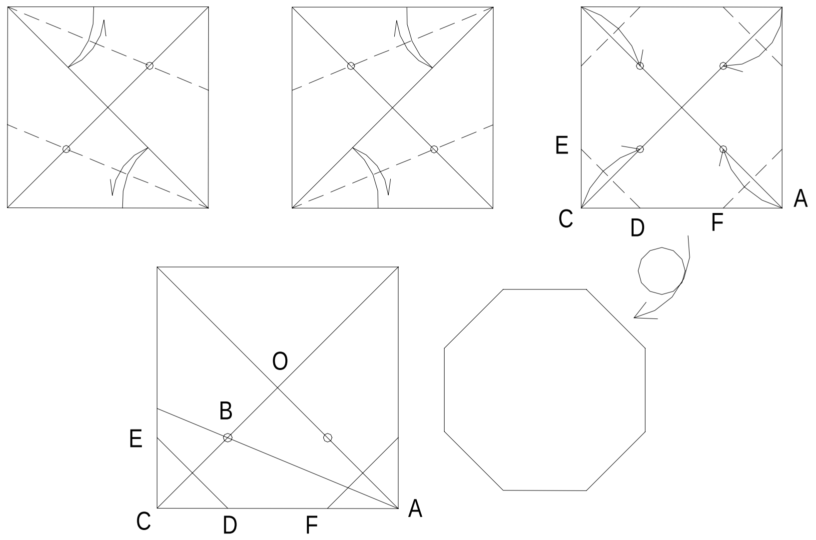
$$\sqrt{2}k \quad - 4k + \sqrt{2} = 0$$

whose roots are:

$$\sqrt{2} + 1 \ \text{ y } \ \sqrt{2} - 1$$

As said at the beginning, square side is 1, so we must ignore the first root for it´s greater than 1. Therefore, it is:

$$k = \sqrt{2} - 1$$



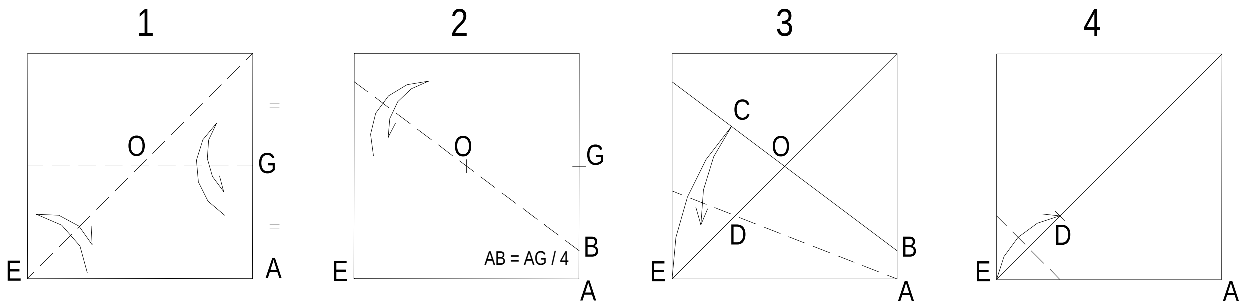
Let´s see the value we get for DF:

$$DF = 1 - 2\left(\frac{\sqrt{2} - 1}{\sqrt{2}}\right) = \sqrt{2} - 1$$

i.e. DF = k what means that the octagon is coherent with the folding process.

10.5 ENNEAGON

The present solution has been chosen as the best among several others, but nevertheless it results imperfect.



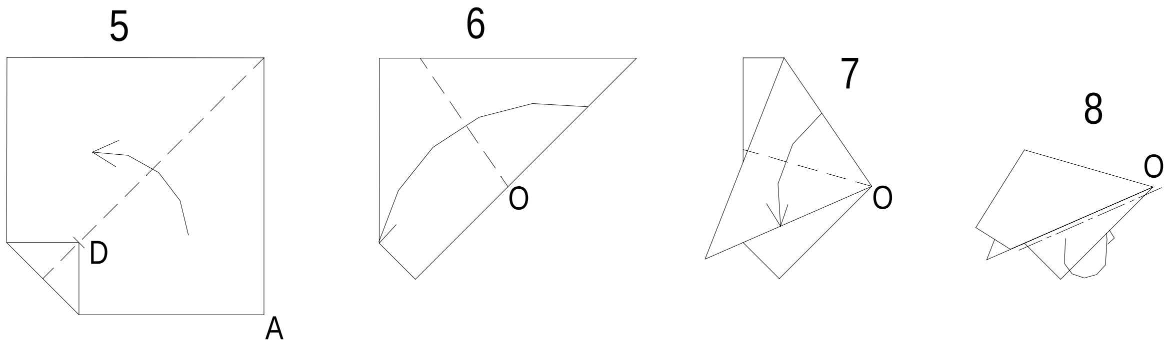
Justification (EA = 1):

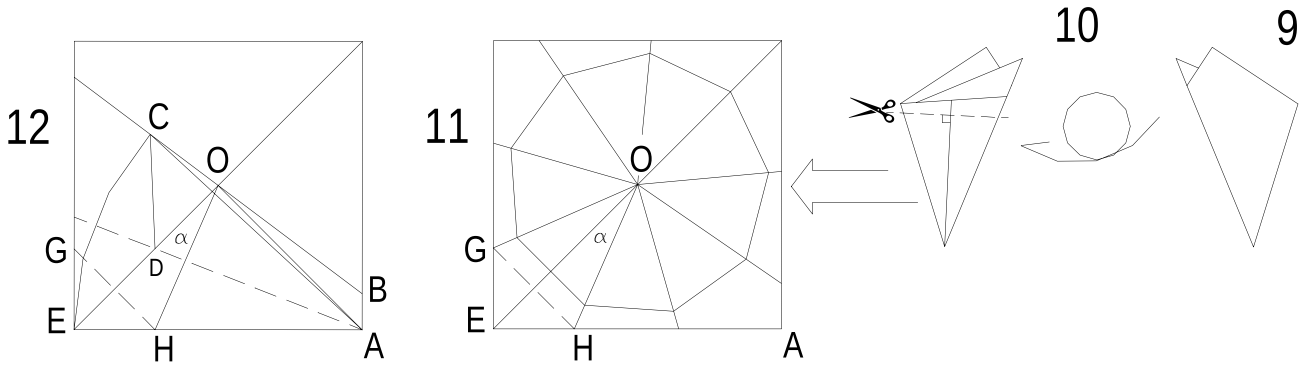
In ΔOAB

GIVEN	OBTAINED
$AB = \frac{1}{8}$	Ang. BOA = 8,1301024º
$OA = \frac{1}{\sqrt{2}}$	
Ang. BAO = 45º	

In ΔOAC

GIVEN	OBTAINED
Ang. COA = 180 – BOA	OC = 0,2949878º
AC = 1	Ang. OAC = 2,3909342º
$OA = \frac{1}{\sqrt{2}}$	





In ΔEDA:

Ang. EDA = Ang. CDA = $180 - 45 - \frac{45 - Ang.OAC}{2} = 113{,}69547$

In ΔOCD:

GIVEN	OBTAINED
OC = 0,2949878	OD = 0,3103317
Ang. COD = 90 – Ang. BOA = 81.869898º	
Ang. ODC = 2 Ang. CDA – 180 = 47,39094º	
Ang. OCD = 180 – Ang. COD – Ang. ODC = 50,739162º	

Enneagon side = GH = ED = $\frac{1}{\sqrt{2}}$ – OD = 0,396775

$\mathrm{tg}\,\alpha = \dfrac{GH}{2\left(OD + \dfrac{GH}{2}\right)} =$ 0,3899744 ; α = 21.304515º

Value of regular enneagon central semiangle = 20º

The base GH becomes a bit greater than the other sides, which in turn are congruent to each other. Same can be said about the angles.

11 STELLATE POLÍGONS

As is well known, to construct a stellate regular polygon with n sides (associated to the division of a circumference in n parts) and species p (pitch between vertices) these conditions are required:

- p > 1
- p and n must be prime between themselves.
- $p < \frac{n}{2}$

With the antecedent conditions, stellate polygons of 3, 4, 6, etc sides are impossible, whereas there can be of 5,7,8, etc.

We are going to focus on the folding of pentagon and heptagon: there is only one stellate pentagon (n = 5; p = 2), but two heptagons (n = 7; p = 2) and (n = 7; p = 3).

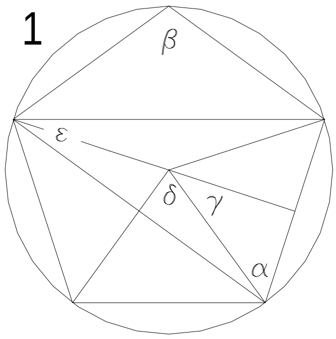
As can be seen, the hexagon can´t be stellate; the only that n = 6 can produce is two opposite overlapped equilateral triangles. Nevertheless we shall study several solutions of what we call hexagonal stars and also those beautiful figures known as *ice crystals*.

Likewise convex regular polygons, the stellate can be perfectly constructed by folding, but we can also find more or less imperfect versions. Besides, a lot of *stars* are offered that, in most of the cases are not regular polygons and, some of them, are not even flat: they may have *a bit of* volume.

11.1 STELLATE PENTAGON (S. FUJIMOTO)

It is a perfect solution obtained from an argentic rectangle.

1



Let´s see first some relations within the convex regular pentagon and between it and the argentic rectangle.

In fig. 1:

$$\beta = \frac{1}{2} 3\delta$$

for β and 3δ *see* the same arc in the circumference, the former from one of its points, and the latter from its center. In like manner it is:

$$\varepsilon = \frac{1}{2} 2\gamma = \gamma = \frac{\delta}{2}$$

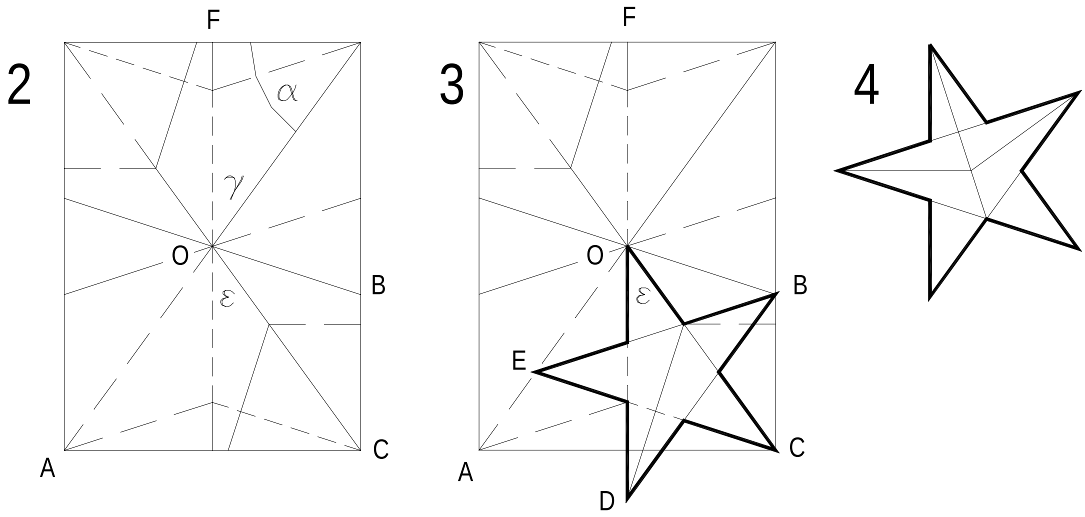
Moreover:

$\beta = 2\alpha$; and being $\beta = \frac{1}{2} 3\delta$, we´ll have: $\beta = \frac{1}{2} 3 \times 2\gamma$; $\beta = 3\gamma$

The antecedent can also be observed directly:

$$\beta = \frac{3\pi}{5} \quad ; \quad \gamma = \frac{2\pi}{10} = \frac{\pi}{5} \quad \text{therefore:} \quad \beta = 3\gamma$$

In the argentic rectangle of fig. 2, and because of its own definition, angles α and γ will appear in the four corresponding right triangles. That is, in the straight angle in O containing the media parallel of the rectangle, are occupied 2γ degrees.



Let´s figure out the difference d = 180 – 2 γ
Looking at Fig. 1 we have:

$$360 = 10\,\gamma \quad ; \quad 180 = 5\,\gamma$$

so it is:

$$d = 5\gamma - 2\gamma = 3\gamma$$

What means that around the center of the rectangle we can trace 10 γ angles by pleat folding of the four we began with.

Let´s see now what is in Fig. 3. Broken line AOBC determines in O and B the β angles of a pentagon whose diagonal is OC (Ang. OBC = Ang. BOF = 3γ).

Vertex D is the symmetric of C with respect to the bisector of ε, and E is the symmetric of D with respect to the bisector of Ang. OBC.

Fig. 4 shows what the resultant stellate pentagon looks like. We may note that the center O of the rectangle becomes one of the points of the stellate pentagon.

11.2 STELLATE HEPTAGON

Begin with the convex heptagon of Fig. 1 making the partial folds of its diagonals. To *fold-stellate* that polygon we would have to be able to gather the paper of triangles like KMN. Since that is not possible in a flat single paper figure, we have to yield some *gatherable paper* by reducing the size of the given heptagon while producing a twist on it.

To produce that torsion we need first to build the small heptagon of side AB (Fig. 2), in this way:

- Fold over each side:
  - Its perpendicular bisector.
  - The perpendicular bisector of half a side (in the figure, anticlockwise sequence).
- The successive intersections of those perpendicular bisectors produce points like A, B and, in consequence, the complete heptagon.

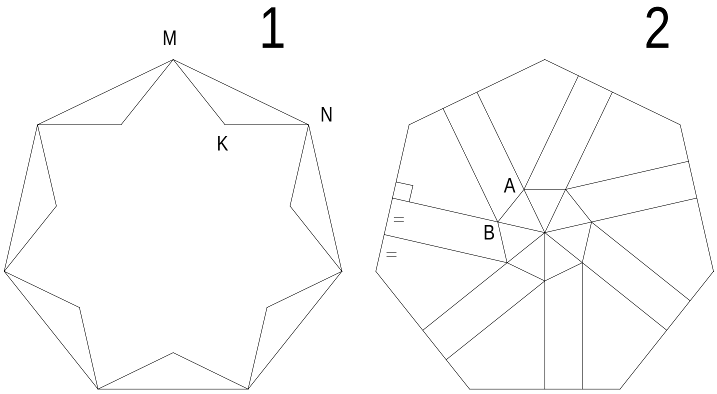
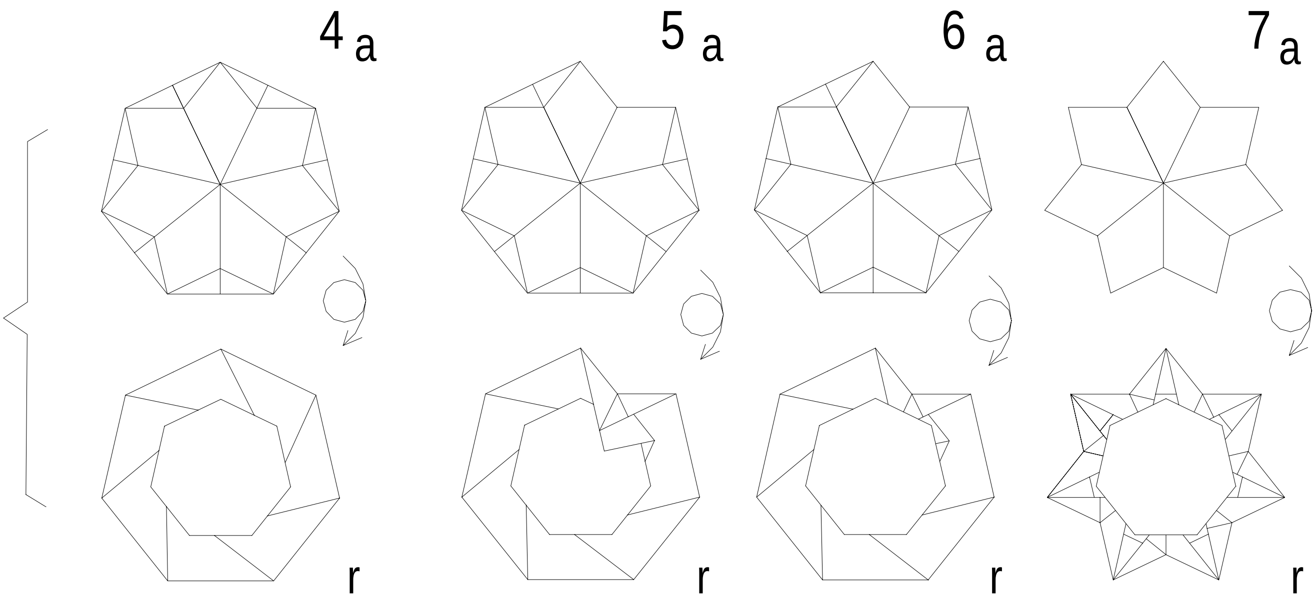


Fig. 3 shows the folding lines to produce the twist, and fig. 4 is the result of the torsion (obverse and reverse).



The rest of the process is easy to follow through Figs. 5, 6 and 7, also showing both faces.

At Fig. 7´s obverse it can be seen the sides (discontinuous) of version n = 7; p = 2. Likewise, in its reverse side it is shown once more the former version and also the version n=7; p = 3 (partly hidden by the small central heptagon).

11.3 STELLATE POLYGONS: FLATTENING CONDITIONS

In connection with Figs. 2 and 3 (Point 11.2) to construct the stellate heptagon, one may question:

The foldings at 1/2 – 1/4 to get the parallels through A, B, are they arbitrary, are they correct under a geometric point of view, are they extensible to the rest of the stellate polygons?

Let´s explore the matter looking (Fig 1) at the polygon of n sides (length L).

In it, $\alpha = \frac{1}{2}\frac{360}{n}$; 2α is the central angle of that polygon ant the segments x, symmetrical about the perpendicular bisector of L, determine the parallel lines to that perpendicular bisector which in turn will condition the twist to flattening.

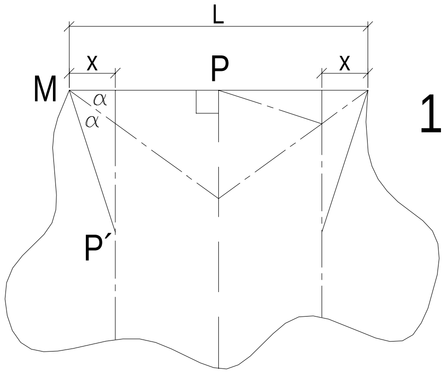
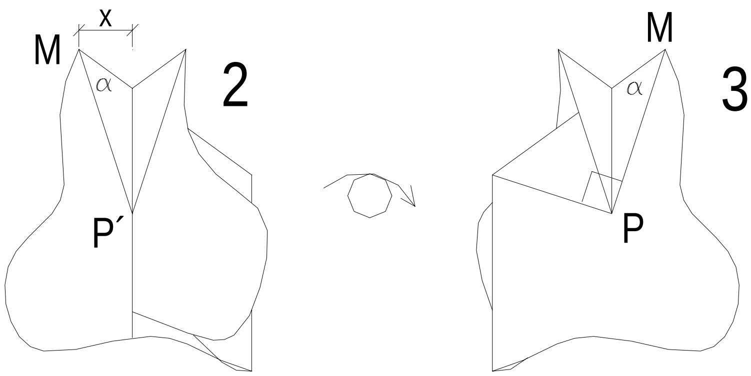


Fig. 2 is obtained by folding flat Fig. 1. To accomplish this, x has to be such that PM = P´M (Fig. 1). Fig. 3 is the reverse side of Fig. 2.



Then it will be:

$$P'M = \frac{x}{\cos 2\alpha}$$

Being $PM = \frac{L}{2}$ we shall have: $\frac{x}{\cos 2\alpha} = \frac{L}{2}$

$$x = \frac{L}{2}\cos\frac{360}{n} \qquad (1)$$

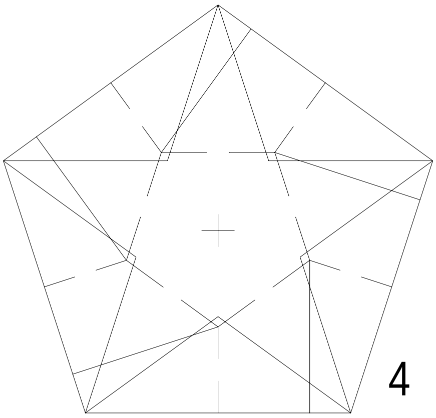
The relation between x and l (side of the small central polygon of n sides around which the torsion is performed), can be found out in Fig. 2 (Point 11.2):

$$\frac{\frac{L}{2} - x}{l} = \operatorname{sen}\frac{180(n-2)}{2n} = \cos\frac{180}{n}$$

$$l = \frac{\frac{L}{2} - x}{\cos\frac{180}{n}}$$

Therefore (1) is the necessary condition to twist till folding flat a stellate polygon. But this is not sufficient: Besides we need that the paper will not interfere within itself. Something similar was already treated in Point 8.2.8.5. Two conditions must be fulfilled: docility and availability; paper has to be docile to flattening and has to be available without any interference.

Let´s consider the aforesaid in connection with the pentagon. If we try to apply (1) to it we´ll come across Fig. 4: paper interference makes it impossible to construct a perfect stellate pentagon (the small imperfection can be perceived).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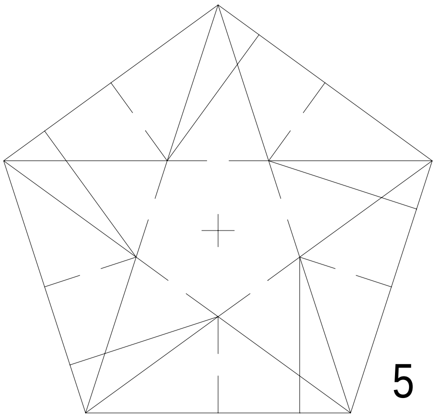
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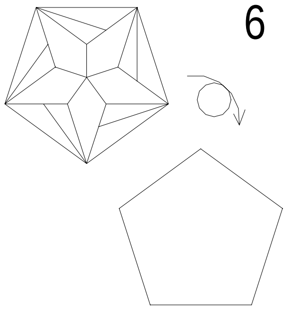
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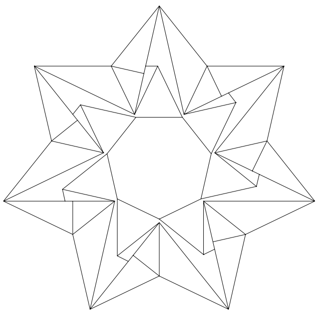
Conversely, if we *cheat* as in Fig. 5, we make the construction possible: the folds take care of that little error and an exempt stellate pentagon (Fig. 6) is obtained jutting out another convex pentagon.

7



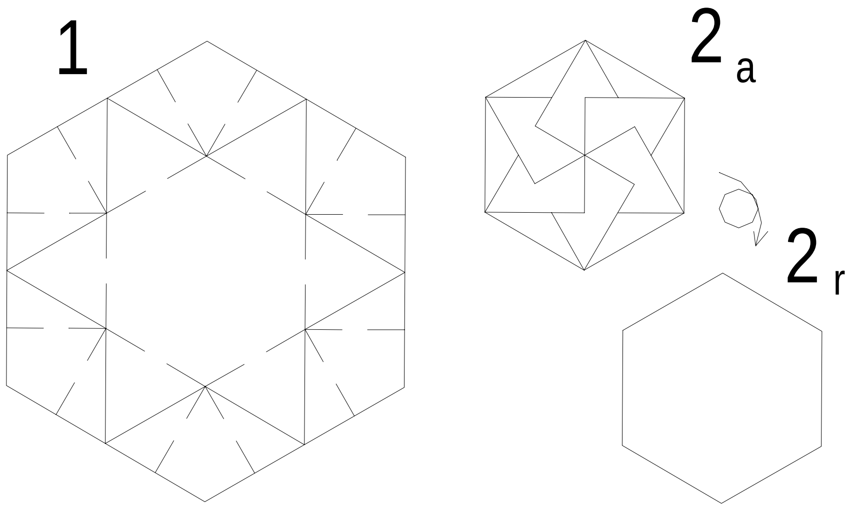
The construction made for the heptagon ($x = 0{,}5\frac{L}{2}$ in Fig.2, Point 11.2) was also a trick. One can compare that value of x with $x = 0{,}6235\frac{L}{2}$ which is obtained for (1) in present Point 11.3. Fig. 7, reverse side, (Point 11.3) is the result of using $x = 0{,}6235\frac{L}{2}$; the process takes care of the error, but you can see the difference between Figs. 7 (Points 11.2 and 11.3).

11.4 HEXAGONAL STAR

VERSION 1 (H. Honda)

It is obtained from the big hexagon of Fig. 1.

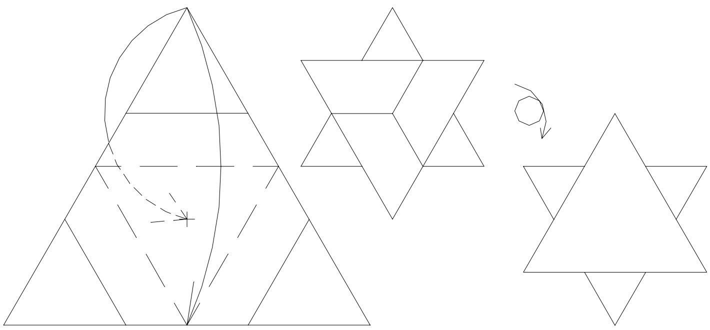
Fig. 2 is the flattened hexagon with 2 a showing the two imbricated equilateral triangles and 2 r, the hexagon in the back.



VERSION 2

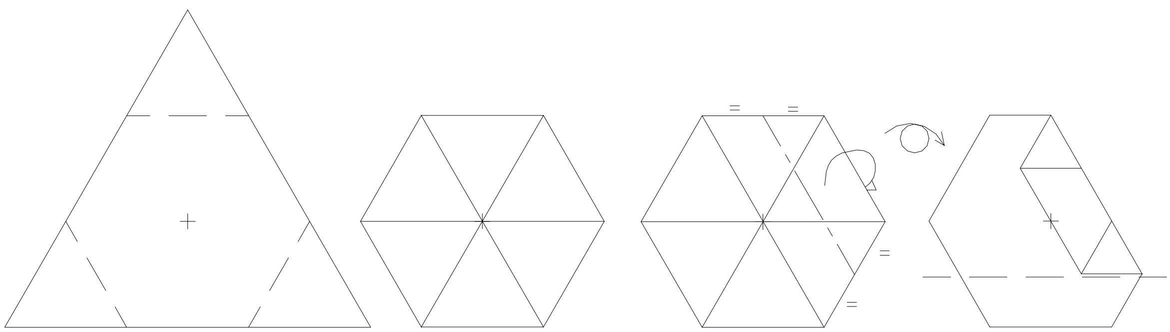
The construction is made out of an equilateral triangle whose center has to be determined beforehand. At the end, the figure becomes interlocked.

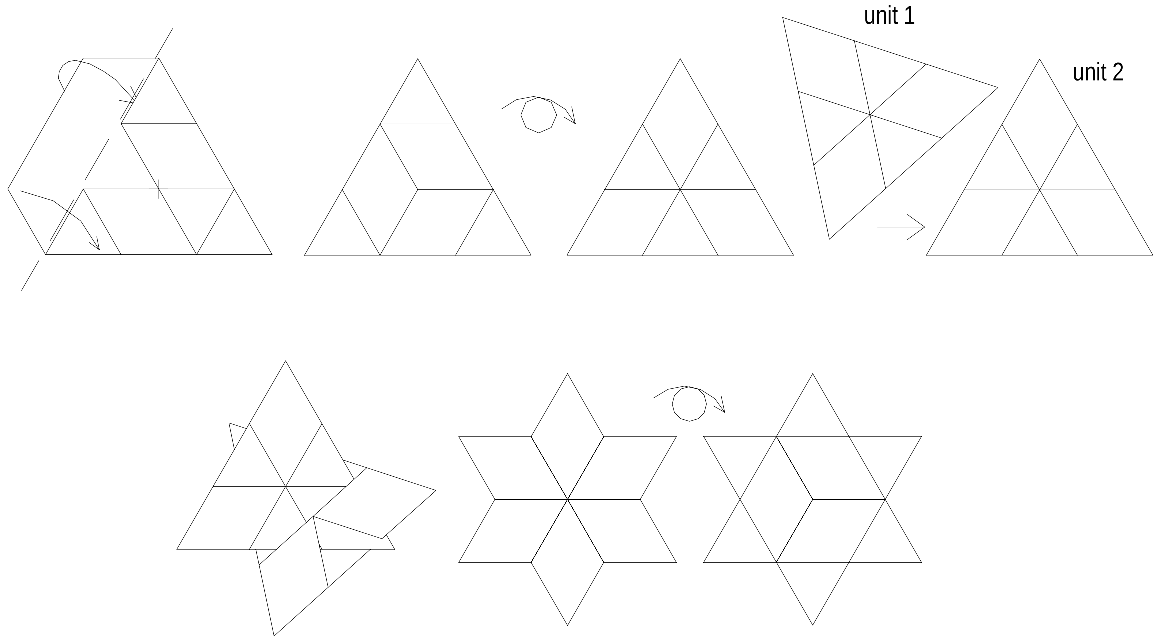
Thus we have a perfect result with this restrain: looking at the obverse we can see the three complete sides of one the triangles. For the other, we must *guess* 1/3 of each side. Looking at the figure upside down, we have the converse effect.



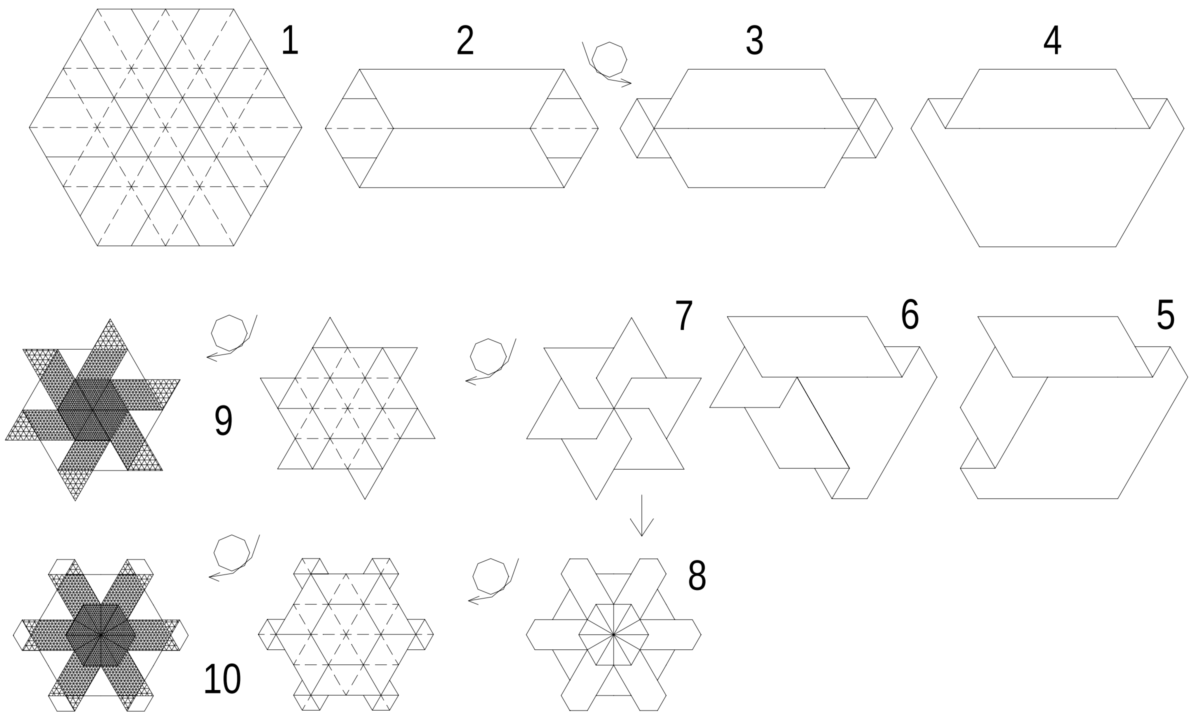
VERSION 3 (F. Rhom)

It is a unit construction that employs two identical units based on the equilateral triangle.





VERSION 4 (Ice crystals)



They are the design of K. Suzuki and start with a regular hexagon piece of paper. Figs. 1 to 7 are the folding process of the latter; Fig. 8 derives from it. Both, 7 and 8 are shown in obverse and reverse.

Suzuki plays with the translucency of folded figures to create numerous versions of the basic ones. The shade intensity on surfaces depends upon the paper accumulated in the folds. Figs. 9 and 10 show that light and shade effect.

When Suzuki faces a geometric folded figure, he studies in it three different ones: the apparent, that which is produced by the translucency and those existing within the interior of folds.

12 TESSELLATIONS

They are decorative surfaces made out of tesserae. Though strictly speaking, these are the small square pieces used in mosaic work, we can generalise and say that a tessellation is composed by combined figures that produce geometric drawings or traceries like in arabesques.

Though in its source, origami had nothing to do with tessellations, we have to honour the paperfolders devoted to this kind of work because eventually they came to the association of both arts.

We can distinguish three dissimilar fields:

- The first is, in fact, a tessellation. It deals with the construction of a unique flat piece of origami that, in turn, fitted with many others alike, can fill a surface as large as we wish, showing repetitive geometrical drawings. As we can imagine it entails a double problem: to design the unit piece and to envisage the large geometric drawing in which the unit will be integrated. When we said a unique piece we did not mean that the unit cannot be made of different colours or even that the paper could not have different colours in either side. If we add that one can play with the unit by translation, turns, symmetries, revolving it upside down, etc. we may arrive to a very much enriched kind of a puzzle.
- The second field starts with a large tracery, all of their lines having to be mountain or valley folded. We can guess how difficult it may be to design such a drawing in order that all partial figures could be folded flat. The process requires gathering and hiding much of the available paper: thus the flattened final figure presents a smaller size than it had the paper we began with. Besides, the final drawing (obverse as well as reverse) has little to do with the original.

  As a matter of fact, this type of construction is not a tessellation strictly speaking for it does not use separate pieces of tesserae. Nevertheless its traceries can be used as the guidelines to copy on them paper cut pieces as tessera-like units, that will fit in the great surface and be reproduced till the infinity.
- The third field looks more like the second rather than to the first. It has the particularity, though, that instead of leading to a flat figure (obtainable as well) it holds a certain volume, shaped as bas-relief that gives a splendid contrast of light and shadow (recall gypsum decorative works in Arab constructions).

Let´s see some examples of each case.

CASE 1A: Forcher´s fish

Figs. 1 to 22 show the fish folding process starting from an equilateral triangle.

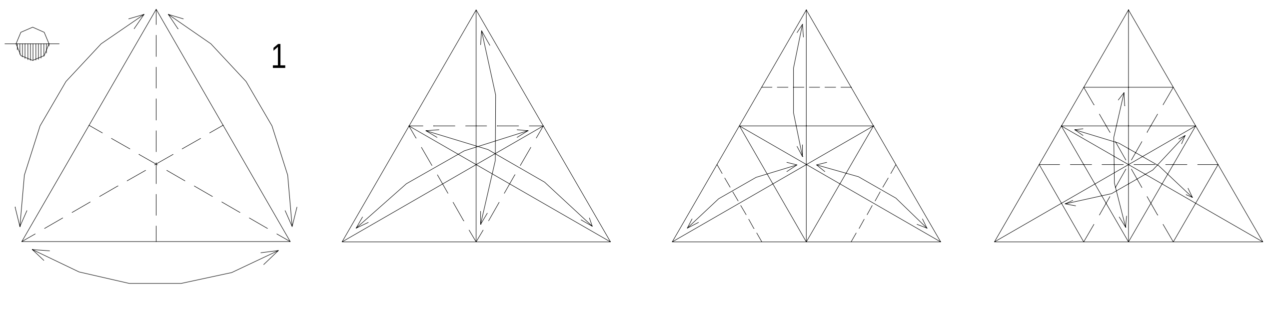
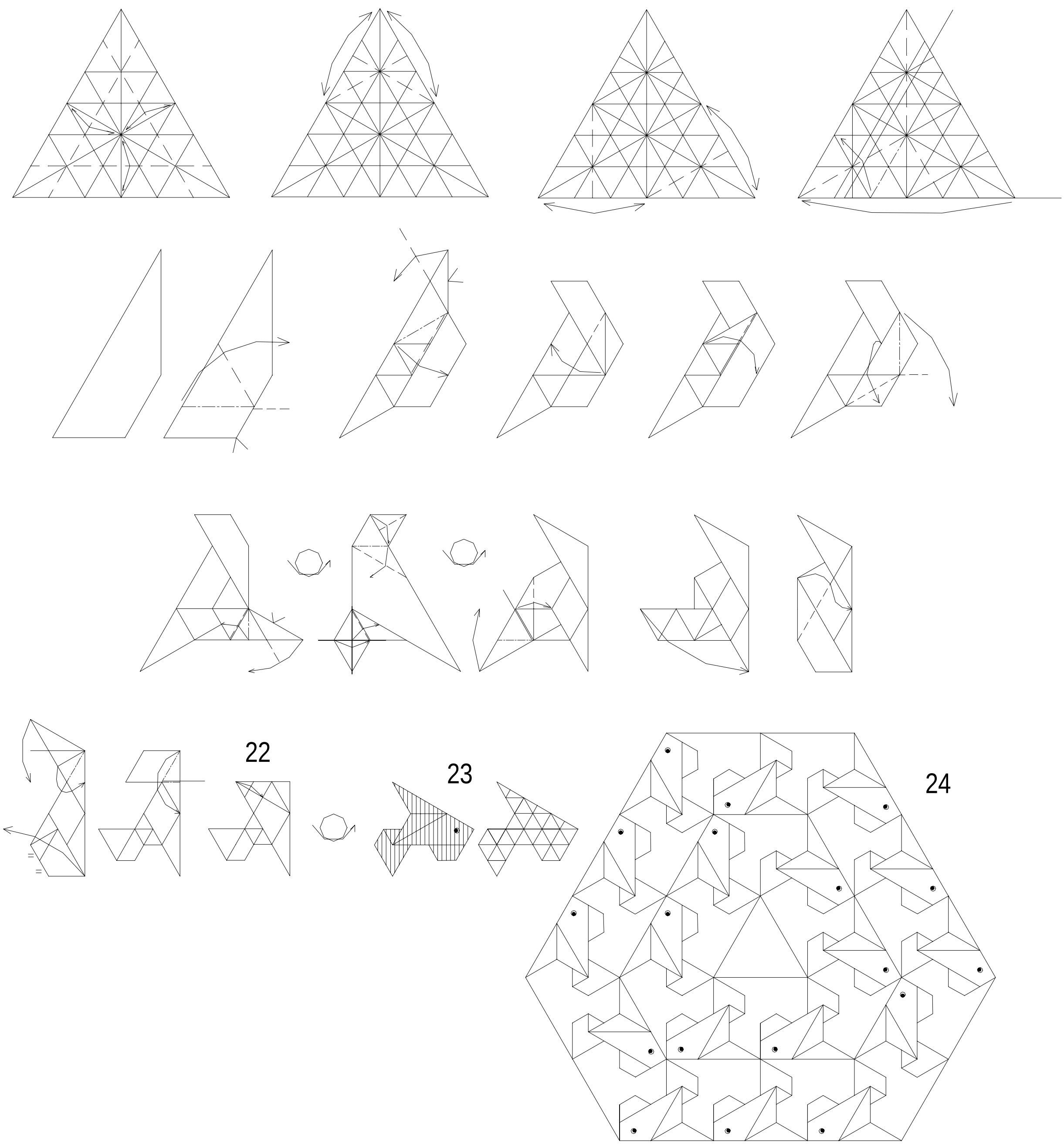


Fig. 23 is the fish we reach at, both, as it looks like and with its triangled surface in case we wish to draw it on a piece of cardboard to be used as the unit of a puzzle. The side of those triangles is 1/12 of the big original triangle. It should be noted that all the angles in the fish are multiples of 30º which is conclusive to fit the units with each others till the infinity.

Fig. 24 is a sample of tessellation: it is one of the many you can construct with the fish.



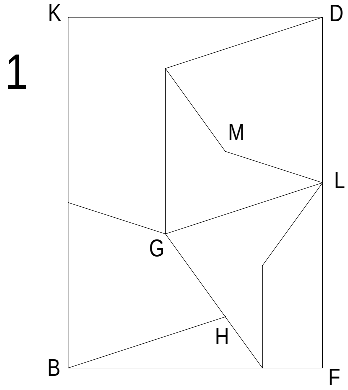
We also see in the center of Fig 24 what I call a *virtual* tessera. To extend the tessellation to the whole plane it´s necessary to play with regular hexagons sided as the small or the great side of the irregular hexagon in Fig. 24; those regular hexagons do not produce any *virtual* tessera in the center.

CASE 1B: Penrose´s tessellation.

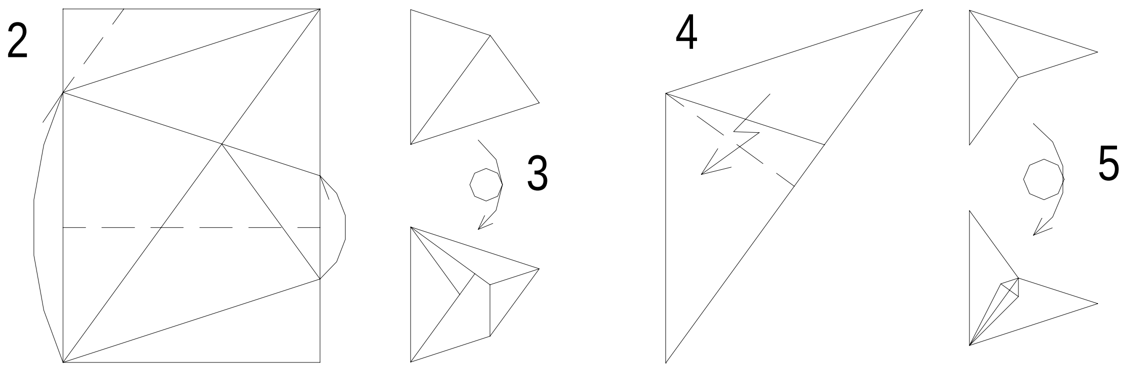
This tessellation, studied by S. Turrión, is based upon two complementary tesserae originated in Fig. 8 (Point 10.1.1). Present Fig. 1 shows a couple of each on the argentic rectangle of said Fig. 8. Thus we can profit to the most the rectangle´s area just in case we want to cut four tesserae. Conway named them a dart and a kite after their shapes, and they complement each other in a rhomb.

The kites are symmetric with reference to EC in that Fig. 8, and the darts do as well with reference to GL in this Fig. 1.

Their peculiar configuration is a consequence of the inherent singularity of the argentic rectangle as can be seen comparing that Fig. 8 and this Fig. 1:



- The small side of both tesserae (this Fig. 1) is the side of the pentagon in that Fig. 8. Likewise, tesserae´s big side is the diagonal of said pentagon.
- As studied in Point 10.1.1 (AN INTERESTING VERIFICATION), the side of a regular convex pentagon divides its diagonal in mean and extreme ratio, i.e., big and small sides of these teserae keep an auric proportion
- In point 10.1.1 we also saw the value of the angles in the argentic rectangle: diagonal and sides 54º and 36º respectively. In consequence, the three acute angles of the kite measure 72º = 180 - 2×54 = 36×2 and the obtuse angle measures 144º = 360 - 3×72 (like the angle of the convex regular decagon).
  The obtuse angle of the dart measures 216º, its opposite 72º and the other two, 36º.



To get the folded tesserae out of an argentic rectangle it is advisable to aim at the largest possible ones to avoid, as much as we can, paper gathering and hiding.

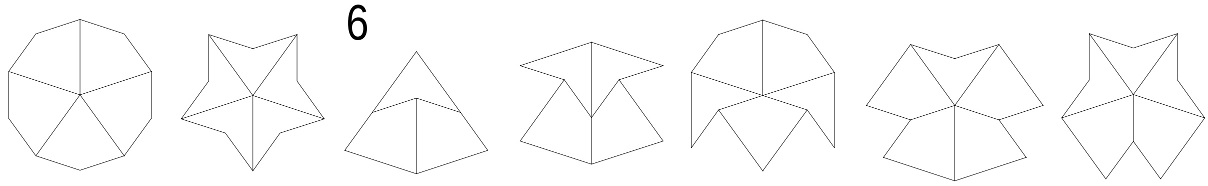
This Fig. 2, derived from that Fig. 8, has the folds for the kite, and this Fig. 4, derived from that Fig. 3 shows the fold lines for the dart (mountain, continuos; valley, dashed).

Figs. 3 and 5 are, to one half scale, the obverse and reverse of both tesserae.

Fig. 6 shows the seven possible combinations forming a perigon under the condition that the adjacent sides of two distinct tesserae have to be always of equal length.

In Fig. 7 there are two tessellations: one of them is rather simple whereas the other renders itself complex as it grows up. In both, some *virtual tesserae* can be seen. One thing is clear about these tessellations: they give the artists the chance to develop their own ingeniousness and to the mathematicians that of disclosing the laws of formation.

6



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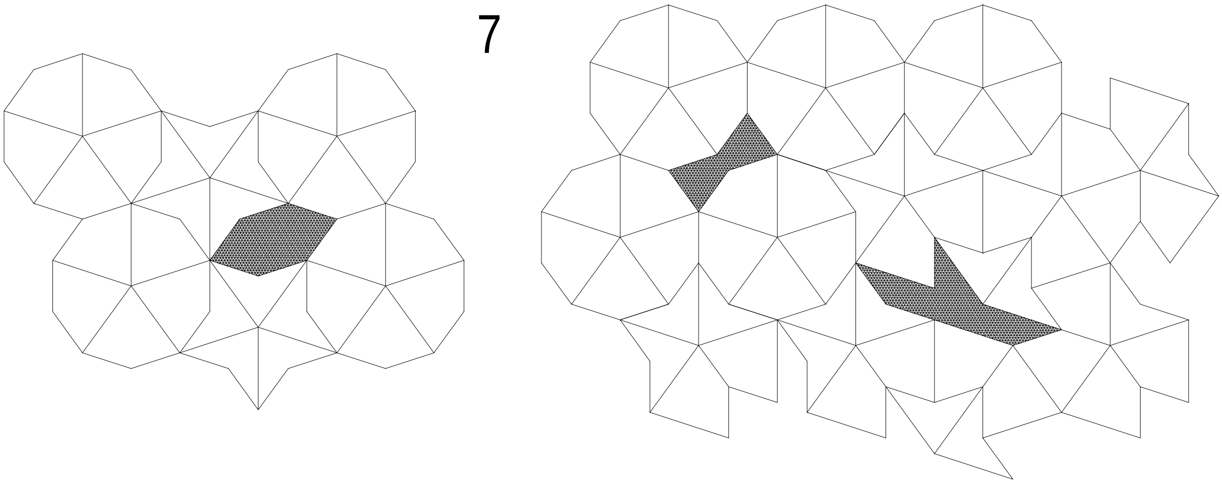
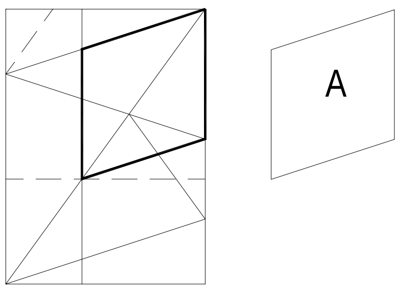


Penrose has also designed another tessellation based as well on the argentic rectangle. It is composed of two rhombic tesserae with like sides and angles as follows:

Rhomb A: 108º; 72º
Rhomb B: 144º; 36º

8



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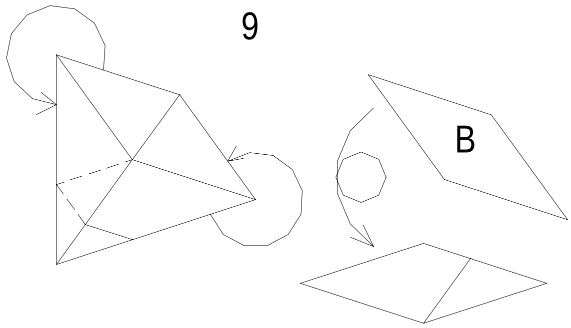
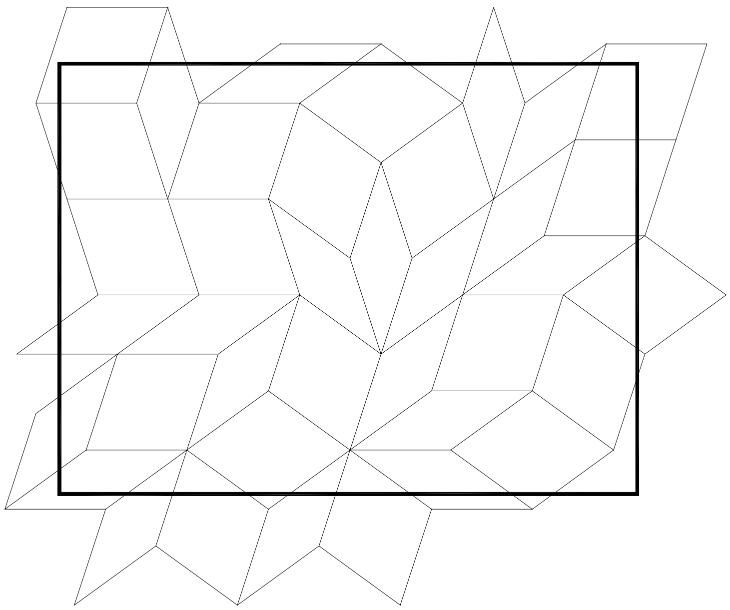


Fig. 8 shows the genesis of rhomb A from Fig. 2; in Fig. 9 we obtain the rhomb B from the kite of Fig. 3. Note that the sides of both rhombs are congruent.

Fig. 10 is a tessellation made out of rhombs A and B.



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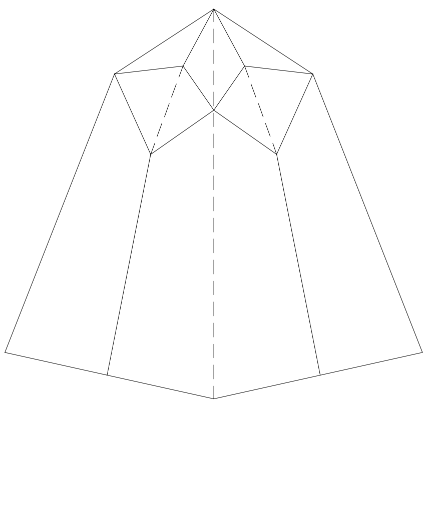
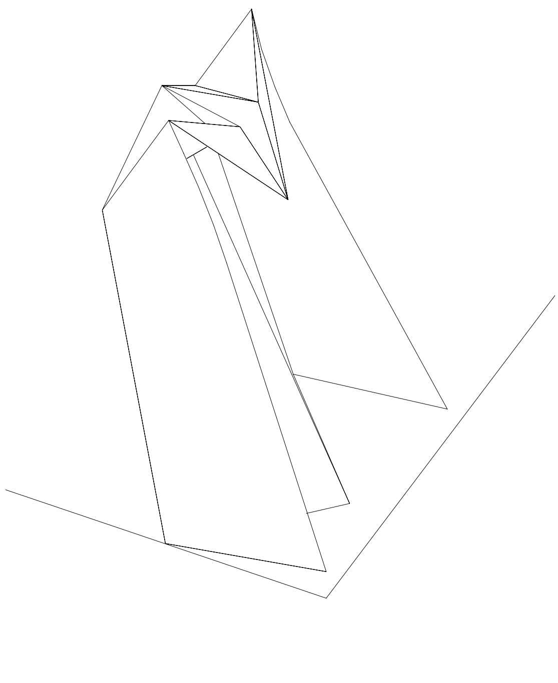
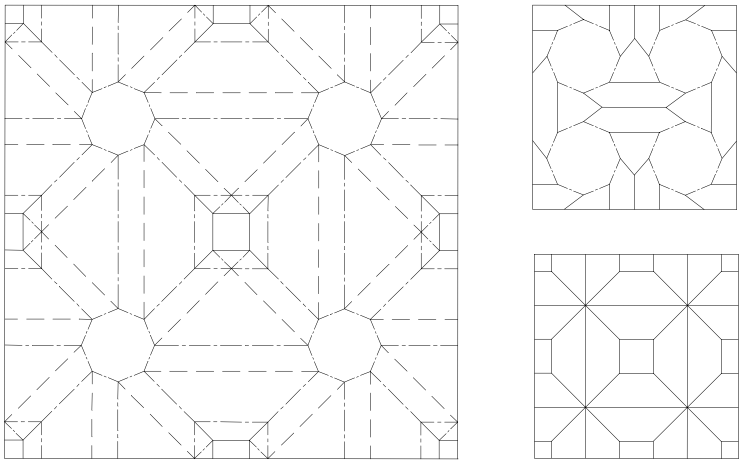
Interlude





CASE 2A: Tessellation by Chris K. Palmer.

Fig. 1 is a tracery among the many and very beautiful developed by the author. Figs. 2 (obverse) and 3 (reverse) are the result of folding flat Fig. 1. The three figures keep in the



1 2 3

drawing their natural proportions. To get the final figure in its protuberances, the octagons of Fig. 1 have to be twisted (recall Point 11.2).

Corners of Fig. 1 are maintained in 2 and 3. One can note that flattening of Fig. 1 is guaranteed after folding because: there is an even number of concurrent lines in each node; the sum of their alternate angles add up to 180º, and finally, paper does not interfere at folding.

These three conditions are common to this type of tessellations, though there may exist apparent exceptions in certain nodes. What happens in those cases is that one extra fold is provided to ease folding operation, but once this is over, that fold remains inoperative.

By merely translating the figures of the three big squares we can occupy the whole plane. That is so because we are playing with fitting angles of 90º, i.e. multiples of 30º like in 1A.

CASE 2B: Tessellation by Alex Bateman

1 2 3

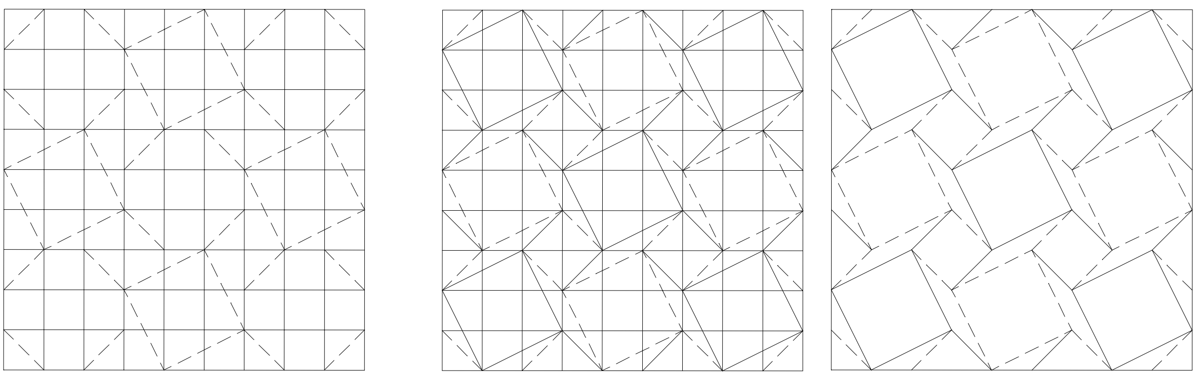
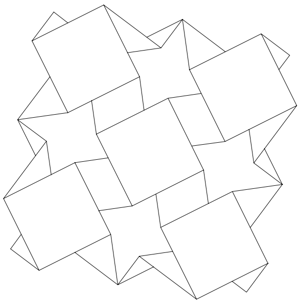


Fig. 1 shows a quadrille as guideline to start with, and all the valley folds. Fig. 2 adds the mountain folds. In Fig. 3 the guidelines have been erased. Fig. 4 is the result of folding, twisting and flattening Fig 3. Fig 5 is Fig 4 seen as translucent (recall version 4 of Point 11.4).

Fig. 6 is a tessellation based on Fig.4 showing the 5 tesserae that are produced when fitting four Figs. 4. Fig. 7 is another tessellation formed by tesserae obtained from Fig. 4. Finally, Figs. 8, 9 and 10 are, likewise, other type of tessellations made with tesserae from said Fig. 4.

4



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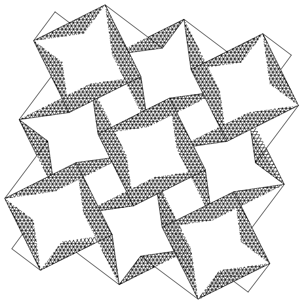

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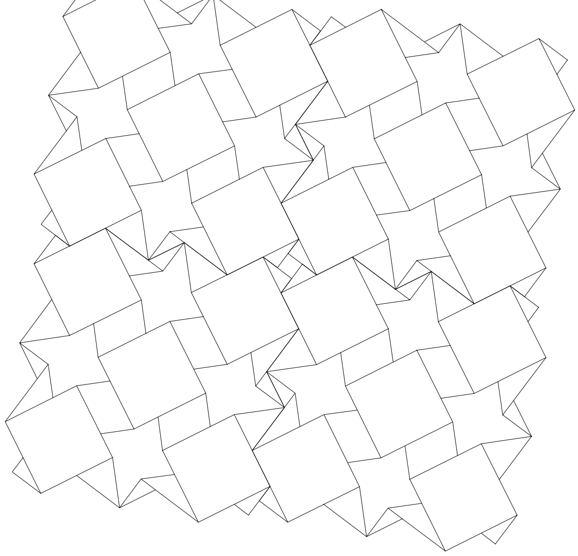

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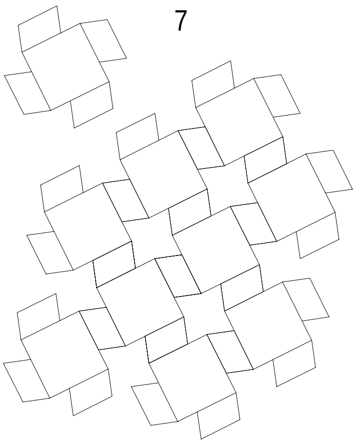

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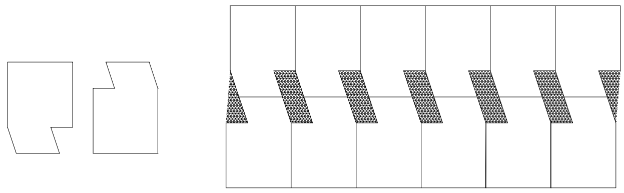

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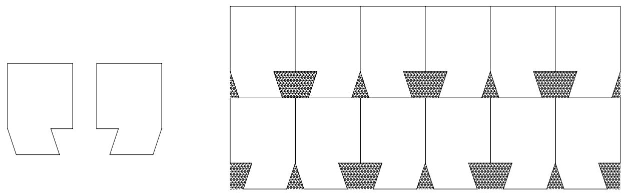

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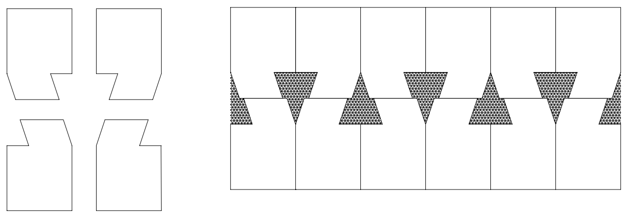
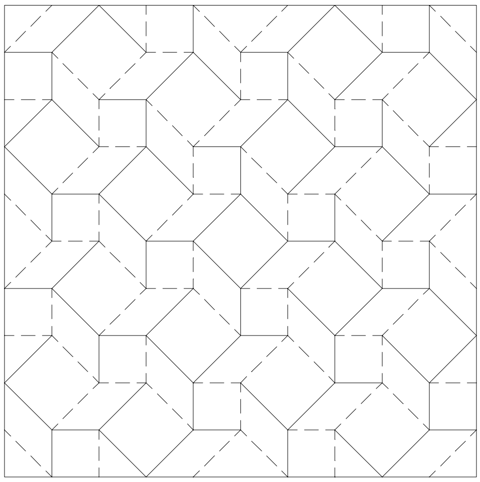


CASE 3: “Mars T” by P. Taborda.

Fig. 1 is the basic tessellation with folding lines (mountain, continuous). Fig. 2 (obverse and reverse) is Fig. 1 after folded, twisted and flattened.

Between rigid Figs. 1 and 2 exists a continuous range of elastic configurations like the last shown. It has been set to a fixed position to make the drawing possible.



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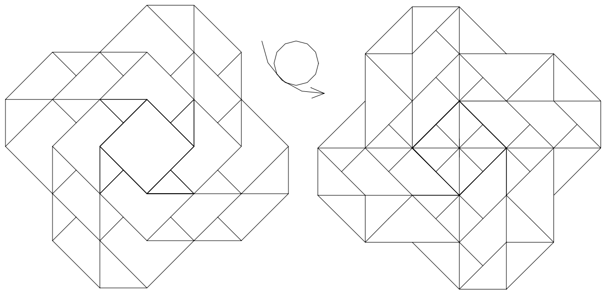
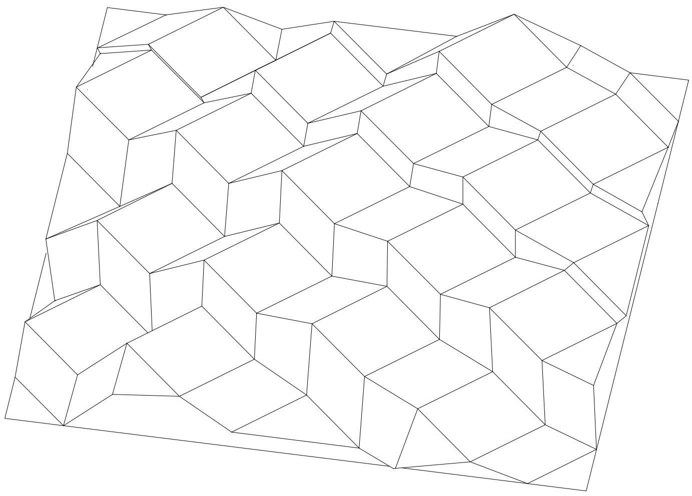
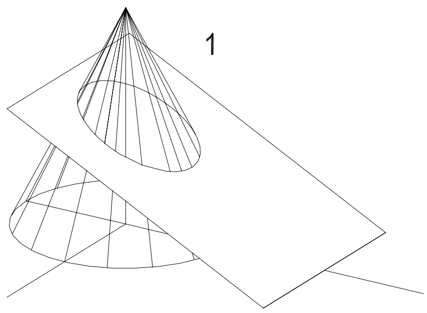


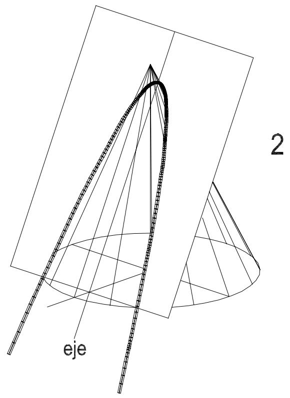


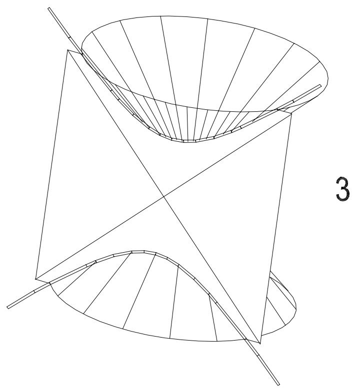
13 CONICS

As it´s well known, they are curves determinated by the intersection of a cone and a plane not passing through its vertex. Therefore they are plane curves as shown in Figs. 1, 2 and 3.

- Ellipse (Fig .1): The intersecting plane is oblique to the cone´s axis in such a way that a close curve is obtained.
- Parabola (Fig. 2): It comes out as an open curve when the intersecting plane is parallel to one of the cone´s generatrices. In turn, that generatrix is parallel to the parabola´s axis.
- Hyperbola (Fig. 3): It is the other open conic, with two branches. The intersection plane has to be parallel to the cone´s axis and cuts both cone´s volumes: The two conventional cones have a common vertex and their respective generatrices lie in opposition on the same straight line; that is what is called a complete cone. On intercepting plane of Fig. 3 we can see the asymptotes of the hyperbola.







<u>REMARKS ON CONICS AND ITS DEGENERATION</u>

If we compare how ellipse and parabola are generated (Figs. 1 and 2), we can imagine that the second is an ellipse whose unseen vertex is the ideal point of its axis

In case we revolve the plane of Fig. 1 till a position perpendicular to the axis of the cone, its section in it becomes a circumference: this is, therefore, an ellipse with equal axes equivalent to the diameter of said circumference.

If the plane passes through the vertex, the ellipse degenerates to one point, the parabola to a generatrix and the hyperbola to a couple of coplanar generatrices.

Let´s see here after the relations that can be issued between these curves and origami.

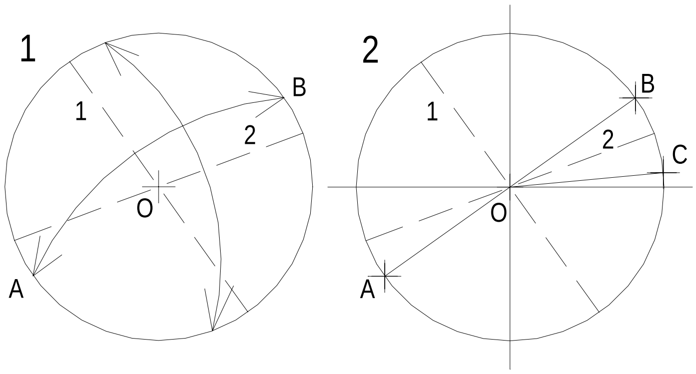
13.1 CIRCUMFERENCE

13.1.1 ITS CENTER

To obtain the center of a cut circumference given as in Fig.1, produce two cross-folds as perpendicular as possible to each other, from edge to edge. The intersection of both folds will be the center.

To produce it, look in Fig. 2 how point A becomes B because of the first symmetry, and B becomes C after the second folding. Those symmetries make that OA = OB = OC. Thus we have three points in a circumference equidistant from another interior one: this point is the center according to circumference definition.

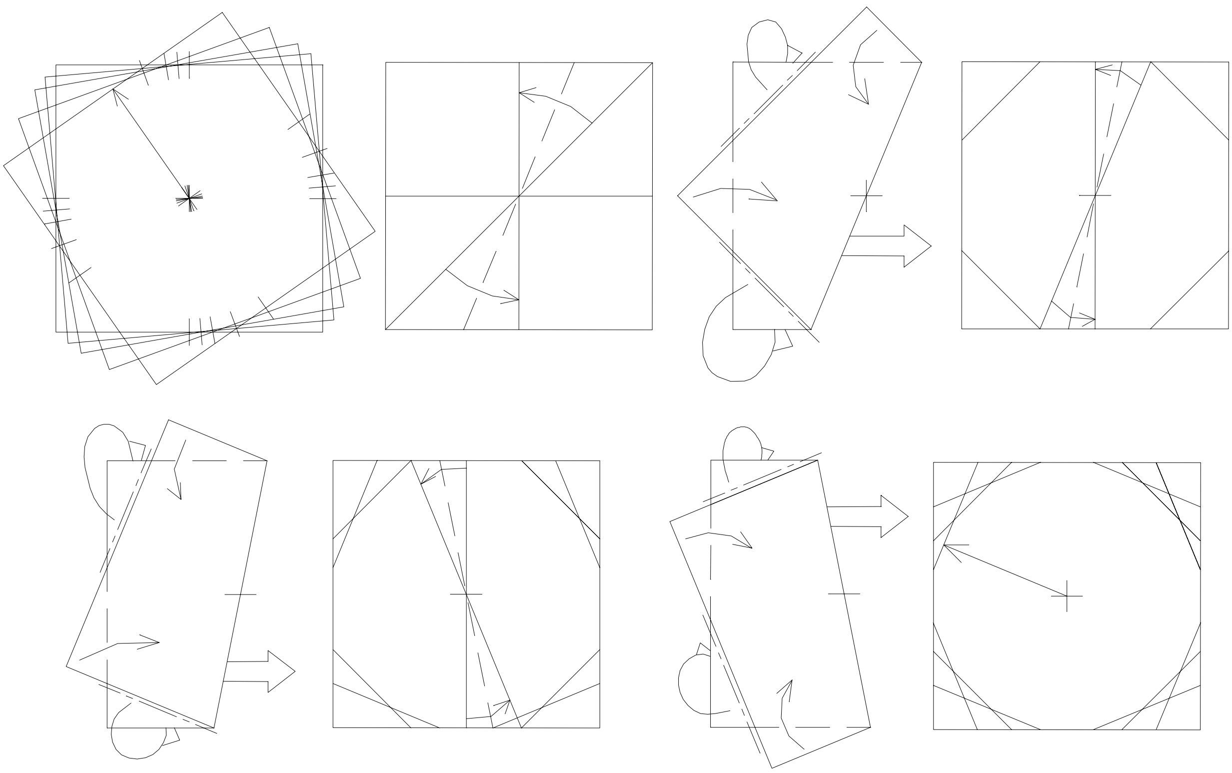
If the circle is not cut, but merely drawn on a piece of paper, the process is the same. The only precaution is to have the circumference heavily marked to ease folding by transparency.



13.1.2 A CIRCUMFERENCE AS THE ENVELOPE OF ITS OWN TANGENTS

INSCRIBED WITHIN A SQUARE

Follow the process here after:

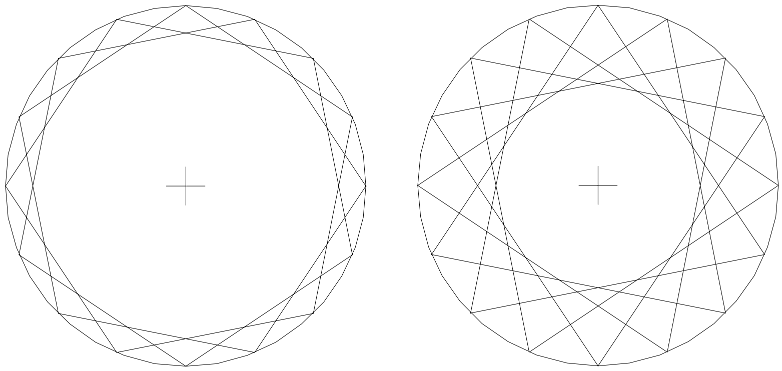


Its foundation is in the first figure: it consists basically in rotating a given square around its center. Thus each turned side is equidistant to the center in the same value, i.e. the apothem length. Then we can see that each side is a tangent to the circumference, and their midpoints are the points of tangency.

The rest of figures show the process, whereas the last one displays the situation of circumference and given square after the three folds were performed. If we go on folding, we can approximate as much as we wish the relation circumference / envelope.

CONCENTRIC WITH ANOTHER GIVEN ONE AND INTERIOR TO IT

The solution is to fold a stellate polygon inscribed in the given circumference. The resulting ring shows in its inner side a convex polygon made by the folding lines: sought circumference.



n = 16 ; p = 3 n = 16 ; p = 5

It is pertinent to recall Point 11 about stellate polygons and Point 9.1 on how to divide a circle. Note that for a greater pitch we get a smaller circumference.

Process is as good for a given circumference cut out of a piece of paper as for other drawn on it. Of course, first case is easier.

13.2 ORIGAMI AND PLÜCKERIAN COORDINATES

One could be disappointed when a curve is defined as the envelope of its tangents: Such may be the case of the circumference in former Point 13.1.2, of the parabola –Point 1.2.4-, or some other cases we shall see here after. In those cases the curve does not appear in its points, as it is usual in Cartesian representation.

Nevertheless we must admit that origami does not play with Cartesian co-ordinates. Its very name gives us the clue: origami, paper folding, i.e., paper (the plane) on which folds are produced (folds are but lines –mountain or valley-).

Then we can agree to the natural representation in origami as made out of straight lines (Plückerian co-ordinates) rather than of points (Cartesian co-ordinates). In origami the point is normally a by-product, i.e., the intersection of two straight lines (two creases) through which, most likely, a new straight line will pass.

What happens in practice when we deal with both, maths and origami, is that we bear a hand to certain instruments. We may recall, e.g. Point 7.11 in which we had to set the co-ordinates of two points in the plane.

Once we know that Cartesian (points x,y) and Plückerian (straight lines u,v) co-ordinates coexist, it will be good to dig out the mutual correspondence that will enable us to do some verifications.

Fig. 1 shows the straight line t on which all its points (x,y) lie. It has *a* as *x* intercept and *b* as *y* intercept, being its equation:

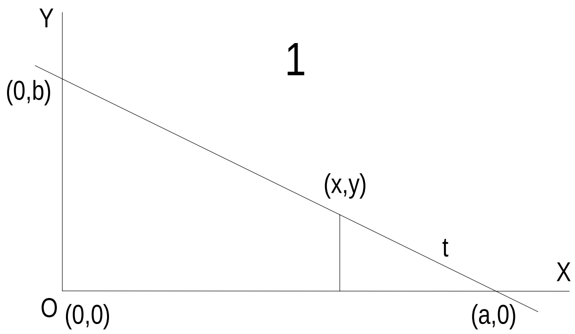
$$\frac{a}{a-x}=\frac{b}{y}$$

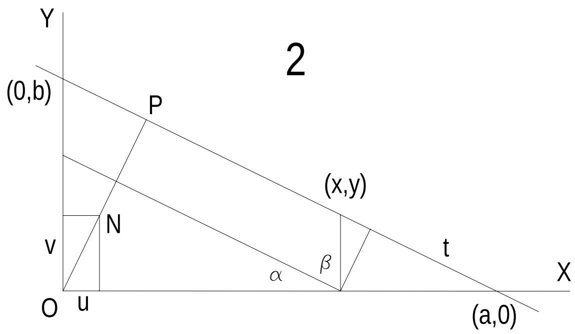
that can be transformed to

$$-\frac{1}{a}x-\frac{1}{b}y+1=0 \qquad (1)$$

Fig. 2 is the same Fig. 1 to which has been added the normal OP with point N (u,v) such that:

$$ON = -\frac{1}{OP} \qquad (2)$$





In same Fig. 2 we see:

$$OP = x\cos\beta + y\cos\alpha \qquad (3)$$

As also it is:

$$\cos\beta = \frac{u}{ON} \quad ; \quad \cos\alpha = \frac{v}{ON} \qquad (4)$$

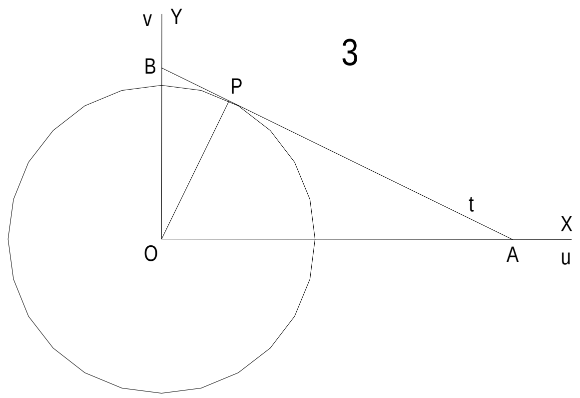
substituting (2) and (4) in (3), we have:

$$OP = x\frac{u}{ON} + y\frac{v}{ON} \quad ; \quad ux + vy + 1 = 0 \qquad (5)$$

(5) is the Plückerian equation of straight line t and (u,v) its covariant co-ordinates. If we compare now (1) and (5) we have the relation between Cartesian and Plückerian equations of t:

$$u = -\frac{1}{a} \quad ; \quad v = -\frac{1}{b}$$

If (5) takes the form of $ux_1 + vy_1 + 1 = 0$, it represents a pencil of lines in the plane with variables (u,v) passing through the particular point (x_1,y_1)

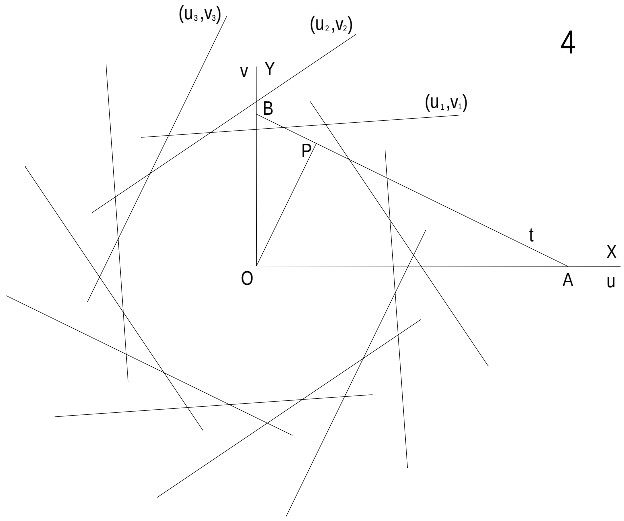


Let´s find the tangential equation (i.e. in Plückerian co-ordinates) of the circumference with center O (0,0) to which t is the tangent on P. For that it´s enough to equalise OP and the radius r of the circumference:

$$OP = r \quad ; \quad -\frac{1}{ON} = r \quad ; \quad -\frac{1}{\sqrt{u^2+v^2}} = r \quad ; \quad r^2 = \frac{1}{u^2+v^2} \quad ; \quad u^2 + v^2 = \frac{1}{r^2}$$

Circumference tangential equation: $u^2 + v^2 = \dfrac{1}{r^2}$ (6)

,, Cartesian ,, ,, $x^2 + y^2 = r^2$



In Fig. 4 we can see the circumference of center O and radius OP represented by several tangents (u_i,v_i).

(6) can be verified in Fig. 3:

$$u = \frac{-1}{OA} \quad ; \quad v = \frac{-1}{OB}$$

$$\frac{1}{OA^2} + \frac{1}{OB^2} = \frac{1}{OP^2}$$

$$\frac{OA^2 + OB^2}{(OA \times OB)^2} = \frac{1}{OP^2} \quad ; \quad AB^2 \times OP^2 = (OA \times OB)^2$$

$AB \times OP = OA \times OB$; $\dfrac{AB}{AO} = \dfrac{OB}{OP}$ which is true after the similarity of ΔAOB and BPO.

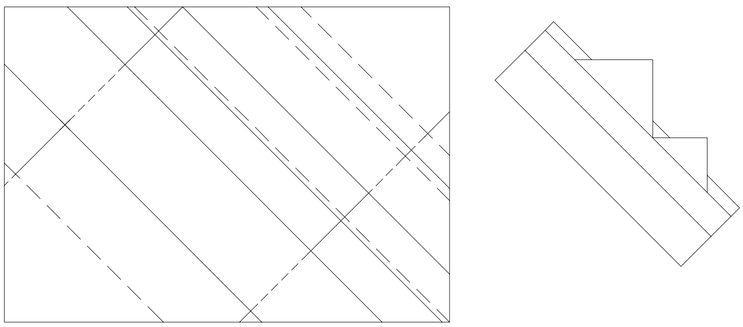
In case one would want to make similar checks with the rest of the conics, here we have their Plückerian equations with respect to the origin, while keeping their characteristic parameters:

Ellipse: $a^2u^2 + b^2v^2 = 1$

Parabola: $pv^2 = 2u$

Hyperbola: $a^2u^2 - b^2v^2 = 1$

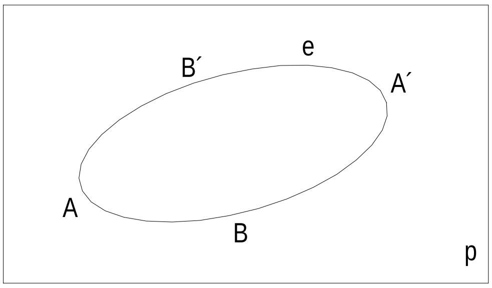
Interlude

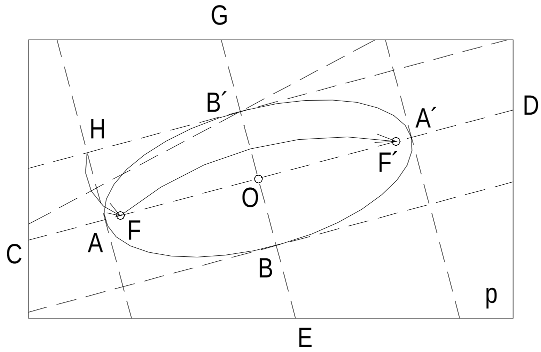


13.3 ELLIPSE

13.3.1 TO FIND ITS PARAMETERS

Let loose ellipse e on paper p; it may be a CAD print. Folding process will be as follows.





1. Likewise circumference (Point 13.1.1):
   A → A´ ; B → B´
   Result: axes AA´, BB´ and consequently center O.
2. C → AA´ ; A → A
   D → AA´ ; A´ → A´
   E → BB´ ; B → B
   G → BB´ ; B´ → B´
   Result: circumscript rectangle to the ellipse.
3. H → AA´ ; B´→ B´
   Result: focus F.
4. F → OA´ ; O → O
   Result: second focus F´.

Note that in the ΔB´OF obtained in step 3, it is:

$$B'F^2 = B'H^2 = a^2 = OB'^2 + OF^2 \quad ; \quad a^2 = b^2 + c^2$$

13.3.2 THE ELLIPSE AS THE ENVELOPE OF ITS OWN TANGENTS

The given data are:

- The measure of its half-axis a.
- The position of its focuses F´, F on the plane.

That amounts to a given circumference with radius 2a and center F´ (director circle cd), and the position of F inside it: distance FF´ is equivalent to 2c (Fig. 1).

Likewise, the principal circle cp (Fig. 3) is given: it has O as center and a as radius.

Here we have the folding process to get the tangents of Fig. 2:

Take any point P on the circumference, to lie on F by fold AB: AB will be the tangent to the ellipse. Then fold F´P: the intersection point T of both folds is the tangency point.

This is the proof:

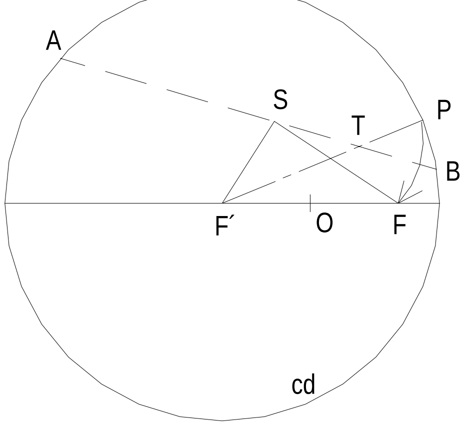
F´T + TF = F´T +TP = 2a (ellipse condition)

So T is a point on the ellipse. For any other point of AB, e.g. S we have:

F´S + SF = F´S + SP > F´P = 2a (ΔF´SP)

It means that any other point on AB but T, does not fulfil the condition of being on the ellipse. Hence AB is the tangent and T its point of tangency.

1



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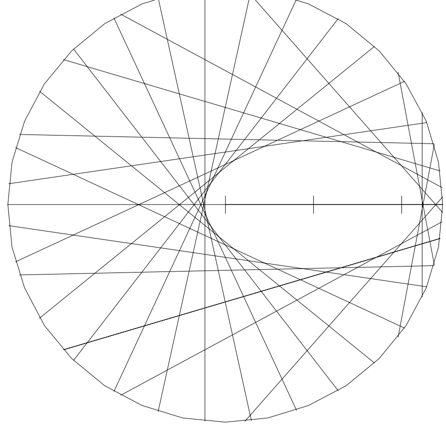

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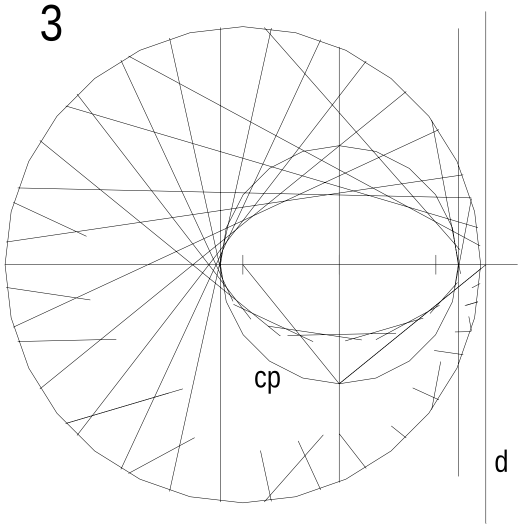
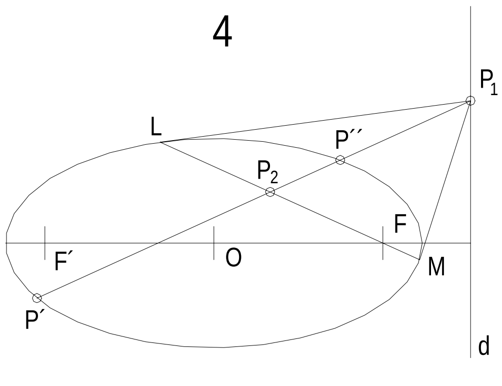


Fig. 3 displays a very special straight line: it is the ellipse directrix d corresponding to focus F. The directrix is a rather unknown line in the ellipse, though not in the parabola. Let´s look at Fig. 4 and see certain things about it.

4



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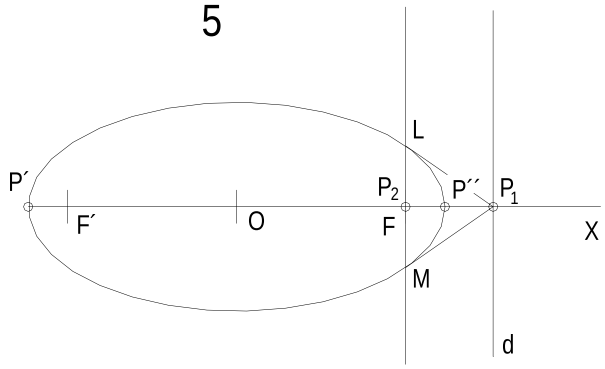


To start with, d is the polar of F.

This means that d is the locus of the poles of all the straight lines passing through F; e.g. P_1 is the pole of LM (polar of P_1) with P_1 L and P_1M tangents to the ellipse.

Under these circumstances, any line through P_1 intersecting its polar and the ellipse produces a set of four points whose harmonic cross ratio is:

$$(P'P''P_1P_2) = -1 \qquad (1)$$

Let´s see the meaning of this expression. Point P_1 of Fig. 4 has slid down in Fig. 5 to the X-axis so (1) will lie on it (look the four rounded points).

Let x_1 be the abscissa of the 1st point P´ in (1), x_2 of P´´, x_3 of P_1 and x_4 of P_2. The cross ratio (1) has the value:

$$\frac{x_1 - x_3}{x_2 - x_3} : \frac{x_1 - x_4}{x_2 - x_4} = -1 \qquad (2)$$

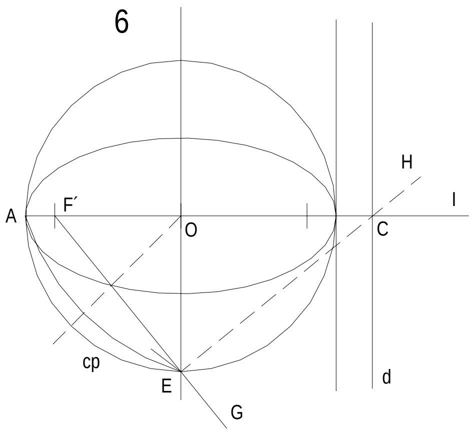
In (2) we know all the abscissas but x_3 to which we assign the value x. Substituting in (2) the known values, we have:

$$\frac{-a-x}{a-x} : \frac{-a-c}{a-c} = -1$$

From which we obtain the distance x between the center O of the ellipse and the directrix d:

$$x = \frac{a^2}{c} \qquad (3)$$

Now we can show the folding process to construct the directrix (Fig. 6):



1- F´ (after step 3, Point 13.3.1) $\rightarrow$ Small axis of the ellipse ; O $\rightarrow$ O
Result: E.
2- EF´ $\rightarrow$ EG ; E $\rightarrow$ E.
Result: EH, hence C.
3- AC $\rightarrow$ CI ; C $\rightarrow$ C.
Result: d

Note that x in (3) is greater than the abscissa of the point symmetric to F about the ellipse´s vertex. These abscissas´ difference is:

$$\frac{a^2}{c} - (a + a - c) = \frac{(a-c)^2}{c}$$

obviously greater than zero.

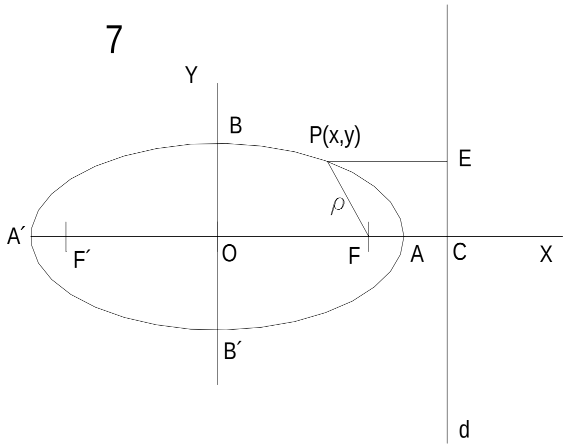
(3) can also be proved geometrically just looking at the right ΔEF´C in which OC = x. In it we have:

$$OE^2 = OF' \times OC \qquad ; \qquad a^2 = cx$$

Here it is another property of the directrix d (Fig. 7): the ratio of distances from any point P(x,y) on the ellipse to the directrix (PE) and to its corresponding focus (r = PF), equals the ellipse´s eccentricity (e = c / a).

To prove it we must consider the ellipse´s peculiar equation that relates the radius vectors and the Cartesian co-ordinates of one point on it, with its basic parameters:

$$-cx + a^2 - a\rho = 0$$



From it we can write $\rho = a - \frac{c}{a}x = a - ex$

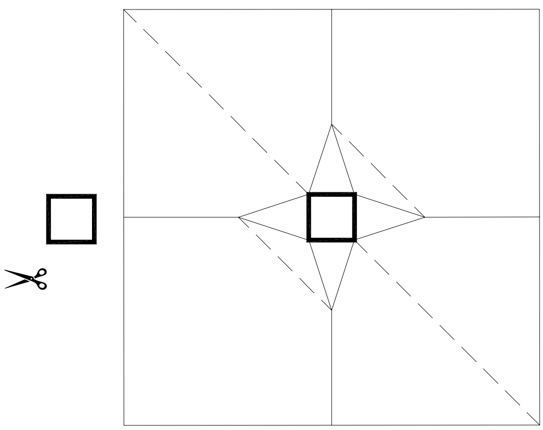
After (3) it is $OC = \frac{a^2}{c} = \frac{a}{e}$, hence:

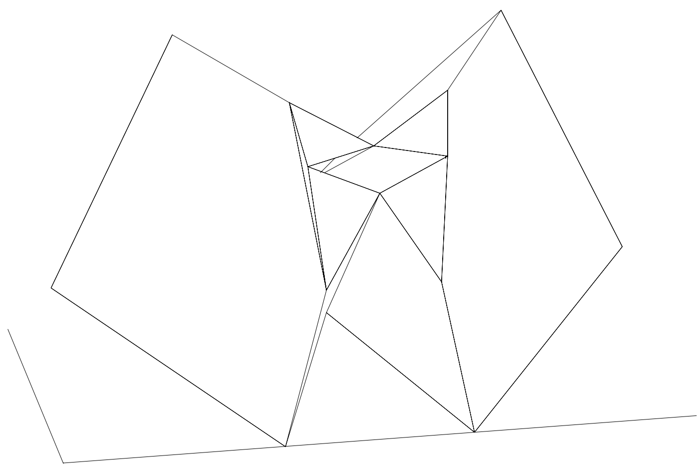
$$PE = OC - x = \frac{a}{e} - x$$

And finally we´ll have:

$$\frac{\rho}{PE} = \frac{a - ex}{\frac{a}{e} - x} = e$$

Interlude

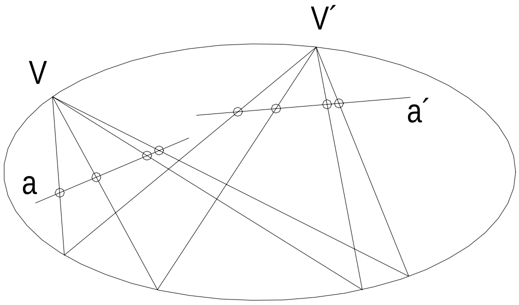




13.3.3 ELLIPSE INSCRIBED WITHIN A RECTANGLE

Let´s recall first the generation of a conic by means of projective line pencils. Fig. 1 shows the pencils V and V´ inscribed in an ellipse in such a manner that the corresponding lines intercept in points lying on the ellipse.

1



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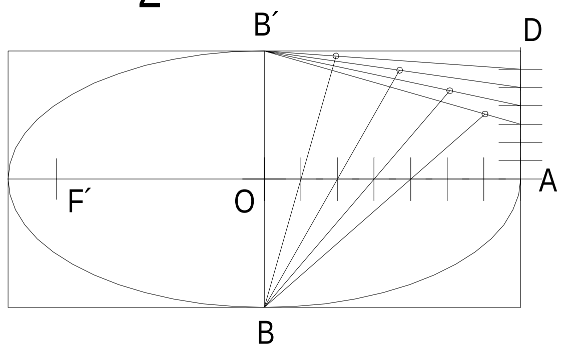


In a figure like that, both line pencils are projective, i.e., when intercepted by any lines like a and a´, produce two sets of four points (rounded in the drawing) with the same value of their respective cross ratios.

To evaluate these cross ratios, see (2), Point 13.3.2. The difference is that the value then found was –1 (a harmonic set), whereas now it is not; cross ratios on a and a´ have to be just of *any* equal value.

It is also pertinent to recall that 5 points define a conic: the center of a projective pencil of lines and the four extremities of those lines.

Fig. 2 is an application of that explained hereinbefore, to a rectangle circumscribing an ellipse. The centers of line pencils will be now the ellipse´s vertices B and B´ (given). Line a will be the half-axis OA and line a´, the segment AD.

Divide OA and AD in an equal number of parts (seven in the drawing) and fold both line pencils. The intersection of corresponding fold lines (rounded in the drawing) are points of the ellipse.

It´s easy to check that both cross ratios are equal:

$$OA \equiv \frac{\frac{a}{7}-3\frac{a}{7}}{2\frac{a}{7}-3\frac{a}{7}} : \frac{\frac{a}{7}-4\frac{a}{7}}{2\frac{a}{7}-4\frac{a}{7}} = \frac{-2}{-1} : \frac{-3}{-2} = \frac{4}{3}$$

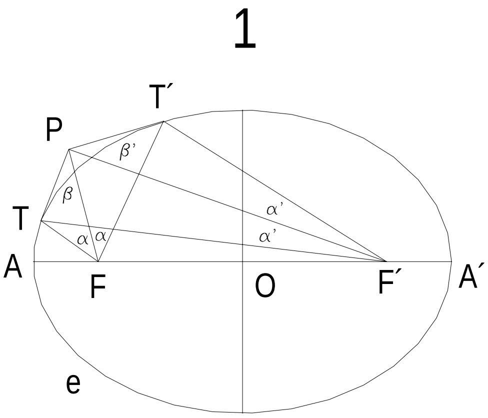
$$AD \equiv \frac{\frac{b}{7}-3\frac{b}{7}}{2\frac{b}{7}-3\frac{b}{7}} : \frac{\frac{b}{7}-4\frac{b}{7}}{2\frac{b}{7}-4\frac{b}{7}} = \frac{4}{3}$$

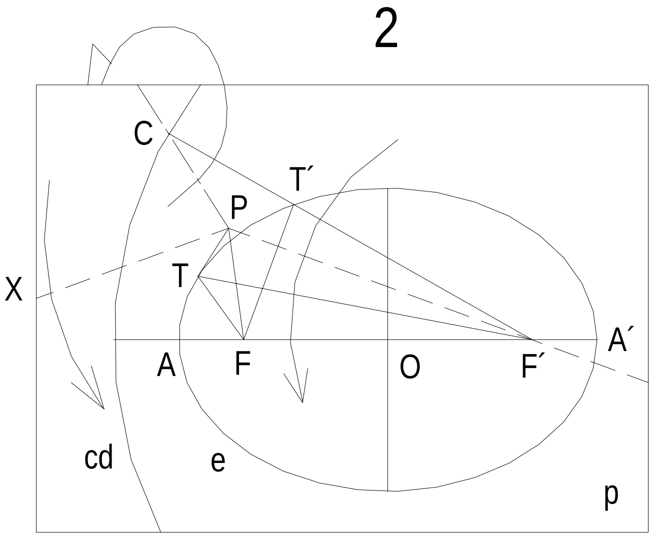
The verification has been made with the first four lines, but we might as well make the set to carry on progressively.

We can get points in the other quadrants following the same procedure, or else, by means of symmetry. Of course, the four vertices of the ellipse belong to it.

13.3.4 ELLIPSE: PONCELET THEOREM.

This theorem is applicable to all of three conics, but here we shall prove it by means of paper folding, for the ellipse only, and just for one of the focuses. It can be done alike for the other one. Its enunciation is as follows (Fig. 1):



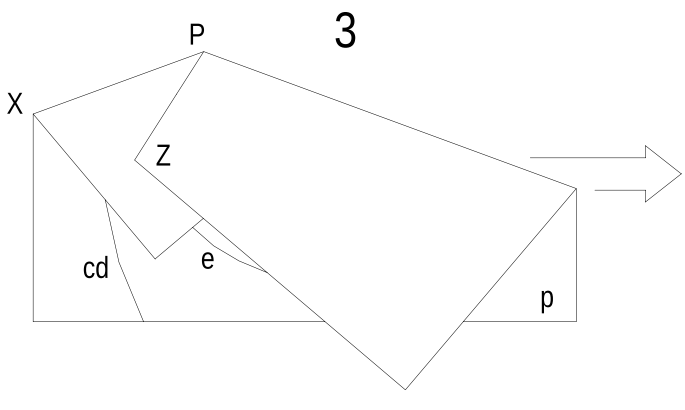


Let PT and PT´ the pair of tangents to ellipse e from the exterior point P:

1- PF´ is the bisector of Ang. T´F´T (a angles are congruent; likewise a´ are also congruent).

2- Angles T´PF´ = b´ and TPF = b are congruent.

Fig. 2 shows ellipse e and its director circle cd (Point 13.3.2) within a paper rectangle p. We can also see three fold lines through P: PX along tangent PT´ (valley fold); PF´, valley fold as well, and PC (mountain fold). These lines extend up to the rectangle´s boundaries.



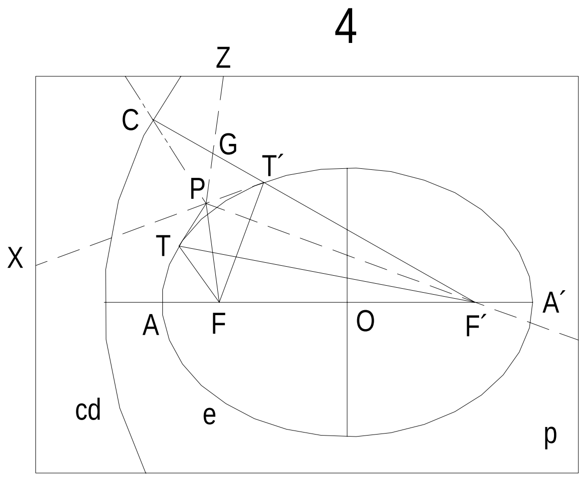


Fig. 3 is the result of folding flat Fig.2. When unfolding Fig. 3 we get Fig. 4, in which we can see valley fold PZ that implies these consequences:

- Ang. CPG must measure 180º - Ang. XPF´ (see Point 8.2.8.5).

- New point G lies on T after flattening because:
  - C $\rightarrow$ F (symmetry about tangent on T´; Fig.1 Point 13.3.2)
  - CP $\rightarrow$ PF (same symmetry)
  - F´G + GC = 2a (radius of cd) = F´T + TF (T, point on the ellipse).
  - Not being so, broken line CGF´ within Fig. 3 would not straighten along CGF´ when unfolding Fig. 4.

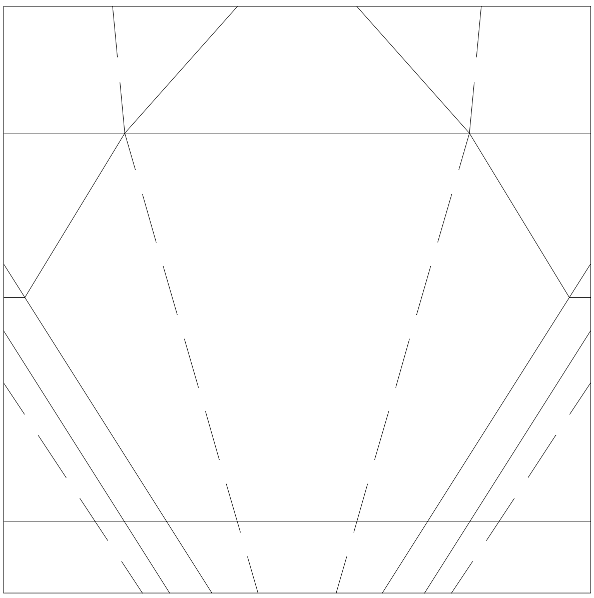
The result is: ΔCGP = ΔFTP after coincidence of their vertices: C $\equiv$ F; P $\equiv$ P; G $\equiv$ T. This means that Ang. TPF = Ang. CPG

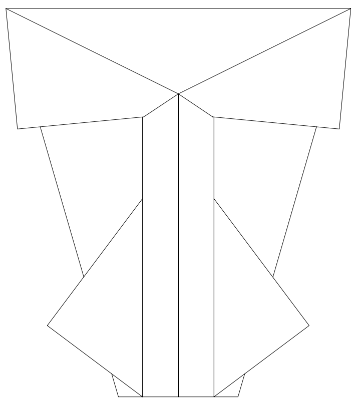
Hence,

Ang. TPF = Ang F´PT´ = Ang. GPC = 180º - Ang. XPF´ (*)
Ang. CPX = Ang. XPF (symmetry about the tangent in T´).
Ang. FPF´= 180º - (Ang. XPF + Ang. T´PF´).
Ang. GPT´= 180º - (Ang. XPC + Ang. CPG)
Ang. F´PG = Ang. T´PG + Ang. F´PT´
Ang. F´PT = Ang. F´PF + Ang. FPT

Therefore Ang. F´PG = Ang. F´PT : it demonstrates the 1st part of Poncelet theorem.
(*) ,, ,, 2nd ,, ,,

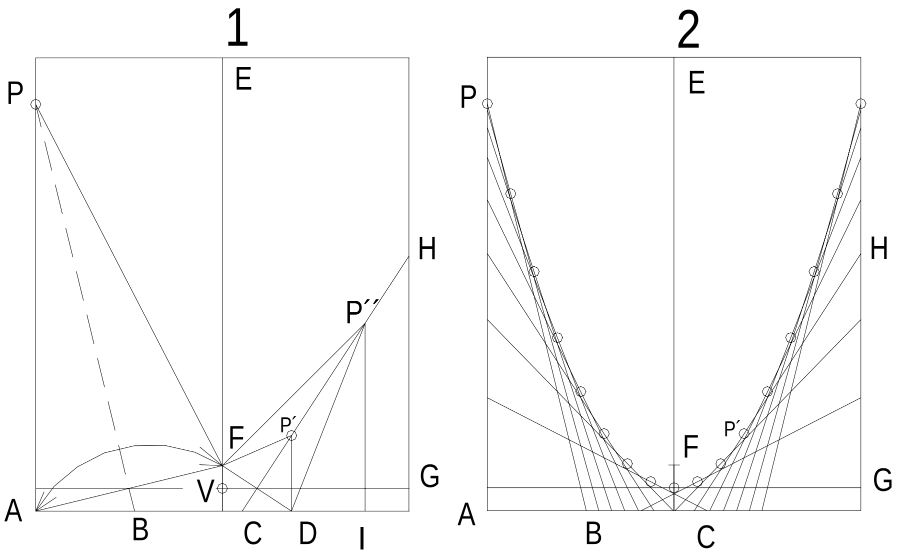
Interlude





13.4 PARABOLA

Although we already dealt with the parabola in Points 1.2.5 and 4 now we shall insist in its properties and folding process.



As we know, the parabola is a conic whose points (P, P´, etc.) are equidistant to a point F (the focus, situated on the axis EF of the parabola) and to a line AD (directrix).

Here it is the folding process (Fig. 1) to get a parabola as the envelope of its tangents:

- To take a point (A, D, etc.) on the directrix.
- To produce lines AF. DF, etc.
- To fold the perpendicular bisectors PB, CH, etc.
- To produce the normals to the directrix through points A, D, etc. i.e. lines AP, DP´, etc.
- To find the intersection points (P, P´. etc.) of these normals and previous perpendicular bisectors PB, CH, etc.
- Lines PB, P´C, etc. are the tangents to the parabola on points P, P´, etc.

It is easy to prove:

Fold line CH (perpendicular bisector of FD) is an axis of symmetry, hence P´ D = P´F. Therefore P´ is on the parabola.

On the other hand, lying P´ on CH, this fold CH has to be a tangent. This is so because any other of its points, e.g. P´´ is at a smaller distance to the directrix than it is to the focus: P´´I < P´´D; in right triangle P´´ I D, a leg is smaller than the hypotenuse.

So, if P´ is the only point on CH belonging to the parabola, it is because CH is the tangent to it on P´.

The process consists then in folding a given rectangle of paper in such a way that the points of its lower side (the directrix) are taken to lie on the focus F. The folds so produced are the tangents to the parabola (Fig. 2 also shows the rounded tangency points).

Note that the vertex V, half way in between focus and directrix, is also a point of the parabola, and the parallel VG to the directrix is the tangent to the conic on V.

13.5 HYPERBOLA

This conic is the locus of the points such that the difference of the distances from them to another two fixed points (focuses F, F´), is constant. See Point 8.2.8.6.

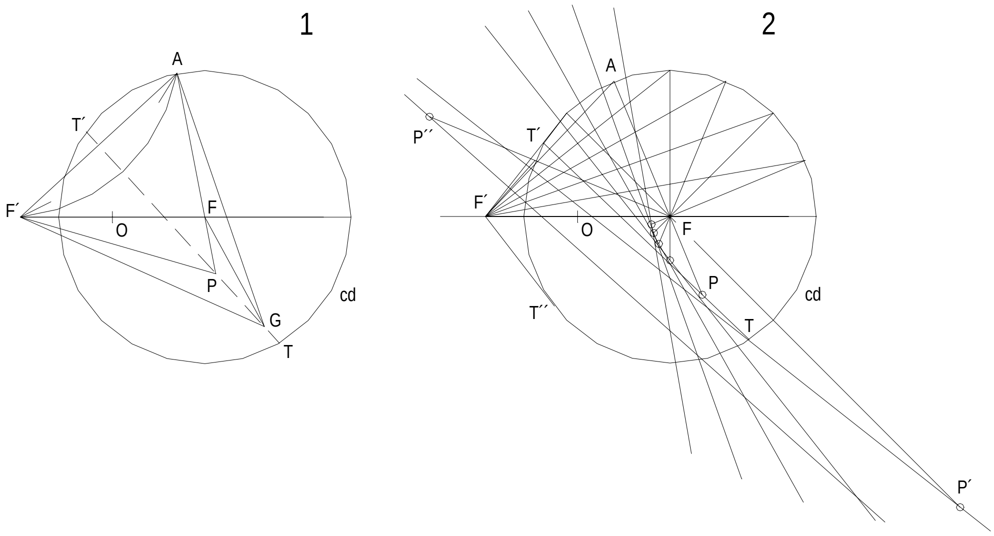


Fig. 1 shows how to get a point P of a hyperbola and its tangent TT´on it, from these data: both focuses F, F´, its center O and its director circle cd with center F and radius 2a.

This is the folding process:

- Bring F´ to lie on any point A of cd: folding line, TT´.
- To produce fold AF till the intersection with TT´.
- Result: point P on the hyperbola and its tangent TT´ on it.

Proof:

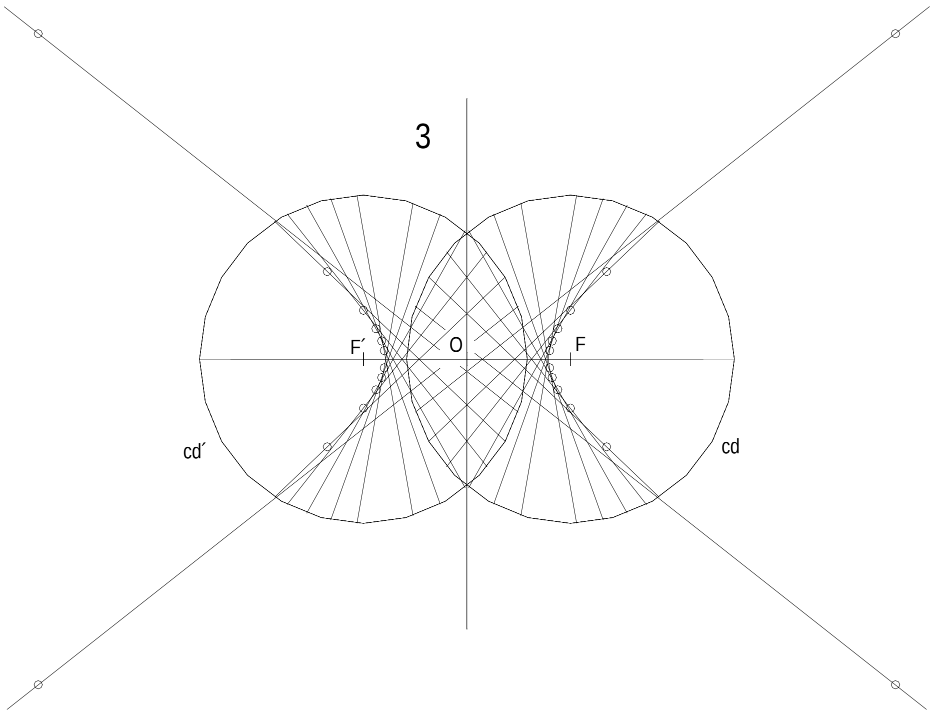
P is in the hyperbola for PF´- PF = PA – PF = FA = 2a.

Moreover, TT´is the tangent on P because any other point but P (e.g. point G) does not fulfil the hyperbola´s condition: GF´ - GF = GA – GF > FA = 2 a (see ΔGAF).

Fig. 2 shows how points and tangents of the lower part of right hand side branch of the hyperbola have been obtained. P´ is the furthermost point. P´´ is also in there and belongs to the left branch, which corresponds to focus F´. Note that the points of the left branch are produced after the points on cd lying on its arc seen from F´ (tangent F´ T´´ is one of its limits).

Thus, there are two ways of constructing the left branch: take as many points in that seen arc as were taken in the rest of cd for the right branch, or else, do the same construction for circle cd´with center F´.

Fig. 3 shows the complete hyperbola (rounded points) with its two branches.



13.6 ANOTHER CURVES

Origami deals with conics as the envelopes of their tangents; likewise we shall study now some other curves, though not conics, under an analogous treatment.

13.6.1 LOGARITHMIC SPIRAL

Its equation in polar co-ordinates is:

$$\rho = ke^{m\omega} \qquad (1)$$

It is represented in Fig. 1 after a hexagon. We can see in it that the angles grow as an arithmetic progression of ratio $\frac{\pi}{6}$ whereas the radius vectors do as a geometric progression with $\cos\frac{\pi}{6}$ as ratio. This correspondence of arithmetic and geometric progressions brings forth logarithms. Let´s find out the value of the constants in (1) to conclude that the spiral we get is actually a logarithmic one. Calling *a* to the apothem of the hexagon we have:

Vertex	r	w
1	a	$5\frac{\pi}{6}$
2	$a\cos\frac{\pi}{6}$	$6\frac{\pi}{6}$
3	$a\cos^2\frac{\pi}{6}$	$7\frac{\pi}{6}$
4	$a\cos^3\frac{\pi}{6}$	$8\frac{\pi}{6}$

Vertex	ρ	ω
i – 1	$a\cos^{i-2}\left(\frac{\pi}{6}\right)$	$(i-1+4)\frac{\pi}{6}$
i	$a\cos^{i-1}\left(\frac{\pi}{6}\right)$	$(i+4)\frac{\pi}{6}$

Substituting in (1) the last four values of ρ,ω, we´ll have:

$$a\cos^{i-2}\left(\frac{\pi}{6}\right) = ke^{m(i+3)\frac{\pi}{6}} \qquad (2)$$

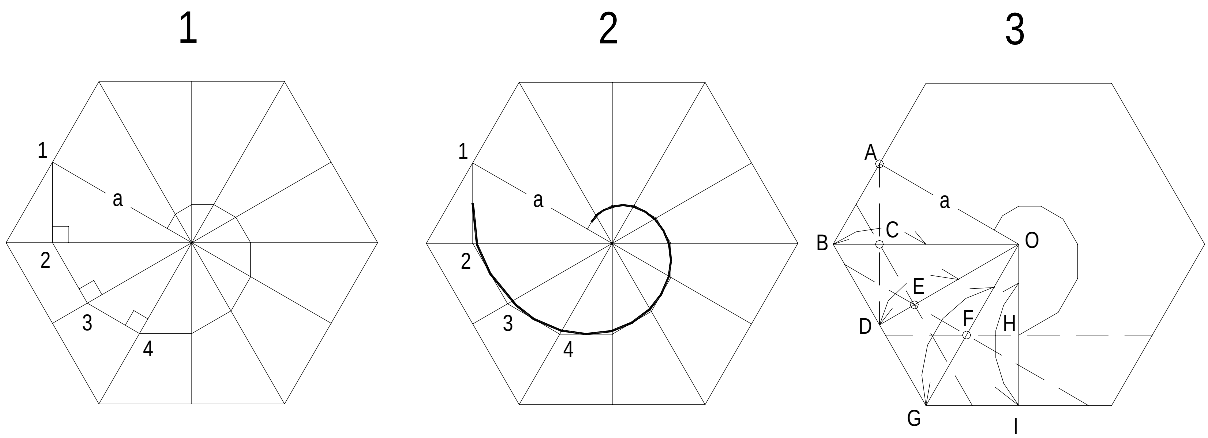
$$a\cos^{i-1}\left(\frac{\pi}{6}\right) = ke^{m(i+4)\frac{\pi}{6}} \qquad (3)$$

Dividing (2) by (3):

$$m = \frac{\ln\left(\cos\frac{\pi}{6}\right)}{\frac{\pi}{6}} = -\,0.2747161 \qquad (4)$$

Taking (4) to (3) and assigning any value to *i* (e.g. *i* = 1), we get the other constant k:

$$k = e^{-5\ln\left(\cos\frac{\pi}{6}\right)} = a \times 2.052801 \qquad (5)$$



Once we know the values of m, k in (1) we may verify that equation in its particular application to Figs. 1,2,3. First one shows the series of tangents to the spiral; the second is the spiral as envelope, and the third one is the folding process that will be as follows.

1. To produce the three diametrical folds of the hexagon (Fig. 1).
2. To fold each radii to lie on their contiguous one to get the six apothems (Fig. 1).
3. OB → OB ; A → A
   Result: AC.
4. OD → OD ; C → C
   Result: CE.

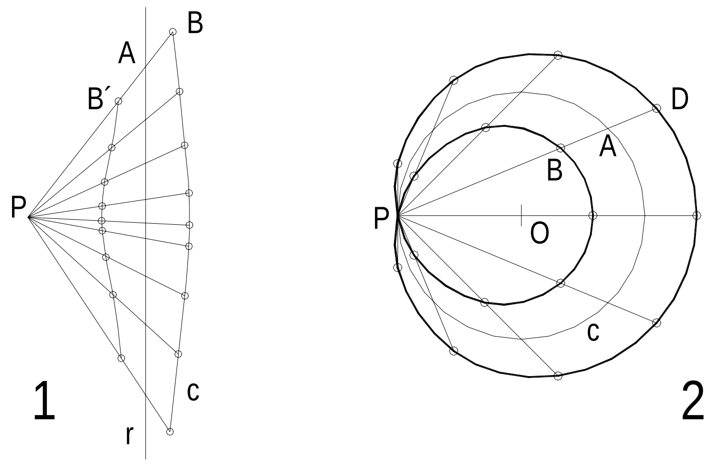
5. OG → OG ; E → E
   Result: EF
6. OI → OI ; F → F
   Result: FH

13.6.2 CARDIOD

A BIT OF HISTORY

Fig. 1 shows the *conchoid of a straight line* named after Nichomedes. Curve c, in its two branches, is determined by line r, pole P and equal segments like AB = AB´.

The *Pascal´s limaçon* (Fig. 2) is the conchoid of a circumference c (radius OP = r) with respect to one of its points P, in such a way that always it is AB = AD < 2r. It also has two branches.



The CARDIOID derives from the *Pascal´s limaçon*, with this condition: AB = AD = 2r. The curve is symmetric but has only one branch. Its shape reminds that of a heart (Fig. 3).

This Fig. 3 shows the rounded points of the cardioid at the extremities of its radius vectors. Being

OP = r ; AB = AP = A´B´ = A´B´´ = 2r

and taking into account the right triangle PAA´, the cardioid polar equation will be:

$$\rho = 2r\cos\omega + 2r$$

$$\rho = 2r(\cos\omega + 1)$$

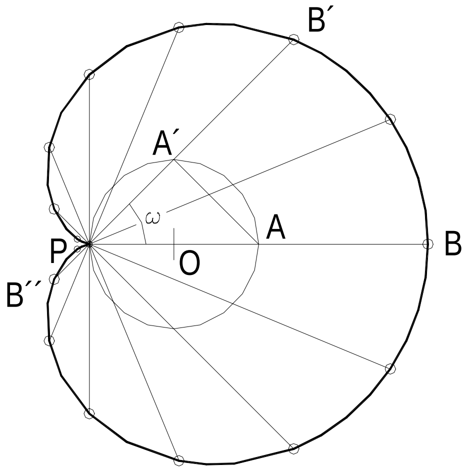
We have drawn Fig. 3 by dividing the circumference with radius OA in 8 equal parts, what means that, as P lies on the circumference, angles ω have these values:

22,5º; 45º; 67,5º; 90º; 112,5º; 135º; 157,5º and 180º

Let´s see now how to make this construction by paper folding.

For that we shall draw (Fig. 4) a circumference with center O and radius 3r; then divide it in 16 equal parts numbering them orderly.

Afterwards we fold the chords between points i → 2i in the circumference, as we´ll see later. Those folded chords will be the tangents to the cardioid, and this will be the envelope of them.



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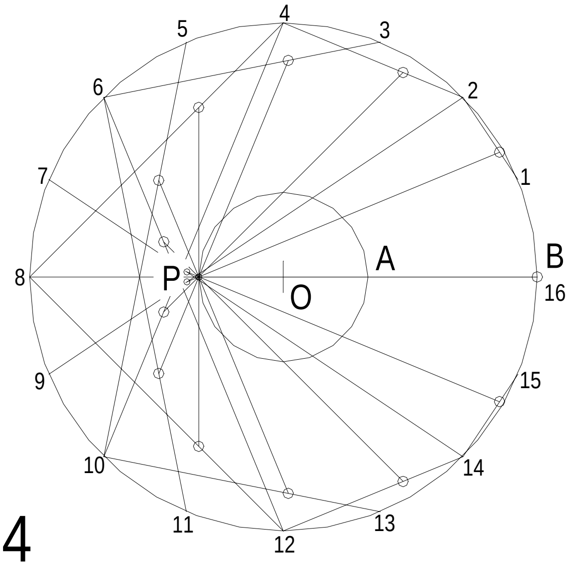


1 → 2
2 → 4
4 → 8
8 → 16 Once we reach maximum point 16, subtract 16 (one circumference) and join to resultant point 0; as this is equivalent to 16, it means that the series started with 1, is exhausted.

Restart a new series from first undisclosed point, in this case the number 3.

3 → 6
6 → 12
12 → 24 (24 – 16 = 8)
12 → 8 From here on this series is exhausted.

5 → 10
10 → 20 (20 – 16 = 4)
10 → 4 Exhausted series.

7 → 14
14 → 28 (28 – 16 = 12)
14 → 12 Exhausted series.

9 → 18 (18 – 16 = 2)
9 → 2 Exhausted series.

11 → 22 (22 – 16 = 6)
11 → 6 Exhausted series.

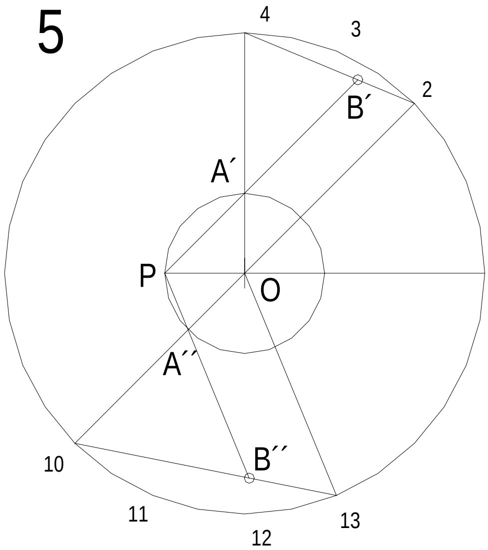
13 → 26 (26 – 16 = 10)
13 → 10 Exhausted series.

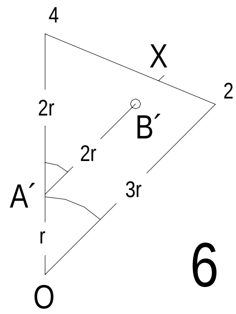
15 → 30 (30 – 16 = 14)
15 → 14 Exhausted series and process.

We overlap now Fig. 3 to the figure just obtained, in order to check how the rounded extremities of the cardioid lie in a very particular way on the respective chords 1,2; 2,4; etc.

We have to prove now that the folding process after Fig. 4 is equivalent to the geometry in Fig. 3. To obviate a tangle of lines we outline in Fig. 5 only the elements of Fig. 4 related to ω = 45º and ω = 292,5º.

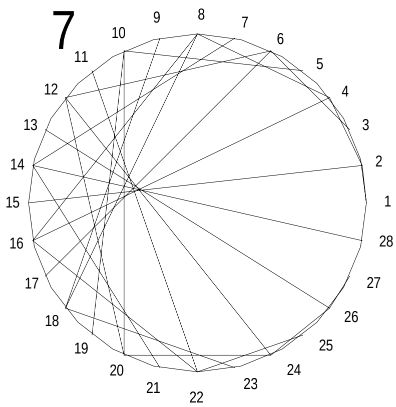
From Fig. 5 we can also deduce that a cardioid is as well an epicycloid generated by a circumference with a diameter equal to the directrix circumference´s (that having O as its center).





We shall focus our reasoning in ω = 45º ; alike result would be reached for any other value of ω. Being the proportions of the segments as shown in Fig. 6, with the marked angles in O and A´ measuring 45º, B´ will coincide with X, bearing the consequence of $B'4 = 2 \times B'2$ and $A'B' = 2r$.

Fig. 7 is a cardioid got after chord folding, for a circumference divided in 28 equal parts.



13.6.3 NEPHROID

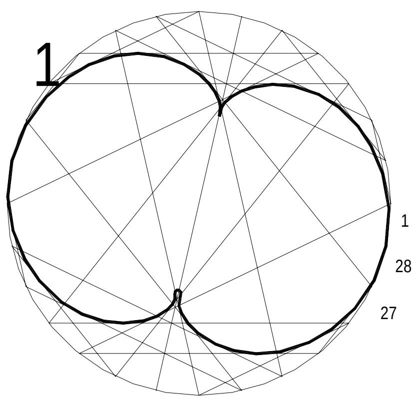


Fig. 1 shows that curve whose shape reminds that of a kidney. It has been obtained like Fig. 7 in point 13.6.2, but applying to chords and foldings a pitch of i → 3i.

14 TOPOLOGIC EVOCATIONS

If we think of topology as the science of surfaces continuity, we are taken to Möbius bands and flexagons, specially studied by Miguel Ángel Martín Monje.

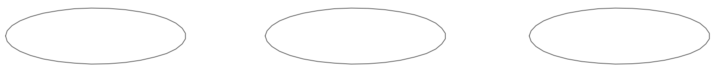
14.1 MÖBIUS´ BANDS

They are paper strips very long as compared with their width, having both extremities glued to each other.

Let´s imagine a band with red (R) obverse and white (B) reverse. We shall call discontinuity limit to any of the two long sides of the rectangle for they are the borders between the red and the white. On the contrary, we´ll assume surface continuity across the small sides for they are stuck.

We shall study several cases. For the sake of clarity, the drawings shown are unidimensional.

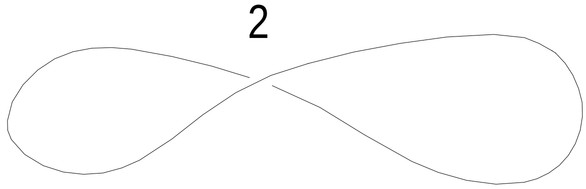
1. If we glue the red extremity over the white one without twisting the band (zero twists), we get a cylindrical surface white in the convex side and red in the concave one. Now, if we imagine an ant walking forward on the white surface without crossing the discontinuity limits, it will give an infinity of turns over the same white surface. If we would have placed the ant on the red surface and made a similar experiment, the ant would walk all the time on the red surface.
   SUMMARY: glue R / B, zero twists, n (even or odd) cuts (parallel to the discontinuity limits and equidistant). RESULT: According to Fig.1, n independent bands, with same length and configuration as the original, having a width of 1 / n.



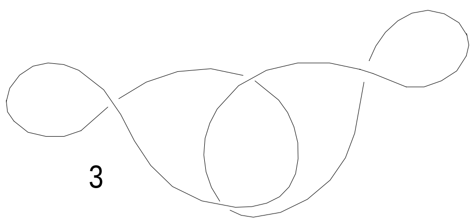
1

2. R / B; even number of twists (Fig. 2 has two); zero cuts. There is no transition from one colour to the other, as in case 1. Length and width are the original.

2



3. R / B; 2 twists; one cut. Result in Fig. 3: two interlaced bands having each of them the same

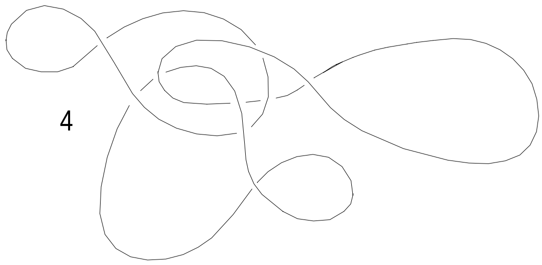


3

configuration of Fig. 2.

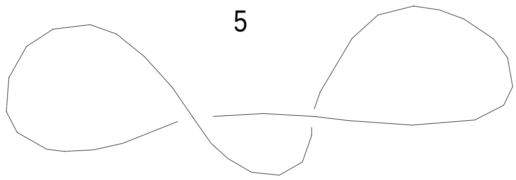
4. Keeping the left-hand side band of Fig.3 as it is, we produce a cut in the other. Result: the left band stands fixed whereas the other is divided in two with a length equal to each of Fig. 3; They appear interlaced between them and with the one in the left.

4



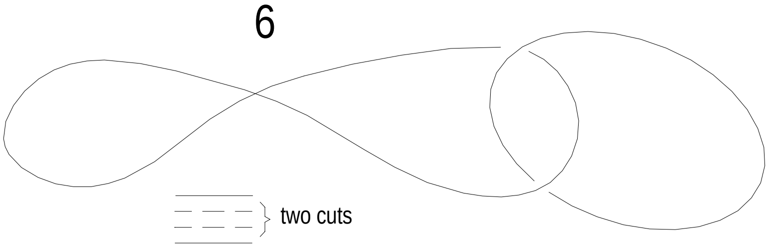
5. Glue B / B; one twist; one cut. Result (Fig. 5): band´s length double than the original and half width. The ant strolls successive and continuously on red and white.

5



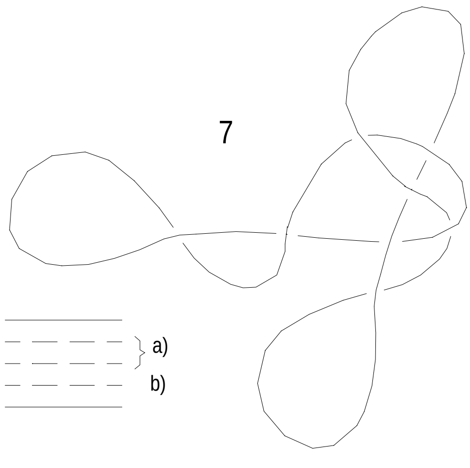
6. Glue B / B; one twist; two cuts produced as an only ontinuous one. Result (Fig. 6): two interlaced bands of the same width (1 / 3 of the original); one of equal length as the original, and double length the other.

6



7. Glue B / B; one twist; three cuts: a) First two cuts are produced as an only continuous one

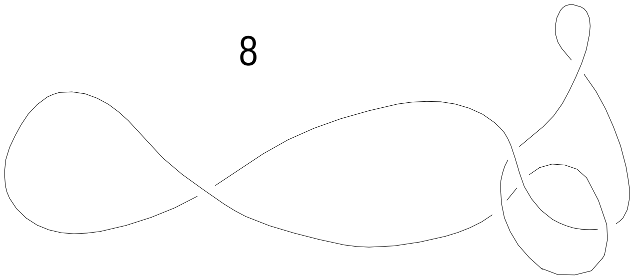
7



with double length than the original, thus resulting in a band with that double length and 1/4 width. b) The other cut that appears in the middle after cut a), has a length equal to that of the original and is independent from a), produces a band of double length than the original, width 1 / 4, interlaced with itself and with the curl a). See Fig. 7.

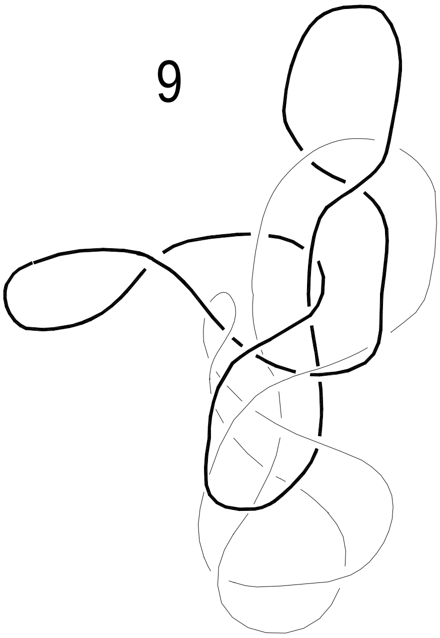
8. Glue B / B; three twists; one cut. Result: one band of double length, half width, and double-interlaced with itself. (Fig. 8).

8



9. Glue B / B; three twists; three cuts. Result: two independent bands, both with a length double than the original and width 1 / 4, interlaced each one with itself and both between themselves. See Fig. 9 where one of the bands is outlined.

9



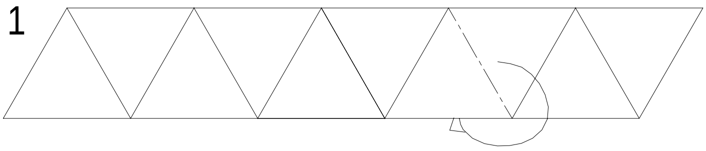
After the foregoing one can imagine how many possibilities the Möbius bands can render. Not needed to add that playing with the length, width and curvatures of the bands we can obtain very beautiful figures.

14.2 FLEXAGONS

We can define the flexagon as a flexable polygon (not merely flexible).

The simplest one is that whose construction is described in Figs.1 to 8. It was discovered by A. Stone while playing with paper strips (5 / 8 ” width) obtained after the equalisation of English and American paper sheets. Incidentally, A. Stone was a British graduand in mathematics at Princeton.

1



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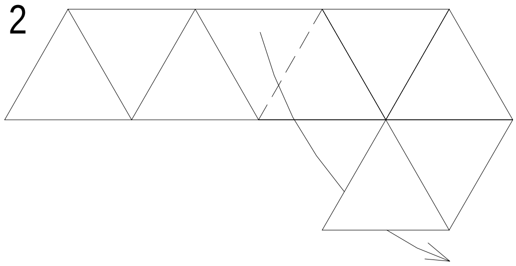
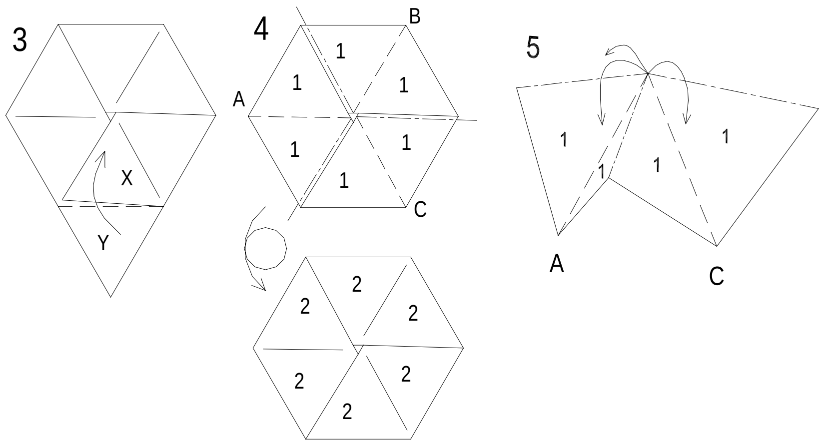


One can understand why we said earlier that a flexagon is alike a Möbius band: looking afterwards how it works, we´ll see its topological connotations observing its endless continuity of surfaces.



Here it is the folding process. Fig. 1 starts with a triangled paper strip as in *Solution 1*¸ Point 8.2.3. Figs. 1 and 2 show the folds to get Fig. 3; this indicates how ΔY has to be folded and stuck over ΔX.

In Fig. 4 we number with 1 the six triangles of the obverse, and with 2 the six of the reverse side. Then we produce in the same Fig. 4 three mountain and three valley folds, bearing in mind that the seen triangles that meet, respectively, in A,B,C show paper continuity whereas this is not the case with the others. That´s why it is a must to perform the folds as indicated.

So we reach Fig. 5 with the beginning of the operation that gets points ABC meeting at the bottom vertex of Fig. 6. Once this operation is over, a new hexagon will appear, surprisingly, with six unnumbered triangles. If we number them with a 3 (Fig. 7), we can check that the reverse of Fig. 7 has numeration 1.

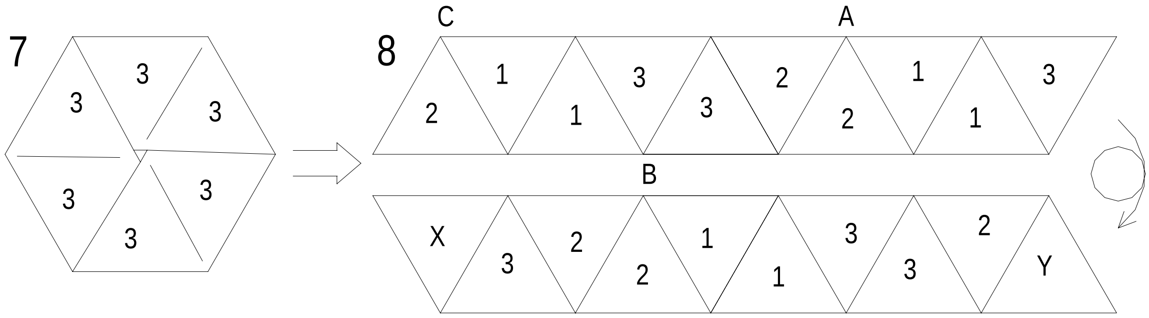
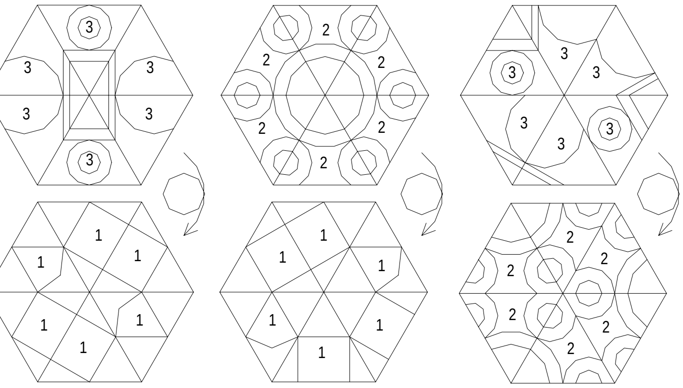


Fig. 8 is Fig. 7 after the triangles XY have been unglued and the rest developed. It shows in its obverse and reverse the corresponding numbers in each triangle (six 1, six 2, six 3), the X, Y triangles and vertices A, B C.

If we manipulate Fig. 7 as we did with Figs. 5, 6, we can see appear hexagons with these combinations in *obverse / reverse*: 1 / 2; 1 / 3; 2 / 3.

Till now we have got a tri-hexa-flexagon: A hexagon based flexagon, with three faces (marked with 1, with 2 and with 3). But we can introduce a certain number of variations.
For example, we can draw on each one of said three faces, three different figures having internal symmetry within the hexagons. When manipulating that flexagon and taking into account obverses and reverses we come across, as before, with two faces 1, two faces 2 and

two faces 3. What is peculiar now, though, is that the two faces 1 are different to each other, and the same applies to faces 2 and 3. Hence, now we have 6 different faces instead of the three we had before. We still have a tri-hexa-flexagon, though. All this can be seen in Fig. 9 which, when unfolded yields Fig. 10 : Both figures evoke a kaleidoscopic vision.



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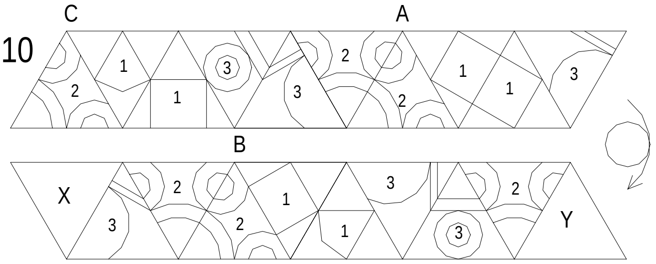

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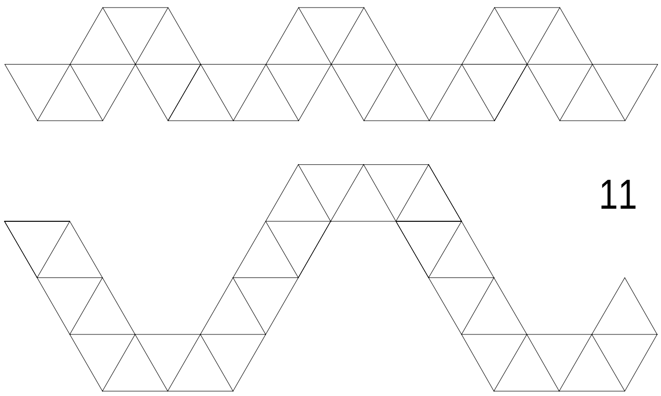
Going a bit further we might say that flexagons have to be constructed but not necessarily with straight paper strips, nor even have to be equilateral triangle based.



11

Fig. 11 (lower side) shows a strip that properly folded will yield an hexa-hexa-flexagon: a hexagonal based flexagon with the possibility of six faces. The upper illustration is the base for another different flexagon. Hence, tri, hexa, octo ... n-hexa-flexagons, can be constructed, though paper accumulation handicaps flexibility.

Looking at Figs. 8 and 10 we can see how difficult may be to define the fold lines and its nature (mountain or valley) in order that the final result will be the wished hexagon (in any case gluing the triangles of both extremities). A. Stone and P. Jackson have developed a rather elaborate graphical process to enable the adequate folding of a given strip.

For the moment we should forget the combinatorial calculus to get the formulas leading to a flexagon mathematical approach: Mathematicians have not yet arrived to define conclusions because of the great number of restrains associated to that constructive process.

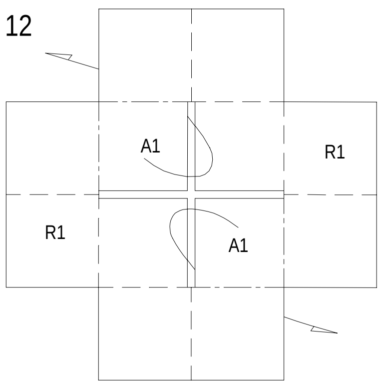
As we mentioned before, the equilateral triangles that are the base for hexaflexagons, may be superseded by squares to produce tetraflexagons. Moreover, flexagons have to be not necessarily plane as those seen hereby: they can be solid, too.

These employ the same *raw material* than the others, i.e., equilateral triangles or squares that eventually configure in combinations of cubes, parallelepipeds or tetrahedrons instead of polygons.

Once we are about jumping from 2 to 3D, we shall exhibit a solid flexagon. These options, as well as open flexagons (enchained rectangles alike folding screens) hold off the initial conception of a strip glued at its extremities.

The one shown in Figs. 12 to 17 is original of R. Neal. Fig. 12 is the starting plan view with the necessary folds to get Fig. 13. From this, and forcing downwards its base on the center, we get the total flattening of the figure with the four faces 1 as shown in Fig. 14. These faces are indicated in Fig. 12 on their respective places: A1 for the obverse and R1 for the reverse.

12



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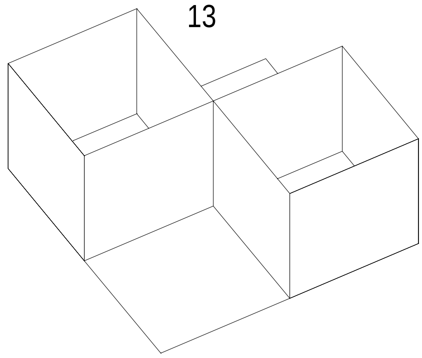

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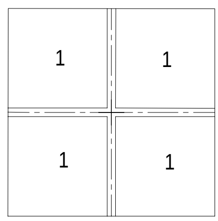

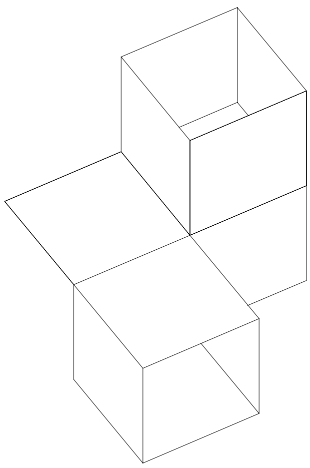

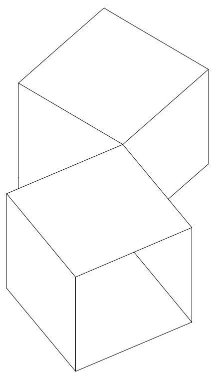

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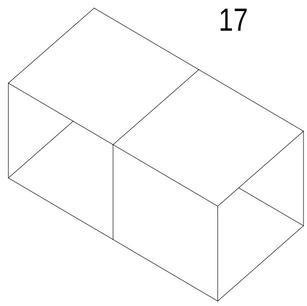


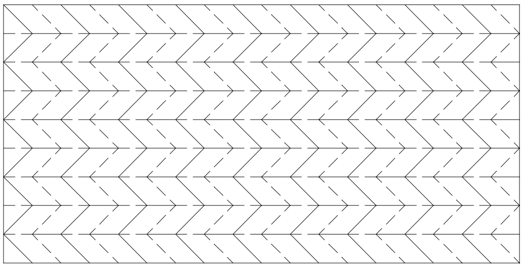
The rest of forms are some of the possible resultant figures after manipulating the former.

15 FROM THE 2nd TO THE 3rd DIMENSION

Everything seen up to now has a bidimensional treatment, though we would have to incorporate certain nuances to this assertion. For example: the paper itself is three-dimensional (2.500 times thinner than the small side of a DIN A4); the added volume when folding flat; the flattvolumelike of unit figures in Point 8.3.4; the volume shaped as a bas-relief of tessellations (case 3, Point 12).

We shall begin with the latter type of figures to jump to the 3rd dimension. Thus, we have in Fig. 1 a development which, when folded as indicated leads to the elastic 3D composition of Fig. 2.

1



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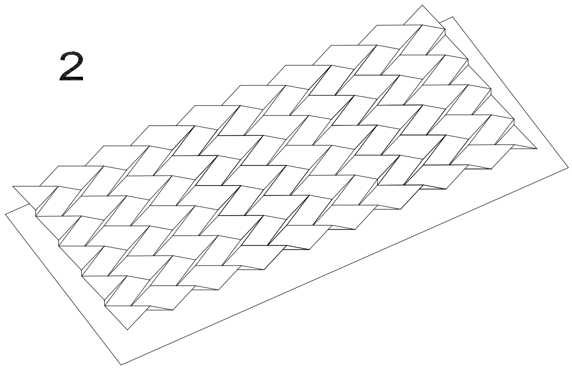
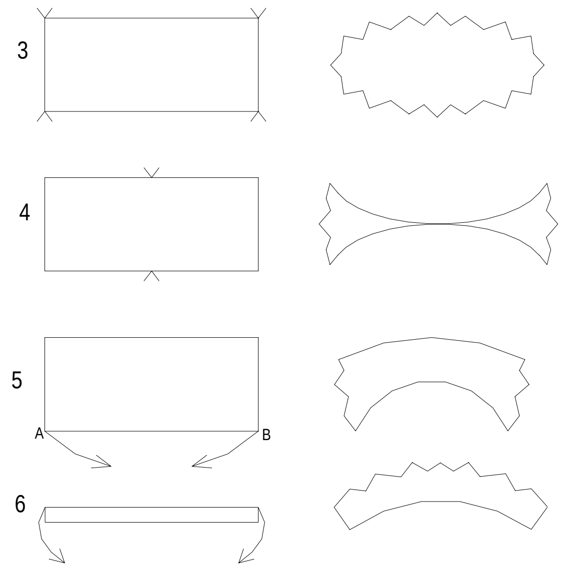


Just because of that elasticity it allows delightful transformations. Let´s see several of them:

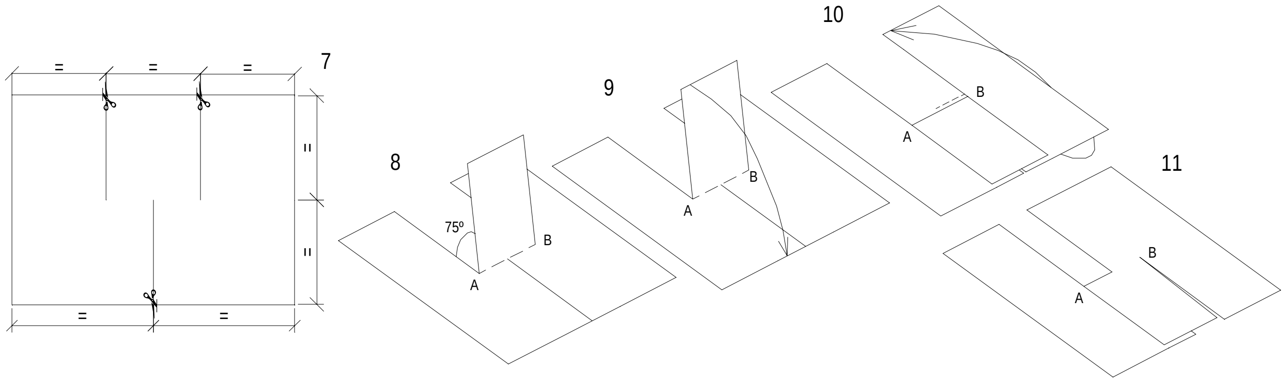
3- Plan view of a quasi-ellipse.
4- Same of a quasi-hyperbola.
5- Face AB adapts itself to the arc of any opposed curve (circumference, parabola, etc.).
6- Quasi-toroidal section.



As a matter of fact it is very easy to pass from the 2nd to the 3rd dimension; it suffices to indicate in 2D that a fold has to be performed not at 180º as usual, but at any other angle specified there.

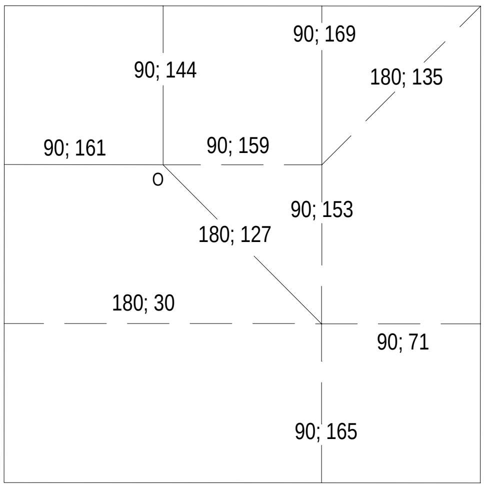
This becomes clear in the transition from Fig. 7 to 8: in Fig. 7 a rectangle receives three cuts and then its central upper part is revolved 75º about AB. Fig. 8 is, in fact, a 3D figure.

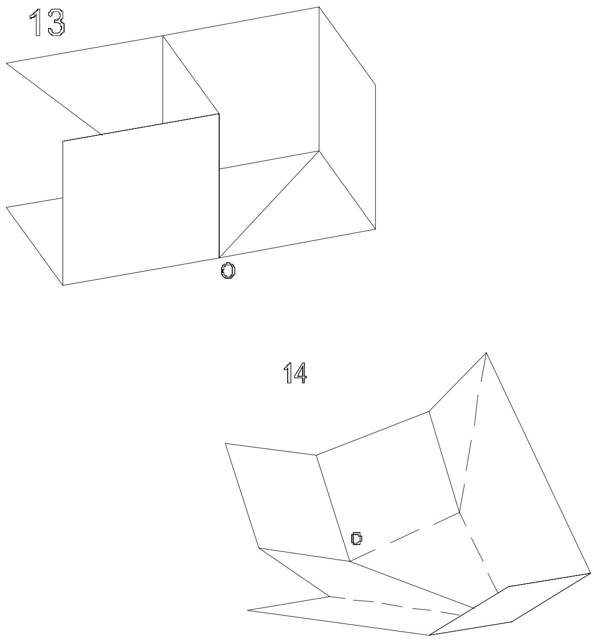
Let us profit of that to show, again in 2D, a paradoxical figure. Fig. 11 results at the end of process 8,9,10: In Fig.7 we made three cuts, whereas in Fig. 11 there appear to be four.



Another manner to indicate the revolving angle, is to denote on the folding line the value of the dihedral angle to build (T. Kawasaki). By so doing, the process data are completed: Fig. 12 shows a pair of angles on each of the hinges; the set of the first is used in Fig. 13, whereas the second is for Fig.14 (only approximate values in this case).

12



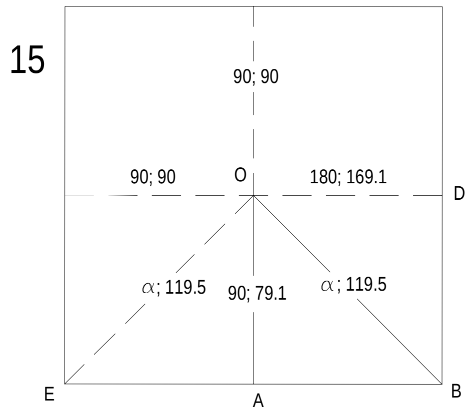


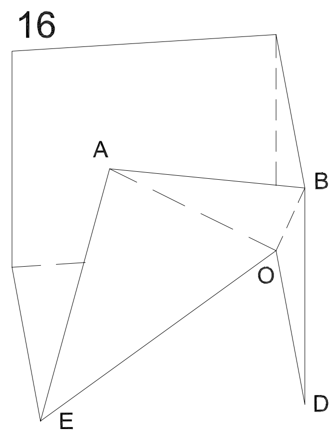
It is pertinent to clarify certain things about the three latter figures.

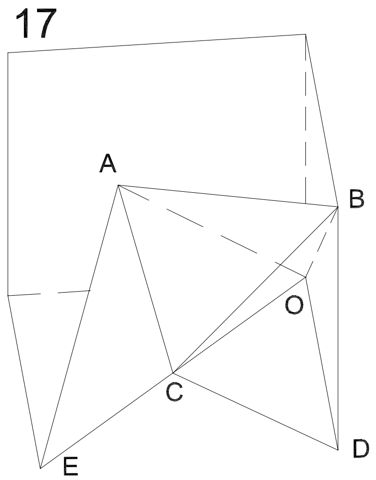
It is evident that in node O of Fig. 12 not all the conditions of Point 8.2.8.5 to flat folding are fulfilled. The fact is that there is not such a flat folding anymore: what we have produced is, actually, a 3D form.

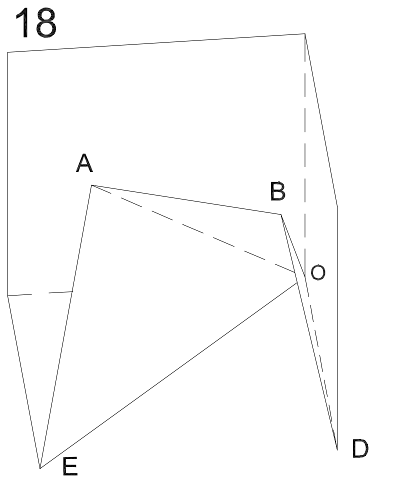
We said before that dihedral angles in Fig. 14 have an approximate value. There is not other alternative: the quadrilaterals of Fig. 12 that are kept as they are in Fig. 13, have, on the contrary in Fig. 14, their vertices subjected to compound revolutions in such a way that the former plane figures, are not such any more. Only paper docility allows that manipulation.

The case of Fig. 15 (also by T. Kawasaki) is different though two pair of values are shown in the hinges: first set for Fig. 16 and second for Fig. 18.









We must clarify two things: a) Mountain or valley folds in Figs. 16, 17 and 18 not always are in correspondence with Fig. 15. The formers are represented after the perspective´s point of view. b) Dihedral angle values in Fig. 15 do not exhibit any sign: they are supposed to have the most significant value offered to the point of view.

Fig. 17 helps to get the two α values for those dihedral angles in Fig. 15:

Dihedral angle in OE = Ang. ACD = 2 Ang. BCD

$$EA = AB = BD = DE = 1 \quad ; \quad CD = \frac{1}{\sqrt{2}}$$

ΔBCD is right angled in D, and in it:

$$\text{tg}\, BCD = \frac{BD}{DC} \quad ; \quad Ang.BCD = Arc\,\text{tg}\sqrt{2}$$

$$\alpha = Ang.ACD = 2Arc\,\text{tg}\sqrt{2} = 109.47^{\circ}$$

Likewise we would obtain the same α value for the dihedral angle in OB.

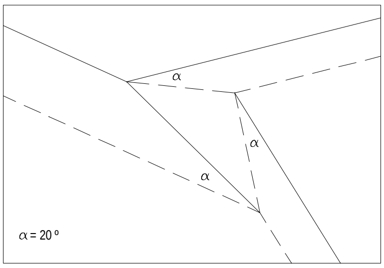
Dihedral angle in OA = Ang. EAB = 90º, verifiable in Fig. 16 after the data offered by the drawing and the programs of calculus that complement CAD. In fact, all dihedral angles have been calculated in this way.

In Fig. 18 one can see how the polyhedron angle O (DBAE) can be revolved about its base DOE from the situation in Fig. 16 till the coincidence of B and E. Note that this can be done freely because the affected figures are triangles instead of the quadrilaterals of Fig. 14.

Another question to add is that Figs. 12 and 15 which, of course, lead to 3D figures, also yield *subfigures* 2D about the hinges with 180º value dihedral angles.

Let´s dig out now in the process 2D → 3D → 2D already seen in case 2A, Point 12. There we played with an octagon; here we´ll do with scalene triangles. Each triad of figures

19



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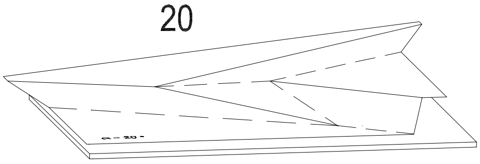

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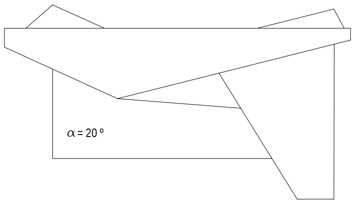

(e.g. 19,20,21) keeps that order 2D, 3D and 2D (a flat configuration).

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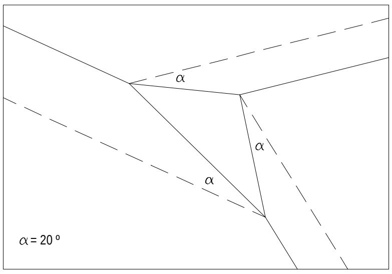

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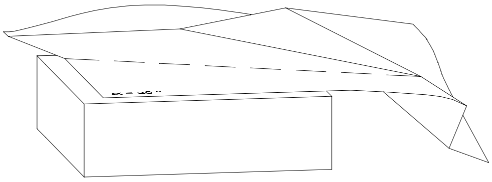

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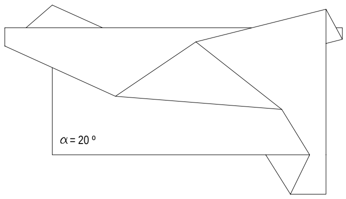

We can note that the four figures at the beginning (19, 22, 25 and 28) are geometrically identical: the central triangle is always the same, and it is subjected to a twist of 40º (α = 20º).

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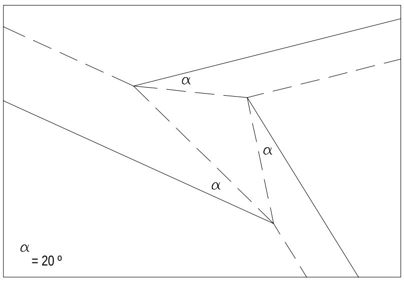

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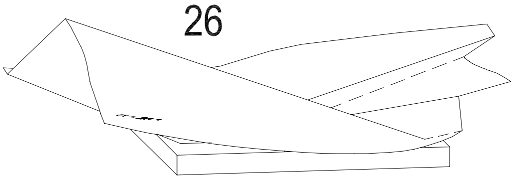

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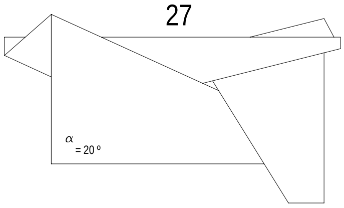
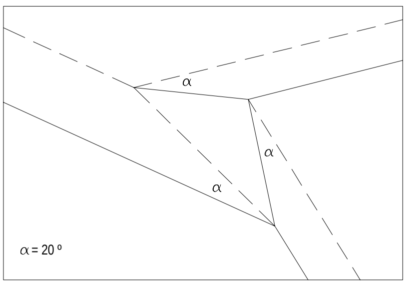


The only difference is the nature (mountain or valley) of folding lines.

28



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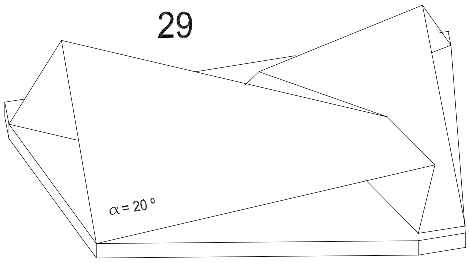

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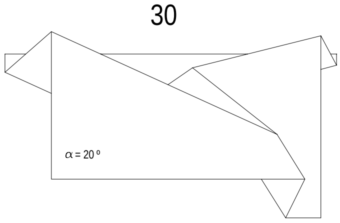


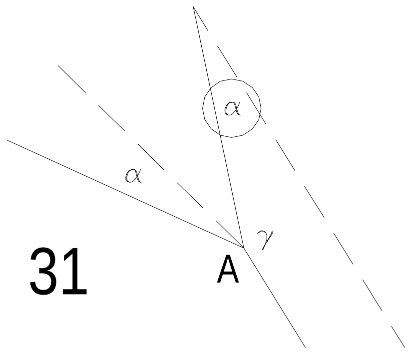
Fig. 31 is a portion of Fig. 28 with some addenda. In it we can see that node A fulfils the flattening condition of alternate angles adding up to 180º. This is so because of the configuration of the α angles adjacent to the sides of the triangle and for the corresponding triset of parallel lines.

In effect, rounded angle α and γ are supplementary for both are interior to the same side of a secant, therefore α and γ will also be supplementary.

In all the cases one can observe the fulfilment of all the flattening conditions by all the nodes.

It should be noted that, though in all the cases flattening is reached, continuous docility (accordion-like) operates only in Figs. 19 and 28. On the contrary, in the rest there appears a forced docility. To keep flat the triangle and the surfaces between parallels, the other surfaces are compelled to adopt the form of a ruled developable surface.

31



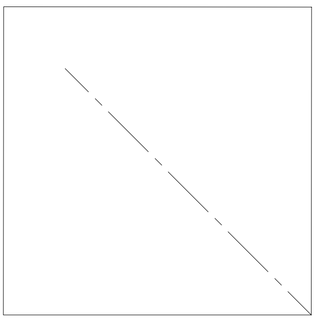
Note that in Figs. 22 and 25 all the sides of the triangles have folds of the same nature, whereas that is not the case for Figs. 19 and 28.

Another peculiarity is this: as we said, the four figures at left are equal geometrically speaking. The four to the right are also externally congruent (we can overlap in coincidence their perimeters), though their internal foldings are all different.

Summarising: with $\alpha = 20º$ it has been possible to get flatness with any combination of mountain / valley folds for the triangle. That is so because $0 < \alpha \leq 30º$. If $30 < \alpha \leq 60º$, flattening will only be possible when there was mountain / valley alternation in the three sides of the triangle. If $60 < \alpha \leq 90º$, flatness is impossible.

Former conditions apply to triangles. For the octagon mentioned before, as the sum of all its angles is 1080º, the limits for α change from 30; 60 to 67,5; 135. In fact, for Fig.1 (case 2 A, Point 12), it is $\alpha = 67{,}5º$ and all the eight sides of the octagon are mountain fold.

In practice, all the cases we have studied till now require that the paper will be subjected



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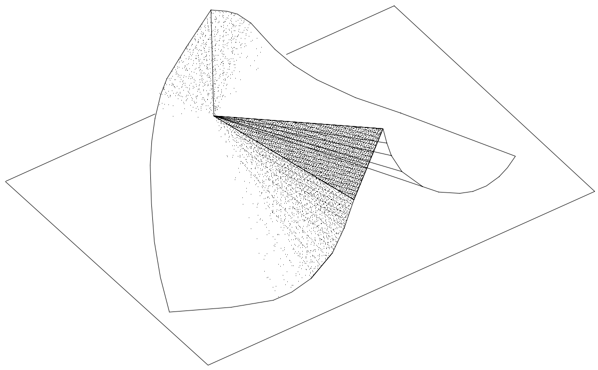
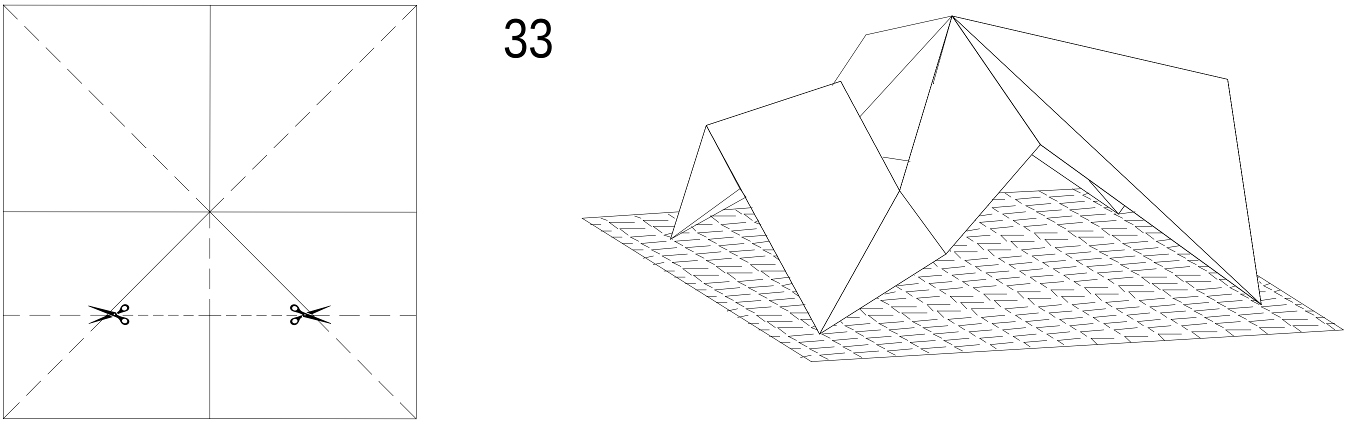


to many folds to give way to 3D from 2D. We have even considered ruled developable surfaces. Right-oh!, but P. Jackson and A. Yoshizawa exploit to incredible limits of beauty the obtention of 3D forms with the minimum of folds and the paper as developable means. To the first of the authors belongs Fig. 32 obtained by a unique fold.

It is not the first time that scissors are mentioned in this book. Now, though, we are going to use a cutter instead, and help out ourselves with a pencil.

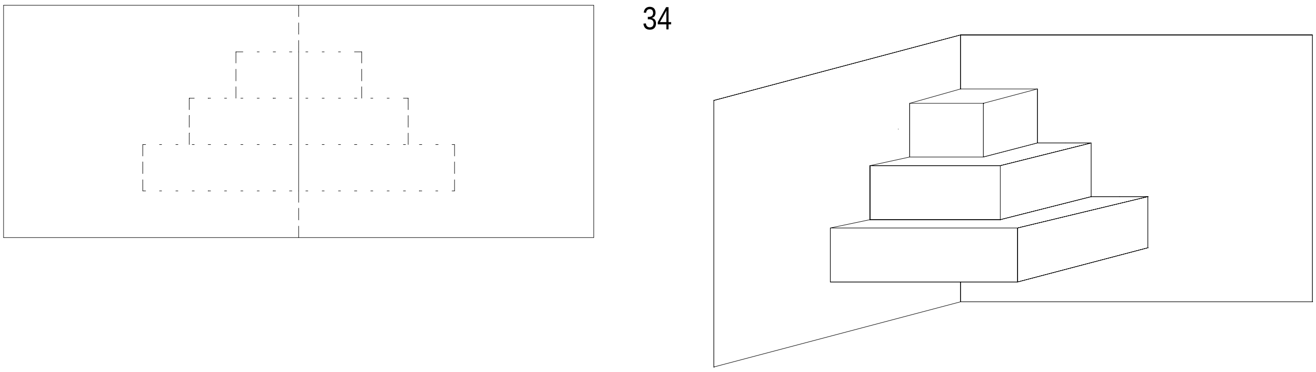
We know that strict paperfolders forbid the use of pencil and scissors, but as this book is not only origami but is also mathematics, we can afford certain licenses.

Basically cutting, in combination with folding and the use of geometric resources such as revolutions, translation, symmetries, etc., will allow us to get 3D forms out of elemental 2D diagrams. The results are figures of a sober beauty apt for architecture or sculpture. As it is common in these arts, the light is a basic ornament of these forms. We cannot forget the contribution of R. Razani or M. Chatani.



33

The process 2D → 3D reminds the infantile tale books in which, when passing a page, a new episode appears with the princess in her garden deployed open in space; or the theatre scene, or the wild forest full of fierce animals: all very rococo to call the *reader´*s attention. Where scissors are not shown, the cut is simply indicated by a dotted line.



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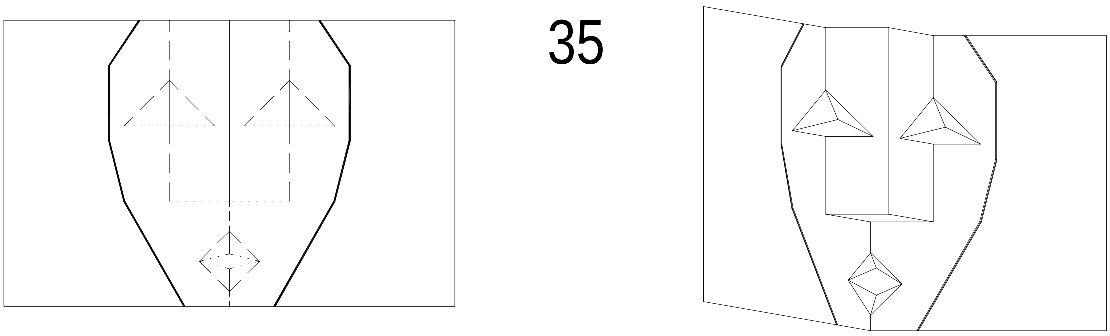
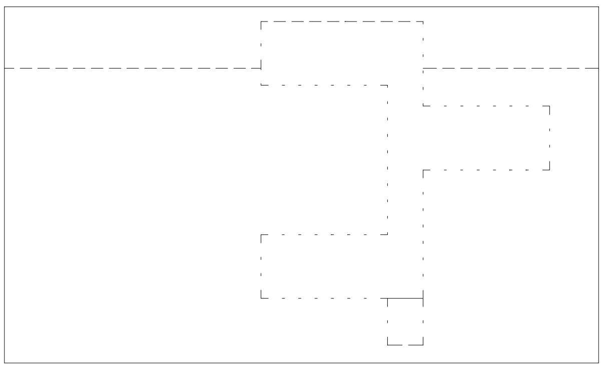
In Fig. 33 we have what could be the cover for square plan, e.g. of a church.



35

Fig. 34 is the simplest we can imagine; and most fascinating.

Fig. 35 is the only example among the last five in which a piece of paper is discarded after been cut. It could be a Picassian mask that can adopt different attitudes.



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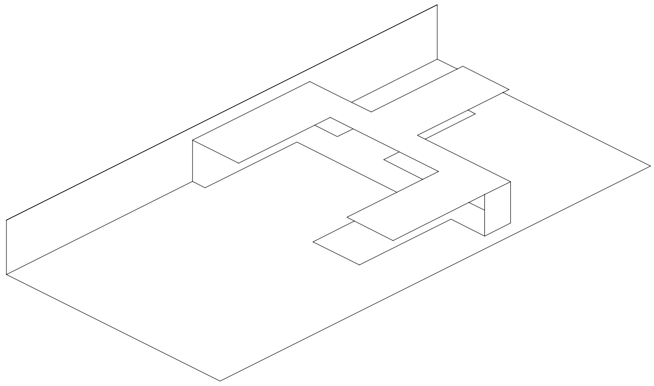
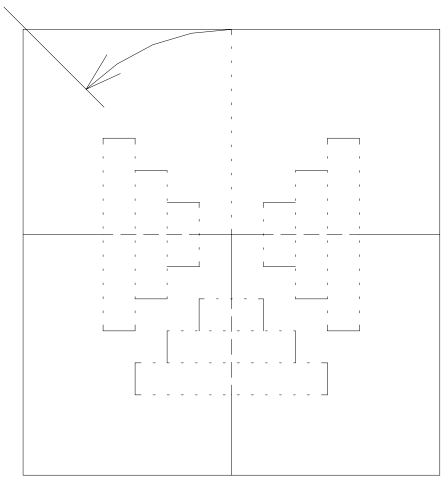


Fig. 36 is as good for furniture design as for mural decoration.



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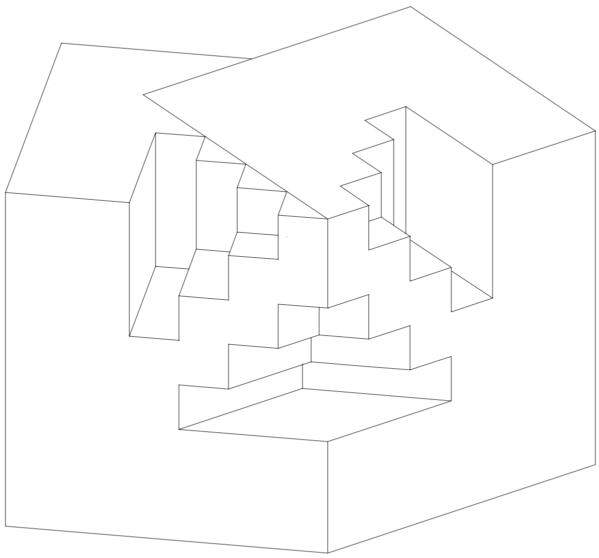


Finally, Fig. 37 is a good example of what straight lines can produce when duly organised.

16 FLATTENING: RELATION BETWEEN DIHEDRAL AND PLANE ANGLES.

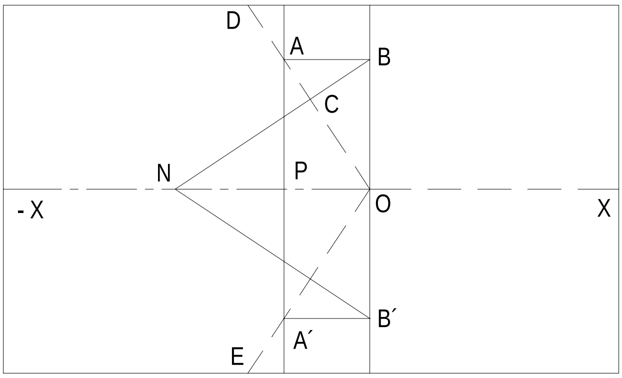
During transition from a plan 2D figure to the fold-flattened one (also 2D), 3D situations are produced bearing certain dihedral angles.

It would seam reasonable to seek the relations between those angles, plane and dihedral, but the task is not neither direct not easy. Nor even for a figure so elemental as Fig.1.

We could figure out the solution by means of a rather cumbersome program of calculus made expressly; it would have to solve the steps given by CAD, so we have decided to employ just CAD.

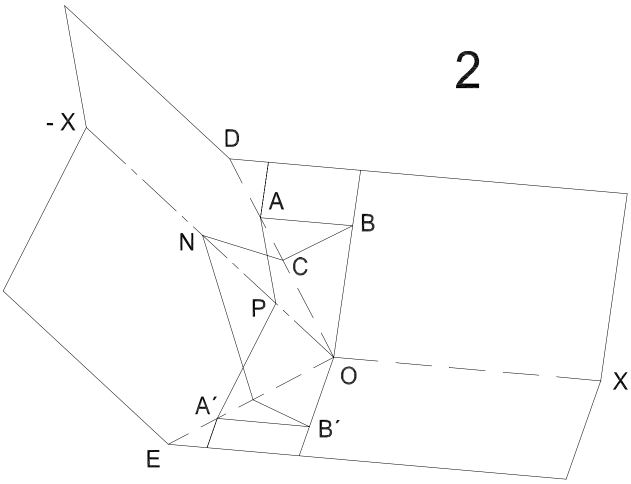
Fig. 2 is the same Fig.1 transformed by revolving the latter about OX a dihedral angle of α = 40º (corresponding to an obtuse angle of 140º). To draw Fig. 2 we have to pursue the steps that follow.

1



In that revolution (Fig. 2), AB is a parallel to OX just like its image A´B´. Hence, for any position during rotation always is AA´= BB´. Then triangles APA´ and BOB´ are congruent for their three sides are, respectively, equal. This entails that Ang. APA´ = Ang. BOB´. These angles measure the values of equal dihedral angles in (- XO) and (XO) for they are respectively in normal planes to (- XO) and (XO).

2



Let´s seek now the other dihedral angles in OD and OE that are congruent because of symmetry. We can see in Fig. 2 that wanted angle BCN is the measure of those dihedral angles.

3

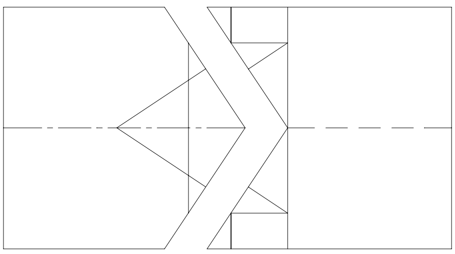
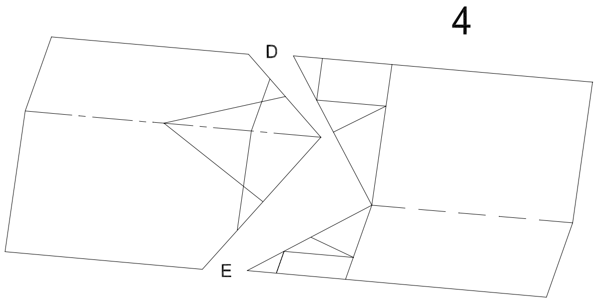


Fig. 3 is Fig.1 splitted out by translation, in two half-figures. In Fig. 4 those half-figures appear folded to the final dihedral obtuse angles of 140º in (XO) and (- XO).

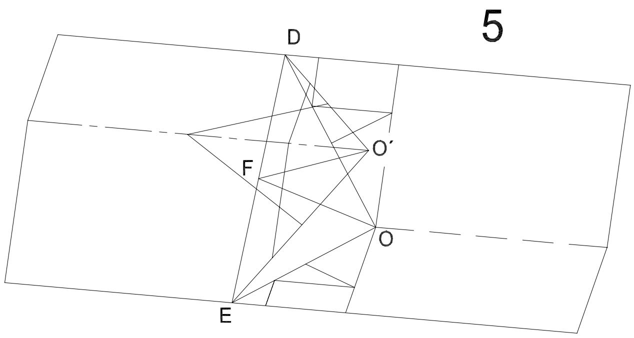


The two 3D half-figures of Fig. 4 are moved to fall in coincidence points D and points E. Fig. 5 is the result.

To pass from Fig. 5 to final Fig. 2 we must revolve the half-figure at left an angle equal to O´FO about DE.

Being F the mid-point of DE, angle O´FO measures the value of dihedral angle determined by planes O´DE and ODE.

With the latter revolution O´ will fall on O, and so we shall get final Fig. 2. Measuring in said Fig. 2 we get: angle BCN = 113,329º, Ang. DOX = 123,597º, obtuse dihedral angle in OX = 140º and Ang. O´FO = 54,4823º. Ang. (–XOX) = 180 – Ang. O´FO = 125,5177º.



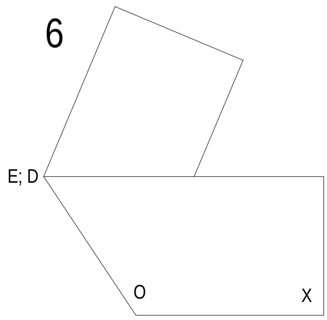


Fig. 6 is Fig.2 led to total flattening.

17 PAPER SURFACES

First of all let´s recall how plane curves may be represented in two different manners (Point 13.3.2, ellipse): by means of points (Fig. 4) or as the envelope of their tangents (Fig.2).

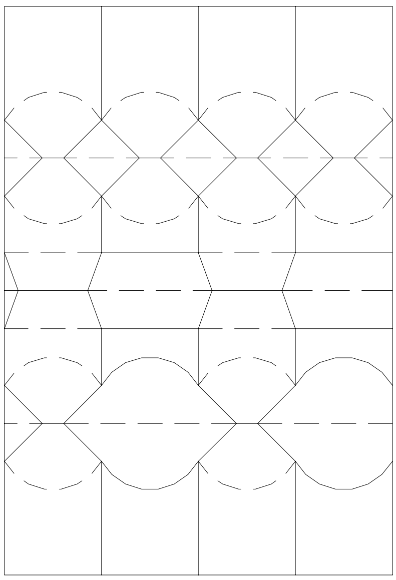
If we make use of an analogy, we shall call *real surfaces* those in which the paper is the set of all the infinity of straight lines contained in it as generatrices (ruled surfaces).

This group covers Figs. 23 and 26 (Point 15), as they are described there. Likewise, Fig. 32 in said Point 15, and the cones shown in Figs. 1,2 and 3 (Point13). We´ll come back on these when dealing with quadrics.

Continuing with the analogy, we shall call *virtual surfaces* those that have to be *guessed* as the envelope of a discrete amount of generatrices, which, in turn, are but intersections (folding lines) of paper planes.

17.1 REAL SURFACES

Let´s add some others to those already mentioned. In first place, Figs. 1,2 and 3, similar to those designed by T. Tarnai.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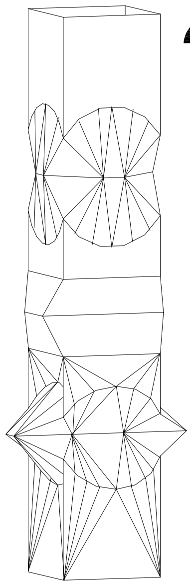
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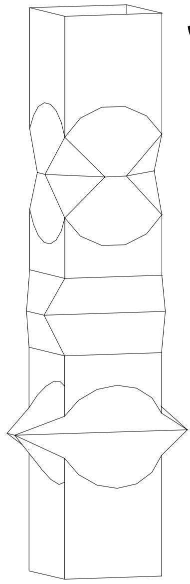
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Fig. 1 is the folding diagram, and Fig. 3 is the result after folding. The latter may induce to think that all the obtained surfaces are flat (distrust of retouched pictures), but it is not so. It is pertinent to set this clear to avoid frustration when constructing forms.

It is evident that the surface of a triangle with straight sides, does is flat. They may also be flat other polygonal surfaces (specially quadrilaterals) when none of their sides is curved.

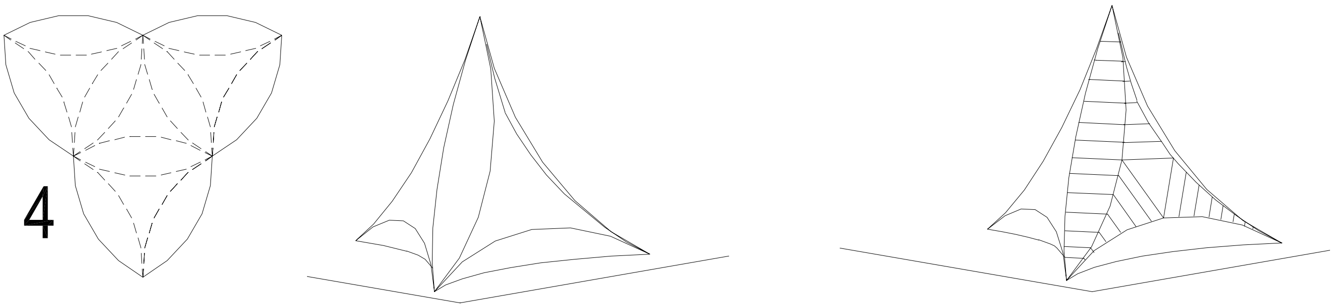
Nevertheless, a paper surface bounded by any curve cannot be a spatial plane surface: the natural paper docility leads to a composition of plane triangulations and conic surfaces made out of straight line generatrices.

The latter has been disclosed in Fig. 2; not really in all the cases but only in those most evident in order not to entangle the figure.

The fact that those surfaces do not become flat does not lessen the forms´ beauty: they may lead to very attractive models for stone sculptures or, with a bit more of difficulty, for steel sheet works.

Fig. 4 is the pseudotetrahedron by T. Yenn. At left we have the folding plan, in the center the solid form, and to the right the detail of ruled surfaces. Except for the small central equilateral triangles, all the rest is made out of ruled surfaces of cylindrical generatrices that give the impression of embossments and depressions.

Here it is an advice to paper folders relative to curve folding: pre-fold is eased with the finger nail pressure.



It is relevant to recall that ruled surfaces with conic and cylindrical generatrices are alike in this sense: all have a common point and rest on a directrix. In the second case, the common point is the ideal point of one of the lines, for, being parallel, all these lines are parallel to any plane containing one of them. A physical point is, obviously, the common point of conic generatrices.

Fig. 7 is another example. I came across it when designing paper strips made with argentic rectangles to construct the perforated pentagonal-dodecahedron. The small rectangles in

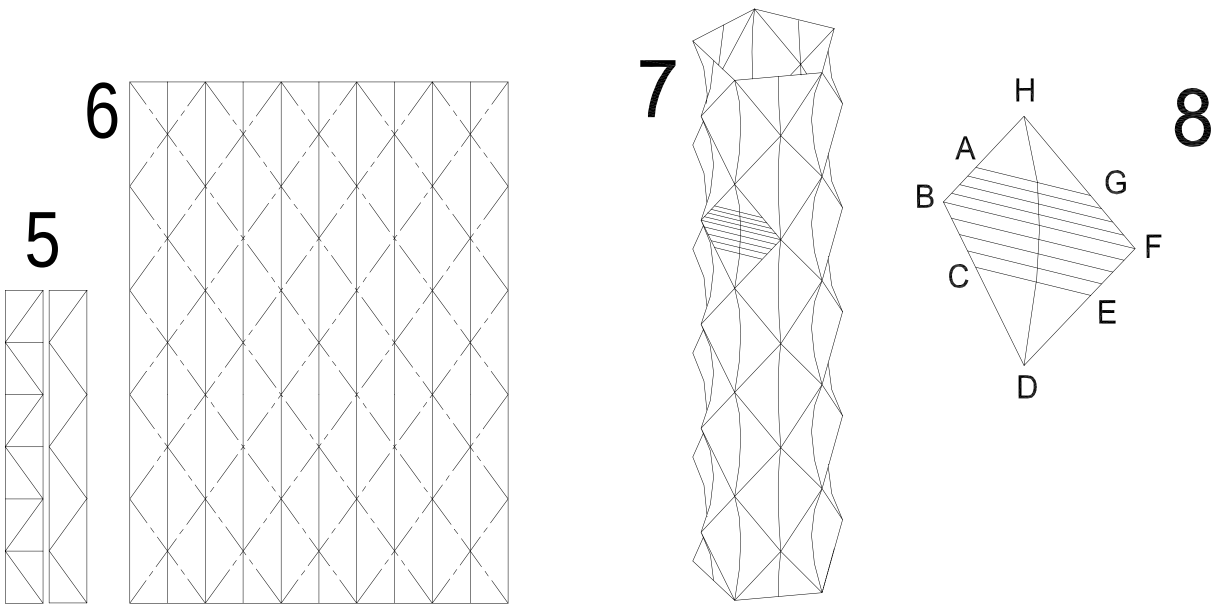


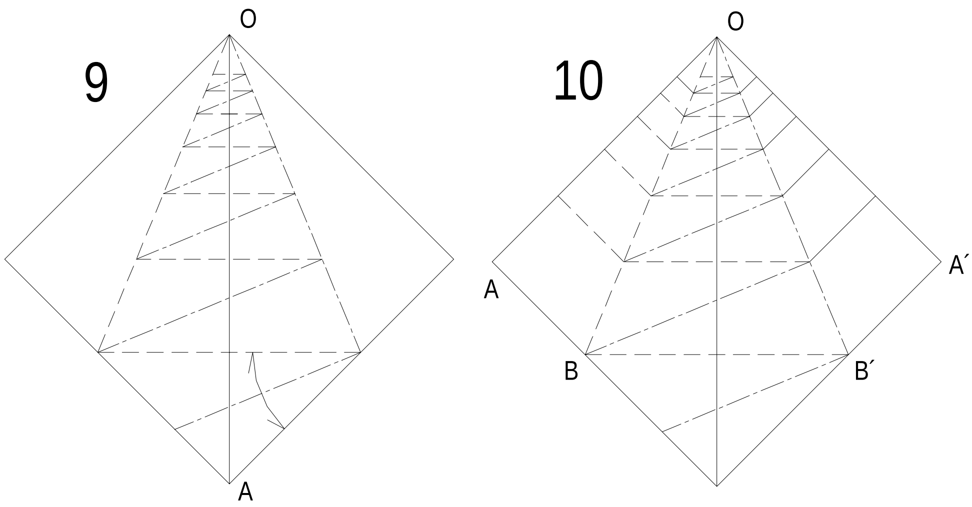
Fig. 5 are argentic ones, and the oblong at right is the unit to draw 6. Folding 6 gives 7. By the way, in all these figures (3,4,7) nothing is said on how to close the form: a practical resource is to provide an extra unit to act as a glued closing lapjoin.

As can be seen, Fig. 8 is an enlarged view of the corresponding *rhomb* of Fig. 7. *Rhomb* BDFH does not exist as such because the quadrilateral is not flat. Physically is made up by:

- two isosceles triangles AHG and CDE.
- the cylindrical surface ABCEFG bounded by the generatrices AG and CE, and the helix arcs AB, BC, EF, and FG. These helices are superimposed to mountain folds

of Fig. 6 (lines BH, BD, FH y FD in Fig. 8): The result is that in vertices B and F certain tensions are produced and the paper replies with a minimum of deformation.

Let´s see another example of a real surface, in this case by A. Ratner ("PAJARITA", special issue 1996). We begin with Fig. 2 (Point 7.15.2) to draw Fig. 9 that in turn is the folding plan for Fig. 11.

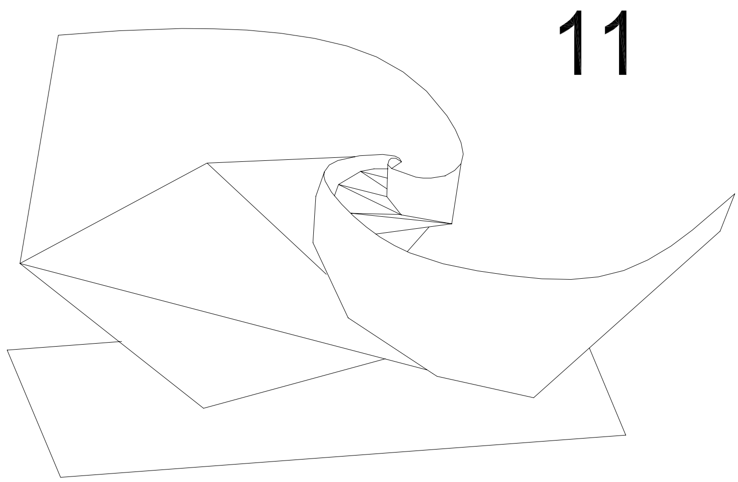


It is advisable to produce in Fig. 9 as many valley folds as possible (horizontal segments), to reach near O (decreasing geometric progression).

Fig. 10 is deduced from Fig. 9: Triangles OAB and OA´B´ will become a pair of quasi-cylindrical surfaces whose generatrices are the respective parallel lines to AB and A´B´.

They are not full cylindrical surfaces because though lines OA and OA´ are free to take its curvature, OB and OB´ are restrained as open polygonal lines.

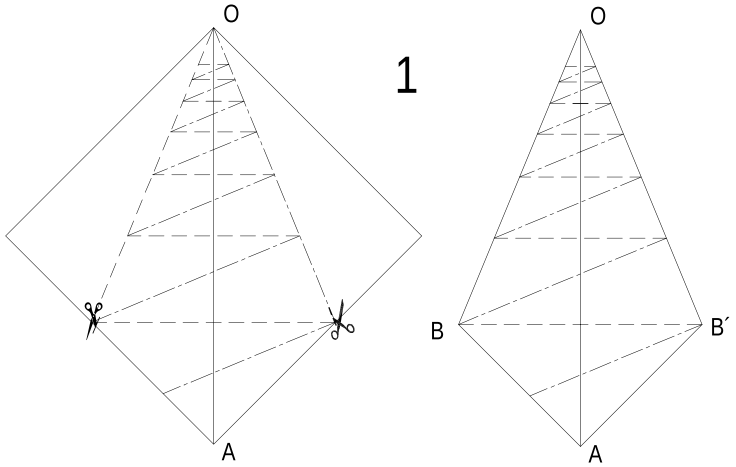
Both surfaces could be named spiroids because their directrices are not plane spirals but helicoidals (Fig. 11)



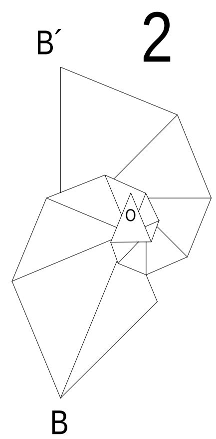
I must say that Fig. 11 is so beautiful in reality that neither a photography nor a perspective can convey to the viewer the harmony it contains: it must be constructed! (what is very easy, indeed).

17.2 VIRTUAL SURFACES

Let´s start with Fig. 9 (Point 17.1) to get Fig.1 of present Point 17.2. Thus we have eliminated both spiroids of Fig. 11 (Point 17.1) to facilitate folding of central part of said Fig. 9.

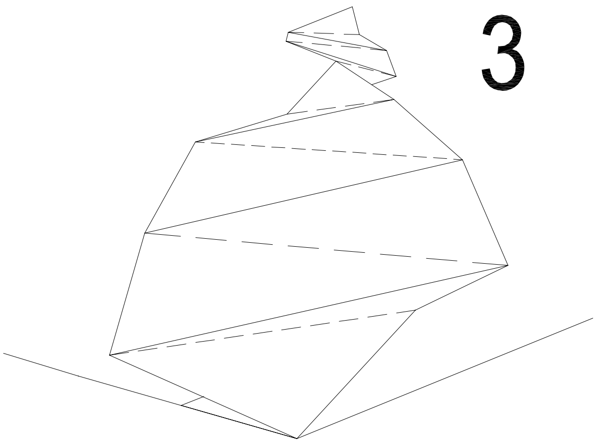


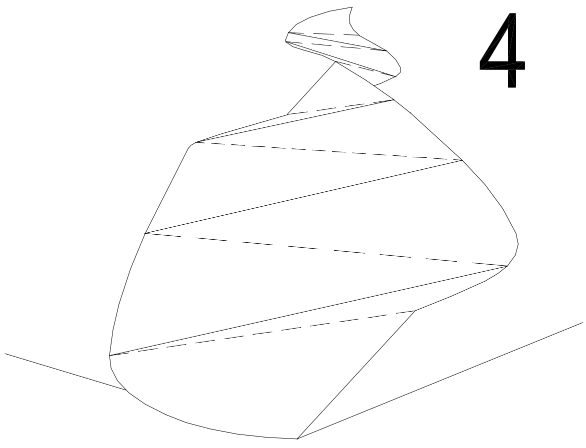
Present Fig. 1 may be folded flat as shown in Fig.2. But it also can be left free-folded in the space producing an elastic form (recall Figs. 1 and 2, Point 15), due to alternate mountain and valley folds. That´s why it is feasible to get a form like that of Fig. 3. In it, all mountain generatrices are parallel to the horizontal plane (what does not mean that they will be parallel between them).



Those generatrices in turn, rest on either open polygonals that tend to the helicoidal spiral curves already seen in Point 17.1.

Fig. 4 shows that pair of curves in such a manner that the virtual ruled surface determinated by those mountain fold horizontal generatrices, is a *quasi-conoid*



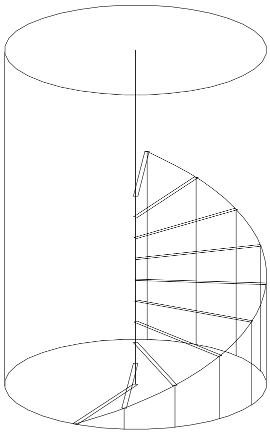


A conoid is a ruled surface with:

- Its own right directrix.
- A director plane not parallel to it.
- Another directrix, curved or straight.

Hence, the surface of Fig. 4 is not a conoid strictly speaking for, though it has the horizontal as director plane, it has two curved directrices instead of having, at least, one straight.

To illustrate this, Figs. 5 and 6 show a couple of conoids: The first is a coiled stairwell and the second is the Plücker´s conoid. Both have the horizontal plane as director. The directrices of Fig. 5 are, the vertical axis of a cylinder and a helix on its surface.



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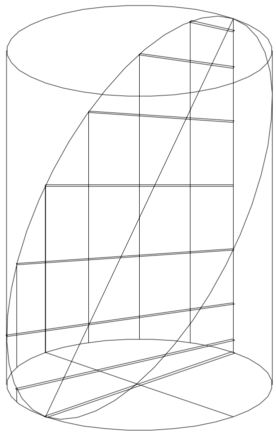


6

One of the directrices of Fig. 6 is a generatrix of the cylinder and the other is an ellipse on the surface of said cylinder having one of its vertices on the straight directrix of that conoid. The generatrices of both conoids are outlined in Figs. 5 and 6.

A CONOID OF PAPER

Fig. 7 is a conoid obtainable by folding. It is a ruled surface whose generatrices rest at equal intervals on the crossed diagonals of two squares. With a common side, these squares form a 90º dihedral angle. The director plane, as shown in Fig. 8 is x = 0.

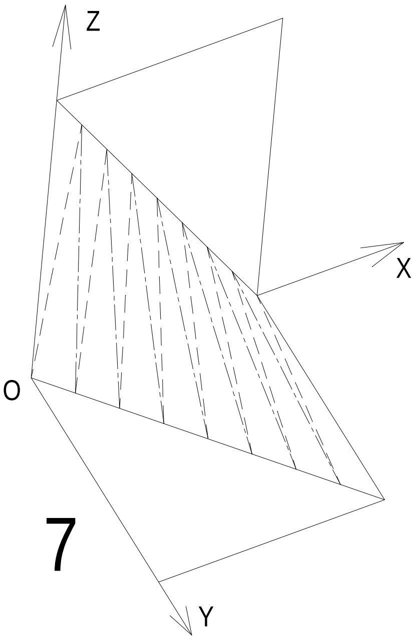
In Fig. 8 we can see the structure required designing a simple program of calculus that figures out the length of mountain and valley folds. That will allow drawing the adjacent triangles of Fig. 9: this figure is the conoid folding plan.

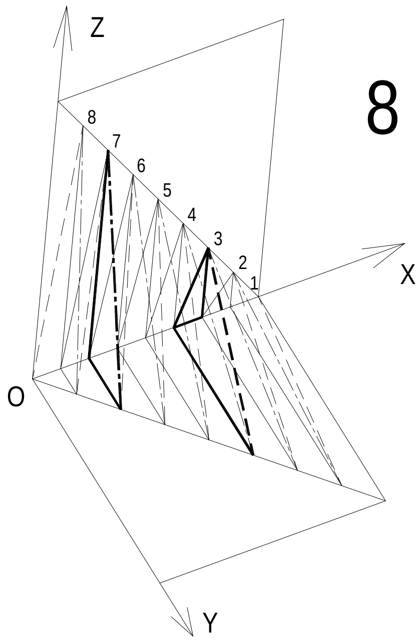
The inputs of that program are: the side of the square and the n parts to divide the diagonal. In the drawing we took n = 8 and the value of the small sides of all triangles of Fig. 9 is $\frac{a\sqrt{2}}{n}$

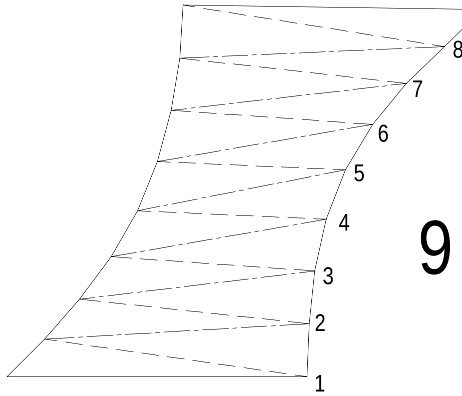
Let us calculate the length of both, a mountain and a valley fold, e.g., for points 7 and 3: m (7) and v (3).

$$m(7)=\sqrt{\left(\frac{a}{n}\times 2\right)^2+\left(\frac{a}{n}\times 6\right)^2} \quad \text{equivalent to:} \quad m(i)=\frac{a}{n}\sqrt{(n-i+1)^2+(i-1)^2}$$

$$v(3)=\sqrt{\left(\frac{a}{n}\times 2\right)^2+\left(\frac{a}{n}\right)^2+\left(\frac{a}{n}\times 5\right)^2} \quad \text{i.e.:} \quad v(i)=\frac{a}{n}\sqrt{(i-1)^2+1+(n-i)^2}$$







Conoids are warped (not developable) ruled surfaces (with straight generatrices). One surface is developable when the planes tangent to it along all the points of any generatrix mix in one only tangent plane (recall the plane tangent to a cone along, of course, of one of its generatrices).

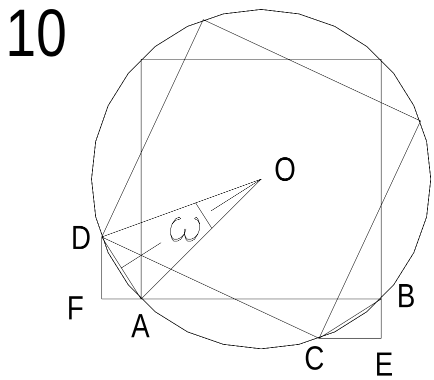
Conversely, in a warped surface, the tangent planes to it in the different points of a generatrix, vary: they revolve from one position to another along the generatrix.

See in Fig. 5 how two tangent planes to the conoid in any generatrix are different depending of which extremity of the generatrix segment we consider as point of contact. Similar remark is applicable to Fig. 6.

It should be made clear that the conoid in Fig. 7 is also a warped ruled surface, and therefore Fig. 9 is but the folding plan that produces a *virtual surface* by gathering some of the paper.

A TWISTED COLUMN (SALOMONICA)

To close the examples of virtual surfaces, we shall study this interesting and beautiful form after N. Nagata.

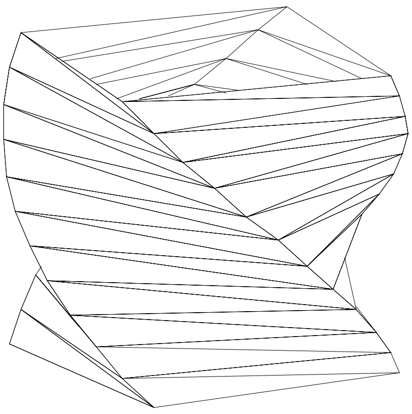


And we´ll do it beginning from the end. CAD produces the generation of the solid figure as follows:

- Start with a square with base AB = a and center O (Fig. 10).
- Rotate it clockwise on its own plane around O, an angle ω.
- Then move it vertically by the amount h.
- Repeat successively the operation n times by rotating ω and moving h.
- Thus we get Fig. 11 which appears with all its folds: AD (mountain), DB (valley) and DC (mountain). Fig. 12 shows them enlarged.

Besides a solid perspective, Fig. 11, is also the clue to the fold plan of Fig. 13.

11



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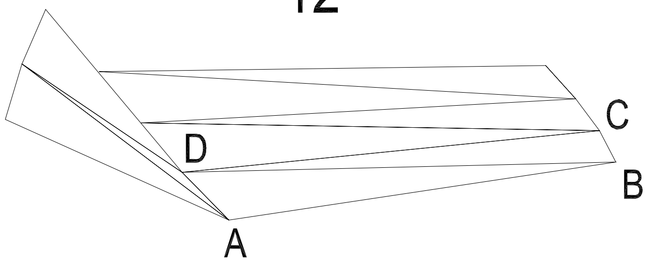
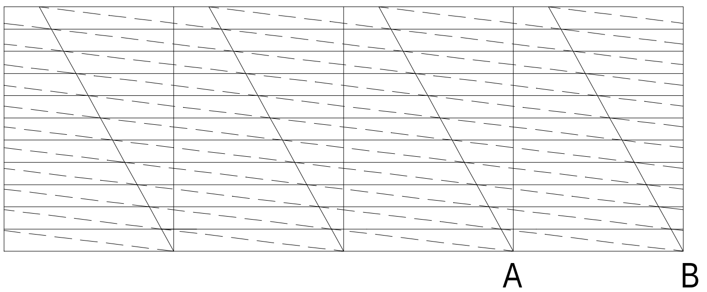


We can note that all we need to draw Fig. 13 is to fix ABD (enlarged in Fig.14). Incidentally, Fig. 11 is fully flattable if properly twisted.

13



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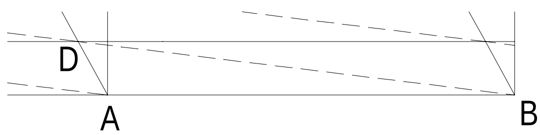


We know the three sides of that triangle: AB is given; CAD in Fig. 11 can read BD and DA. Then from triangle ABD we can complete Fig. 13 by extending and copying lines.
We should note two things:

- The three central vertical lines of Fig. 13 are of no use to construct Fig. 11. Only its bases AB and the other three are needed.
- The projection h´ of AD over the vertical, times n, gives the altitude of the starting paper that is greater than nh. It means that the quadrangled prism originated by Fig. 13 is contracted in its height while twisted to form Fig. 11.

We have seen up to now that CAD allows us to draw Fig. 13 as well as Fig. 11. Then we´ll see the analytical procedure in case CAD would not be available.

Here are the data:

- The side of the square = a.
- The distance between squares = h.
- The angle turned around each time = ω.
- Amount of turns and consequent upward moves given to the square = n. Obviously n may be as great as we wish if the paper size can cope with it.

Let´s look again to Fig. 10 in order to get the co-ordinates of points A, B, C, D

A ≡ (0,0,0) ; B ≡ (a,0,0)

Ang. FAD = $180-45-\left(\dfrac{180-\varpi}{2}\right)=45+\dfrac{\varpi}{2}$

$$AD=2\frac{a}{2}\sqrt{2}\,\mathrm{sen}\frac{\varpi}{2}=a\sqrt{2}\,\mathrm{sen}\frac{\varpi}{2}$$

$$FA=AD\cos FAD=a\sqrt{2}\,\mathrm{sen}\frac{\varpi}{2}\cos\left(45+\frac{\varpi}{2}\right)$$

$$DF=a\sqrt{2}\,\mathrm{sen}\frac{\varpi}{2}\,\mathrm{sen}\left(45+\frac{\varpi}{2}\right)$$

ΔDFA = ΔBEC after transformation by the rotation ϖ.

BE = FA ; CE = DF

$$x_C=a-a\sqrt{2}\,\mathrm{sen}\frac{\varpi}{2}\,\mathrm{sen}\left(45+\frac{\varpi}{2}\right)$$

$$\left.\begin{aligned} x_C &= a\left(1-\sqrt{2}\,\mathrm{sen}\frac{\varpi}{2}\,\mathrm{sen}\left(45+\frac{\varpi}{2}\right)\right)\\ y_C &= -BE=-a\sqrt{2}\,\mathrm{sen}\frac{\varpi}{2}\cos\left(45+\frac{\varpi}{2}\right)\\ z_C &= h \end{aligned}\right\}$$

$$\left.\begin{aligned} x_D &= y_C\\ y_D &= a\sqrt{2}\,\mathrm{sen}\frac{\varpi}{2}\,\mathrm{sen}\left(45+\frac{\varpi}{2}\right)\\ z_D &= h \end{aligned}\right\}$$

Its three sides define ΔABD. Besides, calling p to its half-perimeter, its area will be

$$S=\sqrt{p(p-a)(p-BD)(p-AD)} \qquad \text{and therefore:} \qquad h' = \frac{2S}{a}$$

Once the solid form of Fig. 11 is physically completed, we can check that the shrinkage produced by the twist is equal to $n(h'-h)$.

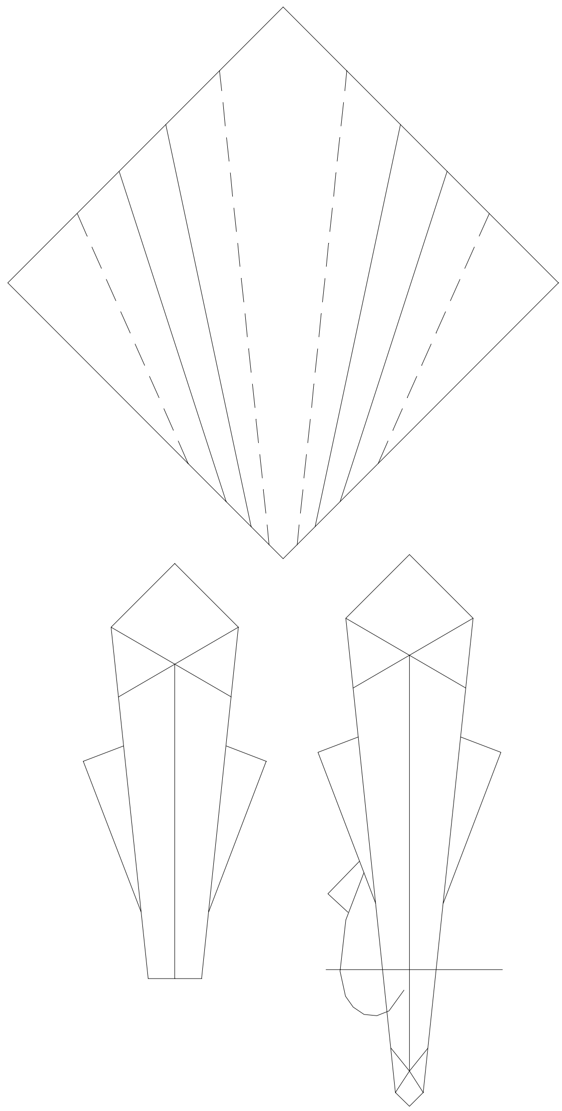
We should recall that h´ is the distance between horizontals in Fig. 13, and h is the same distance taken in the space (sides DC and AB of Fig. 12, which cross each other): h´ > h.

Once we know the co-ordinates of A, B, C, D, it is easy to obtain the length of BD and AD:

$$BD = \sqrt{(x_D - x_B)^2 + (y_D - y_B)^2 + (z_D - z_B)^2}$$
$$AD = \sqrt{(x_D - x_A)^2 + (y_D - y_A)^2 + (z_D - z_A)^2}$$

Fig. 11 has the structure of a screw with four threads in it.

Interlude



18 POLYHEDRA

They are solids bounded by plane faces (polygons, obviously in number of 4 or more).

- Their faces C are those polygons.
- Their vertices V are the vertices of the polyhedral angles.
- Their sides A are the intersection of two faces, forming a dihedral angle.

All polyhedra are governed by the Euler theorem that relates C, V, A. The generalised Euler theorem should be applied to stellate regular polyhedra; we´ll see to it later on. For the rest of polyhedra, here we have the Euler theorem:

$$C + V = A + 2$$

We shall not prove it now. The reader should recall a similar approach given to plane figures in Point 1.3.1.

According to various criteria, which in turn may be related to each others, polyhedra are classified in:

Regular, irregular, pseudorregular, concave, convex, stellate, platonic, archimedean, conjugate, etc, etc.

Regular polyhedra are the five *platonic* ones: all their faces are equal regular polygons. Let´s see why there are but five.

To begin with, their faces can only be equilateral triangles, squares, or regular pentagons, hexagons, etc.

It is obvious that the sum of the plane angles forming a polyhedral angle must add up to less than 360º.

With the 60º of the angle of an equilateral triangle we can construct a polyhedral angle with a maximum of 5 faces (6 would not form a polyhedral angle, but a perigon: 60 x 6 = 360º). Since the trihedral is the smallest possible polyhedral angle, it follows that with equilateral triangles we´ll be able to form polyhedral angles of 3 faces (3 * 60 = 180 < 360); of 4 faces (4 * 60 = 240 < 360); of 5 faces (5*60 = 300 < 360); and no more faces (6 * 60 = 360).

For the square: 3 * 90 = 270 < 360; 4 * 90 = 360. With the square we can construct only trihedral angles.

Let´s see what happens with the regular pentagon (the value of its interior angle is $\frac{180(5-2)}{5} = 108^{\circ}$). We can construct a pentagonal trihedral angle, for 108 * 3 = 324 < 360; four pentagonal faces are too many: 108 *4 = 432 > 360.

It is not feasible to construct a trihedral angle with hexagonal faces (120 * 3 = 360).

Therefore, with the different regular polygons we can construct:

FACE POLYGON	NUMBER OF FACES IN POLYHEDRAL ANGLE	POLYHEDRON´S NAME
EQUILATERAL Δ	3	TETRAHEDRON
	4	OCTAHEDRON
	5	ICOSAHEDRON
SQUARE	3	HEXAHEDRON
PENTAGON	3	PENTAGON-DODECAHEDRON

The latter 5 regular polyhedra, besides the faces, have also respectively equal, sides and angles: dihedral as well as polyhedral.

The *archimedean polyhedra* are obtained by truncating the polyhedral angles of the platonic. It is well known that Archimedes dealt with 13 of them. They are a special source of inspiration for imaginative unit folders.

Irregular polyhedra do not openly conform with to equality conditions of regular polyhedra. Neither do the *pseudorregular*, but not so openly: they may have even a regular appearance. Later on we shall study two of these: the rhombic-dodecahedron and the trapezohedron.

As for *stellate polyhedra,* we shall devote special chapters only to the regular ones.

A polyhedron is named *convex* when the plane containing any of its faces leaves the whole polyhedron in one of its two hemispaces. Conversely it is a *concave* one if its volume is scattered within the two hemispaces. The kneading-trough we´ll see here after is one example of a concave irregular polyhedron.

The conventional –and simplest- classification of polyhedra is: regular and irregular, the prism and pyramid being part of the latter.

A *prismatoid* is an irregular polyhedron bounded by two polygons (bases) situated in parallel planes, and several lateral faces shaped as triangles or trapeziums (in case their four sides were coplanar). If both bases have the same number of sides, the prismatoid becomes a *prismoid.*

If one of the bases of a prismatoid is reduced to a point, the prismatoid becomes a *pyramid*. Therefore a *pyramid* is a polyhedron whose base may be any polygon; its lateral faces (as many as the sides of the base) are triangles that meet at the vertex forming there a polyhedral angle.

The *pyramid* may be named *triangular*, q*uadrangular*, etc. when its base is a triangle, a quadrilateral, etc. Moreover, if the base is a regular polygon and the lateral faces are congruent, we have a *regular pyramid*.

A *prism* is like a prismoid with equal bases; the other faces (lateral) should be parallelograms; their sides will belong to the bases or to the lateral faces.

A *prism* is named *right* if it has its lateral sides at right angles with the base; if not, it is an *oblique* prism.

A *prism* will be *triangular*, quadrangular, etc. according to the polygon of its base. If such a base consists of a regular polygon, we´ll have a *regular prism*. The bases of an *irregular prism* are irregular polygons.

A prism is named *parallelepiped* if its bases are parallelograms: in other words, it will have 6 faces such that each pair of the opposite ones are equal and parallel. A *right parallelepiped* derives from a right prism.

A *rectangular parallelepiped* is a right one whose base is a rectangle.

When a plane oblique or parallel to the base cuts a pyramid, two solids are obtained: a small, new pyramid and a *frustum*.

If a prism is cut off by a plane oblique to its base we get two *truncated prisms*.

Two regular polyhedra are *conjugate* if the number of faces (or vertices) in one is equal to the number of vertices (or faces) in the other: cube and octahedron; pentagon-dodecahedron and icosahedron.

We shall show the mathematical background of the solids to be studied here on. Paperfolding will also play its roll through folding schemes that convey to ultimate figures, also shown in perspective. Folding does not produce interlocked assemblies; on the contrary, it requires the help of glue or sticking paper (transparent) to fix the union laps (not always shown).

18.1 A KNEADING-TROUGH

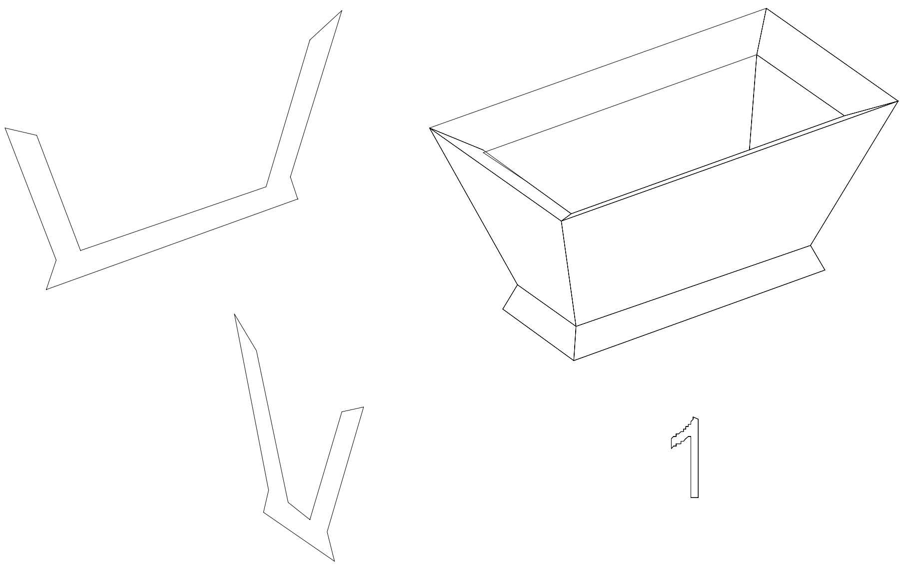
It is an example of an irregular concave polyhedron that fulfils Euler´s theorem: C = 18; V = 20; A = 36: $18 + 20 = 36 + 2$

Fig. 1 is a perspective view with its two orthogonal transversal sections.

The design may be taken as a model to construct a kneading-trough made out of five equally thick boards: the base and four lateral faces. Those are represented in the terminated figure by five empty virtual boards of paper.

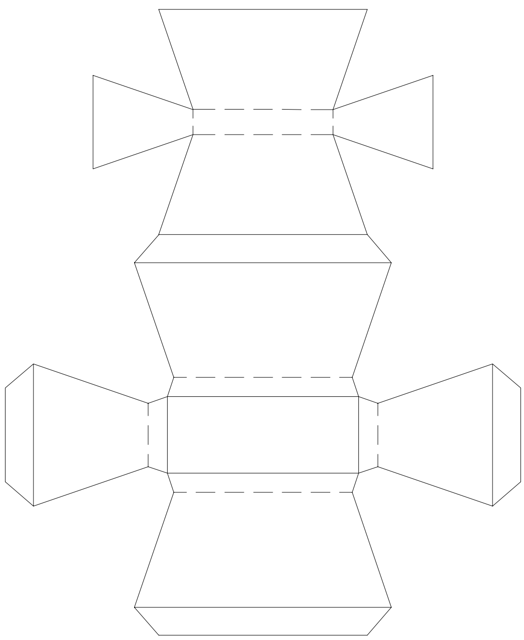
This construction associates geometry and paperfolding, and has these characteristics:

1. It starts with a unique piece of paper.
2. That paper is not a square; instead it takes the form of the mere development of the finished figure (Fig.2).
3. The paper should be cut.
4. Union laps, though necessary, are not shown. The construction requires the use of glue or sticking paper.



1

As can be seen, origami is now present. The licenses expressed here do not affect the essence of paperfolding. In this respect it will be good to remind that the paper used formerly for origami in western countries was rectangular shaped (D. Lister), whereas in Japan the traditional square paper is related to the ancestral square parcels of land used to cultivate rice (K. Ohashi).

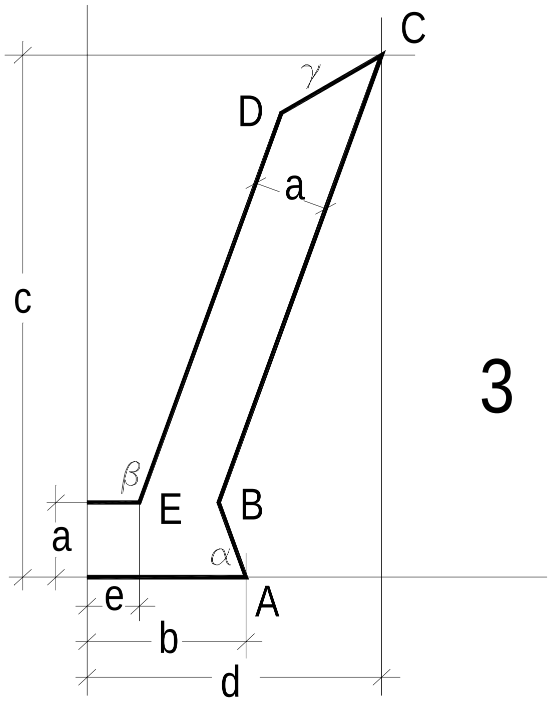


2

The kneading-trough we are dealing with here has been inspired by the KIKUJUTSU traditional Japanese carpentry so well described by T. IWASAKI. This technique makes use of certain special graduated squares able to configure any type of angles. That craft has even assigned specific names to some of these angles. Through a rather awkward application of those tools, the craftsmen can cope with any wood construction such as slants, dovetails, etc.

At present, instead of appealing to the old techniques, I have preferred to use those taken as modern nowadays. CAD, nevertheless is not a panacea. What I mean is that, for example, to figure out the angles in the trapeziums of Fig. 2, we must know beforehand, which is the intersection line of two planes in space. To solve problems like this, I have been forced to develop calculus programs such as those that give the angle formed by two planes, the point of intersection of line and plane, the distance to a plane from a point, etc.

We have to bear in mind that CAD works with points alike the 3D measuring machines; therefore a plane is defined by three of its points.



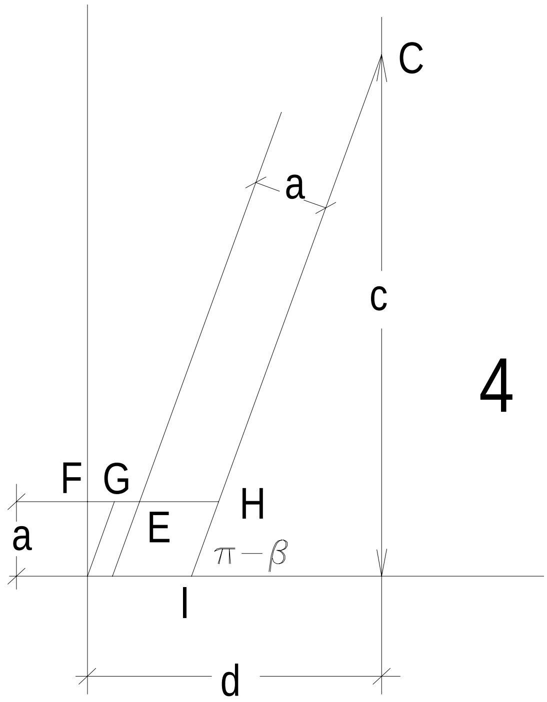


Fig. 3 is a dimensioned half-section of the small cut in Fig. 1. From that we can figure out the length of the segments forming the broken line EDCBA.

$$\left.\begin{aligned} AB\,\mathrm{sen}\,\alpha + BC\,\mathrm{sen}\,\beta &= c \\ -AB\cos\alpha - BC\cos\beta &= d-b \end{aligned}\right\}$$

$$\left.\begin{aligned} -ED\cos\beta + DC\cos\gamma &= d-e \\ ED\,\mathrm{sen}\,\beta + DC\,\mathrm{sen}\,\gamma &= c-a \end{aligned}\right\}$$

$$BC = \frac{c\cos\alpha + (d-b)\,\mathrm{sen}\,\alpha}{\mathrm{sen}(\beta-\alpha)}$$

$$DC = \frac{(d-e)\,\mathrm{sen}\,\beta + (c-a)\cos\beta}{\mathrm{sen}(\beta+\gamma)}$$

$$AB = \frac{c - BC\,\mathrm{sen}\,\beta}{\mathrm{sen}\,\alpha}$$

$$ED = \frac{c - a - DC\,\mathrm{sen}\,\gamma}{\mathrm{sen}\,\beta}$$

RELATION BETWEEN c AND d (Fig. 4)

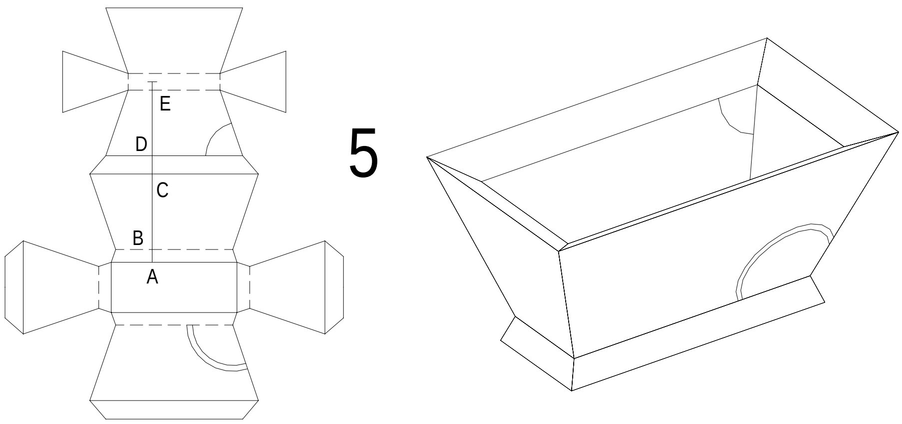
$$EH=\frac{a}{\operatorname{sen}\beta} \quad ; \quad GE=EF-FG=e-a\operatorname{tg}\left(\beta-\frac{\pi}{2}\right) \quad ; \quad \operatorname{tg}(\pi-\beta)=\frac{c}{d-GE-EH}$$

Here are the data needed to construct the kneading-trough:

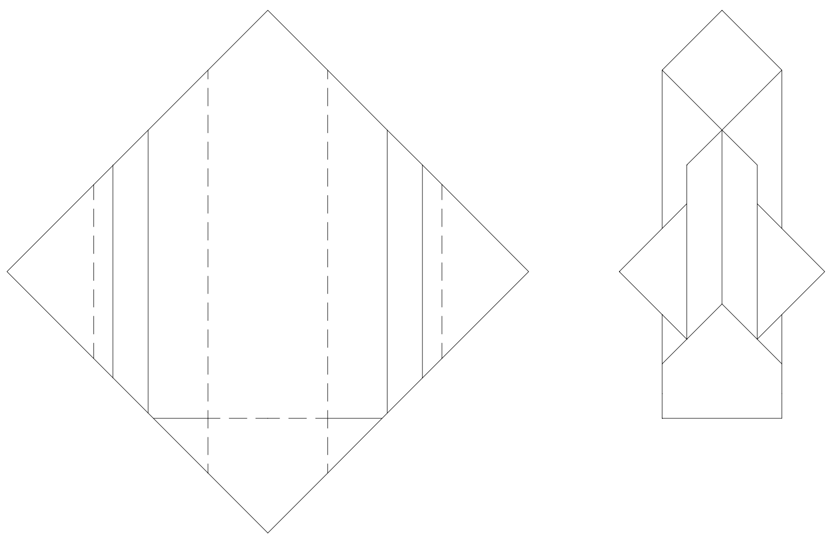
- angles α, β, γ.
- board thickness a.
- bases dimensions e, b, d; height c is a function of d as seen in Fig. 4.
- the longitudinal dimensions associated to transversal section of Fig. 3.

Let´s see now how we can draw some of the trapeziums of Fig. 2.

Fig. 5 shows, within Fig. 2, the broken line ABCDE and a pair of angles in the trapeziums, also marked in the duplicated Fig. 1. CAD gives the value of those angles by actual measurement in Fig. 1.



Interlude



18.2 PYRAMIDS

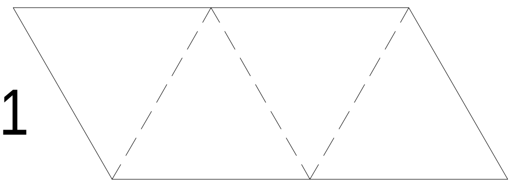
We are going to study various regular pyramids and one irregular, using when possible, for their construction, the Solution 1 of Point 8.2.3 to generate equilateral triangles.

18.2.1 TRIANGULAR PYRAMID

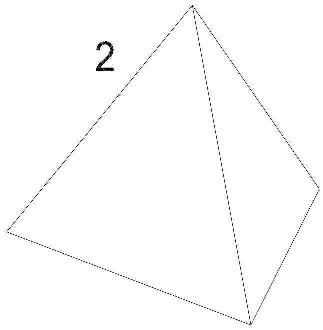
18.2.1.1 TETRAHEDRIC

Fig. 1 is the folding scheme and Fig. 2 is the obtained solid. It is evident that the resulting pyramid is a *tetrahedron* since its three lateral faces are equal and also equal to the base.

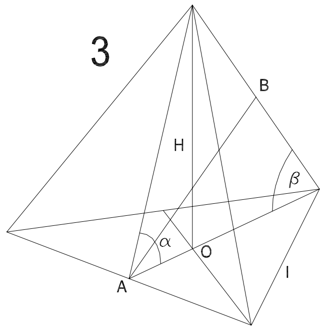
1



2



3



To draw Fig. 2 by means of CAD (Point 18.1 showed to which extent CAD is an origami tool and not a mere ornament) we must know the value of the α angle in the tetrahedron (Fig. 3), for CAD usually plays with plane revolving.

The same requirement will be put forward with other polyhedra and it will have to be satisfied in each occasion.

In the tetrahedron of Fig. 3 we have:

- side = l
- altitude of one of its faces $h = \dfrac{l\sqrt{3}}{2}$
- distance AB between two opposite sides:

$$AB = \sqrt{\left(\frac{l\sqrt{3}}{2}\right)^2 - \left(\frac{l}{2}\right)^2} = l\,\frac{\sqrt{2}}{2} = l0{,}7071067$$

- altitude of tetrahedron: distance from the *pyramid´s* vertex to the center of its base = H
- dihedral angle of two faces: angle formed by two segments h meeting on the same side = α.
- angle formed by two segments l and h meeting on the same vertex = β.

$$H = \sqrt{l^2 - \left(\frac{2}{3}h\right)^2} = l\sqrt{\frac{2}{3}}$$

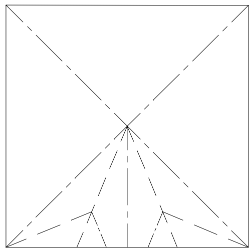
$$\alpha = \mathrm{arctg}\,\frac{H}{\frac{h}{3}} = \mathrm{arctg}\,2\sqrt{2} = 70{,}528779^{\circ}$$

$$\beta = \mathrm{arctg}\,\frac{H}{\frac{2h}{3}} = \mathrm{arctg}\,\sqrt{2} = 54{,}73561^{\circ}$$

18.2.1.2 OF TRI-RIGHT-ANGLED VERTEX

From a square of side l (Fig. 1), we get a virtual pyramid (I name so, in this case, the pyramid lacking its base) with these characteristics (Figs. 2, 3):

1



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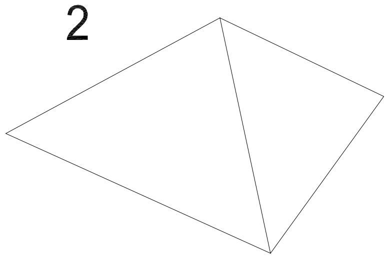
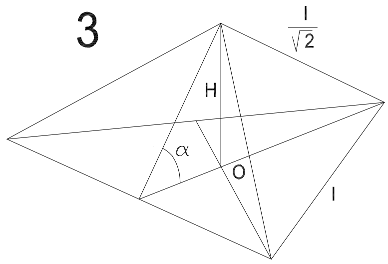


3



- Its base is an equilateral triangle of side l and altitude $h = \frac{l\sqrt{3}}{2}$
- Its three equal lateral faces are isosceles right triangles. The vertices of their right angles coincide with the pyramid´s vertex; their legs are the pyramid´s lateral sides and measure $\frac{l}{\sqrt{2}}$

CALCULATION OF DIHEDRAL ANGLE α

Altitude of pyramid $H = \sqrt{\left(\frac{l}{\sqrt{2}}\right)^2 - \left(\frac{2}{3} l \frac{\sqrt{3}}{2}\right)^2} = l \times 0{,}4082483$

$\alpha = Arc\,\mathrm{tg}\,\frac{H}{\frac{h}{3}} = Arc\,\mathrm{tg}\,\sqrt{2} = 54{,}735613^{\circ}$

We should note that this angle α is equal to β in Fig. 3, Point 18.2.1.1.
The folds in lower triangle of Fig. 1 allow pyramid interlocking.

18.2.2 QUADRANGULAR PYRAMID

18.2.2.1 VIRTUAL QUADRANGULAR PYRAMID

It is quite defined by its vertex, the four base´s vertices, two full lateral faces and the other two semi-full ones; it is lacking the base.

The starting rectangle, according to Fig. 1 is a DIN A4 with sides 1 (the small) and $\sqrt{2}$ (the large). Fig. 1 shows the folds previous to final folding performed to Fig. 2: pleat its large sides in such a way that the distance between its endpoints will be 1.

Thus we obtain the complete folding diagram of Fig. 3 and hence the pyramid of Fig. 4. The construction requires that both pleats in the semi-full faces, will be fixed.

The final pyramid has these characteristics:

- The side of the square of its base is 1
- The diagonal of this square is $\sqrt{2}$

- The lateral side (see Fig. 1) is: $\frac{1}{2}\sqrt{1+\left(\sqrt{2}\right)^2} = \frac{\sqrt{3}}{2}$
- The pyramid´s altitude is (Fig. 5): $H = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{1}{2}$
- The dihedral angle whose side is the side of the pyramid´s base is: $\alpha = 45^o$ (it pertains to an isosceles right triangle)

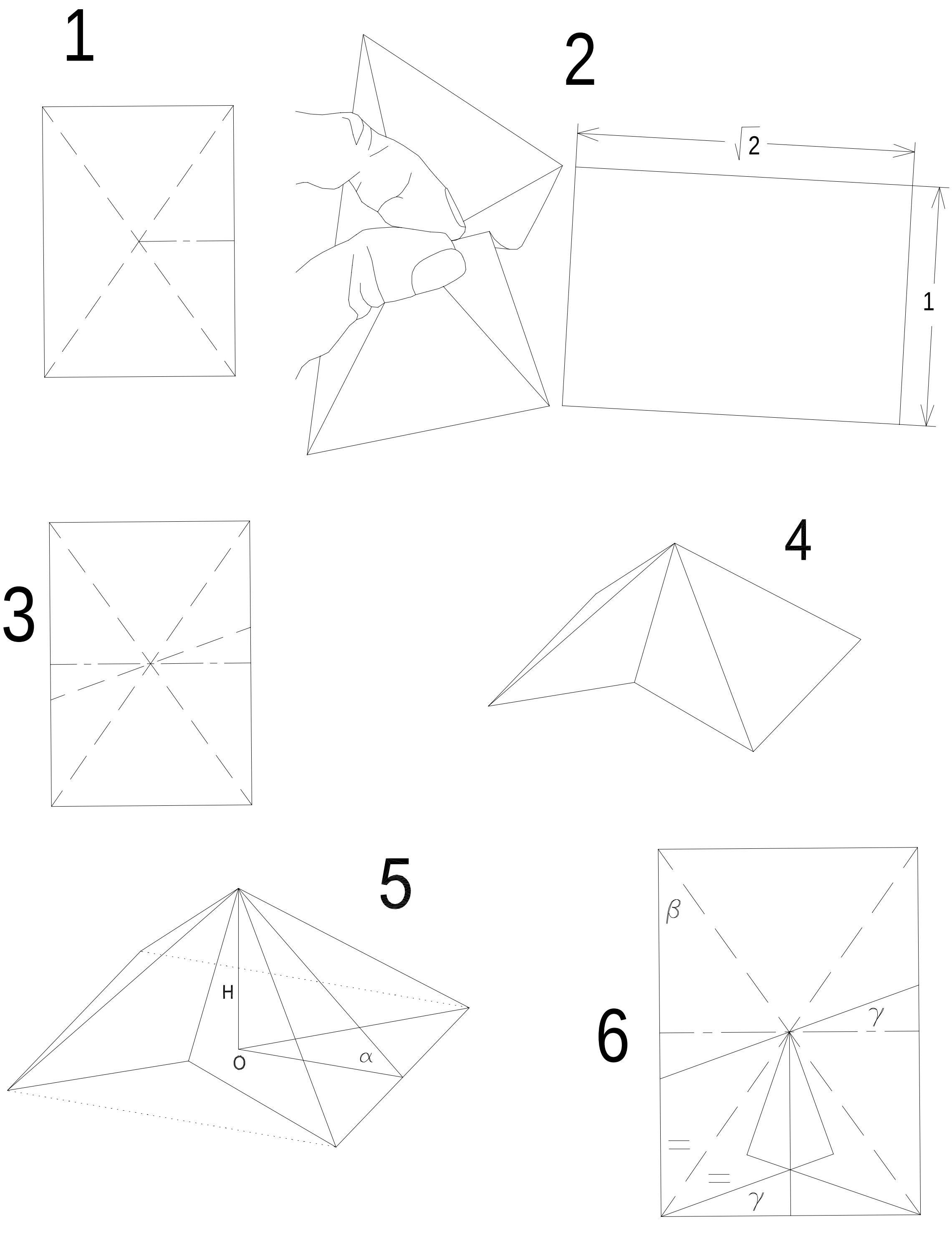


Fig. 2 shows, as said, how to pleat fold, at the discretion of the folder (but with a great accuracy), the large sides of the original rectangle. Fig. 6 shows how to get the pleat angle γ:

$$\gamma = 90 - 2Arc\,\mathrm{tg}\frac{1}{\sqrt{2}} = 19{,}471221^{\circ}$$

18.2.2.2 EQUILATERAL-TRIANGLE QUADRANGULAR PYRAMID

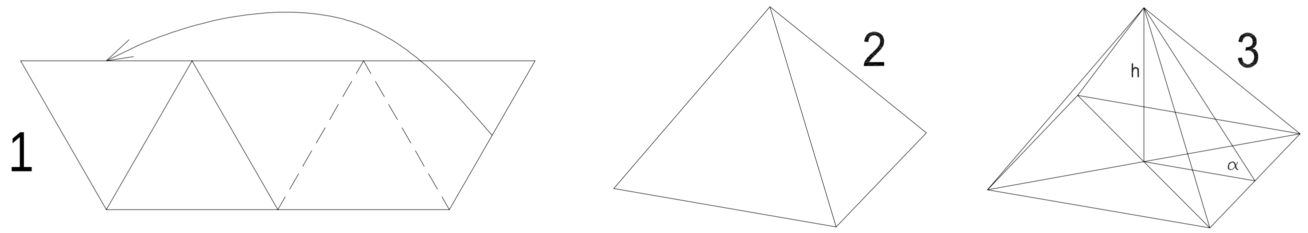
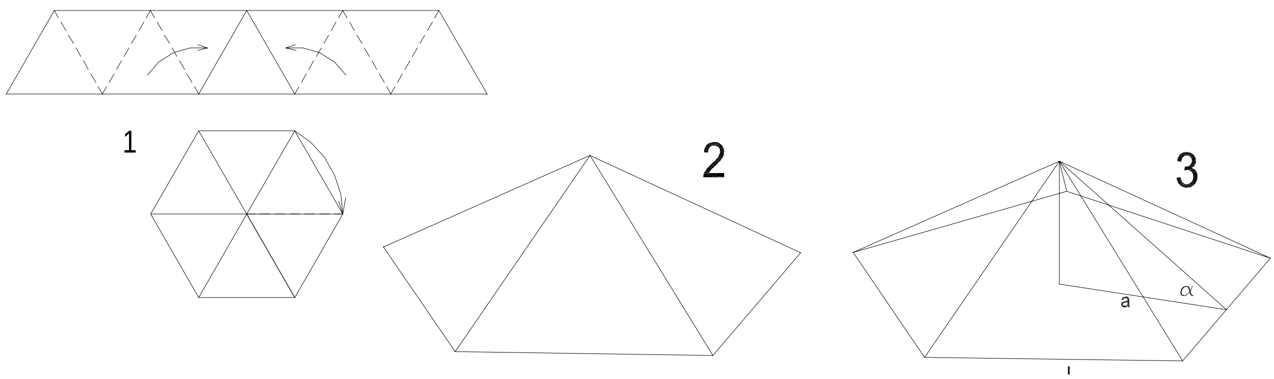


Fig. 1 is the folding diagram and Fig. 2 is the pyramid we get. In Fig. 3 we can calculate the altitude h and dihedral angle α, the side of the equilateral triangle being the unity.

$$h = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{4}} = \frac{\sqrt{2}}{2} = 0{,}7071067 \quad ; \quad \alpha = \arccos\frac{1}{\frac{2\sqrt{3}}{2}} = \arccos\frac{1}{\sqrt{3}} \quad ; \quad \alpha = 54.735613^{\circ}$$

(equal to the α angle in Point 18.2.1.2)

18.2.3 PENTAGONAL PYRAMID

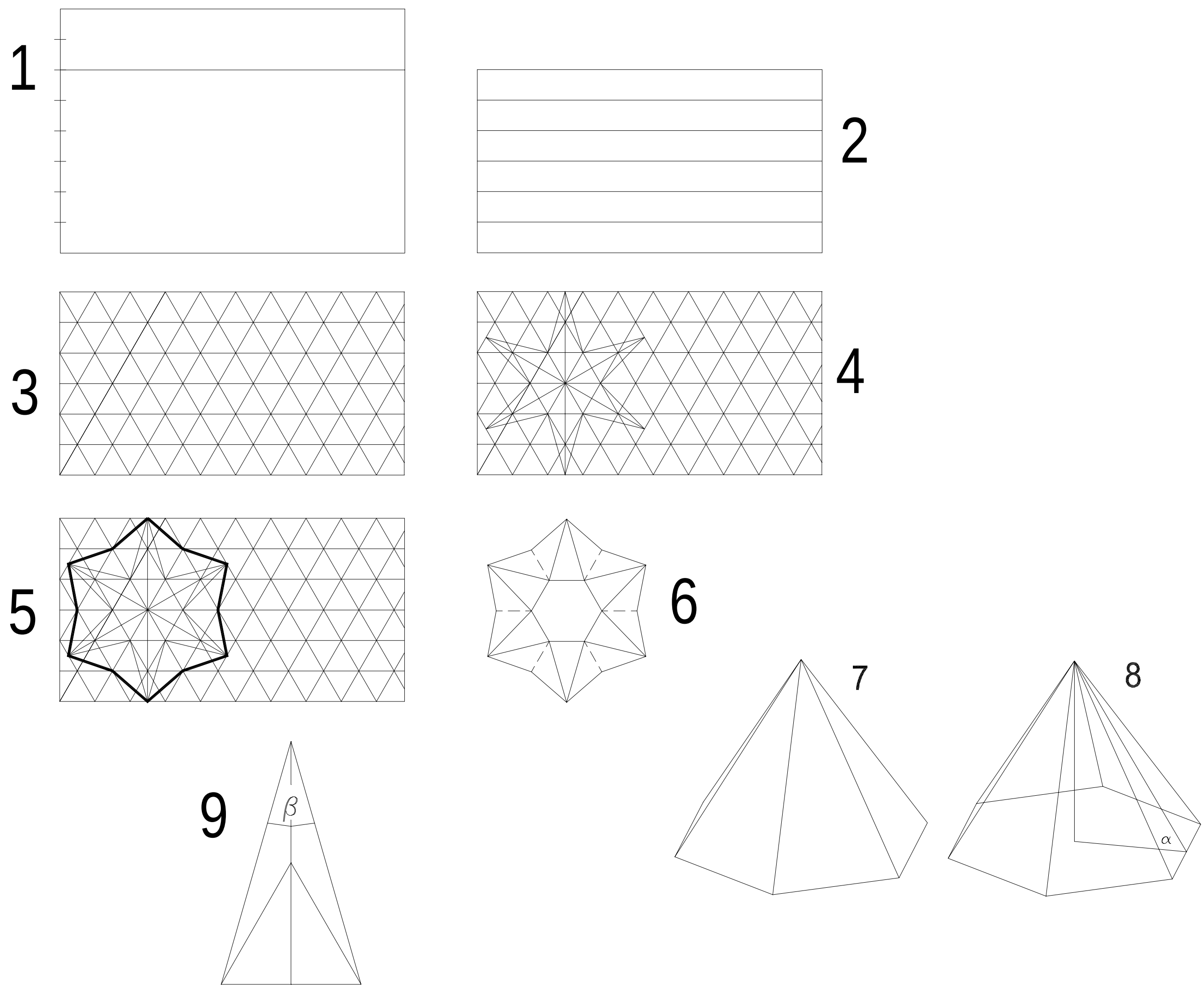


This pyramid is also lacking its base. Its lateral faces are equilateral triangles of side l (same as the side of the base pentagon). The apothem of said pentagon is calculated in Point 18.6.1 though Fig. 3 makes evident its value.

Fig. 1 is the folding diagram which is worked out in two steps: in the first place we form a hexagon; then, the upper trapezium that appears is rotated 60º around the center of the hexagon, as shown. While performing this operation the figure is filled out to attain its pyramidal volume (Fig. 2).

$$a = \frac{l\,\mathrm{tg}\frac{108}{2}}{2} \quad ; \quad \alpha = \arccos\left(\frac{l\,\mathrm{tg}\,54}{2} : \frac{l\sqrt{3}}{2}\right) = 37{,}377368^{\circ}$$

18.2.4 HEXAGONAL PYRAMID



The folding process beginning in Fig. 1, is self-explanatory. Fig. 5 shows how to cut along the solid line to get Fig. 6. After folding the latter, we obtain pyramid 7.

As said in Point 18, a hexagonal polyhedral angle must be constructed with plane angles smaller than 60º. Fig. 9 shows a lateral face whose altitude is twice that of the equilateral triangle having a side equal to the base hexagon (see Fig. 4). This means, in our case, that the apothem of the pyramid is double of the base´s apothem. Bearing this in mind we can figure out angles α and β:

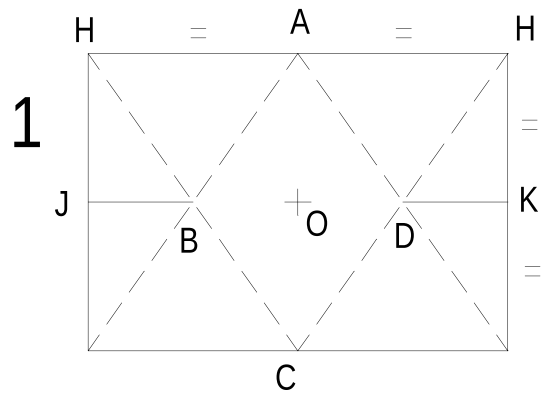
$$\alpha = \arccos \frac{1}{2} = 60^{\circ} \qquad ; \qquad \beta = 2\,\mathrm{arctg} \frac{\frac{l}{2}}{2\,\frac{l\sqrt{3}}{2}} = 32{,}204228^{\circ}$$

18.2.5 RHOMBIC PYRAMID

The one to be studied now is an irregular pyramid whose base is a rhomb having its diagonals in the ratio $\sqrt{2}:1$. We obtain it from a DIN A rectangle with its small side equal to1 (Fig. 1).

Folding accordingly we get the mesh-like pyramid of Fig. 2. Its base is the rhomb ABCD whose diagonals are:

$$AC = 1 \quad ; \quad BD = \frac{HH}{2} = \frac{\sqrt{2}}{2} \qquad \text{therefore } \frac{AC}{BD} = \sqrt{2}$$



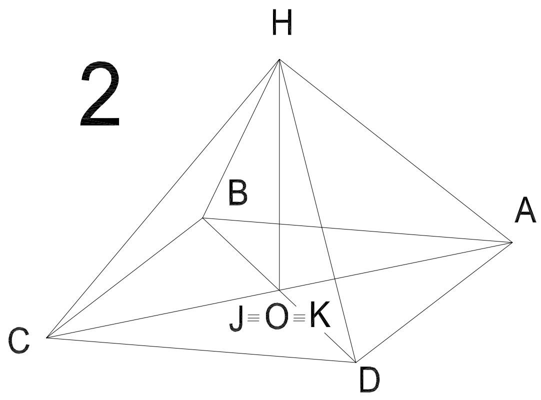
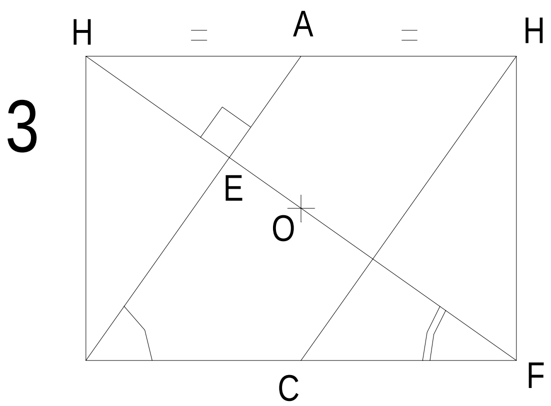
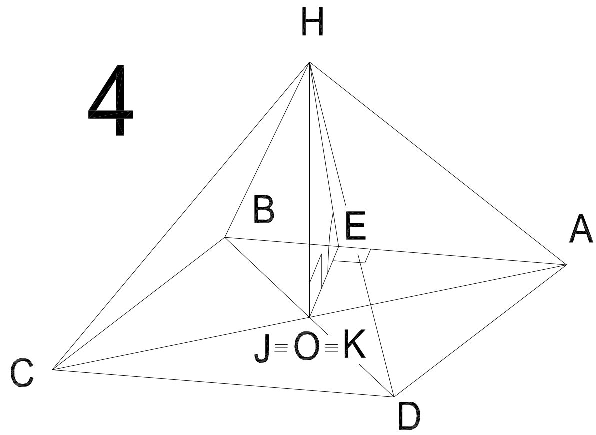


Fig. 3 shows the right angle E:
$$Ang.E = 180 - \left(\operatorname{arctg}\frac{1}{\frac{\sqrt{2}}{2}} + \operatorname{arctg}\frac{1}{\sqrt{2}} \right) = 90$$





On the other hand we know (Point 9.8) that HE = $\frac{1}{3}$ HF, hence

$$HE = \frac{1}{3}\sqrt{1+2} = \frac{\sqrt{3}}{3} = 0{,}5773502 \qquad \text{(Figs. 3 and 4)}$$

Besides, folding to Fig. 1, points J and K will lie on O (Fig. 4).

Consequently $h = HJ = HK = \frac{1}{2}$ are the altitude of the pyramid; (HD = HB) < (HA = HC)

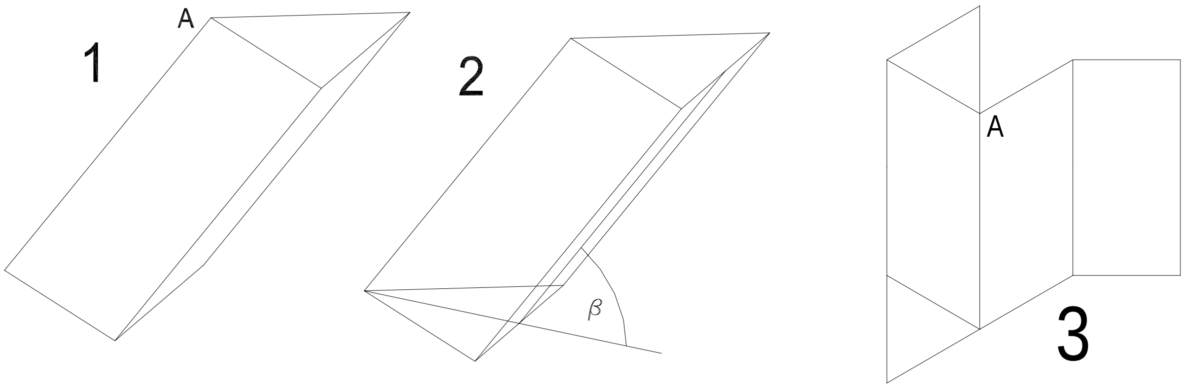
In Fig. 4, h = HO. Likewise, Ang. HEO (formed by a triangular lateral face and the rhombic base) measures:

$$AngHEO = \operatorname{arcsen}\frac{HO}{HE} = \operatorname{arcsen}\frac{1\times 3}{2\sqrt{3}} = \operatorname{arcsen} 0{,}8660254 = 60^{\circ}$$

18.3 PRISMS

Let us construct an oblique equilateral triangular prism whose lateral sides form with the base an angle β = 54,735613º (see point 18.2.2.2) having the attitude shown in Fig. 2.

Those lateral sides will measure twice the base´s side. Under these circumstances we have the prism of Fig. 1 whose folding scheme is Fig. 3.



18.4 TRUNCATED PRISM

Let´s truncate the prism of Point 18.3 by a plane through A forming with the base an angle twice the value of 90 – 54,735613. That angle is the same formed by two opposite lateral faces in the quadrangular pyramid with vertex A. That plane divides the prism in two solids: the upper one which is the above mentioned pyramid (already studied in Point 18.2.2.2), and the lower one: the truncated prism as such (see Fig. 1).

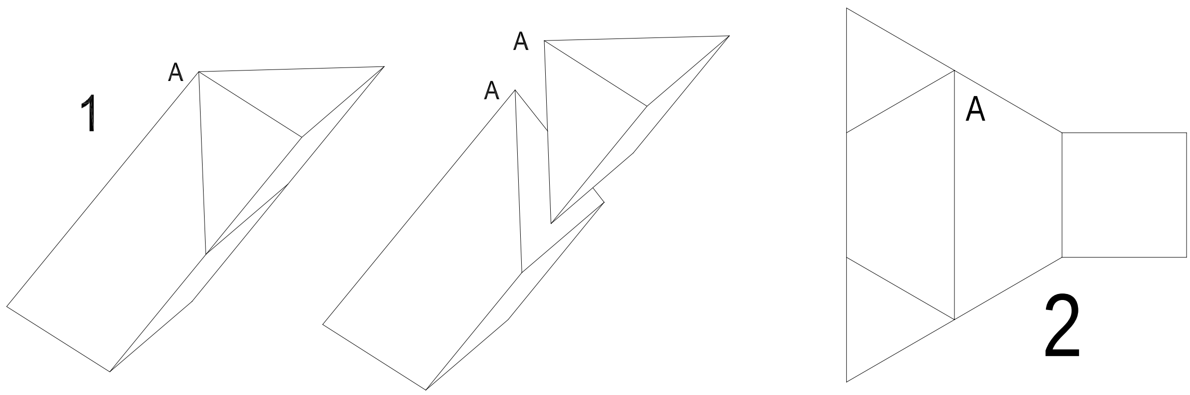


Fig. 2 is the development to construct the truncated prism.

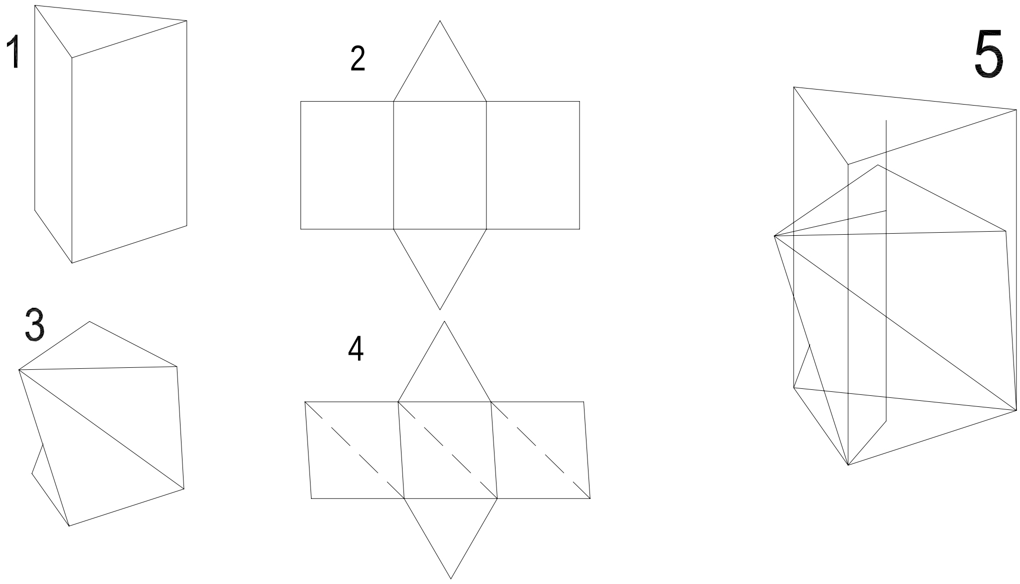
18.5 PRISM TORSION (obtention of PRISMOIDS)

When we studied the so-called SALOMONICA COLUMN in Point 17.2, we already saw how to get the piling up of partial twists in a quadrangular prism. There we pointed out two ways of construction: CAD and analytical. Both are basic in present study, but we shall stick to the first.

18.5.1 TRIANGULAR PRISMOID

Fig. 1 is a right prism whose bases are equilateral triangles. Fig. 2 is its folding scheme. All five figures are not to the same scale.

Twisting 40º clockwise the upper base of Fig.1 with respect to its lower base, the altitude of the prism is reduced to 70 %. By so doing (CAD) we get Fig. 3.



To draw the folding diagram of Fig.4, the only thing needed is to measure by means of CAD the sides of the eight triangles of Fig. 3.

We must pay attention to the fact that the altitude of prismoid 3 is not available in Fig. 4, but in Fig. 3: it is the distance between the centers of its bases.

In Fig. 5 we can see, overlapped, both Figs.1 and 3 in order to clarify the process.

We may observe that the process above can be fulfilled regardless of the bases of the prism. Actually it is better to have them to afford a more consistent solid.

Let´s consider now the problem of twist to flattening both bases on the same plane.

It is not possible to twist flat the prismoid of Fig. 3 because of the interference of its three valley-fold diagonals. It is required that the prismoid´s altitude fulfils certain conditions, as we shall see here after.

Let´s observe what happens in Fig. 7 that is a flattened prismoid without bases. Fig. 8 is a meshlike version of Fig. 7 showing the twist angle to make clear the process. The vertices of the bases are, respectively, ABC, A´B´C´.

In a folding diagram like that in Fig. 4 we see the six lateral triangles whose sides will be named in this manner: l (side of the base´s triangle); p (the small side); g (the great side). We can see that one of the angles adjacent to the base l is acute and the other is obtuse. The same configuration have the triangles in Fig. 8 (for example BCC´).

Let´s see now how the paper is arranged around vertex C. Each angle will be named after the letters of the sides including the angle. Therefore:

$$lp - pg + gl = 60º$$

$$lp + pg + gl = 180º \quad \text{(the three angles of a triangle, together)}$$

Adding up we have: $lp + lg = 120º$

This means that to fold flat an equilateral-triangle prismoid it is a must that the angles formed by a base´s side and the valley and mountain folds adjacent to it will add up to 120º.

There follows that values lg / lp must be between the limits 30 / 90 and 0 / 120. Under those conditions the intersection of sides g / p will determine the prismoid´s altitude that guaranties the folding flat of the solid.

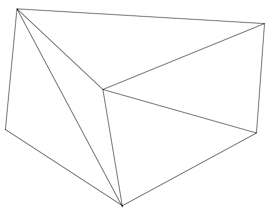
The foresaid permits to draw the folding diagram 9 that in turn may produce the solid of Fig. 6. In this we can measure the prismoid´s altitude and the initial torsion before folding flat. In Fig. 7 we can measure the twist to flatten.

We should note that the various combinations of angles lg / lp convey to different forms for flat Figs. 7 and 8: there is not a unique solution.

Hence, the process will be as follows:

- To fix the base´s triangle with one unit as side.
- To choose the pair of angles, e.g.: lg = 25º; lp = 95º
- To get the development to Fig. 9
- To draw Figs. 7 and 8 to measure the torsion angle to folding flat (in this case it happens to be 50º).
- Fold to 9 to get the solid of Fig. 6.
- Refer solid 6 to a co-ordinate trihedron.
- Measure in it the lower base´s co-ordinates and those of one of its upper base´s vertices.
- With those co-ordinates, draw Fig. 6. In it we can measure the initial altitude of the prismoid (it comes out to be 0,48) and the initial twist angle with respect to the right prism (equal to 7,8º).

6



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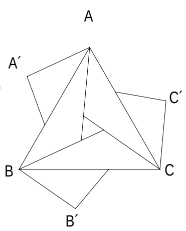

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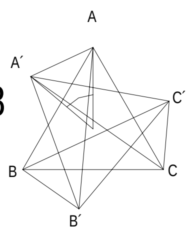

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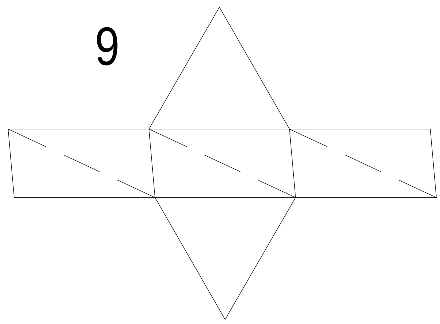
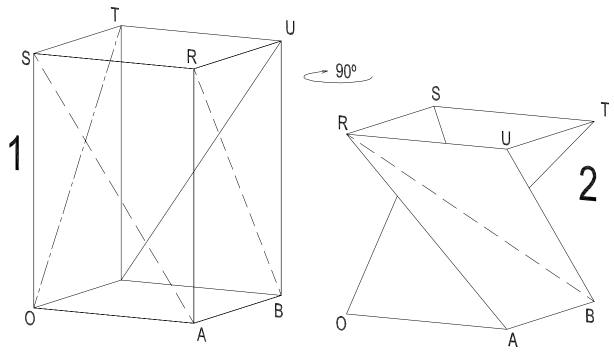


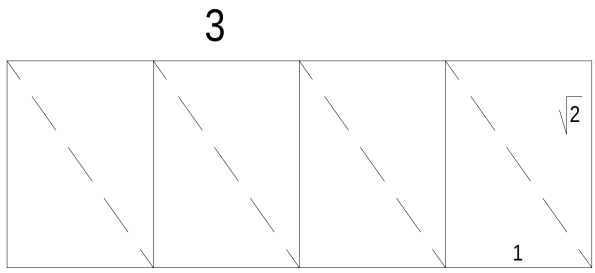
18.5.2 QUADRANGULAR PRISMOID

Let´s start with a quadrangular right parallelepiped (Fig. 1) having A4 rectangles as lateral faces. Fig.3 is its folding diagram. Fixing tight its lower base, we subject the upper one to a clockwise twist of 90º. The result is the quadrangular prismoid of Fig. 2 with all the vertices in a cube.

The valley sides of the prismoid meet in the center of the cube. If AB = 1 is the side of the base´s square, we´ll have:

$$OA = 1 \quad ; \quad AR = \sqrt{2} \quad ; \quad RB = \sqrt{3}$$



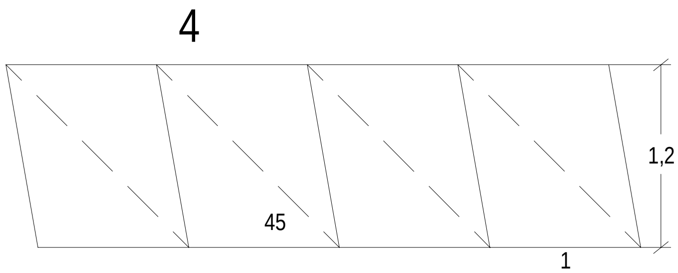


This indicates that RB, in Fig. 2, is the cube´s diagonal. From the center of the cube to its 8 vertices we can adjust 6 quadrangular pyramids like the one seen in Point 18.2.2.1 (lateral side equal to $\frac{\sqrt{3}}{2}$). Two of them are complete (those with the same bases as Fig. 1); the rest are virtual ones.

The shrinkage produced when passing from Fig. 1 to Fig. 2 is $\frac{1}{\sqrt{2}}$. The interference between the valley-fold diagonals impedes the folding flat of the solid. The 90º twist produced between figs. 1 and 2 is not docile (bellows-like) but in collapse mode (snap-like).

Let´s see some variations.

The development of Fig. 4 (gl = 45º; $\sqrt{2} > a > 1$; a = 1,2) produces a prismoid with a natural twist from the right prism. It is docile to an extra twist (total rotation of 69º for Fig. 5). The final prismoid altitude is h = l = 1



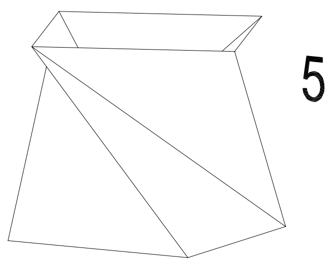
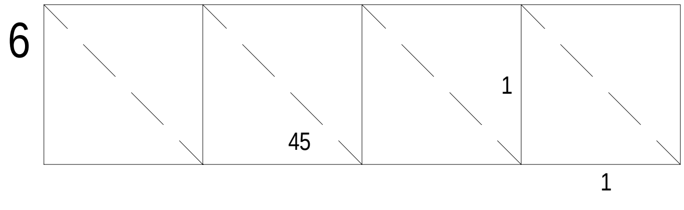
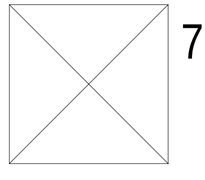


Fig.6 is the development of a prism such that after folding flat produces Fig. 7. The transition from one to the other takes place in the collapse mode. The collapse is produced around the center of Fig. 7. After the 90º rotation, upper and lower base coincide.

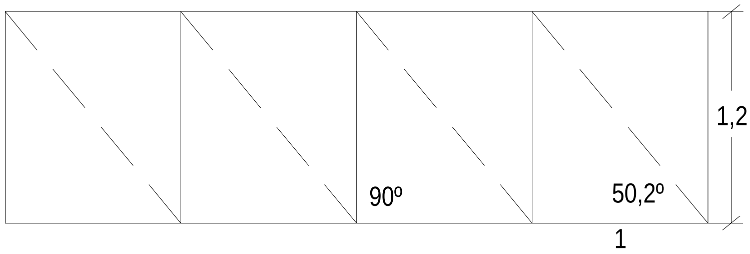


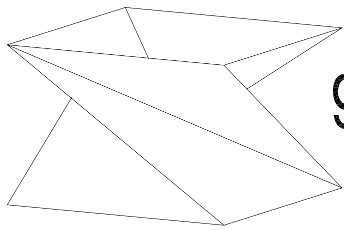


A variant of Fig. 4 is Fig. 8 that in turn produces Fig. 9.

It is not docile but it is collapsible to a virtual right quadrangular parallelepiped (instead of the cube of Fig. 2). The result is a prismoid with altitude $h=\sqrt{a^2-1}=0{,}66$ (a = 1,2). It has its four valley-fold diagonals in contact.

8





9

After all seen till now, one could think that the condition to fold-flat a quadrangular prismoid is that the angle lg = 45º. That is not true, though.

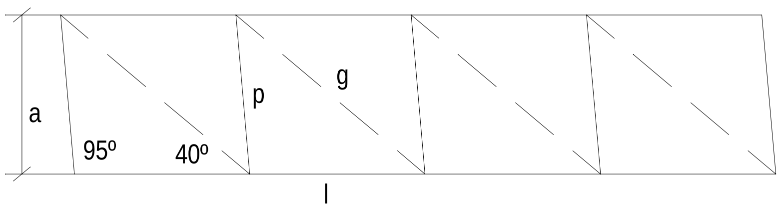
Let´s look over Fig. 10 that exhibits these conditions:

l = 1 ; lg = 40º ; lp = 95º

hence:

$$\frac{p}{\mathrm{sen}\,40}=\frac{1}{\mathrm{sen}\,135} \quad ; \quad p = 0{,}909$$

$$a = p \,\mathrm{sen}\, 95 = 0{,}9056$$



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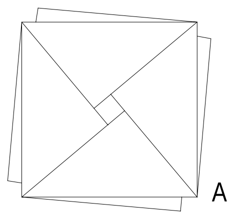


11

The passage from Fig. 10 to 11 is not docile but accepts collapsing to fold-flat. If we analyse the paper arrangement around vertex A (Fig. 11), we have:

lg – gp + pl – ll = 0

Being ll = 90 and lg + gp +pl = 180 (the sum of the angles of triangle lpg), we end up with:

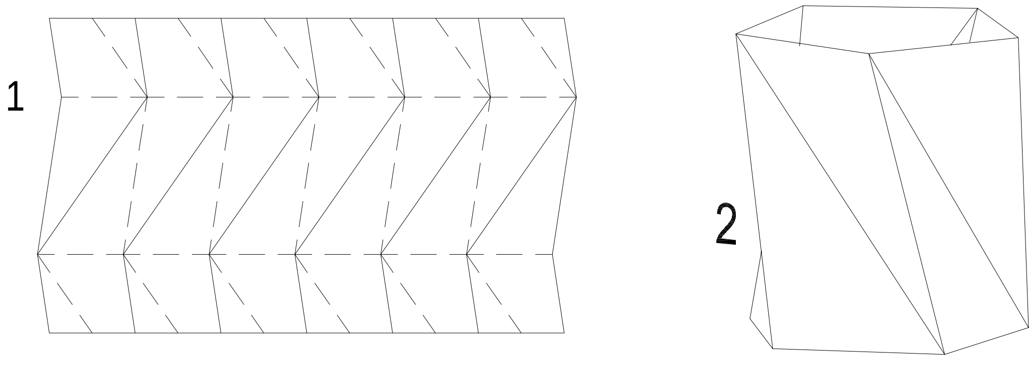
lg + lp = 135

What means that the condition to fold-flat is this: the sum of the angles adjacent to l must be 135º. Fig. 10 accomplishes that condition. Besides angle lp has to be greater than 90º.

Fig. 10 is lacking the bases to allow observing the small interior square appearing in Fig. 11. This square may vary in size according to the chosen combination of angles.

18.5.3 PENTAGONAL PRISMOID

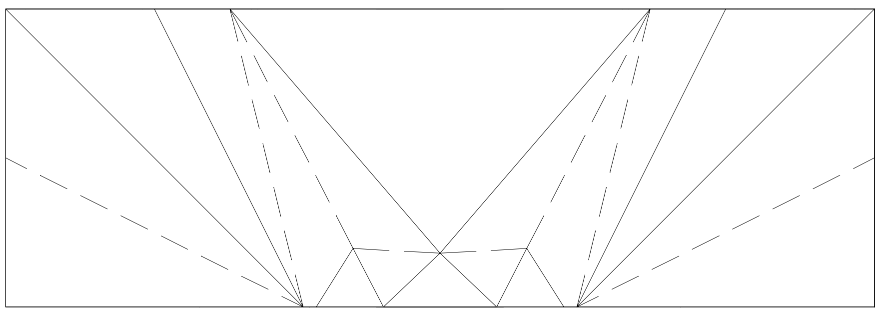
Up to here, the prismoids we have considered were lacking, in general, their bases and the necessary elements (flaps, interlocks, etc.) to conform them tight. In the present occasion, the folding diagram of Fig. 1 is an example of how to close the lateral surface of a prismoid by pocketing (though bases are not shown either).

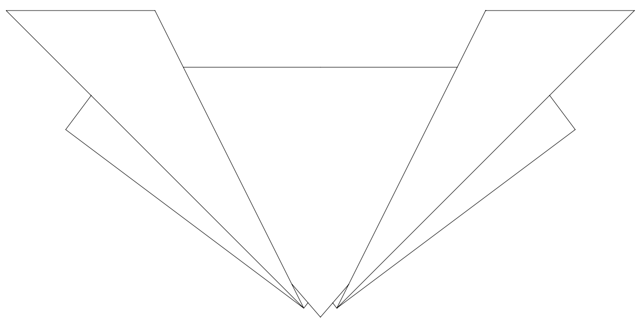


The orientation of segments gp and their folding mode (mountain or valley) are symmetric to those shown herebefore, because now it is put to sight the paper obverse.

The twist between bases of Fig. 2 is 30º and the resultant prismoid´s altitude is 1,76 times the pentagon side.

Interlude





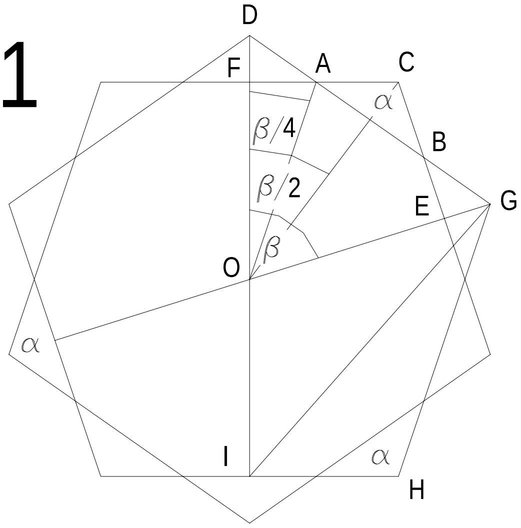
18.6 REGULAR POLYHEDRA

Before digging out about them, we shall analyse certain RELATIONS given within the pentagon, the pentagon-dodecahedron and its conjugate, the icosahedron, to have them at hand whenever necessary.

18.6.1 RELATIONS 1 (dodecahedron)

Fig. 1 shows two congruent regular pentagons rotated 36º one with respect of the other.

1



Let´s figure out the value of some singular segments as a function of the pentagon side l.
l = side of pentagon
r = radius ,, ,,
a = apothem ,,

α = 108 (see Point 11.1) ; $\beta = \dfrac{360}{5} = 72$

$$\left.\begin{aligned} r^2 &= \frac{l^2}{4} + a^2 \\ a &= r \operatorname{sen}\frac{\alpha}{2} \end{aligned}\right\}$$

hence:

$$a = \frac{l \operatorname{tg}\dfrac{\alpha}{2}}{2} \quad ; \quad r = \frac{l}{2\cos\dfrac{\alpha}{2}}$$

$$AC = \frac{r-a}{\cos\dfrac{\alpha}{2}} \quad ; \quad AB = 2a \operatorname{tg}\frac{\beta}{4}$$

In triangle GIH we have:

$$GI = \frac{l - \dfrac{l}{2}}{\cos\varphi} = \frac{l}{2\cos\varphi} \qquad \text{being} \qquad \operatorname{tg}\varphi = \frac{2\operatorname{sen}\dfrac{\alpha}{2}}{l - \dfrac{l}{2}}\sqrt{l\frac{l}{2}} = \frac{4\operatorname{sen}54}{\sqrt{2}}$$

Summarising:
l = l ; a = 0,6881909 l ; r = 0,8506508 l ; AC = 0,2763932 l ; AB = 0,4472135 l
GI = 1,248606 l ; FD = r – a = 0,1624599 l

In Fig. 2 we can see the relation between side and diagonal of a pentagon:

$$d = 2l\operatorname{sen}\frac{\alpha}{2} = 1{,}618034\,\text{l}$$

Fig. 3 shows the same pentagon of Fig. 2 associated with another one in which the side is the former´s diagonal. Being similar both pentagons, we´ll have:

$$\frac{d}{l} = \frac{D}{d} \quad ; \quad D = \frac{d^2}{l} = 4l\operatorname{sen}^2\frac{\alpha}{2} = 2.618034\,\text{l}$$

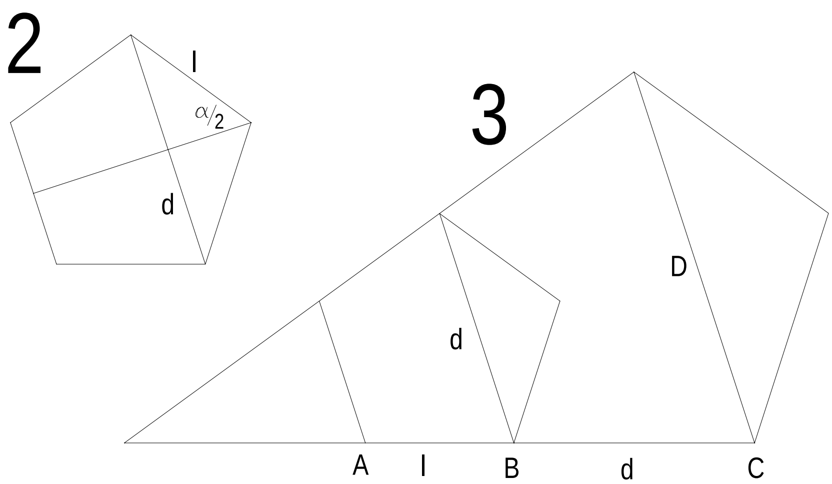
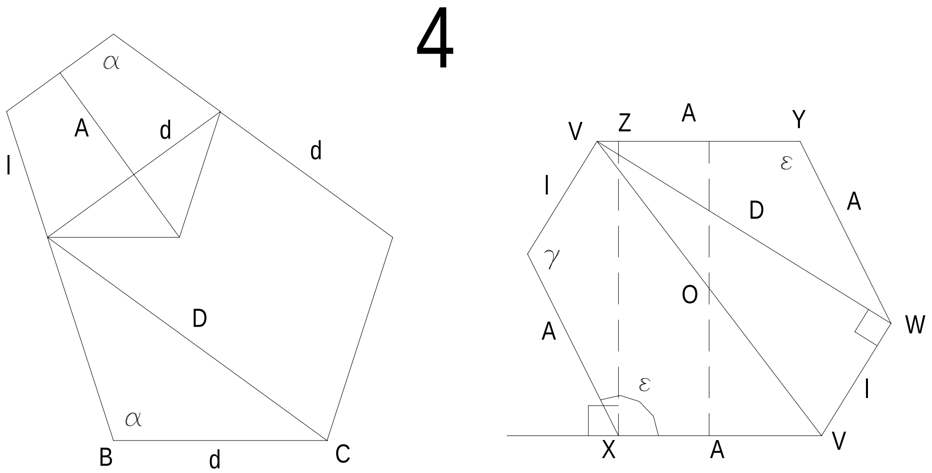


Fig. 4 will allow us to find the dihedral angle ε formed by two adjacent faces of a dodecahedron, as well as its diameter. At left we have two pentagons like those in Fig. 3 and, associated with them, segments l, d, D and A. The value of the latter is:

A = a + r = 0,6881909 l + 0,8506508 l = 1,5388417 l



The figure at right is a hemispherical section of a dodecahedron; in it, D is the diagonal of the great pentagon at left (see Fig. 3, Point 18.6.3). To draw that section we shall start by ΔYVW whose three sides are given. In it we get ε:

$$\text{sen}\frac{\varepsilon}{2} = \frac{D}{2A} \quad ; \quad \varepsilon = 116{,}56505\ ^{\text{o}}$$

The figure at right is symmetric with respect to XY; points V in it are vertices of the dodecahedron. After all that we can deduce:

- the angle γ in the irregular hexagon with sides A, l:

$$180\ (6-2\ ) = 2\ \varepsilon + 4\ \gamma \quad ; \quad \gamma = \frac{720-2\times 116{,}56505}{4} = 121{,}71748$$

- the proof that angle VWV is a right one:

$$\text{Ang. VWV} = \gamma - \frac{180-\varepsilon}{2} = 121{,}71748 - \frac{180-116{,}56505}{2} = 90$$

- the value of VV which is the dodecahedron´s diameter; its mid-point O is, obviously, the polyhedron´s center

$$VV = \sqrt{D^2 + l^2} = l\sqrt{2{,}618034^2 + 1} = 2{,}8025171l$$

Hence the dodecahedron´s radius is $\frac{VV}{2} = 1{,}4012586l$

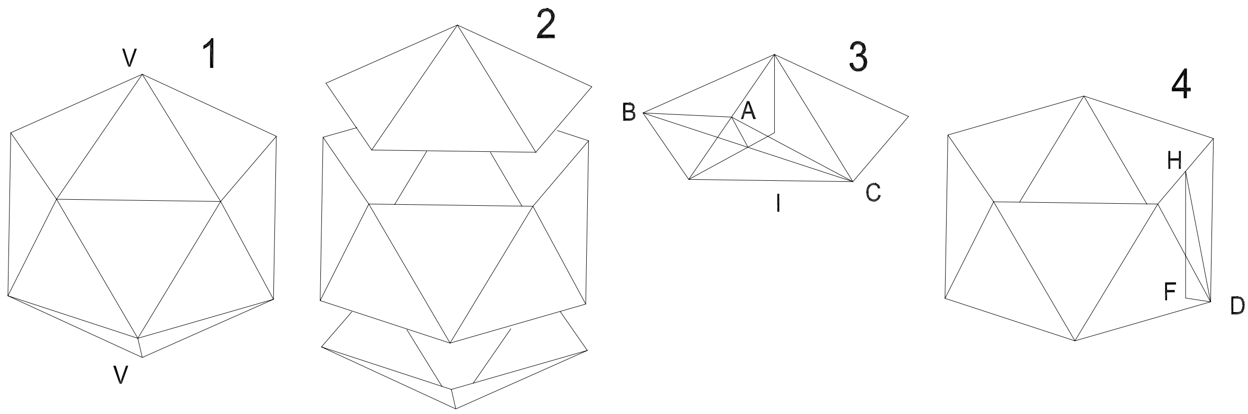
- the proof that distance XY is equal to D

$$XY = l + 2A\cos\frac{\varepsilon}{2} = l\left(1 + 2 \times 1{,}5388418\cos\frac{116{,}56505}{2}\right) = 2{,}618034l = D$$

In the same figure we can verify that VZ = r – a has the same value as FD in Fig. 1.

18.6.2 RELATIONS 2 (icosahedron)

Let´s note in Fig. 2 how the icosahedron of Fig. 1 is constructed: it is made up, in first



place, by two domes (the upper and the lower one) same as that in Fig. 2 of Point 18.2.3; one is rotated with respect to the other an angle of 36º (see Fig. 1, Point 18.6.1). In second place, by a belt of 10 equilateral triangles: its upper and lower pentagonal bases coincide with the respective bases of the associated domes.

In Fig. 3 we can get the value of the dihedral angle formed by two adjacent faces of the icosahedron: it is angle BAC, being BC the diagonal of the dome´s base pentagon and AB = AC the altitude of one face of said icosahedron.

$$Ang.BAC = 2Arc\,\text{sen}\,\frac{BC}{2AC}$$

as BC = 1,618034 l (see Point 18.6.1), and AC = $\frac{l\sqrt{3}}{2}$, the result is:

Ang. BAC = 138,18971º

Either in Figs. 1 or 2 we can see that VV, the icosahedron´s diameter is the sum of: two domes´ altitude (h) plus the belt altitude. Fig. 3 also shows that the dome´s altitude is the vertical leg of a right triangle whose other leg is the radius of its pentagonal base, and its hypotenuse is one side of the icosahedron.

$$h = \sqrt{l^2 - 0{,}85065081^2 l^2} = 0{,}5257311$$

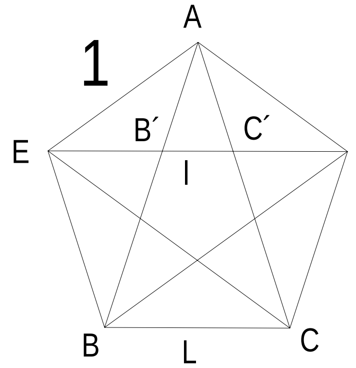
Fig. 4 gives the belt´s altitude HF which is the great leg of right ΔHFD whose hypotenuse HD is the altitude of one face of the icosahedron; the small leg FD has the same length as FD in Fig. 1, Point 18.6.1.

$$HF = \sqrt{\left(\frac{l\sqrt{3}}{2}\right)^2 - 0{,}16245991^2\, l^2} = 0{,}8506508l$$

hence, the icosahedron radius will be:

$$\frac{VV}{2} = \frac{1}{2}\left(2\times 0{,}5257311 + 0{,}8506508\right)l = 0{,}9510565l$$

18.6.3 RELATIONS 3 (stellate pentagon)



Finally we shall study some relations associated to the stellate pentagon; they will be of interest in connection with the regular stellate polyhedra.

In Fig. 1 we see two convex regular pentagons (consequently, similar). One of them is interior with side l (and diagonal d); the other is exterior with side L (and diagonal D). Therefore we´ll have:

$$\frac{l}{L} = \frac{d}{D} \qquad (1)$$

thinking in similar Δ ABC, AB´C´, this implies that

$$AC´ = d = \frac{L}{2\cos\frac{108}{3}} = 0{,}618034L$$

which leads to the new relation:

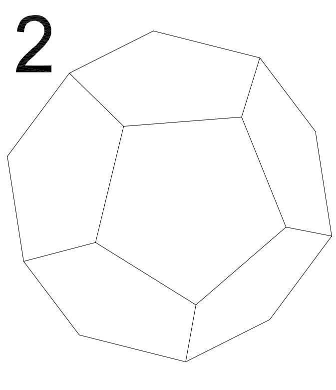
$$D = l + 2d \qquad (2)$$

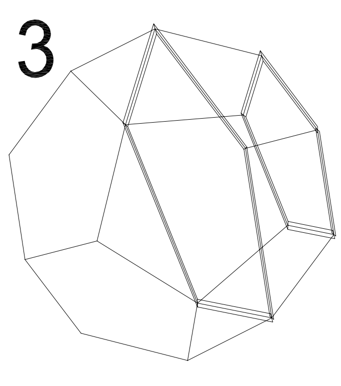
From system (1) (2) we can obtain the values d,l as a function of D,L.

$$d = \frac{D^2}{L+2D} \quad ; \quad l = \frac{DL}{L+2D}$$

Recalling Point 18.6.1 we can also write:

$$\frac{d}{l} = \frac{D}{L} = 1{,}618034$$





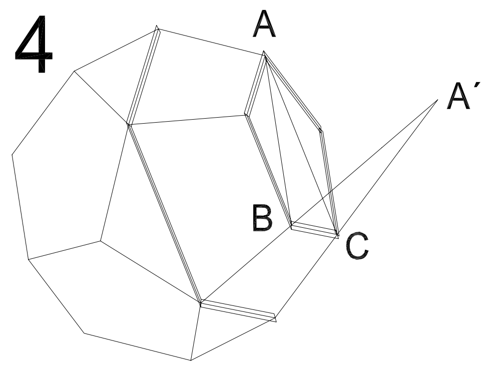


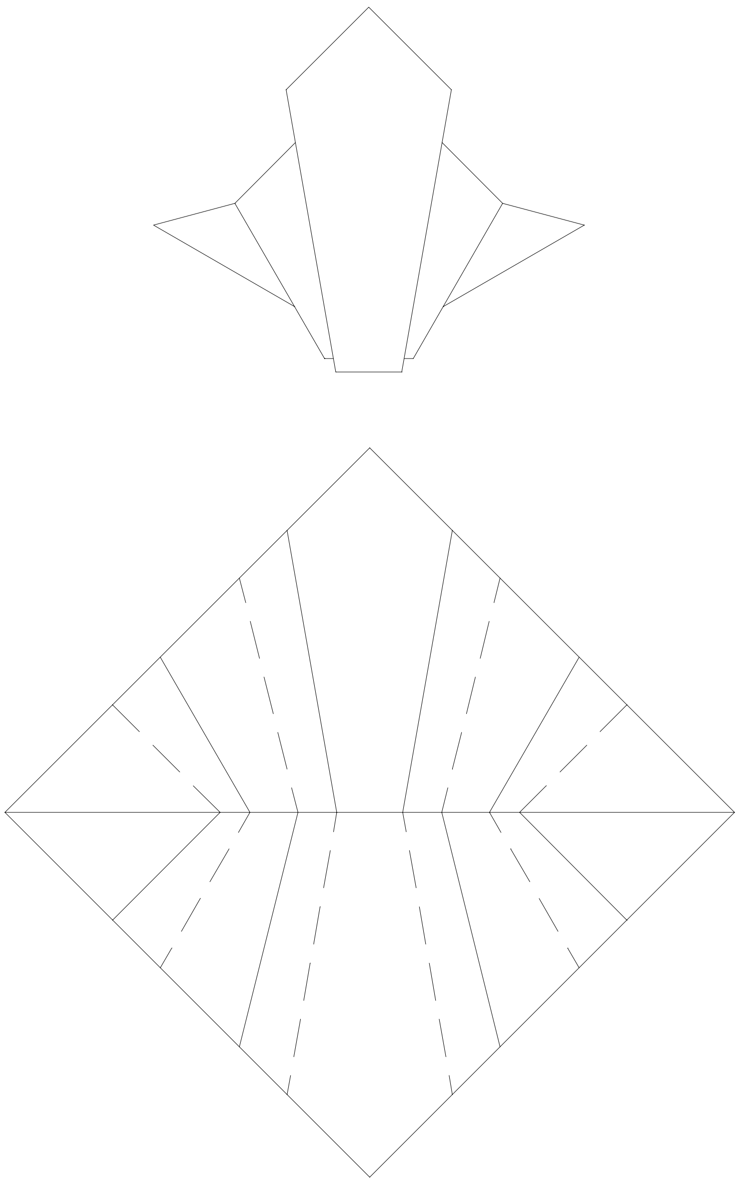
Fig. 2 shows a regular dodecahedron; in it we can see that each face pentagon is homothetic to another one with side d. Both pentagons are outlined in Fig. 3. Let´s look for the center of homothecy. First of all we observe in Fig. 1 that the three angles in B are equal since all of them see the same chord L (same arc of circumference) from the same point B.

As Ang. EBC = 108, it is: Ang. ABC = 72 ; 180 – Ang. EBC = 72

Therefore, angles ABC and EBC are supplementary.

Fig. 4 adds to Fig. 3 the ΔABC of Fig. 1, both, seated on the pentagonal face, and revolved around BC to form ΔA´BC. Consequently A´ is the wanted center of homothecy: the vertex of two pentagonal pyramids having parallel bases. Of course AB = A´B.

Interlude



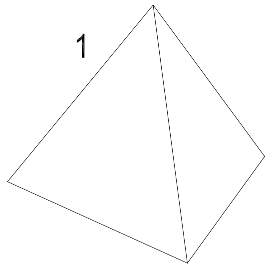
18.7 TETRAHEDRON

C = 4 ; V = 4 ; A = 6

Pyramidal TETRAHEDRON

Fig. 1 is the same obtained in Point 18.2.1.1. As a special triangular pyramid, we omit now its folding diagram.

1

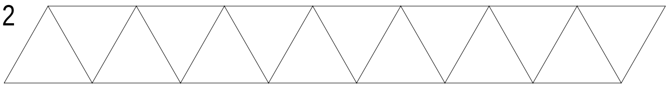


Wound up TETRAEDRON

It is similar to the latter that starts with a paper strip containing just only four equilateral triangles; this, on the contrary, is based in a triangulated paper strip with many more equilateral triangles.

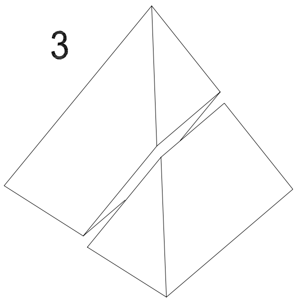
Fig. 2 shows that strip having 14 triangles to guarantee an effective final interlock by pocketing the winding end; 8 triangles in the strip also allow the closing of the tetrahedron, although more precariously.

2



It is up to the folder to decide the folding mode (valley or mountain) to reach the final target.

3



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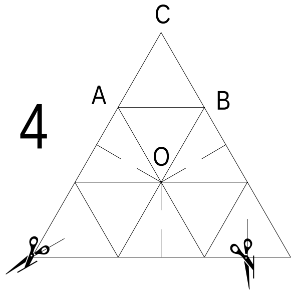


Bi-truncated prism TETRAHEDRON

It is configurated by the union of two equal truncated prisms like those of Point 18.4: They are positioned crosswise, with the square faces in coincidence (Fig. 3).

It is evident that the result is a tetrahedron because the slope β = 54,735613 (Points 18.3, 18.4) of one of the truncated prism faces is the same as its opposite side´s which in turn is the slope of the side of one tetrahedron (Point 18.2.1.1). In addition we should recall that the greater side of the truncated prism is double of its bases´ side: consequently, when completed the assembly, two sides of the truncated prism base add up to one side of the tetrahedron.

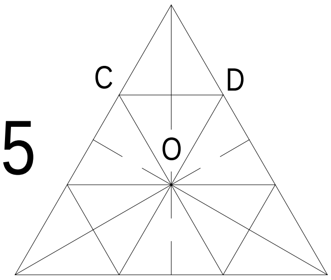
Ex-triangle TETRAHEDRON

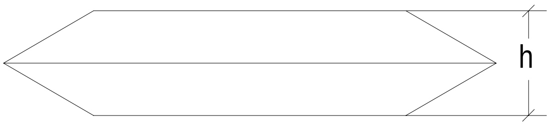
Start with an equilateral triangle of center O (Fig. 4) and fold it as shown. The three OAB type triangles will become the lateral faces, and the three overlapped ABC type will form the base. To fix the assembly interlock the two last folded triangles by means of the cut lines.

Skeletonlike TETRAHEDRON

It is made out of 4 corners as vertices (Fig. 5) and 6 cardboard rails as sides (Fig. 6); rail ends, to 60º.

5





6

The resultant tetrahedron will show only vertices and sides (Fig. 7) and has to be fixed by gluing or by sticking paper.

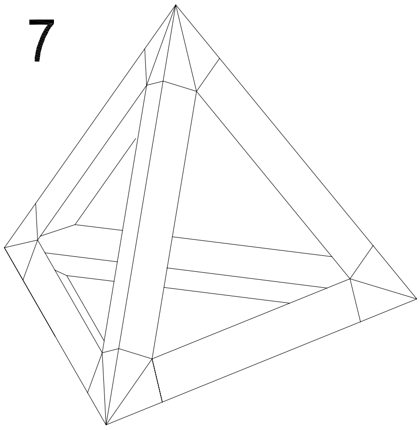
The relation between Figs. 5 and 6 is this: if l is the side of the triangle with center O, it will be:

$$CD = h = \frac{l}{3}$$

We can assign any length to the rails (of course, the same for the six) as long as we get the right proportion in the tetrahedron.

The rails opening to assembly will be fixed to the dihedral angle of the tetrahedron´s faces (α angle in Point 18.2.1)

7



18.8 CUBE

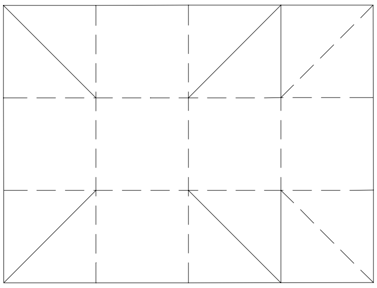
C = 6 ; V = 8 ; A = 12

Perhaps it is the cube (or hexahedron) the polyhedron that has inspired more paper-folded solutions, both in quantity and originality. Now we shall outline only some simple though singular creations and shall also focus on some *cubic* questions specially related to mathematics.

18.8.1 Ex-rectangle CUBE

We start with a folding diagram like that in Fig. 1 and end up with the cube of Fig. 2 fully closed on itself.

1



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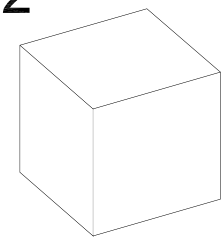


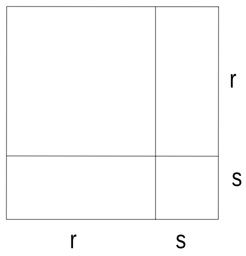
18.8.2 The CUBE of the sum of two numbers

We know that the square of a binomial r + s (Fig. 1) is (see Point 7.2.1):

$$(r+s)^2 = (r+s)(r+s) = r^2 + s^2 + 2rs$$

I.e., the sum of two different squares and two equal rectangles.

1



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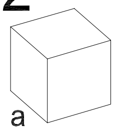

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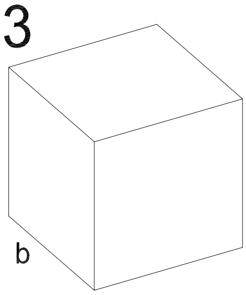


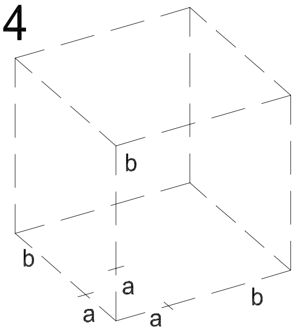
Let´s see now, as an analogy, what is the value of the cube of binomial a + b. Let the cube of side a (Fig.2), and the cube of side b (Fig. 3).

Algebraically we know that

$$(a+b)^3 = (a+b)(a+b)(a+b) = a^3 + b^3 + 3a^2b + 3b^2a$$

that is, the cube of side a + b (Fig. 4) may be considered as made up by both cubes of Figs. 2 and 3, three parallelepipeds same as Fig. 5 and another three like that of Fig. 6.

4



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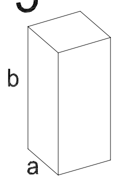

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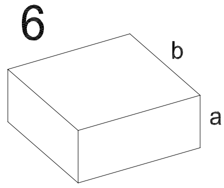

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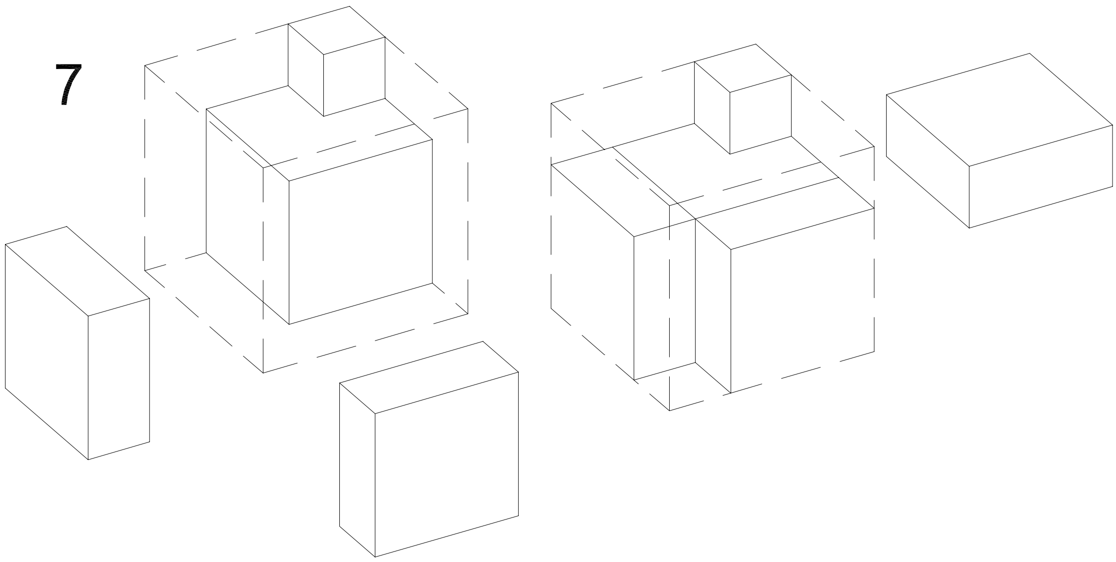

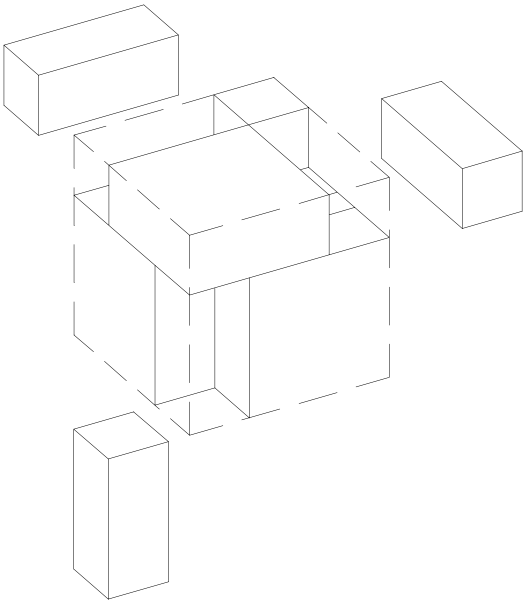

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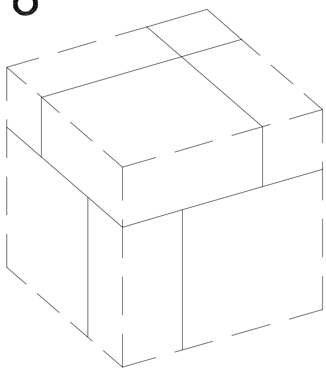
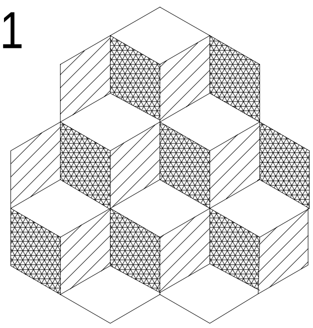


Figs. 7 to 8 show how the assembly is completed.

18.8.3 Magic CUBE, by Jeremy Shafer.

In my opinion, this is the most fascinating discovery I have come across along the whole art of origami. The reason: it is simple, beautiful and original. Before I describe its 3D nature, I shall indulge myself of a 2D digression.

1



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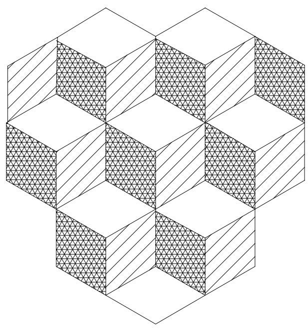


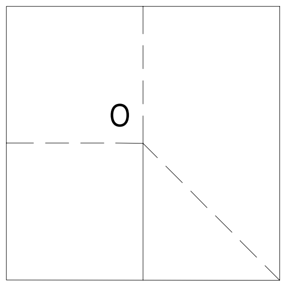
Fig. 2 is the same Fig. 1 after 180º rotation. Both are plane and look like a tessellated floor.

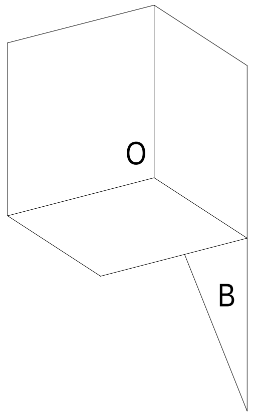
The figures appear to be composed by six hexagons plus some rhombs: two of them white, two shady and two dark, to add up to a total of 24 rhombs. We may notice that 3 equal size rhombs make a hexagon. This will be seen again when dealing with the aragonite’s twin crystal.

Not with much concentration one can see 6 cubes in Fig. 1 and 7 in Fig. 2. Being congruent both figures, one can actually see 6 or 7 cubes regardless of the figure we look at, but it requires a greater concentration to see 7 cubes in Fig. 1 and 6 in Fig. 2.

Up to now we have disclosed the passage from 2D, to some virtual cubes. What offers the Jeremy Shafer´s cube is the virtual passage from a concave tri-rectangle trihedral, to a convex cube, both in the 3-D mode. Besides, it adds a prodigious virtual movement of this virtual cube as well as a beautiful illumination effect. The process simplicity is evident from the folding diagram of Fig. 3 that in turn produces Fig. 4.

3





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To overcome the eventual virtual ambiguity of Fig. 4, the best is to get that figure done and look from top, down to concave vertex O. The observer should take Fig. 4 holding by means of his thumb and forefinger, the overhanging triangle B. It is also required that the interior faces of the trihedral be homogeneously lit and, at least, one of them would show a certain light contrast with respect to the others.

Under these circumstances, the experiment is simple: To look intensively at O while closing one eye. Concentrate in that vision till the moment you see that vertex O as a convex one (it´s easy): by so doing, the trihedral will become a convex cube. If at that moment the observer moves his head sideways, he will see how the *solid* tips clearly around vertex A. The *cube* appears like a lantern, recalling that of Goya´s picture "*May 3 shootings*". Besides, its movement seems to be a matter of magic since the observer is conscious of the fact that he is – holding tight the solid.

18.8.4 CUBE half (or double) the volume of other.

The Greek already knew that this problem could not be solved by means of a ruler and compasses. At present we shall see three different origami based solutions: the first one is an approximation but includes an exact version. The other two are exact and have to do with matters already dealt with before.

SOLUTION 1

It has the peculiarity that no one of both cubes is constructed, but the difference solid; a rectangle is the base for the folding diagram.

Its inexactness is a consequence of the fact that the rectangles sides are divided into three and four equal parts respectively for the sake of simplicity (see how Fig. 1 is designed in Point 18.8.1). By so doing we obtain side L of the great cube. Being l = 3 / 4 L the side of the small cube, the thickness of the difference polyhedron is L / 4.

If the volume of greater cube is double of the smaller, we´ll have:

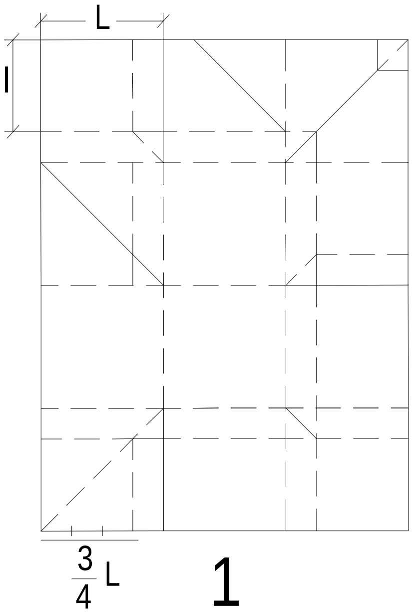
$$L^3 = 2l^3 \quad ; \quad L = \sqrt[3]{2} \times l = 1{,}259921 l \qquad \text{(wanted relation)}$$

$$L = \frac{4}{3} l = 1{,}333333 l \qquad \text{(obtained relation)}$$

the error: $1{,}333333^3 = 2{,}3703702 \neq 2$ (indicator of *double*)

Fig. 1 is the folding diagram for the preceding conditions showing some diagonals and the orthogonal segments L, l, L / 4 (or 3 / 4 L).

Fig. 2 is the result of folding Fig. 1. In Fig. 1 we can see the relation between sides L and l as well as the error produced in the thickness of the difference solid; that error is because l is smaller than needed for the exact solution.



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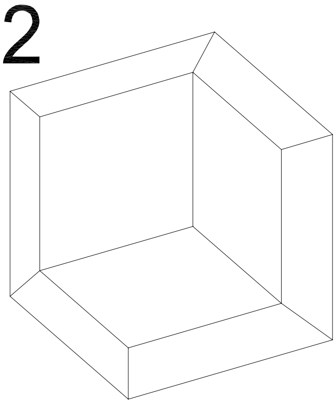
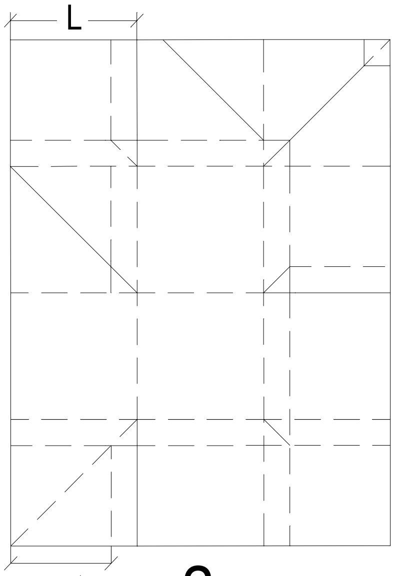
2



Till now we have treated the inexact solution. In case we wish to change easiness for exactitude, all we have to do is this:



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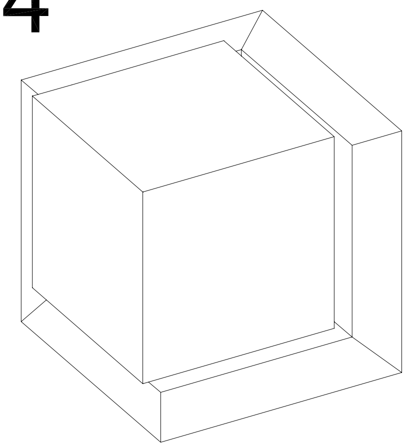
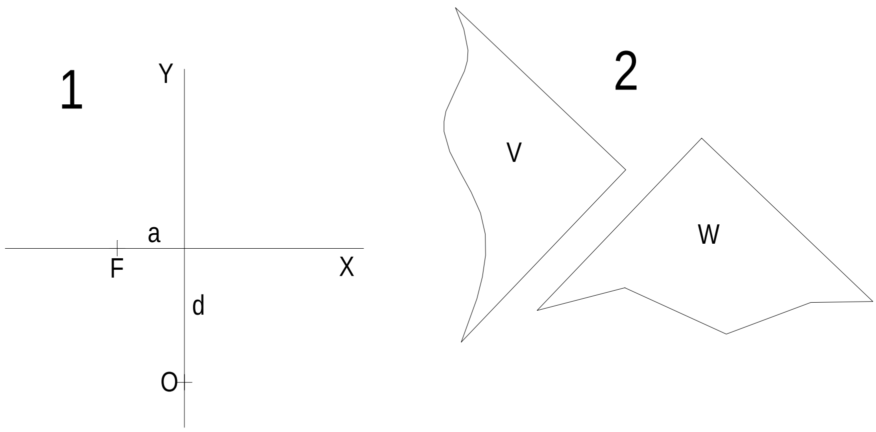


- Start with the same paper rectangle keeping within it the 12 squares of side L.
- Increase l from 3 / 4 of L up to $\frac{1}{\sqrt[3]{2}}L$ i.e., from 0,75 L to 0,7937005 L.
- To obtain $\frac{1}{\sqrt[3]{2}}L$, use any of solutions 2 or 3 and transport its value to Fig. 3 as a paper folded segment.

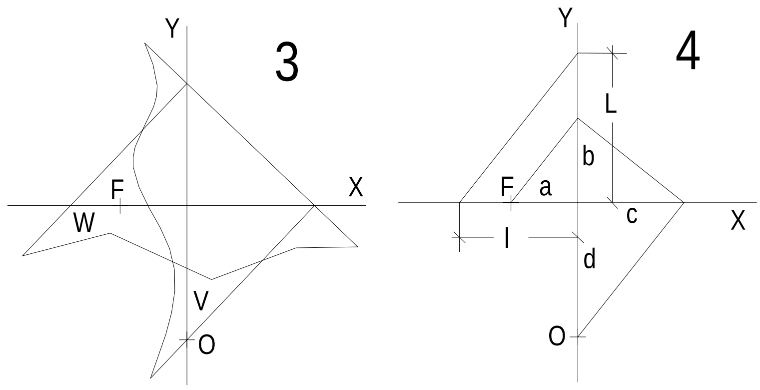
- Once Fig. 3 is folded we get a solid similar to that of Fig. 2; this is shown in Fig. 4 gathering to itself the cube of side l: This cube has obviously the same volume as the solid obtained from Fig. 3, and just half of the cube of side L.

SOLUTION 2

It is based on the orthogonal spiral of powers by H Huzita, as disclosed in Point 7.14.3. It is a matter of finding segment l such that its relation to another given segment L will be $\frac{L}{l} = \sqrt[3]{2}$. The process is as follows:



- To set a co-ordinates system XY (Fig.1) fixing in it points O (origin) and F (final) at distances d,a to the co-ordinates origin, respectively. Condition: a = d / 2.
- To have available a pair of papers V,W (Fig. 2) with right angles.
- By try and error (a maximum of three attempts will be enough) get (Fig. 3):



To lie a side of V on O in such a way that the vertex of its right angle will rest on axle X. The other side of V will intersect axle Y in a point where we shall position the vertex of right angle W.

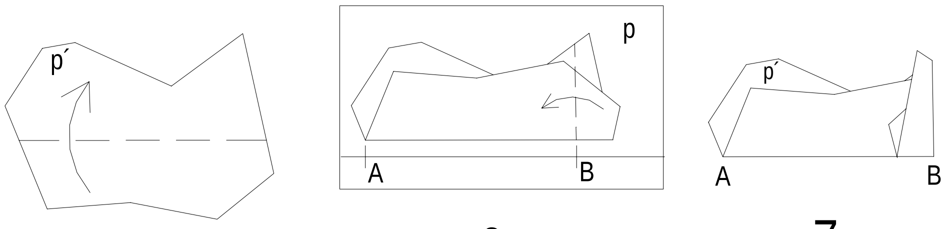
One of the sides of W will lie on the latter side of V, and the other must pass through F. If that would not be the case, try out a new configuration: it is easy because the process is fast convergent. Eventually, we get an orthogonal broken line starting at O and ending at F. It has its two right angles lying on both co-ordinate axles (Fig. 4).

That Fig. 4 gives:

$$\left.\begin{aligned} &b^2 = ac \quad ; \quad c^2 = bd \\ &b^4 = a^2c^2 \end{aligned}\right\} \; b^4 = a^2bd \quad ; \quad b^3 = a^2d = 2a^3 \quad ; \quad \left(\frac{b}{a}\right)^3 = 2 \quad ; \quad \frac{b}{a} = \sqrt[3]{2} = 1{,}259921$$

Said Fig. 4 shows as well how to get l, the small ube side: to take L (greater ube side) along axle Y, and then producing a parallel to the hypotenuse a,b as indicated.

The only problem left is how to transport a segment by means of origami. The process



5 6 7

was already suggested in Points 1.1 and 1.2. Fig.6 shows the piece of paper p in which segment AB to be transported, has being produced. Then cut another paper p´ and fold it to (Fig. 5). Afterwards, make Fig. 5 to coincide with Fig. 6 producing in p´ a normal fold in front of B.

Fig. 7 shows how p´ incorporates segment AB ready for transport (e.g. to Fig. 3 of SOLUTION 1)

SOLUTION 3

The Greek were already mentioned in connection with this problem. We can add now that Hipocrates of Chios (430 B.C.) proved that its solution is associated to the intersection of two parabolas.

In Point 1.2.4 we saw how a parabola is produced by paperfolding. Now we shall jump straight away to the wanted solution.

As before, let l be the side of a given cube and L the wanted side of another cube whose volume is double of the former´s.

Let the equations of two parabolas

$$\left.\begin{aligned} x^2 &= ly \\ y^2 &= 2lx \end{aligned}\right\}$$

From Fig. 1 it is evident that both pass through the origin O. L is given by the abscissa of the other point of intersection.

Solving the former system, we have:

$$\frac{x^4}{l^2} = 2lx \quad ; \quad x^3 = 2l^3 \quad ; \quad x = \sqrt[3]{2}l \quad ; \quad x = L$$

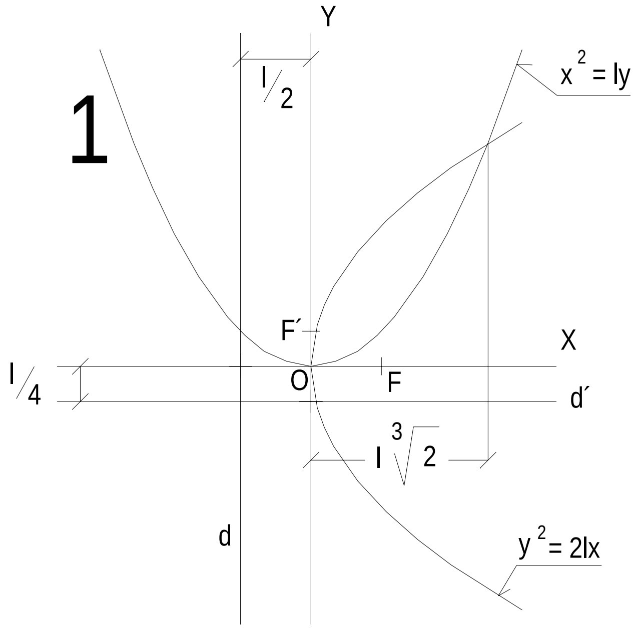
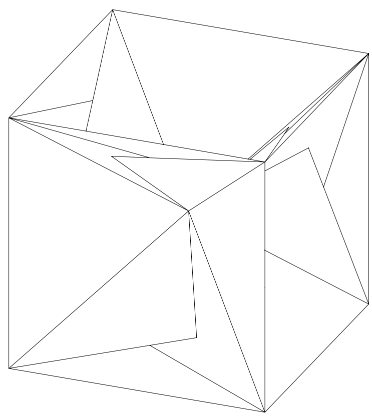


Fig. 1 also shows both parabolas with their foci and directrices respectively equidistant from the co-ordinate origin.

18.8.5 A laminar CUBE.

I give this name to a cube made out of sheets, and determined by its center and sides respectively: in consequence it is only materialised by the center, vertices, sides and diagonals. The sheets form some pyramids alike to those studied in Point 18.2.2.1:



they are leant against to each other.

In that Point we saw how the length of the pyramid lateral side was $\frac{\sqrt{3}}{2}$. Now it´s clear that the double of that side is equal to the cube´s diagonal $\sqrt{1^2+1^2+1^2}=\sqrt{3}$

18.8.6 Diophantine CUBES

The Greek mathematician Diophantus will give us back up for teamwork with paperfolding.

The question is this: How to find 4 cubes whose sides will be represented by 4 consecutive natural numbers in such a manner that the greater cube will have a volume equal to the sum of the other three.

If x is the side of the smaller cube, the imposed condition will be expressed:

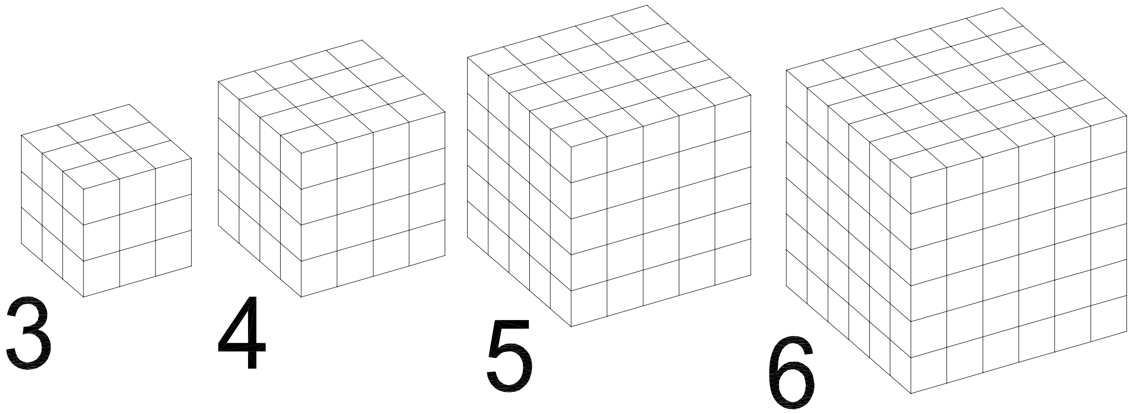
$$x^3 + (x+1)^3 + (x+2)^3 = (x+3)^3$$

developing and simplifying this expression, we get the following third degree incomplete equation:

$$x^3 - 6x - 9 = 0$$

We can resolve this equation in three different ways:

- Through conventional algebraic means.
- By paperfolding (see Point 7.11 for a resultant values of a = 1, b = 0; c = -6; d = -9).
- Applying a diophantine criterion.



The latter consists in outlining the condition, implicit in the original approach, that the solution must be a positive integer. As the smaller number of this kind is 1, we can produce a table with the successive values of x from 1 on, relating them to those taken by both members of equation $x^3 = 6x + 9$ which is the same established before.

x	x^3	6x + 9	Δ
1	1	15	-14
2	8	21	-13
3	27	27	0
4	64	33	31
5	125	39	86
6	216	45	171

....

We can also see in the table how the difference between both members of the equation evolves. From x = 1 up to 3 the difference decreases to zero (which denotes that x = 3 is the solution), whereas from x = 3 on, the difference increases: this means that there are no more solutions. The reader can check that 3 is also the result obtained by folding according to Point 7.11.

The exercise will consist then in the construction of small cubes (Point 18.8.1) of side 1 up to a quantity of 432 (teamwork). With half of the total we shall construct cube 6 and with the other half, cubes 3, 4 and 5. Once the four big cubes 3, 4, 5 and 6 have being built by using all the small ones, we can verify that the original condition has being fulfilled:

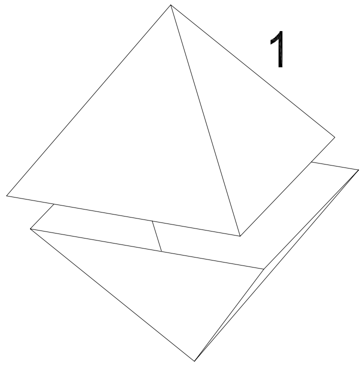
$$3^3 + 4^3 + 5^3 = 6^3 \quad ; \quad 27 + 64 + 125 = 216$$

18.9 OCTAHEDRON

C = 8 ; V = 6 ; A = 12

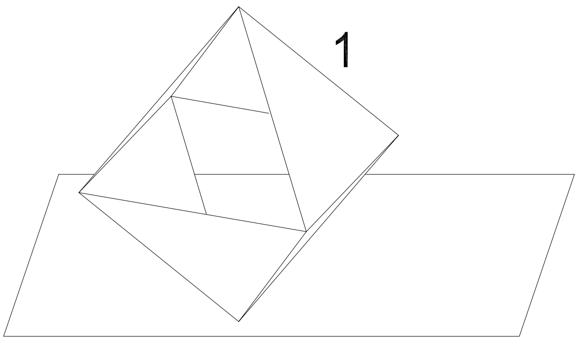
18.9.1 Bipyramidal OCTAEDRON

According to (Fig. 1) it is made out of two opposed equal quadrangular pyramids (see Point 18.2.2.2).



18.9.2 Wound up OCTAHEDRON

Start with a triangulated paper strip of 26 equilateral triangles (most likely we shall have to join several individual strips). Every crease has to be docile to mountain as well as to valley fold indistinctly. Start from one end of the strip to get an octahedron like that shown in Fig. 1 which lacks two opposite faces. Continue winding the strip over the octahedron till the moment the last triangle can be pocked into the corresponding slot. Less than 26 triangles may lead to a precarious structure; with more than 26 we face a problematic construction because of paper accumulation.

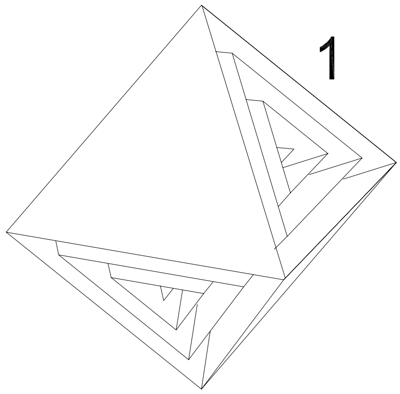


18.9.3 Ex-tetrahedron OCTAHEDRON

Fig.1shows an octahedron built from a tetrahedron whose 4 vertices have been flattened out by pleat folding. Fig. 3 is the tetrahedron folding diagram including the fold lines needed to get Fig. 1.

To be able to draw Fig. 1 we have to get the value of H, the distance between two opposite faces of an octahedron (see Fig. 2).

Fig. 4 is a view of the tetrahedron before its vertices have been flattened out.



$$a = \frac{l}{3} \times \frac{\sqrt{3}}{2} \quad ; \quad H = l\sqrt{\frac{3}{4} - \frac{3}{36}} = l0{,}8164965$$

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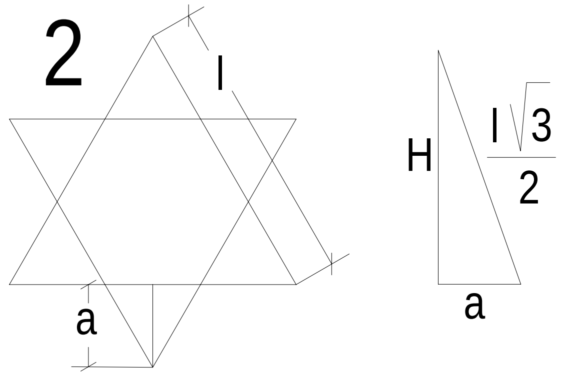

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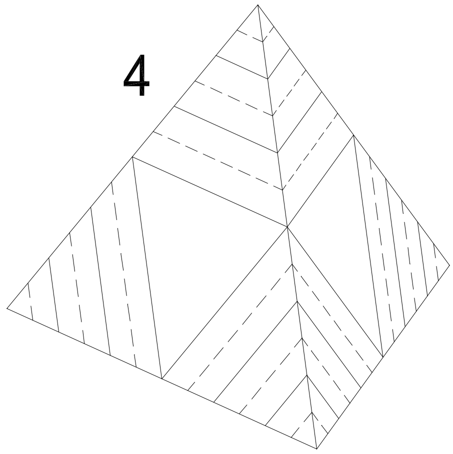

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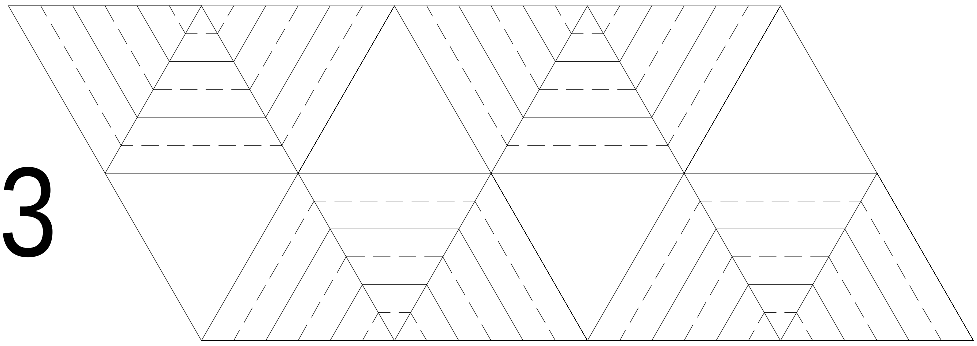
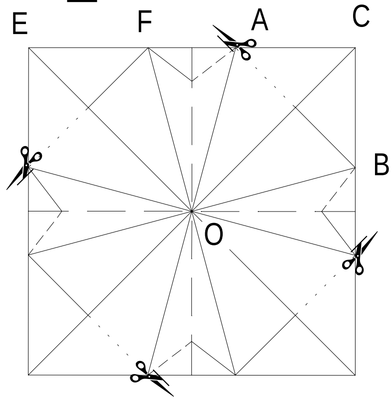


18.9.4 OCTAHEDRON made of two interlocked domes

1



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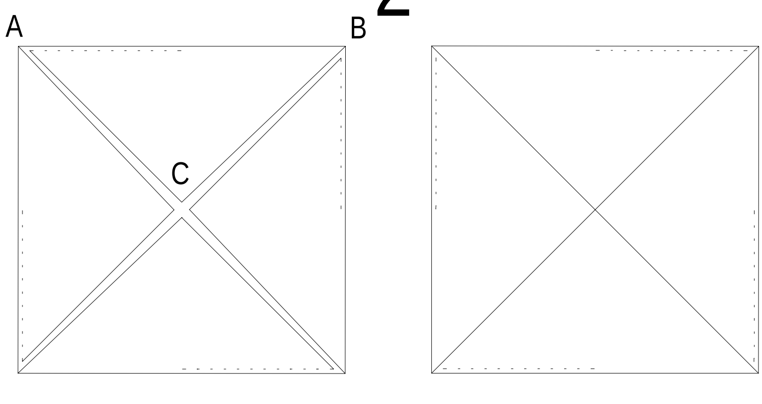
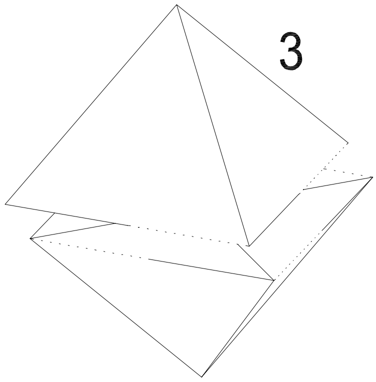


Start with a square (Fig. 1) and divide the perigon at O, in 12 angles of 30º. Divide in halves the four angles type AOF. Fold all the lines with the exception of the four segments type OE. Cut as shown.

By so doing we get a quadrangular pyramid shaped as a dome (in fact a half-octahedron). Repeat to get a second dome.

The bases of those pyramids are shown in Fig. 2 (Figs. 1 and 2 are not to the same scale); in it we can see 4 cuts indicated by dotted lines. Finally place one base against the other, turn around and interlock both domes (Fig. 3).



18.9.5 Skeletonlike OCTAHEDRON

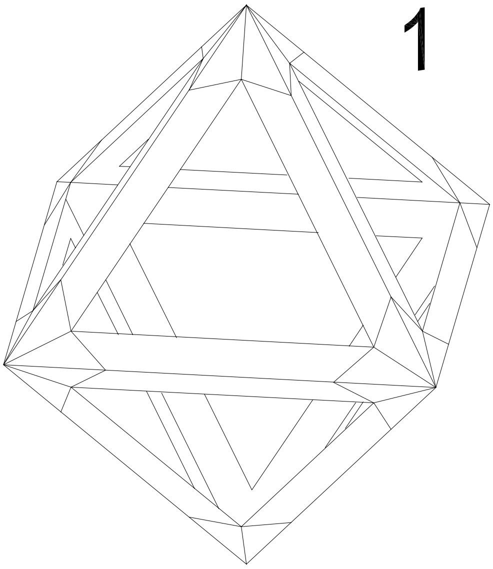
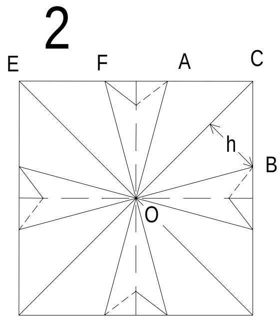


Fig. 1 is the finished octahedron (internal folds of corners are not shown for the sake of simplicity). It has a structure similar to that of the tetrahedron in Fig. 7, Point 18.7: to construct this octahedron we require 6 paper corners for the vertices (Fig. 2), and 12 cardboard profiles for the sides (Fig. 3).

Vertices should be glued for fixing in areas such as OAF and also where Figs. 2 and 3 meet. The crease of Fig. 3 will lie under OC in the assembly.

Profiles to Fig. 3 may have any length (as long as all of them are equal) but they should keep the right proportion with the corners.



Folding angle of profile 3 is 109,47123º, twice as much as angle α in Point 18.2.2.2.

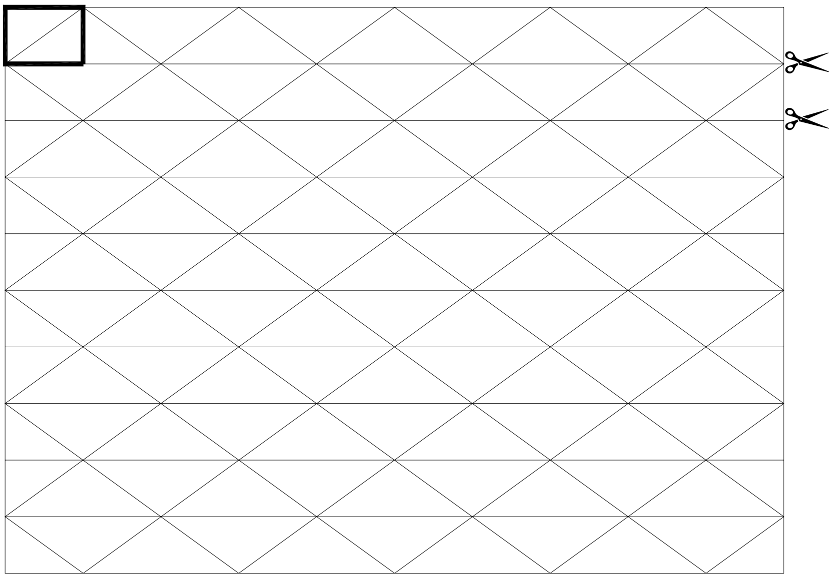
18.10 Perforated PENTAGONAL-DODECAHEDRON

C = 12 ; V = 20 ; A = 30

To optimise paper use, start with an argentic rectangle close in size to an A4, and follow up this process:

1- To divide both sides of that argentic rectangle in 10 equal parts to get 100 small argentic rectangles like the one outlined in Fig. 1.
2- Zigzag draw the diagonals of those 100 rectangles and then clear out their small sides. (Recall Point 10.1.3).

3- All the diagonals will be mountain fold except those affected by a module change (see later). Therefore these diagonals should be pre-folded within the great rectangle in order to ease the process.



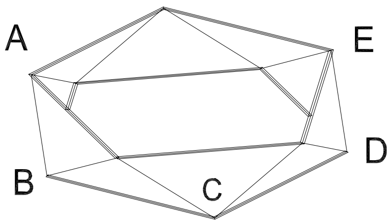
1

4- Cut 10 strips; join them in such a way that two extreme diagonals will overlap keeping all the diagonals as mountain folds. Hence we have produced a strip made out of 90 isosceles triangles.

5- To facilitate the process, begin with half of the joined strips, wind up the polyhedron and add new strips when needed.

6- The diagonals in the strip will become the sides of the polyhedron, and the bases of the isosceles triangles will be the diagonals of its pentagonal faces.

2



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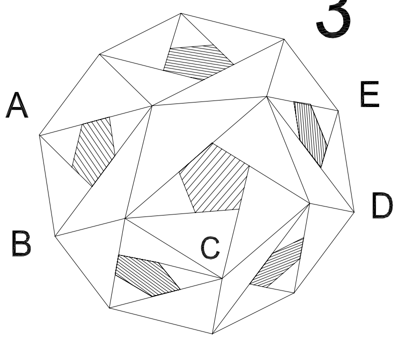

7- From one extremity of the great strip, build an enclosed module having 5 + 5 bases of the corresponding isosceles triangles. Glue the module´s beginning and end to each other making sure that first and last diagonal coincide. In Fig. 2 we can see that module showing those bases set off. It has the vertices ABCDE common with the corresponding ones in Fig. 3.

8- Each one of the modules is a part of the dodecahedron and is determined by two sections parallel to the base of the polyhedron: that base changes as the paper strip is wound around.

9- To continue with a second module, turn over the strip twice on a pair of successive diagonals. When constructing the new module we can observe the coincidence of some diagonals of first and second module: glue them to fix the structure.

10- Turn over again the strip twice to construct a third module. Glue also as before the coincident diagonals. The obtention of last module requires to double-turn over the strip, twice.

11- As the process progresses, add new sections to the main strip, to reach the end. In the meantime we can observe the appearance of the small, hollow pentagons centered within the pentagonal faces of the polyhedron. Eventually, discard the left over piece of strip. Fig. 3 shows how the interior of the dodecahedron is put in shade to avoid the details of inner foldings.

18.11 ICOSAHEDRON

C = 20 ; V = 12 ; A = 30

Begin with an A 4 rectangle (Fig.1) dividing it horizontally in 8 equal parts; discard the two upper portions. Triangulate according to (Fig.2) discarding also the right hand side shown

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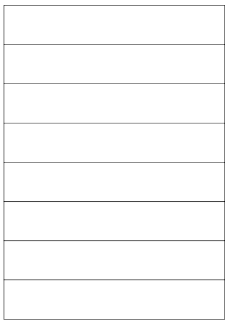

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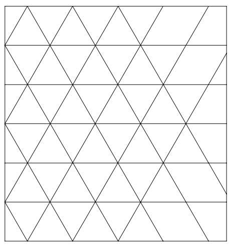

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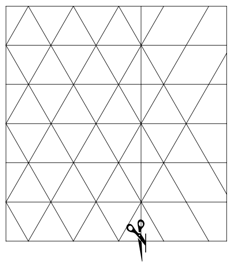
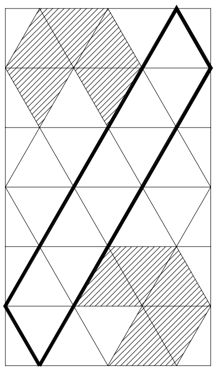


in (Fig. 3). Fig. 4 shows, outlined, the 10 triangles forming the icosahedron´s belt, and, in shadow, the 5 + 5 triangles that eventually will make up the two opposite domes (recall Point 18.6.2)

4



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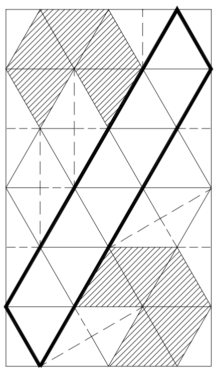

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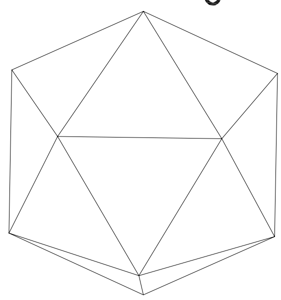


Folding to Fig. 5 we get the icosahedron 6 that must be duly fixed.

18.12 STELLATE REGULAR POLIHEDRA

To try to explain them we shall apply to the stellate polygons (Point11).

Then we said that the species was the indicator of the number of turns to give around the circumference to end up at the starting vertex after all the sides were generated.

We considered as vertices those lying on the circumference, and not the interior intersections of sides.

A convex regular polyhedron can be inscribed within a sphere in such a manner that if the polyhedron is projected from the center over the sphere, we get on this sphere a network of spherical polygons covering, without any overlapping at all, the whole spherical surface.

One stellate regular polyhedron can also be inscribed in one sphere, but if we try to project its faces on it, we observe that the spherical polygons obtained do overlap each other with the consequence that the sphere is covered by them more than once.

The amount of times ε that the sphere is covered upon after that projection, is the species of the stellate regular polyhedron; ε is a function of the number of its faces, vertices and sides, and also of the species e of its faces (whether they are convex or stellate polygons), and of the species E of the polyhedral vertices (likewise they may be convex or stellate).

This fact led Euler to enunciate his generalised theorem that is expressed:

$$\varepsilon = \frac{Ce + EV - A}{2}$$

We can note that this theorem is an extension of that disclosed in Point 18 for the convex polyhedra.

We may recall that regarding stellate polygons, neither triangle, square or hexagon could produce this kind of polygons. Likewise in the case of the convex regular stellate polyhedra, only the convex regular dodecahedron and icosahedron can generate them.

Even so, and because of internal restrains, we end up with only four types of stellate regular polyhedra. From here on, we shall study them in detail just assigning an identification number to each of them. The reason is not to lead the reader to confusion because of the fact that initial and final configurations may be equal or different with regard to the number of faces, in several cases. Therefore, out of the five platonic polyhedra, we only get four stellate regular polyhedra.

18.12.1 STELLATE REGULAR POLYHEDRON nº 1

- We start with a dodecahedron of side BC = L (Fig. 1)
- Having a pentagonal face as base, we build one stellate pyramid with H as vertex.
- H is the center of homothety represented by A´ in Fig. 4, Point 18.6.3.
- Let´s figure out the faces of the stellate polyhedral angle determined by H and the stellate pentagon of one face (Fig. 1).

From Points 18.6.3 and 18.6.1 we infer:

BH=1,618034L ; $l = L\frac{d}{D} = L\frac{0{,}618034L}{1{,}618034L} = 0{,}381966L$ (side of the small pentagon within the stellate one)

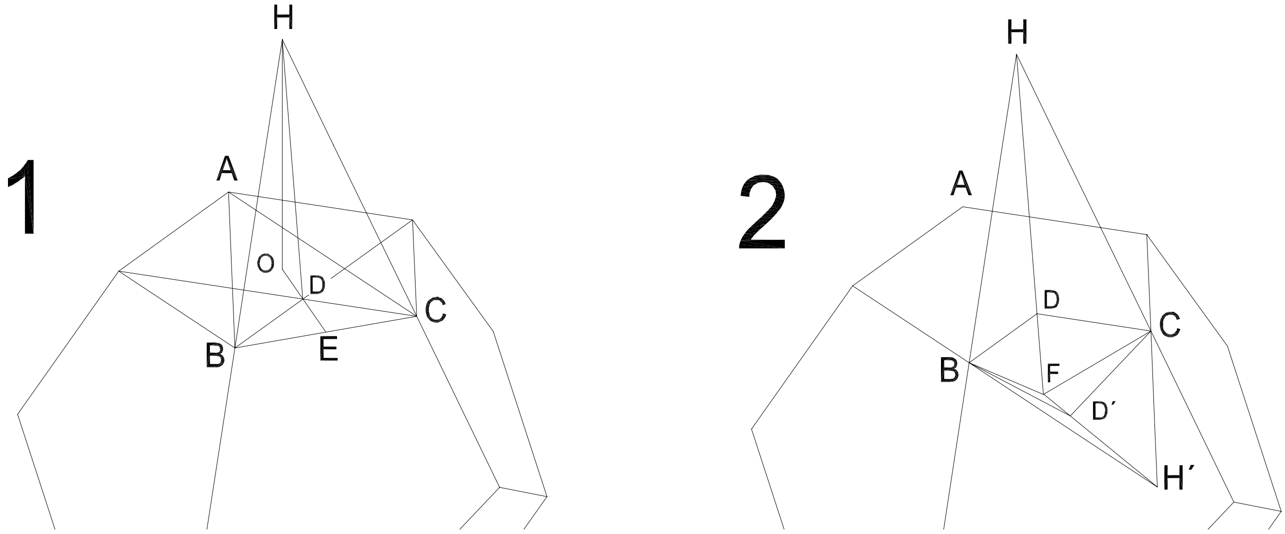
$OD = 0{,}8506508 \times 0{,}381966L = 0{,}3249196L$ (radius of former pentagon)

$OB = 0{,}8506508L$ (radius of pentagon with side L)

$OH = \sqrt{BH^2 - OB^2} = 1{,}3763819L$; OE = 0,6881909 L (apothem of pentagon with side L)

$HD = \sqrt{OH^2 + OD^2} = 1{,}4142135L$ (curiously HD = $\sqrt{2}L$)

$BD = DC = 0{,}618034L$ (equal value than AC´, Point 18.6.3)



With the information gathered till now (the three sides HB, BD, DH of the triangles forming the lateral faces) we are in the position to construct stellate pyramids with H as vertex and a stellate pentagonal base. These pyramids can be placed on the faces of a convex dodecahedron that in turn will be used as the auxiliary structure needed to build the wanted stellate icosahedron.

Fig. 2 shows, besides the triangles with bases DB and DC, and H as upper vertex, the triangles BD´H´ and D´CH´ associated to the other facial pentagon having also BC as a side. We can note in it that those mentioned triangles intersect in the interior of the dodecahedron taken as auxiliary structure. So we ought to find out the length of segments type DF and FC. Fig. 3 will help us for the former and Fig 4 for the latter.

To make up Fig. 3 we should draw ΔHOD (right triangle whose legs we know) and segment OE. Producing HE we get Ang. OEO´ = 116,56505º (ε in Point 18.6.1). The bisector of Ang. OEO´ passes through the center of the dodecahedron, therefore the pyramid in H` will be the symmetric of pyramid H with respect to the plane formed by the side through E, i.e. BC, and said center.

Coming back to Fig. 3, ΔEDF is determined because

DE = OE – OD = 0,3632713 L ; Ang. E = 116,56505 / 2 = 58,282525

$$\text{Ang. D} = \operatorname{arctg}\frac{OH}{OD} = \text{arctg } 4{,}2360692 = 76{,}717478º$$

from what we get that Ang. F = 45º and hence Ang. HFH´ = 90º. Consequently:

$$\frac{DF}{\operatorname{sen} 58{,}282525} = \frac{DE}{\operatorname{sen} 45} \quad ; \quad DF = 0{,}3632713\text{ L } \frac{0{,}8506508}{0{,}7071067} = 0{,}437016L$$

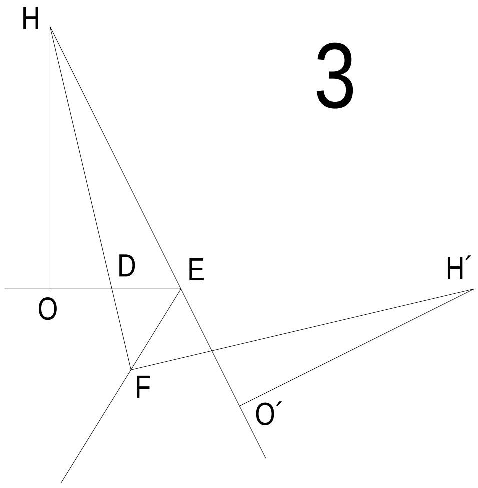
$$HF = HD + DF = 1{,}4142135\text{ L} + 0{,}437016\text{ L} = 1{,}8512296\text{ L}$$

Now we can make up Fig. 4 drawing in first place ΔHDC whose three sides we know:

$$HD = \sqrt{2}\text{ L} \quad ; \quad HC = BH \quad ; \quad DC = BD$$

Then we shall produce HD to get vertex F (DF being also known). This way we get ΔHFC: with 10 of these triangles we construct the polyhedral angle in H over one face of the auxiliary dodecahedron. Extending the operation to its 12 faces we get the complete stellate polyhedron we are looking for.

As we can note, the auxiliary dodecahedron has enable us to draw Figs. 1, 2, 3, 4 but it does not allow the materialisation of the stellate polyhedron because of the above mentioned interference.



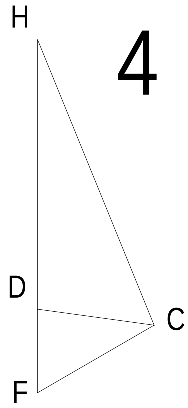
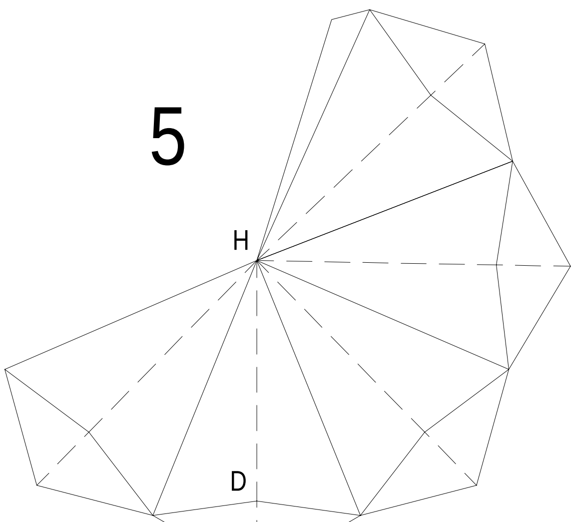
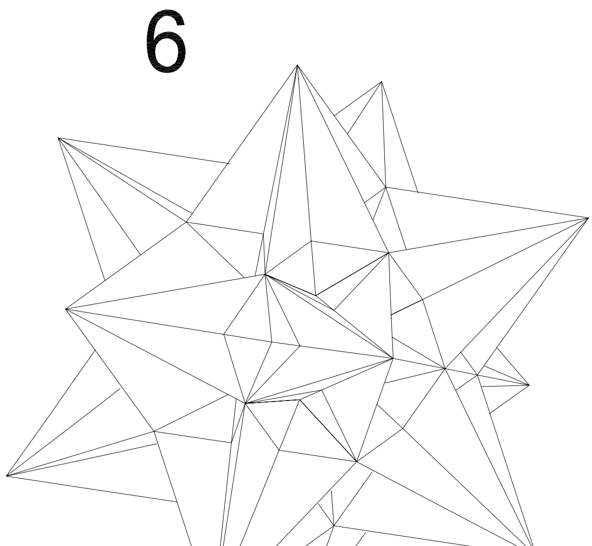


Fig. 5 is the folding diagram for the polyhedral angle in H (only the lateral closing lap joint is shown). Segments type DC are kept as an indication of the stellate pentagons pertaining to the auxiliary dodecahedron. While assembling the 12 elements to Fig. 5, all the dihedral angles will appear automatically. Fig. 6 is the finished stellate polyhedron.





This polyhedron has 12 Vertices (V = 12), one for each face of the auxiliary dodecahedron: all of them are stellate polyhedral angles; moreover, they are pentagonal, i.e. with E = 2 (a stallate pentagon has also species 2).

Out of each vertex start 5 sides making a total of $5\times12=60$; but since each side is common to 2 vertices, we´ll have: $A=\frac{60}{2}=30$; 30 sides in the end.

Let´s compute the faces. From each vertex start 5 equilateral triangles common, in turn, to 3 vertices; therefore $\frac{5\times12}{3} = 20$ faces: C = 20. These 20 faces are equilateral triangles, consequently convex with species e = 1.

Summarising, the obtained polyhedron has:

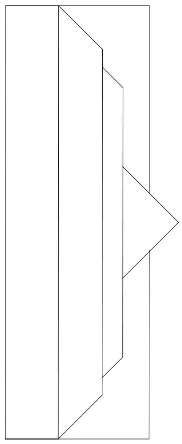
C = 20 ; e = 1 ; V = 12 ; E = 2 ; A = 30

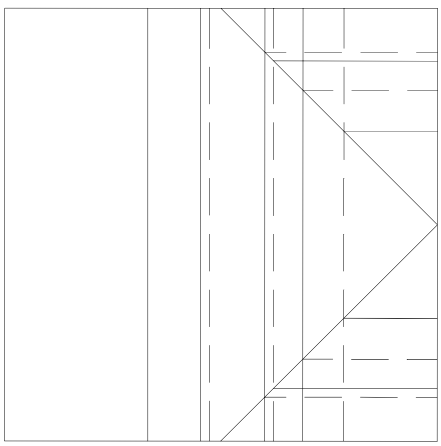
Euler´s theorem for stellate polyhedra gives its species ε:

$$\varepsilon = \frac{Ce + EV - A}{2} = \frac{20\times1 + 2\times12 - 30}{2} = 7$$

Therefore, polyhedron nº 1 is an icosahedron (C = 20) with triangular faces, of 7th species (ε = 7), and 12 pentahedral angles (V = 12) of 2d species (E = 2).

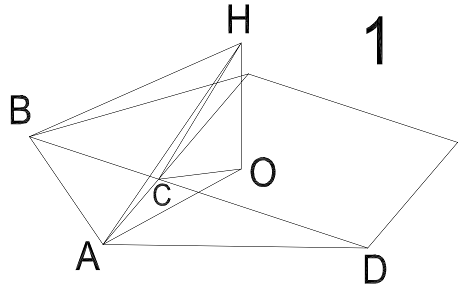
Interlude

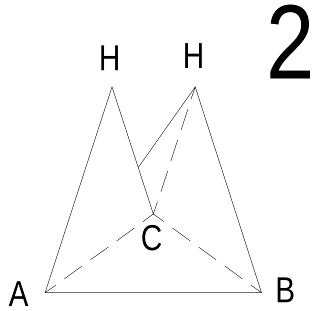




18.12.2 STELLATE REGULAR POLYHEDRON nº 2

The auxiliary starting polyhedron is, in this case, a convex icosahedron with side AB = L (fig. 1); ABH is one of its facial equilateral triangles; the pentagon of center O is the base of its upper dome (Point 18.6.2). ABDH are some of its vertices.



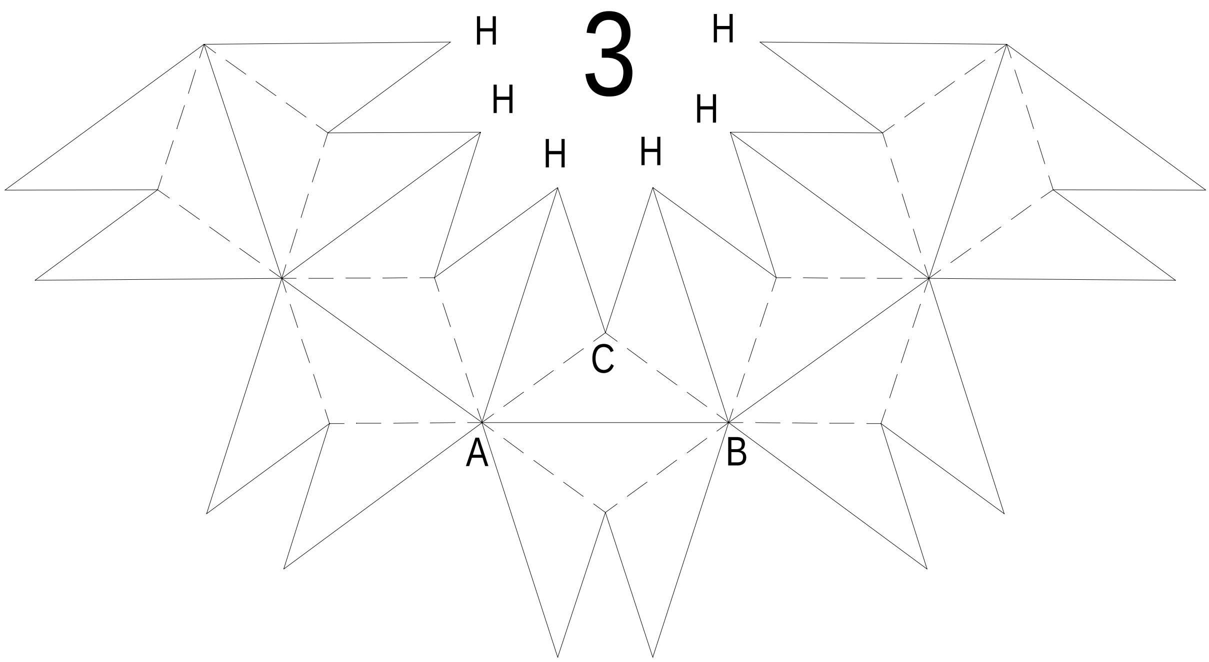


OA = 0,8506508 L (r in Point 18.6.1)
OH = 0,525731 L (h in Point 18.6.2)
OC = 0,3249196 L (OD in Point 18.12.1)

HC = $\sqrt{OC^2 + OH^2}$ = 0,618034 L (what shows that HC = CB = CA, Point 18.6.3).

Fig. 2 is the folding diagram for trihedron in C, ABH (closing lap join is shown).

Fig.3 is the folding diagram (with no lap joints at all) of dome in H (Fig. 1), plus the 5 triangles associated to it (Point 18.6.2). Of course, that Fig. 3 does not develop equilateral triangles, but the trihedral angles to Fig. 2.

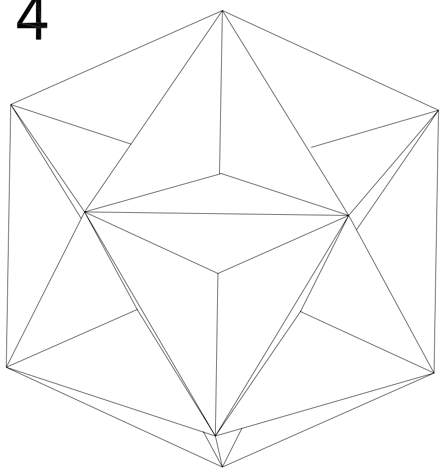


Two assemblies like that obtained with Fig.3, set in opposition, give what appears to be an icosahedron. But it is worthwhile to analyse it closely to see what it is like, actually, the stellate polyhedron we have got (Fig.4).

This stellate polyhedron nº 2 has 12 vertices, as the starting icosahedron; we have to ignore the concave vertices of sank trihedrals because they do not lie on the sphere of reference. Therefore we have 12 vertices composed by stellate pentahedral angles (V = 12); i.e. they have species E = 2: we may recall (Point 11) that the species of a stellate pentagon is also 2.

The faces are planes concurrent in the vertices, i.e. convex pentagons. These pentagons have species e = 1 (they are not stellate).

4



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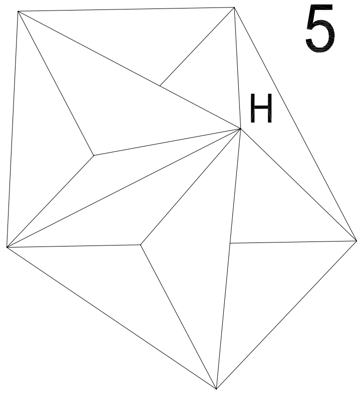


5 of those planes pass through each vertex: 5 × 12 = 60. But as each one of these 60 planes is common to 5 vertices, we end up with 12 faces (60 : 5). C = 12.

Number of sides: 5 per vertex; making a total of 5 × 12 = 60 sides. The sides of the sunk trihedral must be ignored. But as each side is common to a pair of vertices, the result will be 30 sides (60 : 2). A = 30.

Summary. The polyhedron we have obtained has these features:

C = 12 ; e = 1 ; V = 12 ; E = 2 ; A = 30

The species ε for this polyhedron nº 2, according to Euler´s general theorem is:

$$\varepsilon = \frac{Ce + EV - A}{2} = \frac{12\times1 + 2\times12 - 30}{2} = 3$$

Therefore what we have got is a dodecahedron (C = 12); stellate of 3rd species (ε = 3); with 1st species (e = 1) convex pentagonal faces; having V = 12 stellate pentahedral angles, i.e. of 2nd species (E = 2).

Fig. 5 is another view of the dome in H to make it clearer.

18.12.3 STELLATE REGULAR POLIHEDRON nº 3

It is similar to nº 1 (Point 18.12.1). The difference consists in the dissimilitude of their polyhedral angles; though in both cases they are pentahedral, in nº 1 they are stellate whereas in nº 3 they are convex (E = 1). We start with an auxiliary convex dodecahedron for nº3 as we did for nº 1.

ΔBCH in Fig. 1 is the same as the one equally named in Fig.1, Point 18.12.1. Present Fig. 1 is the folding diagram of the pentagonal pyramid (only one lap joint is shown) to be set on each face of the auxiliary dodecahedron; therefore we need 12 of these pyramids to get polyhedron nº 3.

Fig. 2 is the complete polyhedron nº 3. One can see that it has 12 vertices (V = 12) being E = 1 as said before.

5 stellate pentagons start from each vertex, i.e. of e = 2.

There are 5 × 12 = 60 faces, each of them common to 5 vertices; therefore they produce 60 / 5 = 12 faces: C = 12.

In turn, from each vertex start 5 sides, each of them common to two vertices, which gives 5 × 12 / 2 = 30; i.e., A = 30.

The species ε according to Euler is:

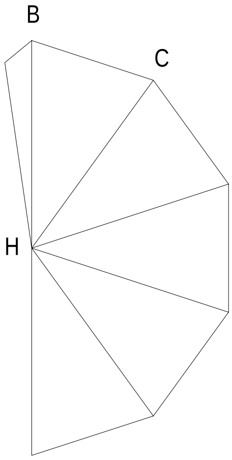
$$\varepsilon = \frac{Ce + EV - A}{2} = \frac{12\times 2 + 1\times 12 - 30}{2} = 3$$

To resume: polyhedron nº 3 has:

ε = 3 ; C = 12 ; e = 2 ; V = 12 ; E = 1 ; A = 30

What means that we have got a dodecahedron (C = 12) with stellate faces (e = 2) and convex pentahedral angles (E = 1); its species is ε = 3.

1



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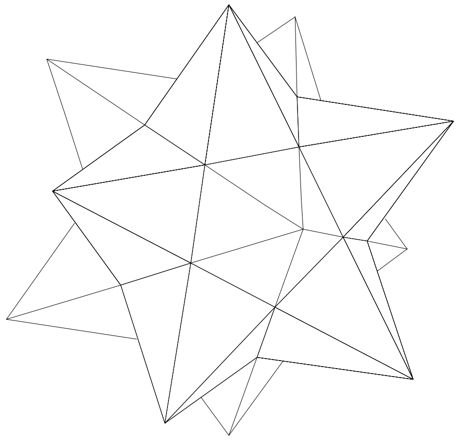


We shall raise now the following exercise: Let´s consider polyhedron nº 3 not as a stellate one, but as a mere irregular polyhedron in order to apply to it Euler´s basic theorem. In that case we shall have:

Faces: C = 5 × 12 = 60

Vertices: we shall add to the 12 extreme points, the vertices of the 12 pyramid´s bases, i.e. the 20 vertices of the auxiliary dodecahedron. V = 12 + 20 = 32.

Sides: The stellate polyhedron had 30 sides, but in the polyhedron we are considering now, each one of them produces 3: one on the base of the pyramid and two lateral sides of another two pyramids, hence: A = 30 × 3 = 90

Applying Euler´s basic theorem we have:

C + V = A + 2 ; 60 + 32 = 90 + 2

18.12.4 STELLATE REGULAR POLYHEDRON nº 4

The starting auxiliary polyhedron is the icosahedron shown in Fig. 1 as a wirework version. In it we can see the outlined triangular face ABC that is homothetic to triangle A´B´C´ (dashed line). The sides of this triangle are the diagonals of the pentagons that in turn have as side the icosahedron´s side L.

That homothety with center at H, is segregated in Fig. 2 where one can see how segments AA´; BB´; CC´ are also sides of the icosahedron.

Consequently, to construct the wanted polyhedron nº 4, all we need is 20 pyramids like HABC (Fig. 2), to set on the faces of the auxiliary icosahedron.

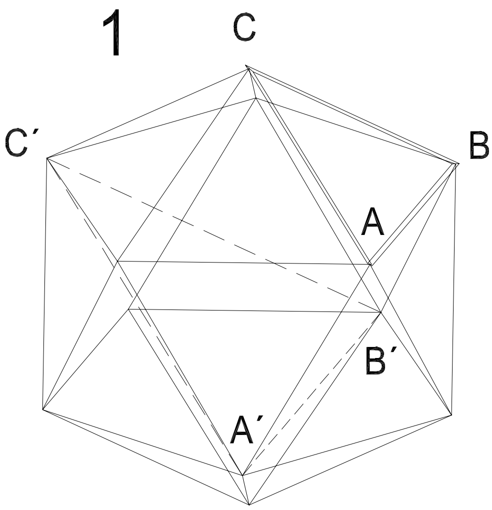
Let´s figure out the lateral side HB in the pyramid HABC (Fig. 2):

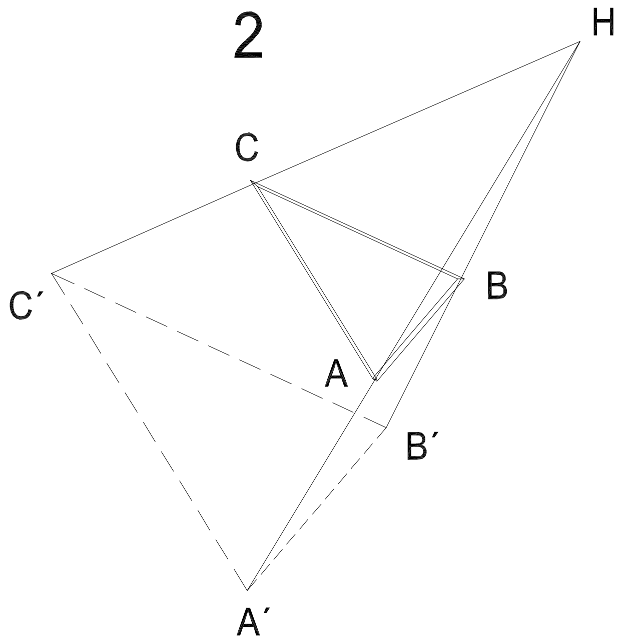
$$\frac{HB'}{HB} = \frac{B'C'}{BC} = 1{,}618034 \qquad \text{(Point 18.6.1)}$$

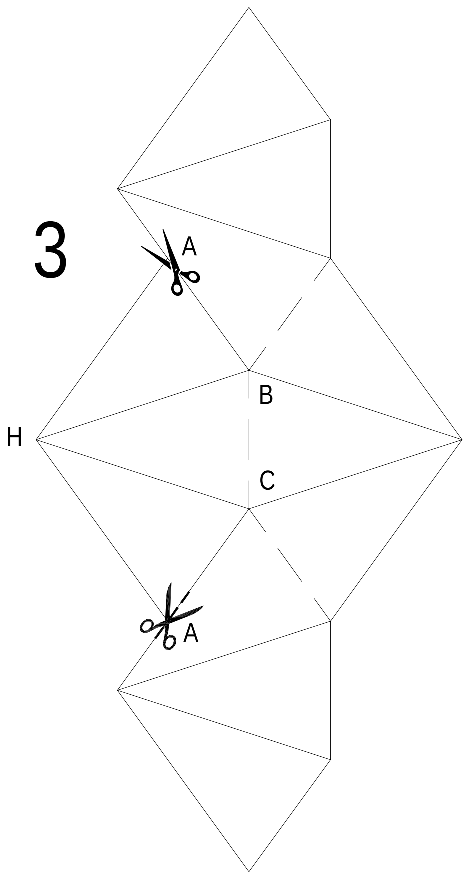
$$HB = \frac{HB'}{1{,}618034} = \frac{HB + L}{1{,}618034} \qquad \text{hence:} \quad HB = 1{,}618034\ L$$

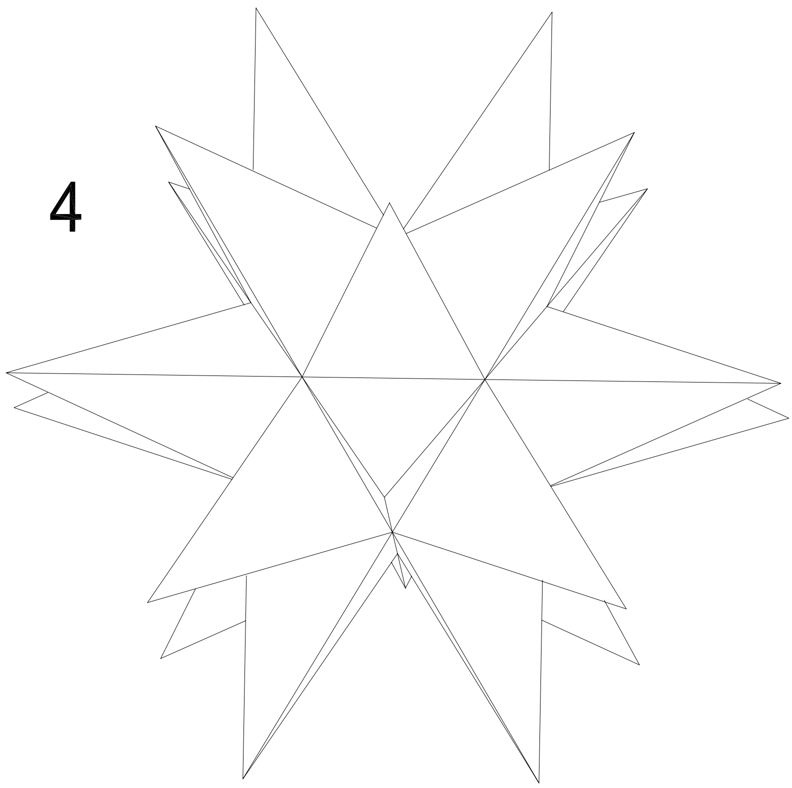
This means that the pyramid´s lateral side HB has equal length than the pentagon´s diagonal B´C´ (see Fig. 1), which reminds what happened to the pyramids of polyhedron nº 3.

Fig. 3 is the folding diagram to get 4 pyramids according to Fig. 2. We need 5 blocks like that of Fig. 3 to construct polyhedron nº 4. It happens, though, that the configurations of









Figs. 1 and 3 are not fully compatible despite of the obvious fact that $4 \times 5 = 20$. At the end of the building process we require to split out the last Fig. 3 in use to get a pair of leant against to each other pyramids and, in addition, two more isolated ones.

Finally we get polihedron nº 4 as shown in Fig. 4. To draw it we have used a very simple contrivance: each pyramid is symmetric to other adjacent to it, with respect to the plane formed by the base´s side they have in common and the center of the icosahedron.

In the polyhedron nº 4 we can see 20 vertices (V = 20) made of trihedral angles, i.e. of species E = 1 (obviously a trihedral angle cannot be stellate).

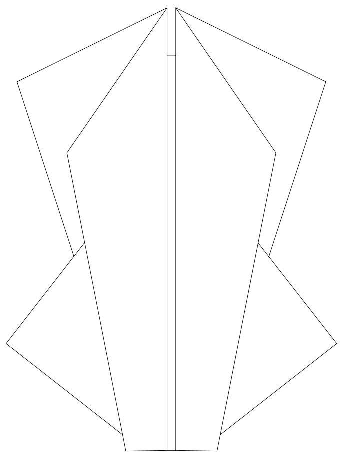
Out of its vertices start 3 faces that are stellate pentagons (e = 2): 3 × 20 = 60 planes. But as each of those planes are common to 5 vertices, we´ll have: 60 / 5 = 12 faces (C = 12).

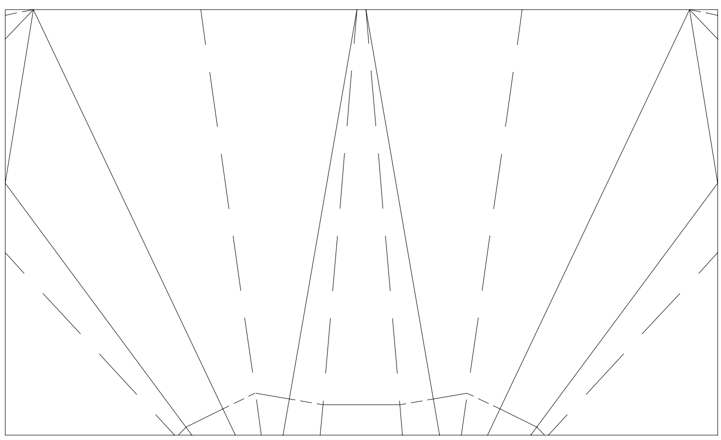
As far as the sides is concerned, since 3 of them start out of each vertex, and being each of them common to 2 vertices, the result is $A = \frac{3\times 20}{2} = 30$; (A = 30).

Summary: The regular stellate polyhedron nº 4 is a dodecahedron (C = 12) with stellate pentagonal faces (e = 2); 20 vertices (V = 20) which are convex polyhedral (E = 1) having a trihedral configuration; the resultant polyhedron has species ε = 7:

$$\varepsilon = \frac{Ce + EV - A}{2} = \frac{12\times 2 + 1\times 20 - 30}{2} = 7$$

Interlude





18.13 PSEUDORREGULAR POLIHEDRA

All their faces are equal (as in the regular polyhedra), or symmetrical; they are not regular, though: they are irregular polygons. We are going to consider two examples with a common octahedric base.

Another common feature is that they represent the crystalline structure (regular system) of some nesosilicates type $Me_2Me_3(SiO_4)_3$: Me may be Al, Fe, Mg, Cr.

18.13.1 RHOMBIC-DODECAHEDRON

In the Cosmo Caixa Museum at Alcobendas, Madrid, one can see a perfect example of a green garnet (there exist garnets of different colours) crystallised as a rhombic-dodecahedron.

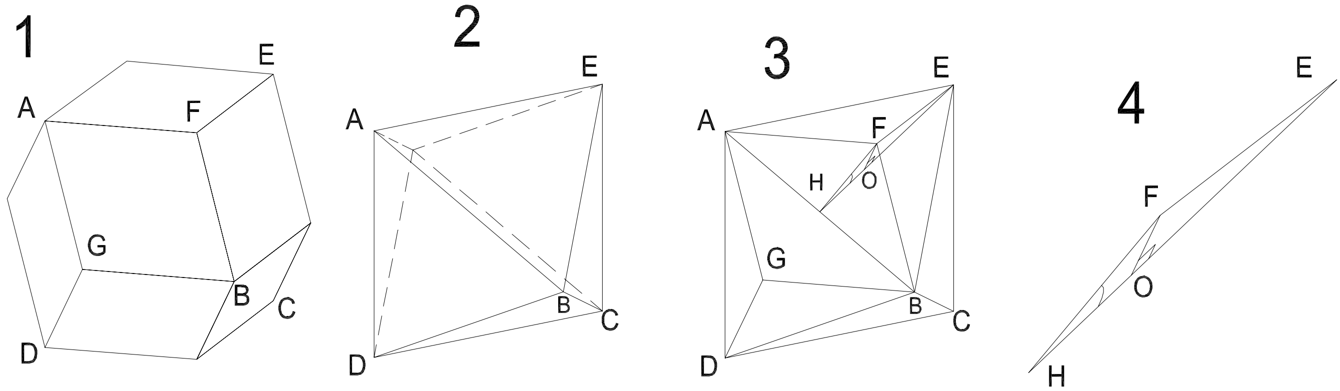
It is shown in Fig.1 and has the following characteristics:

V = 14; there are two groups of vertices: 6 acute polyhedral angles (4 faces) corresponding to the acute angles of the rhombic faces (and in direct relation to the 6 vertices of the basic octahedron). Besides, 8 obtuse polyhedral angles (3 faces) corresponding to the obtuse angles of the rhombic faces and closely related to the 8 faces of the basic octahedron.

C = 12; the 12 faces are equal rhombs with diagonals in the ratio of $\sqrt{2}$. All the dihedral angles formed by the faces are also equal. This leads to what could be named a *regular* rhombic dodecahedron. On the other hand, if large to small rhombic diagonals´ ratio is different from $\sqrt{2}$ but smaller than $\sqrt{3}$, we get the so-called *irregular* rhombic-dodecahedron: 10 equal basic faces plus 2, also equal to each other, but consisting in rhombs different from the other 10. This will hold true as long as the dihedral angles of those 10 faces will have the value of $2\arcsen\left(\frac{RHOMB\acute{}sGreaterDiagonal}{2RHOMB\acute{}sAltitude}\right)$.

A = 24; all the sides are equal to the rhomb sides. Therefore a side has one extremity on an acute vertex and the other on an obtuse one.

$$C + V = A + 2 \quad ; \quad 12 + 14 = 24 + 2$$



In Fig. 1, ABCDE are the viewed acute vertices of the rhombic-dodecahedron; F and G are obtuse vertices. Those 5 acute vertices have being segregated into Fig. 2 to set on the starting octahedron. Fig. 3 is the clue to understand the relation between the rhombic-dodecahedron and $\sqrt{2}$, and hence, with the DIN A rectangle. As we can see, Fig. 3 is Fig. 2 after adding to it the sides in F and G of Fig. 1. By means of that we have constructed the triangular pyramids F ABE and G ABD. The former is cut by the plane EFH (normal to diagonal AB): in it is represented the altitude FO of said pyramid. Fig. 4 is a partial enlargement of Fig. 3 to allow calculation of angle in H.

To figure out that angle, let´s set up first the hypothesis that a rhombic face has L as the great diagonal and L / $\sqrt{2}$ as the small one. In that case, we can write (Figs. 3 and 4):

$$Ang.H = \arccos\left(\frac{OH}{FH}\right) = \arccos\left(\frac{L\sqrt{3}}{2}\,\frac{1}{3} : \frac{L}{\sqrt{2}\times 2}\right) = \arccos\left(\frac{\sqrt{3}\times 2\sqrt{2}}{6}\right) = 35{,}26439^{o}$$

If the hypothesis holds true, the half-rhombs AFB and AGB (Fig. 3) will be coplanar, i.e., Ang. FHG = 180º.

Recalling (Point 18.9.5), that the dihedral angle between faces of an octahedron (in this case that of side AB in Fig. 2) measures 109,47123º, mentioned angle Ang. FHG will measure:

$$109{,}47123 + 2 \times 35{,}26439 = 180$$

what proves that the hypothesis is true.

Finally, let´s construct a paper rhombic-dodecahedron by three different methods. In all the cases we shall use rhombs obtained from DIN A rectangles. Fig. 5 shows how to fold one of those rectangles to get a rhomb whose diagonals keep the ratio $\sqrt{2}$. Fig. 6 is the disposition to be given to the 12 rhombs that will form the polyhedron. Fig.7 suggests a modular solution with interlocked rhombs.

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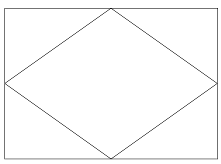

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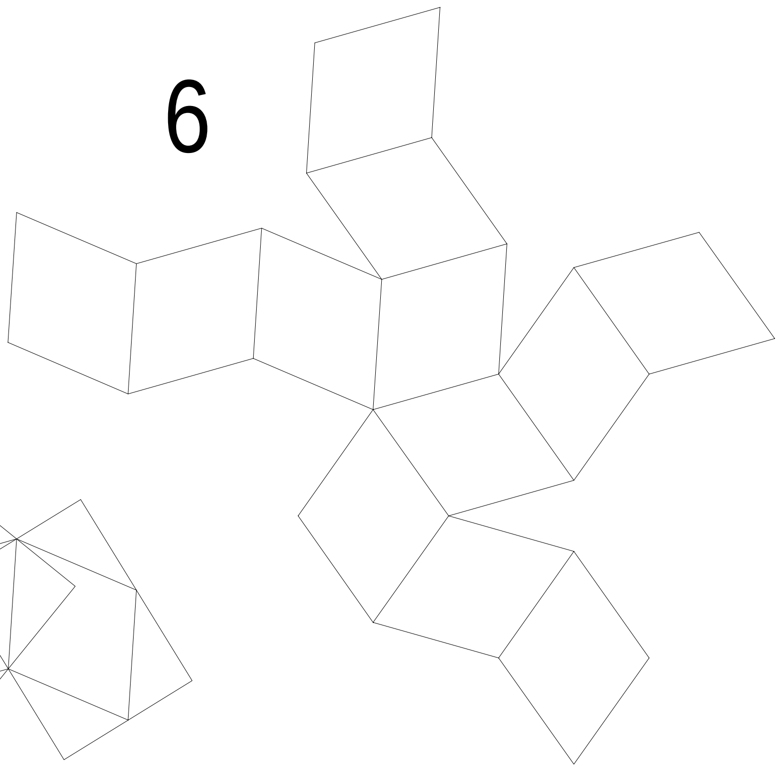

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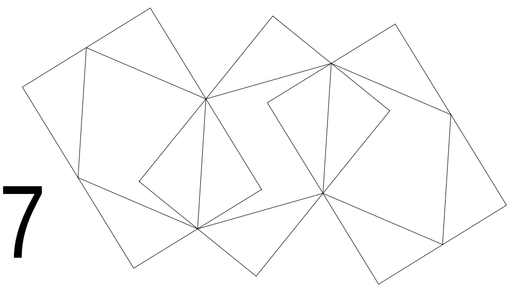


Recall Point 18.9.4 to perform that interlocking. Fig. 5 is a module and we need 4 groups of 3 interlocked modules like the one in Fig. 7. Then interlock the 4 groups with each other.

Before proposing the third method, we shall figure out the value of the dihedral formed by two rhombic-dodecahedron´s faces. Fig. 8 is the same Fig. 1 after its side FB has been cut by a plane normal to it through G (Fig. 1). So, in Fig. 8 A´E´ is equal to the diagonal AE (parallels between parallels) and A´J = E´J (altitudes of two adjacent rhombs); therefore, Ang. A´JE´ measures the wanted dihedral.

If we assume that the dimension of a rhomb´s small diagonal is 1, the great will measure $\sqrt{2}$ and the side

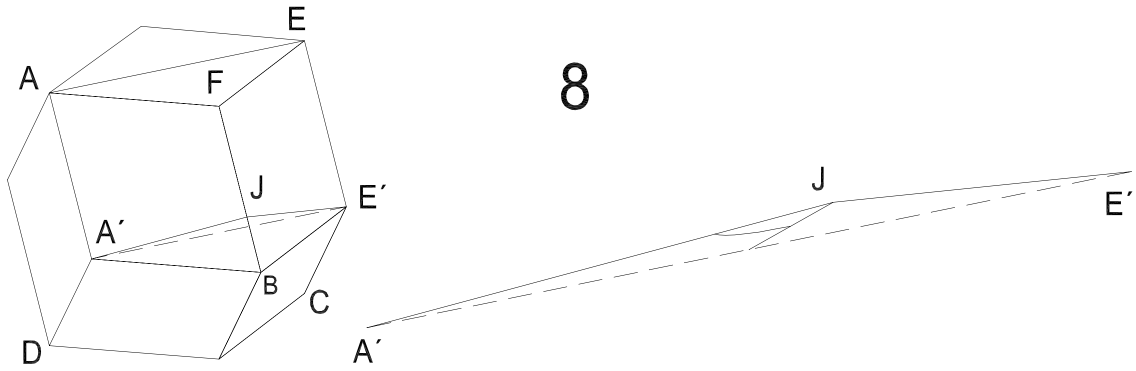
$$\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}=0{,}8660254$$

On the other hand the rhomb´s altitude is equal to its area divided by its base (a side)

$$A'J=\frac{1\times\sqrt{2}}{2\times 0{,}8660254}=0{,}8164965$$

Hence the measure of angle A´JE´ will be:

$$Ang.A'JE'=2\,\mathrm{arcsen}\left(\frac{\sqrt{2}}{2}:0{,}8164965\right)=120^{\circ}$$



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We must say that Fig. 5 is not the only way to get a “DIN A” rhomb. Another procedure was already disclosed in Fig. 1, Point 18.2.5: one is as good as the other, but we have to bear in mind that starting from rectangles of the same size, we end up with distinct, tough similar, rhombs: $\sqrt{2}$ is the ratio of similarity.

Carrying on with Point 18.2.5, we saw there that 60º was the angle formed by the base of the pyramid and one of its lateral faces. It means that if we join together two of those pyramids by setting in common two lateral faces, the rhombic bases will form an angle of 120º in the new figure.

In consequence, that new figure holds two rhombic faces of a rhombic-dodecahedron since we got to know that 120º is also the angle formed by two faces of this polyhedron. Besides, the common vertices of those two pyramids coincide with the center of the rhombic-dodecahedron: of course, the radii of the rhombic-dodecahedron are different depending whether we join its center with an acute or an obtuse vertex. It happened in the rhombic pyramid as regards to its lateral sides.

Fig. 9 shows the 4 pyramids with center O (that of the rhombic-dodecahedron) associated to the acute vertex C of Fig. 1. At left we can see in shade the only seen rhomb.

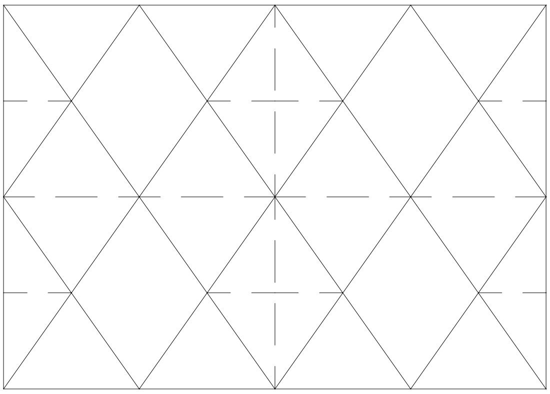
To construct the polyhedron we ought to have 12 pyramids, which, in turn, are obtained this way:

From Fig. 10 we get an acute vertex formed by 4 rhombic faces: it is shown in Fig. 9.

Folding Fig. 11 we get two more pyramids; it´s a matter of redoing this folding four times to complete the 12 pyramids we need.

Two comments: first, the two triangles forming the facial rhombs seem to be equilateral but they are not; second, the rhombic-dodecahedron obtained in the last process is more consistent than the others since its interior is reinforced by 12 pyramids.

10



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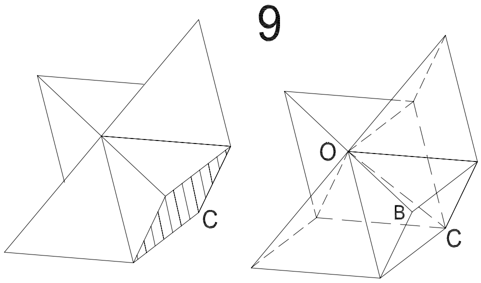

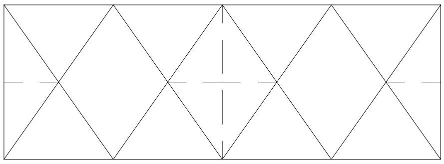

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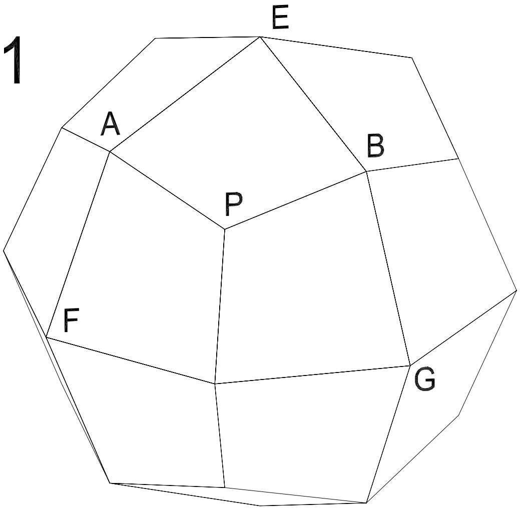
TRAPEZOHEDRON

It is a pseudorregular polyhedron with 24 trapezoid shaped faces, 26 vertices and 48 sides (half of them large and half small). So it fulfils Euler´s theorem: 24 + 26 = 48 + 2.

It is also a form under which the pyrite (iron disulphide S_2Fe) crystallises.

Out of the 26 vertices, 18 are equidistantly inscribed on the three orthogonal maximum circles of a sphere. That sphere circumscribes an octahedron whose 6 vertices are among the 18 mentioned above. These 18 vertices are equidistant from the center of the polyhedron: that distance R is greater than the other 8 vertices´; these 8 vertices are associated to the 8 faces of the octahedron.

1



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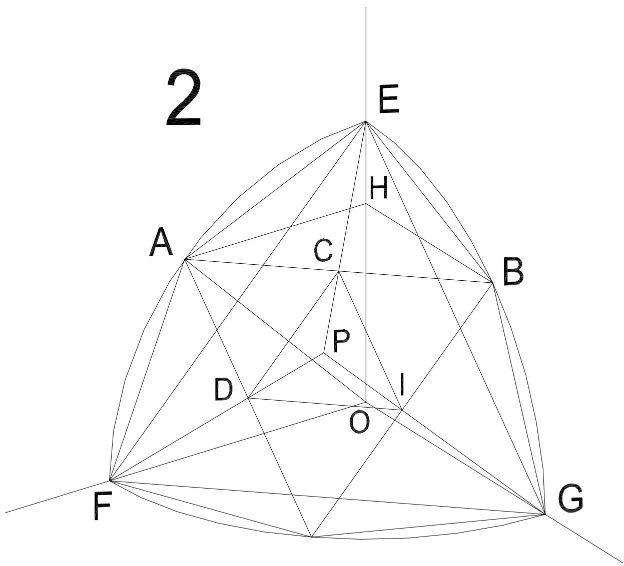


The dihedrals formed by the polyhedron´s faces are all equal (regardeless of the length of their sides) and its measure is 138,118º.

The polyhedron is shown in Fig. 1; we can see in it the section EAFGB that has been segregated and taken into Fig.2: it is integrated there within one of the 8 co-ordinate trihedrals. It follows that the polyhedron results inscribed in a sphere of radius R = OE = OF = OG = OA = OB. Let´s determine the face AEBP as a function of R; if R = 1, R_A = 1 and R_P = 0,9473.

ΔAHB and AHO in Fig. 2 are congruent for they are both isosceles right triangles with a leg in common. Therefore, their hypotenuses will be equal: AB = AO = R = 1.

$$AC = CD = DA = \frac{AB}{2} = \frac{1}{2}$$

Thus we have settled ΔCDI which is homothetic to ΔEFG. Let´s find now the center of homothety P.

As $FE = \sqrt{2}$ and the angle in A of a regular octagon measures 135º, we can write:

$$AE = \frac{\frac{\sqrt{2}}{2}}{\operatorname{sen}\frac{135}{2}} = 0{,}7653668$$

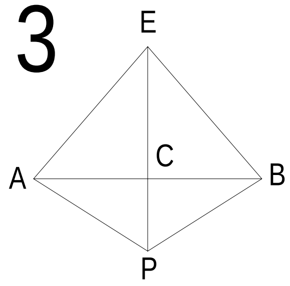
The ratio of homothety being expressed by:

$$\frac{FE}{CD} = \frac{EP}{CP} \quad ; \quad 2\sqrt{2} = \frac{EC + CP}{CP} \quad ; \quad CP = \frac{EC}{2\sqrt{2} - 1}$$

$EC = \sqrt{AE^2 - AC^2} = 0{,}5794708$; hence CP = 0,3169231

With the information gathered till now we can draw Fig. 3 that is a face of the trapezohedron: we know its two diagonals and the intercepts of both of them. The sides of the trapezohedron are AE (already known) and AP whose value is:

$$AP = \sqrt{AC^2 + CP^2} = 0{,}5919799$$



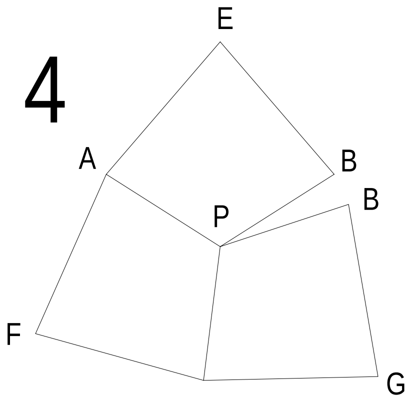
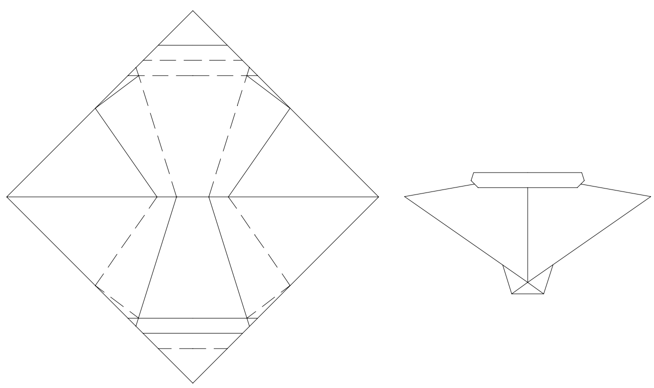


Fig. 4 is the folding diagram of one of the 8 sections that make up the whole trapezohedron (recall Fig. 2).

We must say that the trapezoid of Fig. 3 does not enjoy the auric proportion seen in the Penrose tesserae (case 1B, Point 12).

Interlude

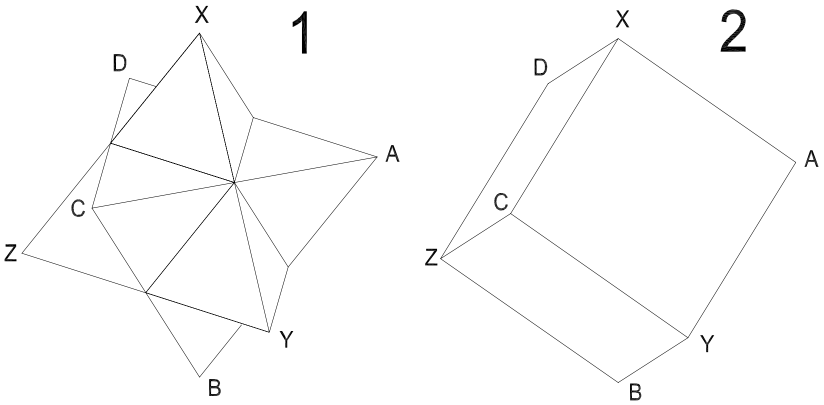


18.14 MACLES

In crystallography are so named the twinned crystals oriented in such a way that one can overlap the other if properly moved, rotated or subjected to a symmetry. The last two operations have as reference what are called macles´ planes or axles. As a matter of fact, macles are steps in a crystal growing process.

18.14.1 TETRAHEDRIC MACLE

It is shown in Fig. 1. In it, two equal tetrahedra inter-penetrate each other: one is the ABCD; of the other we can see its vertices XYZ.



It seems to be a stellate polyhedron, but is not (see Point 18.12). One of the tetrahedra may become the other under this process: to get its symmetric with respect to one of its faces, to move 1 / 2 of its altitude and rotate it 60º.

The macle´s folding diagram is not shown for we consider it made up of a big tetrahedron and four small ones centred on the faces of the big; the small tetrahedra have a side half of the big´s. See Point18.2.1 to construct a tetrahedron: it is obvious that the paper strip width to produce the big tetrahedron is double than the one needed for the small ones.

The macle has $4 \times 2 = 8$ vertices belonging, in turn, to a cube (Fig. 2). If L is the side of the big tetrahedron, the cube´s side is the distance between two opposite sides of that big tetrahedron: its value is $\frac{L\sqrt{2}}{2}$ (see Point 18.2.1.1).

The blende, zinc sulphide (SZn), crystallises in the cubic system, tetrahedric mode, polysynthetic macles.

18.14.2 MADE OF CUBES

Figs. 1 and 2 show the aspect of a multiple macle of these characteristics:

- Big cube of side AB = L.
- Small cubes with sides $\frac{L}{\sqrt{2}}$.
- Their relative position is such that section ABC is an equilateral triangle.

The conditions that follow define the small emergent cube whose folding diagram is Fig. 3:

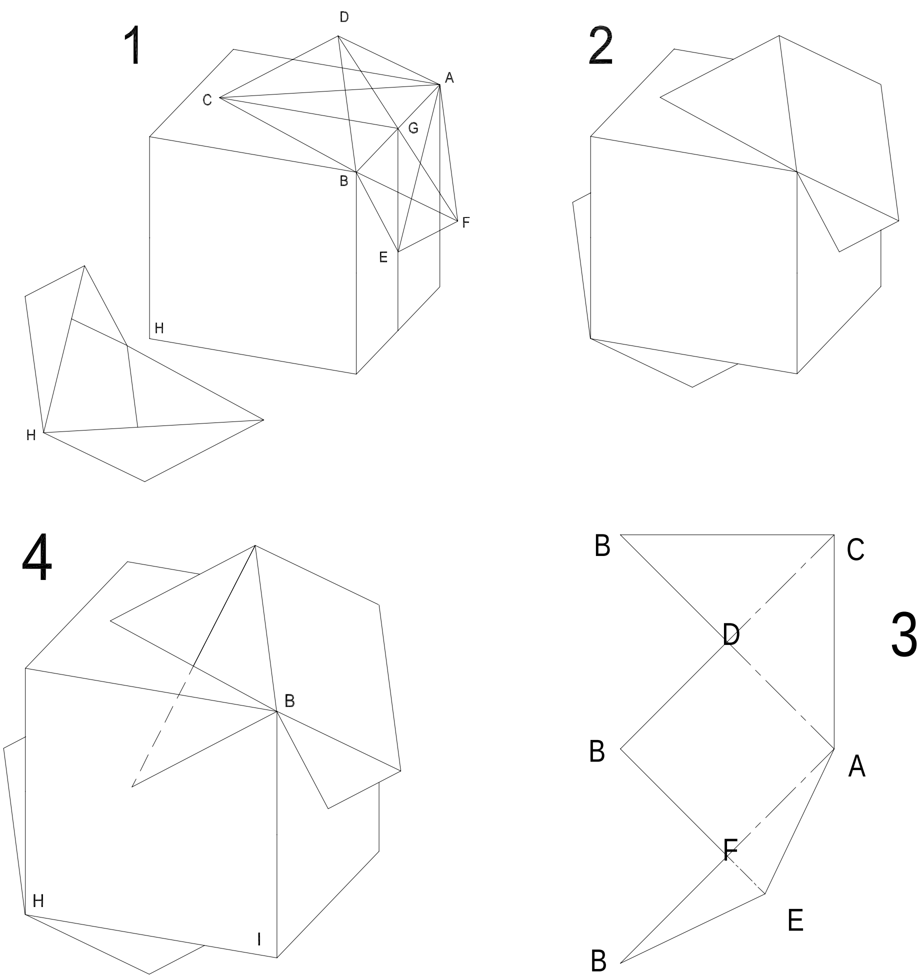
- ☐ AD = $\frac{L}{\sqrt{2}}$ = 0,7071067 L is the side of square ADBF.
- ☐ Triangles BCD, ADC and ABD are isosceles, congruent and right-angled (Point 18.2.1.2); their three sides are given: hypotenuse CA = L, and both legs AD = $\frac{L}{\sqrt{2}}$.
- ☐ Triangles CDG and GFE are similar: all their angles are, respectively, equal:

  $\frac{CD}{DG} = \frac{GF}{FE}$; DG = GF ; $FE = \frac{DG^2}{CD} = \frac{L^2}{4\frac{L}{\sqrt{2}}} = L\frac{\sqrt{2}}{4}$ = 0,3535533 L

- ☐ Triangles BFE and AFE and properly defined: the former is right angled in F; BF = $\frac{L}{\sqrt{2}}$;

  $FE = L\frac{\sqrt{2}}{4}$, hence $BE = \sqrt{\left(\frac{L}{\sqrt{2}}\right)^2 + \left(\frac{L\sqrt{2}}{4}\right)^2} = L\frac{\sqrt{10}}{4}$ = 0,7905694 L

- ☐ With all that information we can draw Fig. 3 which is the folding diagram of the cube emerging through the dihedral of side AB.

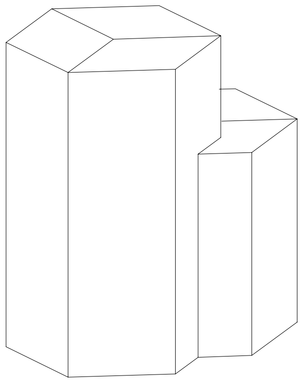


- ☐ Revolving that emerging cube 180º around side AB, and moving it, we get another small emerging cube; this is shown at bottom left of Fig. 1 before its association to the big cube by coincidence at H.
- ☐ Fig. 4 shows how a further growth of upper small cube will determine the appearance of that cube through the face BHI.
- ☐ We have chosen a simple macle in order to ease the construction. To fold the big cube, see Point 18.8.1.
- ☐ The fluorite, consisting of calcium fluoride (CaF_2) occurs in beautiful twin cubic crystals of various morphologies.

18.14.3 ARAGONITE

Calcium carbonate ($CaCO_3$) discovered in Molina de Aragón, Spain. It crystallises in macles of hexagonal aspect, though it consists in orthorhombic crystals, see Fig. 1. Fig. 2 is the macle´s folding diagram.

1



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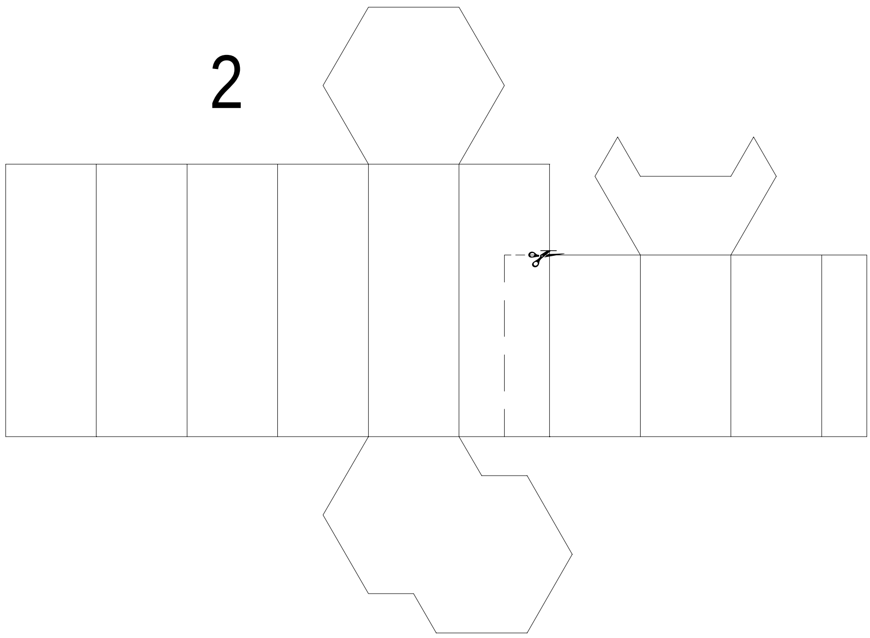
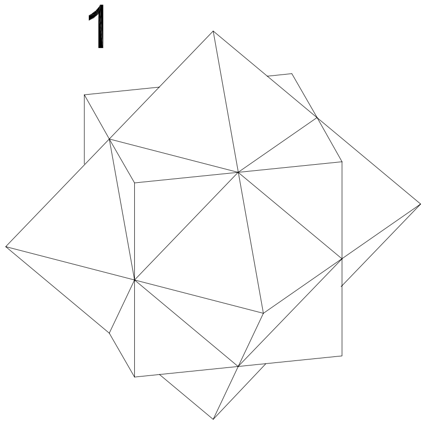


18.14.4 CUBE-OCTAHEDRIC MACLE

1



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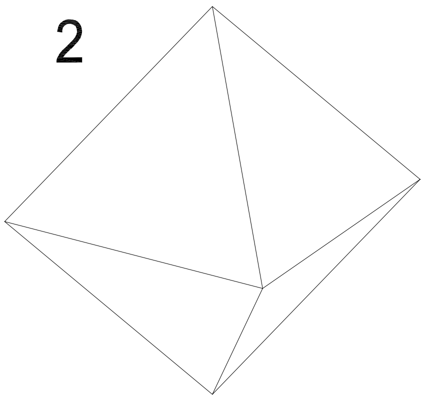

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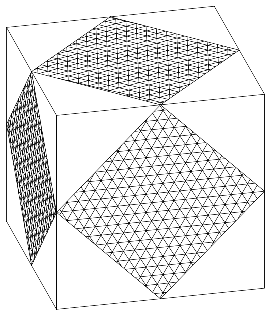
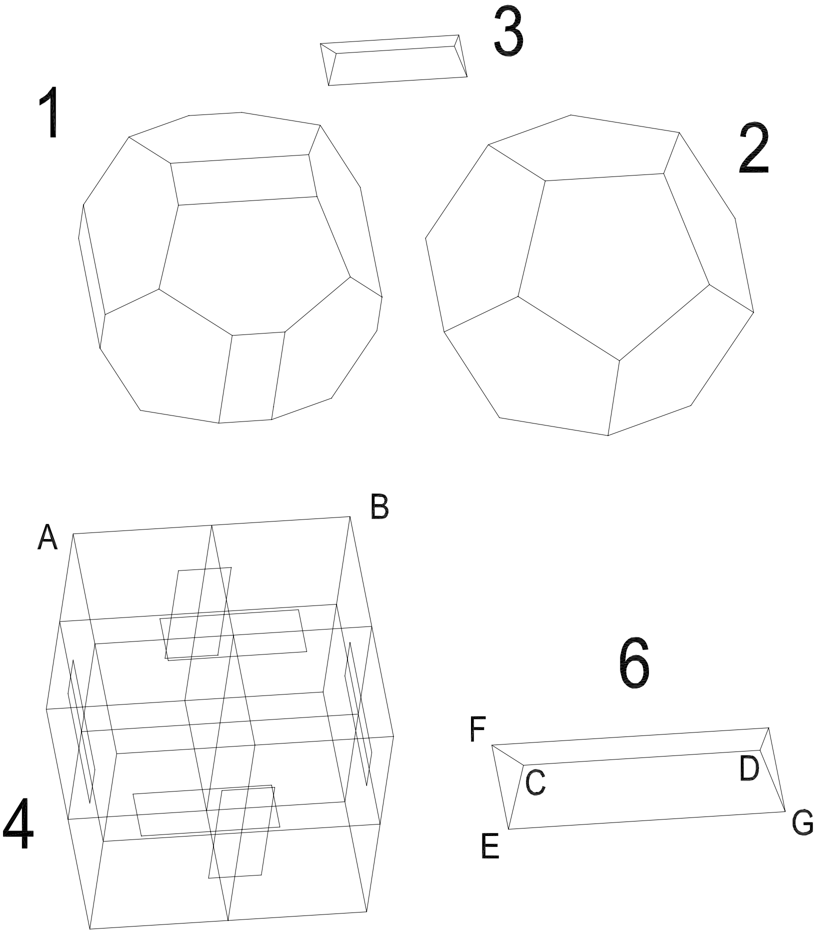


Fig. 1 shows an example made out of one octahedron of side L to Fig.1 (Point 18.9.1) plus 8 tri-right-angled vertex pyramids (Point 18.2.1.2), centered on the corresponding faces of the octahedron. Fig. 3 evidences those pyramids associated to the reference cube (the interior of the cube appears in shade to set off those pyramids on its vertices). From the figures we can deduce the measure of cube´s side: $\frac{L}{\sqrt{2}}$.

The galena, lead sulphide (PbS), among a great variety of combinations (cubes, octahedra, rhombic-dodecahedra), occurs in macles of tabular aspect frequently with the shapes of octahedra and hexahedra.

18.14.5 PYRITOHEDRON nº 1

The pyrite crystallises in the regular system mainly in shape of cubes, pentagonal-dodecahedra or in combinations of both of them. At least 25 different shapes are known.



The first we are going to deal with is shown in Fig. 1 and is the result of chamfering the sides of a pentagonal-dodecahedron like the one in Fig. 2. Discarded wedge shaped chamfers are represented in Fig. 3. One can see that the chamfer may be performed in many ways, but in this occasion we have chosen that which produces in the wedge an oblique section like the triangle ABC of Fig. 1 (Point18.6.1).

If we look at present Fig. 1 we´ll see that it consists in 18 faces: 12 equal irregular hexagons coincident with the faces of the polyhedron of Fig. 2, and 6 equal rectangles, which are the bases of the respective wedges.

Those 6 rectangles are centered on the faces of a cube as can be seen in Fig. 4. Each pair of opposite rectangles is homothetic and determines two planes parallel to the sides l in Fig. 4 right, Point 18.6.1; l is the pentagon´s side of the dodecahedron in Fig. 2.

The distance between those planes is the side AB of the cube in present Fig. 4. We can figure it out from Fig. 5 that is an enlarged detail of said Fig. 4 right.

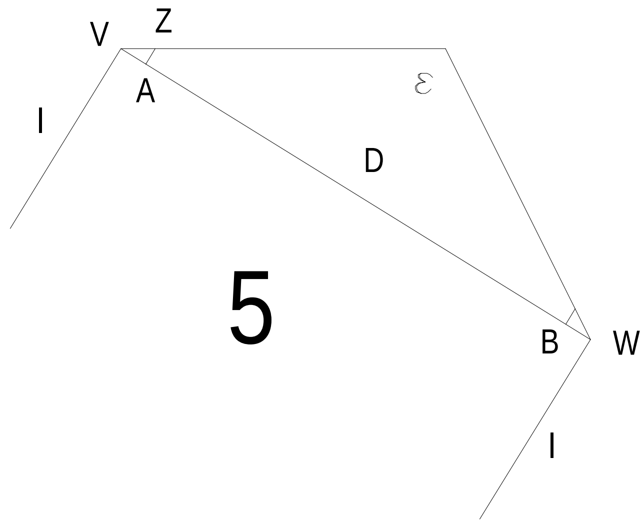
$$AB = D - 2\times FD\cos\left(90-\frac{\varepsilon}{2}\right) = (verPunto 18.6.1) =$$
$$= l\left(2{,}618034 - 2\times 0{,}1624599\cos\left(90-\frac{116{,}56505}{2}\right)\right) = 2{,}3416407l$$

After what we have seen, it is evident the coexistence in the macle of a hexahedron and a pentagonal-dodecahedron.

Before drafting the pyritohedron´s folding diagram (Fig. 7), we shall dig out on how the wedge of Fig. 3 is constructed; see enlarged Fig. 6:

CD = l = side of the pentagonal-dodecahedron.
Ang. ECD = Ang. ECF = Ang. FCD = 108º
CE = CF = DG = AC (Point 18.6.1) = 0,2763932 l
EG parallel to CD

Now we can draw Fig. 8 that is the folding diagram for a hemi-pyritohedron; we require two of them to be mounted in opposition. As each produces 4 rectangles and we only need a total of 6, eventually we must discard 2 of those rectangles.



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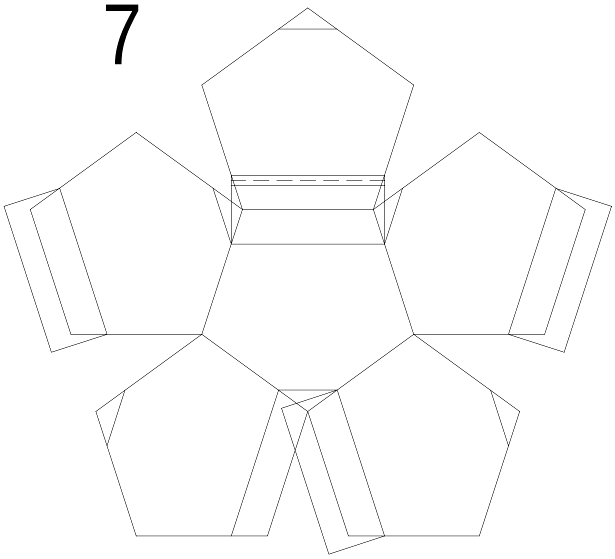
Previous to Fig. 8 we have drawn Fig. 7 which is one hemi-dodecahedron´s folding diagram including the lines needed to transform the dodecahedron into a pyritohedron. In Fig. 8 have disappeared the lines of Fig. 7 not needed anymore. Note the required pleat fold of Fig. 8.

7



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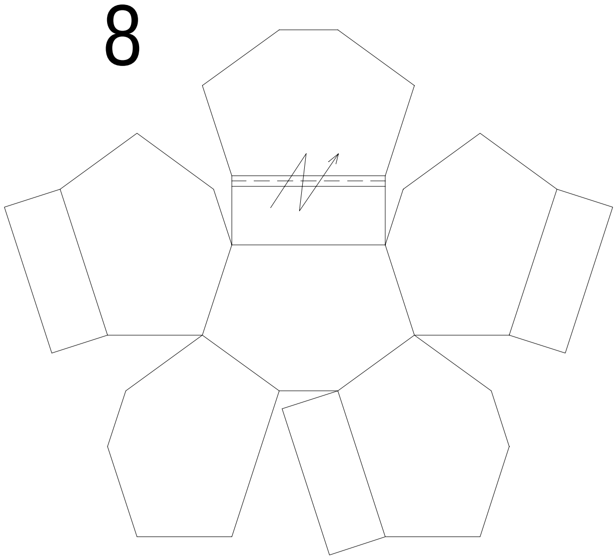
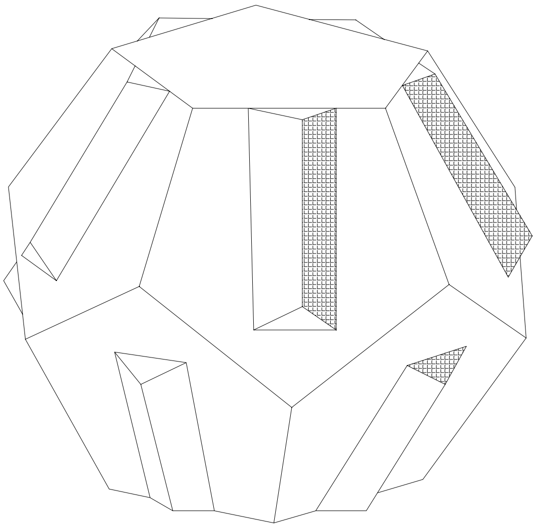


18.14.6 PYRITOHEDRON nº 2

This pyritohedric macle consists in a couple of pentagonal-dodechedra keeping the following relations (Fig. 1):

1



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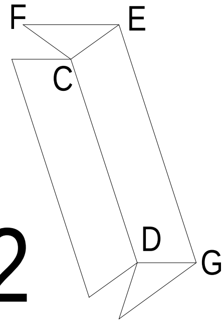


Each pentagonal base (upper and lower) is in parallel planes. One polyhedron becomes the other by rotating the former 36º around the axis determined by the centers of those bases (see Point 18.6.1). By so doing we may see how the second polyhedron emerges through the first one´s faces.

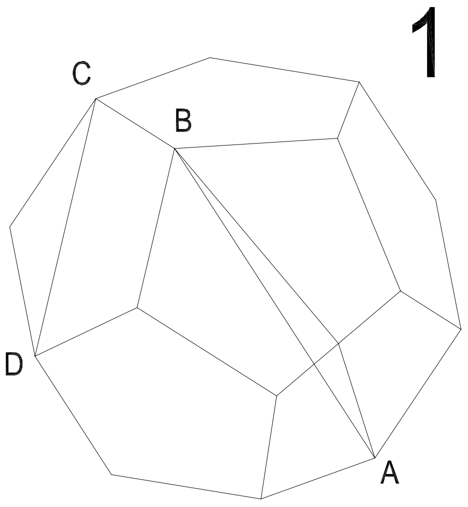
Fig. 1 shows in blank the first polyhedron whereas the second looms up: remark in shade what is partially seen of one of the 10 lateral pentagonal faces of that second polyhedron.

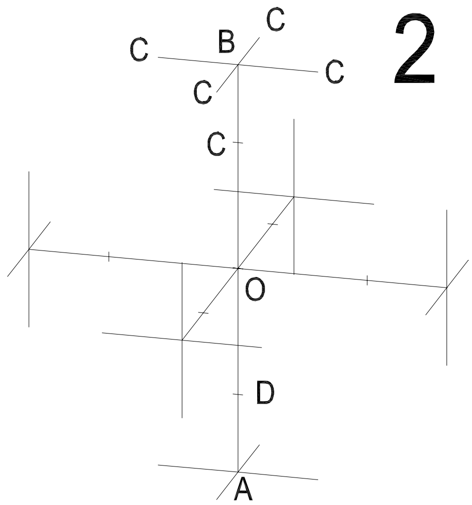
At the end, what looms up out of the first dodecahedron are 10 wedges (actually, truncated triangular prisms) like those of Fig. 6, Point 18.14.5. We are using now the wedges then discarded.

Present Fig. 2 is the folding diagram of the wedge, once we have available all the information needed (see Point 18.14.5). To note that the three angles in D are pentagonal of 108º.

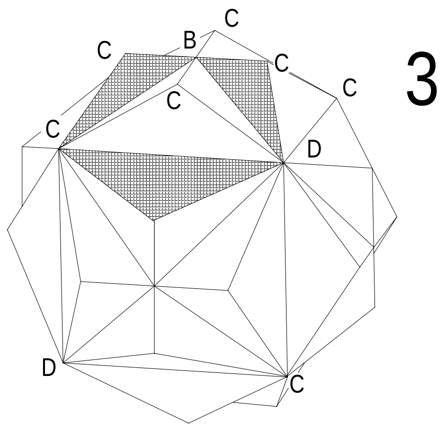
18.14.7 THE IRON CROSS

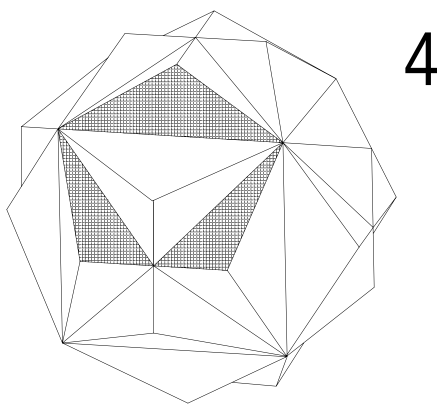
Also known as IRON ROSE, is a complement macle. The pyrite often crystallises in that mode.





To study it we shall start with the pentagonal-dodecahedron of Fig.1 to be associated to Point 18.6.1: here, AB will be what was called there great diagonal D; obviously, BC is the side l of the polyhedron, and CD is one of the facial pentagon´s diagonals then named d.





In Point 18.6.1 we saw that D – d = l, which is the peculiar property we are going now to take advantage of. It will permit us to draw Fig. 2 that consists of a set of trirrectangular crosses of these characteristics:

- Its center O coincides with the dodecahedron´s.
- The great arms type AB have the measure of AB (Fig. 1; the great diagonal AB mentioned before).
- The segments type DC measure as much as the diagonal DC of Fig. 1; they will determine a cube type CDCD (see Fig. 3).
- The small crosses type BCCCCC have their arms made by segments half of the side BC in Fig. 1.

- Joining the extremities of the small crosses and its center B to the vertices of the mentioned cube, we get the macle as can be seen in Figs. 3 or 4.
- In both Figs. 3 and 4 we can observe the shaded pentagonal faces forming both dodecahedra, direct and inverse.

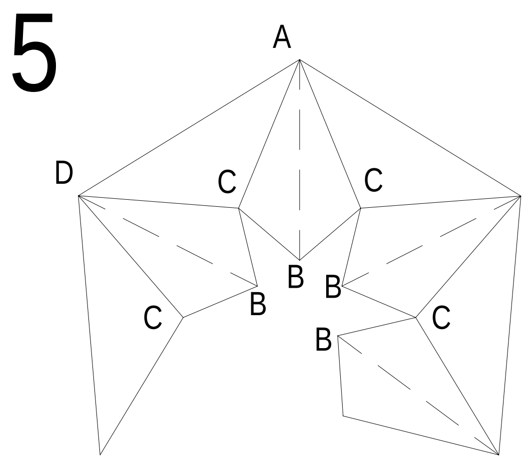


Fig. 5 is the folding diagram pertaining to one of the 6 bodies to be attached to the respective faces of the cube already mentioned.

Every triangle in Fig. 5 is determined:

CA = CD = l, is the side of the original polyhedron.

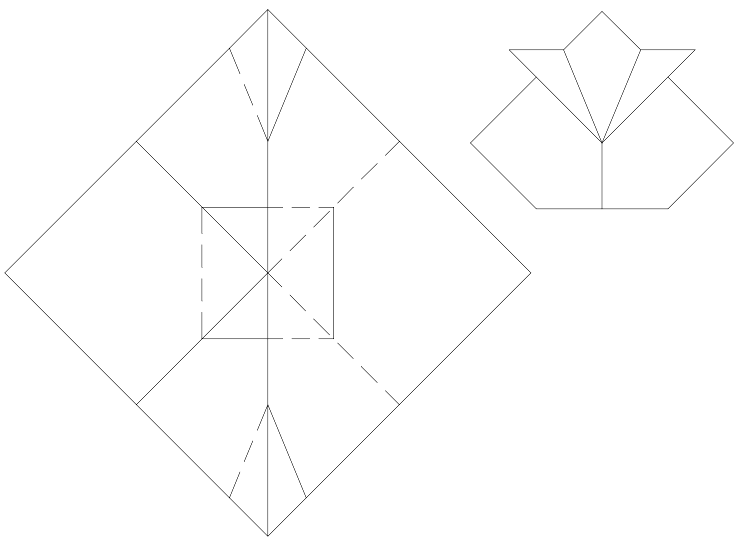
AD = 1,618034 l, is the facial pentagon´s diagonal: it will be the side of the starting cube to be previously constructed.

CB = l / 2

AB = 1,248606 l (GI of Point 18.6.1; see also DB in Fig. 3).

To finish, we must insist that to materialise the Iron Cross, the starting polyhedron we want is a cube with faces CDCD rather than a dodecahedron.

Interlude

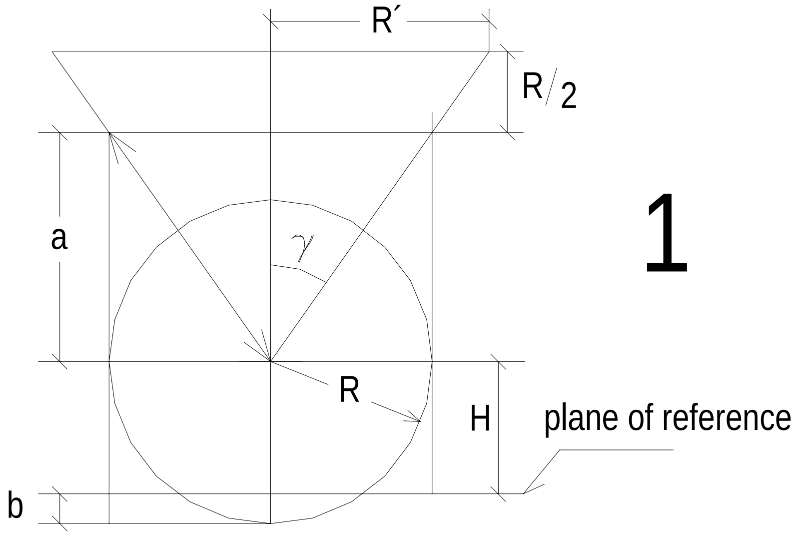


19 ROUND BODIES

Strictly speaking they are the cylinder, cone and sphere. We shall construct them by means of interlocking (fix them with adhesive tape if needed, the cylinder in particular). Just only a paper flexion is required to get cylinder and cone.

Cylinder and cone bases will be virtual (intersections between each other or with the plane of reference). The sphere will be in the laminar mode to Donovan A. Johnson´s design, hence, virtual (recall Point 18.8.5 as an analogy).

We shall associate the three bodies to form a geometric set. Fig.1 is a section of that set whose characteristics are based on the fact that the laminar sphere has R as radius.



Here we have those characteristics:

- The sphere rests on the reference plane: it looks like flattened in a value b.
- A well-fit cylinder that in turn rests on the plane of reference too, will cover the sphere.
- An inverted cone, whose vertex coincides with the sphere´s center, is tangent to the three trirrectangular circles defining the sphere. Besides, it intercepts the cylinder just at its upper base. The cone base is situated at a distance R / 2 over the mentioned cylinder´s upper base.

SPHERE

As said, its radius is R and will consist in three circles also of radius R (Fig. 2). It is required to perform the cuts and folds as indicated and then to introduce circles b and c into the a. We must recall that circle b should take a square shape to go into circle a. Finally undo the foldings and dress the set. Three circles that intersect each other in a trirrectangular mode define that spherical set.

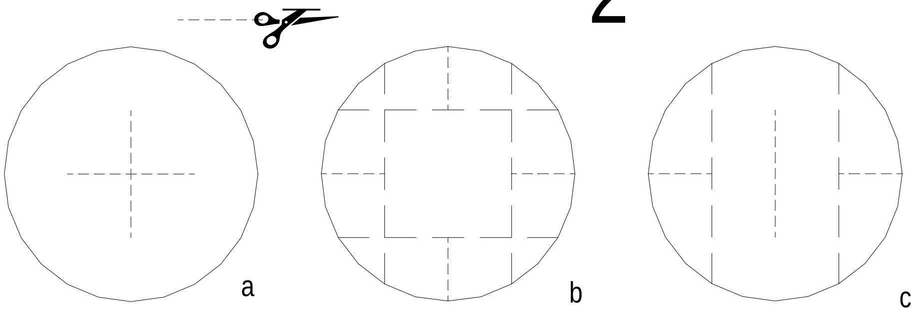
CYLINDER

Fig. 3 is the cylinder development with the indicated dimensions.

CONE

Idem cylinder (Fig. 4).

2



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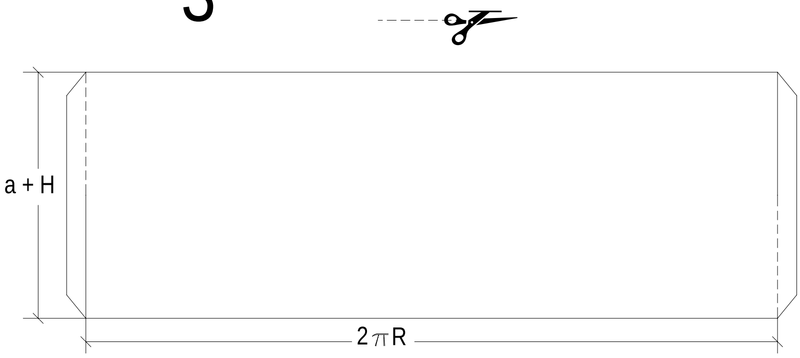

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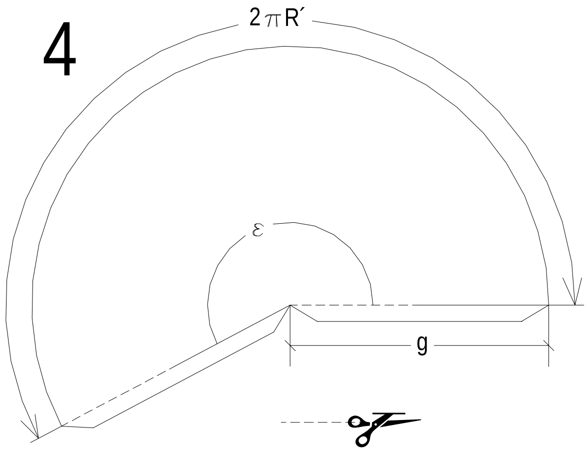
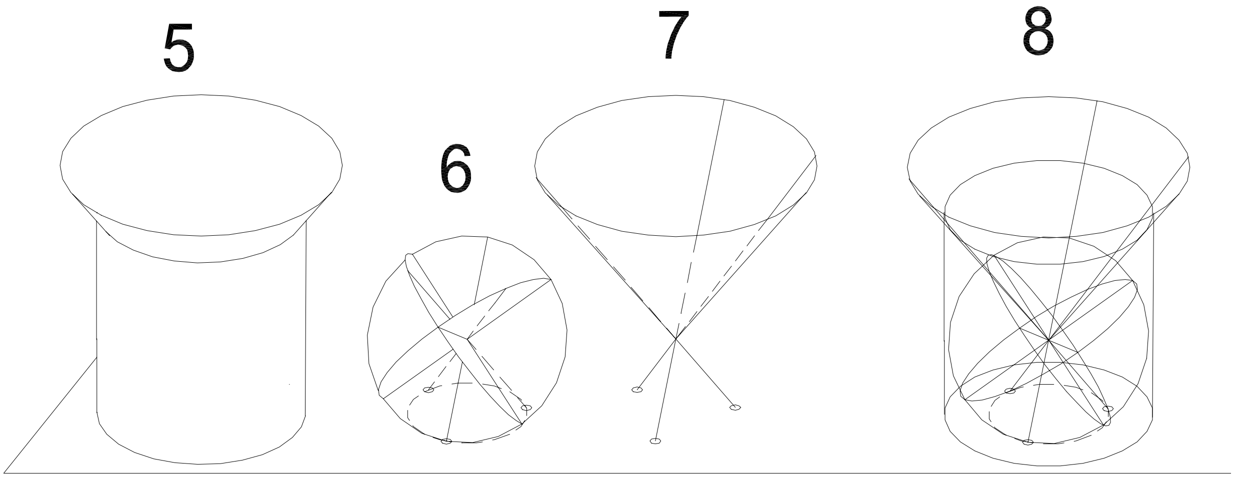


The relations between the magnitudes we have to play with are disclosed below. We need them to construct the round bodies set.



- H is the altitude of a tetrahedron of side R (see the tetrahedron with vertices OAB in Point 18.13.2); its value is:

$$H = R\sqrt{\frac{2}{3}} = 0{,}8164965R \qquad \text{(see Point 18.2.1.1)}$$

- Therefore, flattening b is:

$$b = R - H = 0{,}1835035R$$

- Cone´s semiangle γ is the complement of angle β as calculated in Point 18.2.1.1:

$$\gamma = 90 - 54{,}73561 = 35{,}26439^{\circ}$$

- An angle of 54,73561º is also formed by any of the three circles defining the sphere, and the reference plane. The sphere seats on this plane through the dashed circle shown in Fig. 6.
- In order to have the cylinder just reaching the cone surface, the cone´s altitude *a* up to the upper base of the cylinder will be:

$$a = \frac{R}{\operatorname{tg}\gamma} = R\sqrt{2} = 1{,}4142135R$$

- The radius of the cone´s exterior base has to be:

$$R' = \left(a + \frac{R}{2}\right)\operatorname{tg}\gamma = 1{,}3535534R$$

- Cone generatrix is:

$$g = \frac{R'}{\operatorname{sen}\gamma} = 2{,}3444232R$$

- The developed angle of the conic surface is (Fig. 4):

$$\varepsilon = \frac{2\pi R'}{g}\,radians = \frac{2R' \times 180}{g} = 207{,}8461^{\circ}$$

- Total altitude of cylinder:

$$a + H = 2{,}2307101R$$

- The circumference length of cylinder base is:

$$2\pi R = 6{,}2831853R$$

Fig. 5 is the final set as seen from the outside. Fig. 6 represents the sphere with its seating circle as explained above. Fig. 7 shows the cone with its three generatrices that are the lines of tangency to the three circles of the sphere: they are, in turn, three sides of the abovementioned tetrahedron. Finally, Fig. 8 has in it all the lines required for full representation; as we can see, there are many of them: that is the reason why it has been decomposed in the previous figures.

20 PAPER FLEXIBILITY

It seems reasonable to touch upon this matter, for it is inseparable from origami. I do dare say that along this book we only have dealt with paper flexibility, strictly speaking, but in three occasions: while studying Möbius bands (Point 14.1), and in former Point 19 when constructing the cylinder and the cone.

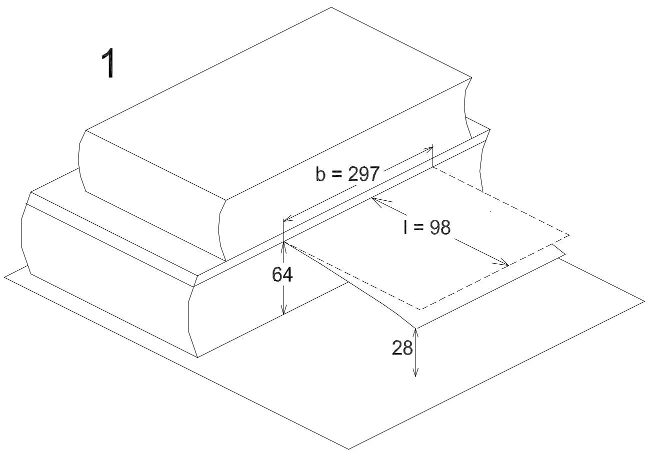
The Spanish word for origami (paperfolding) is *papiroflexia* but in my opinion it would be worthwhile coining the Spanish neologism *papiroplegia*, which is closer to folding than to flexing. Of course I am aware that the *flexia* (flexing) is always previous to the *plegia* (folding), but at the same time it seems a misuse to take the latter for the former indistinctly.
Note: The English suffix *plegia* (with a totally different meaning) exists in Spanish spelled as *plejia.*

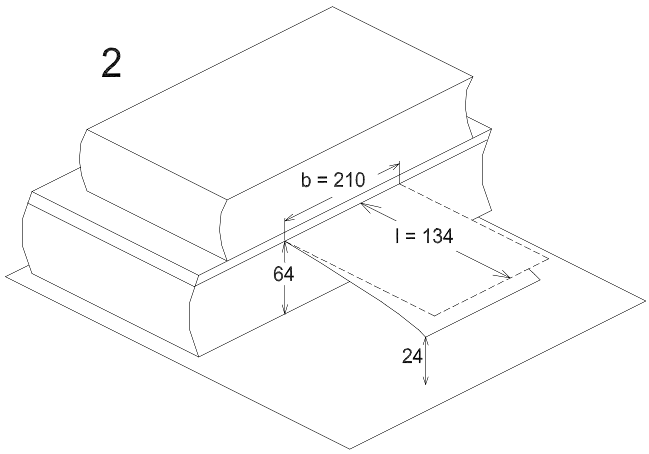
20.1 HOOKE´S LAW

These preliminary considerations have led me to disclose, elementally though rigorously, the flexibility of a piece of paper. Therefore I have decided to approach the study of its modulus of elasticity, the Young´s Modulus E that we find in Hooke´s law. It is well known that this law is the fundamental of materials resistance.

The first thing that came to my mind was to support a sheet of paper on the respective edges of two books to allow it to flex freely as a beam resting on both extremities and subjected to its own weight. Next, I should apply all the measurable data to the differential equation of the elastic line in order to get E.

Soon I realised though, how problematic it was that configuration. My research conveyed me to the “Handbook of pulp and paper technology” where I could find an improved setup of my experiment: instead of having a beam resting on both extremities, the beam was somehow embedded and left to be freely projected in the cantilever mode; look to Figs. 1 and 2 to see surmounted the difficulties inherent to double resting.





The resolution of the already mentioned differential equation bears on the formula that gives the maximum sagita (at its free end) of a cantilever beam that has embedded the other end and is subjected to a uniform load along its entire length:

$f = \dfrac{ql^4}{8EI_z}$; from which we get E: $E = \dfrac{l^4 q}{8I_z f}$

with:

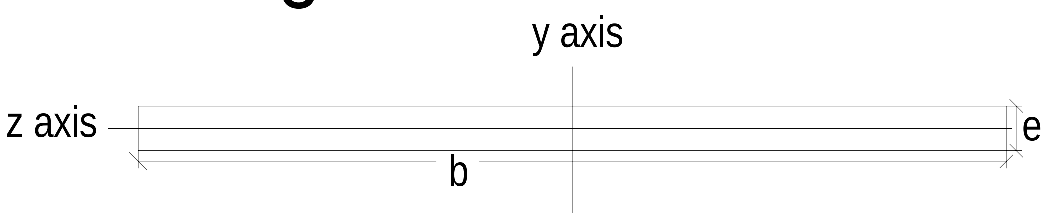
l, free paper length in mm.

q, kilograms / mm as paper´s unitary own weight along the l dimension.

I_z, the momentum of inertia of the paper beam section with respect to its axis z, expressed in mm^4 (see in Fig. 3 the section of the paper sheet).

f, the paper´s free flexing sagita expressed in mm.

3



Figs. 1 and 2 show the approach of both experiments, in accordance with each of the axes of the Din A4 rectangular paper.

We have to take into consideration that, in any piece of paper it is predominant the direction in which that paper has been rolled: i.e. the paper is an anisotropic material because it has different molecular orientation and therefore different mechanical behaviour depending on the rolling direction. Hence, we shall have two different E values.

Let´s see how q and I_z have been obtained, since they are not shown in Figs. 1 and 2.

To obtain the paper thickness (Fig. 3) we took 900 grams of A4 sheets that came out to be in a quantity of 181 with a pile height of 20 mm; hence $e = \dfrac{20}{181} = 0{,}11mm$.

At the same time we´ll have:

$$q_1 = \frac{0{,}9}{181 \times 210} = 2{,}3677979 \times 10^{-5}\, Kg / mm$$

$$q_2 = \frac{0{,}9}{181 \times 297} = 1{,}6742006 \times 10^{-5}\, Kg / mm$$

According to Fig. 3, the momentum of inertia of that paper section is:

$$I_z = \frac{be^3}{12}$$

Therefore:

$$I_{z1} = \frac{297 \times 0{,}11^3}{12} = 0{,}0329422mm^4 \quad ; \quad I_{z2} = \frac{210 \times 0{,}11^3}{12} = 0{,}0232925mm^4$$

Obviously, the value of sagitas is:

$$f_1 = 64 - 28 = 36mm \quad ; \quad f_2 = 64 - 24 = 40mm$$

Substituting values we have:

$$E_1 = \frac{98^4 \times 2{,}3677979 \times 10^{-5}}{8 \times 0{,}0329422 \times 36} = 230{,}19922 Kg/mm^2 = 23.020 Kg/cm^2$$

$$E_2 = \frac{134^4 \times 1{,}6742006 \times 10^{-5}}{8 \times 0{,}0232925 \times 40} = 724{,}20347 Kg/mm^2 = 72.420 Kg/cm^2$$

Here there are several E values corresponding to different materials to be compared:

MATERIAL	E (Kg / cm^2)
Pine	96.000
Holm oak	108.000
Beech	180.000
Lead	50.000
Glass	700.000
Forged iron	2.000.000

20.2 THE π NUMBER

The construction of a cylinder (Point 19) profiting of paper flexibility will help us to focus closely on the π number.

Then we saw how, to properly dress the cylinder, it was required to interlock and glue the lap joints, besides adjusting inside it a cone and a sphere.

The reason was that under those conditions, the cylinder does not yield spontaneously a circular section because of the discontinuity of paper flexibility in the vicinity of the lap joints. The best way to lessen that effect is to construct a tube by wrapping up the paper around itself with as many layers as possible: the greater the number of layers, the higher the cylindrical (circular) precision.

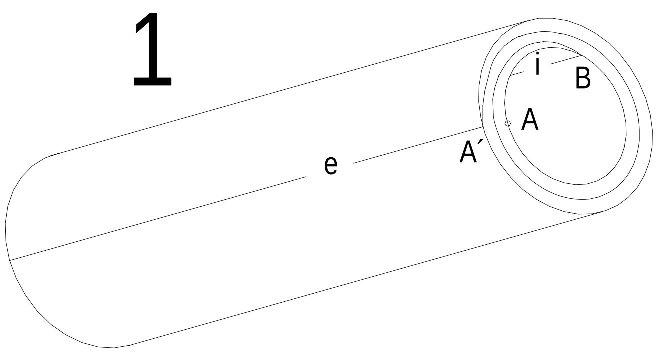


Fig. 1 shows a tube obtained by rolling a paper three times around itself, plus an arc AB. Since the circumference length is πd, we should divide that length by its diameter to get π. The process asks for these precautions:

- Make sure that edge e is properly fixed on the tube´s outer surface (use glue, an adhesive tape, etc.).
- Every layer of paper must seat tight against each other as well as the remains AB over the last one.
- To know the paper thickness. For copying paper we take a = 0,11 mm (see Point 20.1).
- To know also the paper´s initial development. In our case it is A´B = 297 mm since we start with a DIN A4 rectangle.
- To measure the exterior tube diameter with the greatest possible accuracy. We know how difficult it is to reach a certain precision because of the inherent cylindrical inaccuracy associated to a reduced number of layers, and also due to the scarce resolution of a measuring

ruler. That´s why we recommend to perform several measures and then to get the mean value. The best of all is to roll the paper in as many layers as possible and then to measure the outer diameter by means of a calliper fitted up with a vernier (and of course, to obtain the mean value of several measures).

- To measure also the remains AB. This can be easily done by using a small piece of paper introduced underneath it, and then rectified. What matters, as was recommended earlier is that part AB, as well as the rest of the layers will be tight fixed without any play at all.
- To take into consideration that length A´B – AB equals n circumferences of which, the outer one has d as diameter, being d – 2a, d – 4a, etc. the successive diameters of the others.

Therefore we can write:

$$A'B - AB = \pi d + \pi(d-2a) + \pi(d-4a) + \pi(d-6a).....+\pi[d-2(n-1)a]$$

$$\pi = \frac{A'B - AB}{nd - 2[a + 2a + 3a + ... + (n-1)a]}$$

At the denominator´s subtrahend we can find the sum of all the terms of an arithmetic progression whose value is (Point7.15.1):

$$\frac{a(n-1)n}{2}$$

Therefore:

$$\pi = \frac{A'B - AB}{nd - a(n-1)n}$$

In our case we take for given:

A´B = 297 ; AB = 13,5 ; n = 4 ; d = 23 ; a = 0,11 with this result:

$$\pi = \frac{297 - 13{,}5}{4 \times 23 - 0{,}11(4-1)4} = 3{,}1264 \qquad (1)$$

Comparing that result with the π value displayed in a pocket calculator (* 3,1415927) we could be tempted to feel a sort of frustration; nevertheless, there is not any reason for discouragement.

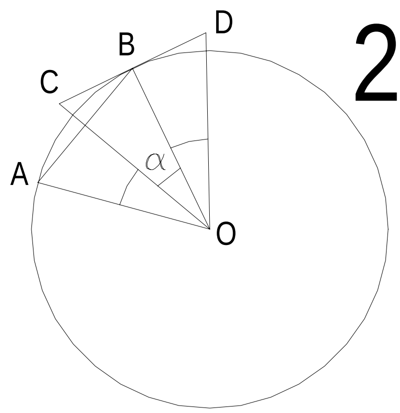
- Most likely, it is when measuring d where we introduce the main error: note that if we had taken 22,643 instead of 23 mm, the resulting value for π would have been *.
- The purpose of the experiment is to offer the order of magnitude of π. To obtain it very close to exactitude is highly difficult because it is an irrational number.
- In that respect we have to admit that the big error is in expression (1) which takes the form of

$$\pi = \frac{28350}{9068} \qquad (2)$$

- The latter expression represents a rational number (what π is not) which, in turn, might take three different configurations: an integer, in the event of exact division (it´s not our case), and a periodic or mixed-periodic fraction. The expression (2) belongs to one of the latter two.
- The man is in search of π over 4.000 years. By Euclid's time it was well known that the value of π should be confined between 3 and 4. The reason: 3 is the ratio between an inscribed hexagon´s perimeter and its diameter; on the other hand 4 is the ratio between a circumscribed square´s perimeter and its diameter.
- Modern computers have made it possible to get π with more than 100.000 significative digits. It is a matter of time and memory applied to develop series such as ASN (x) (in turn obtained as an inversion of the sine series), in spite of its slow convergence. Or the series of the ATN(x). In any case, series development is the procedure to add significative figures to

an irrational number by means of the addition of rational summands which looks rather contradictory, but in line with our experiment.

To finish, and in accordance with the Euclidean inequality 3 < π < 4, we shall now consider the circumference as the limit of n sides polygons´ perimeters: first, as the lower limit of a circumscribed polygon, and second, as the upper limit of an inscribed one when n tends to infinity.



2

In Fig. 2 we find that AB is the side of an inscribed polygon of n sides and CD is that of a circumscribed one; in both cases the radius of the circumference is r = OA = OB and the central angle is 2α, being $\alpha = \frac{360}{2n}$. Therefore we´ll have:

$$AB = 2r\,\mathrm{sen}\,\alpha \quad ; \quad CD = 2r\,\mathrm{tg}\,\alpha$$

As 2πr is the length of the circumference, it will be:

$$2rn\,\mathrm{sen}\frac{360}{2n} < 2\pi r < 2rn\,\mathrm{tg}\frac{360}{2n} \quad ; \quad n\,\mathrm{sen}\frac{360}{2n} < \pi < n\,\mathrm{tg}\frac{360}{2n}$$

Let´s see how this inequation looks for different polygons.

n = 10	3,0901699 < π < 3,2491970
n = 50	3,1395260 < π < 3,1457334
n = 100	3,1410759 < π < 3,1426266
n = 1.000	3,1415875 < π < 3,1416030
n = 10.000	3,1415926 < π < 3,1415928
n = 50.000	3,1415927 < π < 3,1415927

(n = 50.000 produces overflow in a calculator screen)

Having all previous considerations in sight, it does not look so discouraging the result that may be obtained for π by means of a rolled paper tube.

21 QUADRICS

They are conic generated surfaces (see Point 13). That generation is associated always to a combination of conics in the role of directrices and generatrices, respectively.

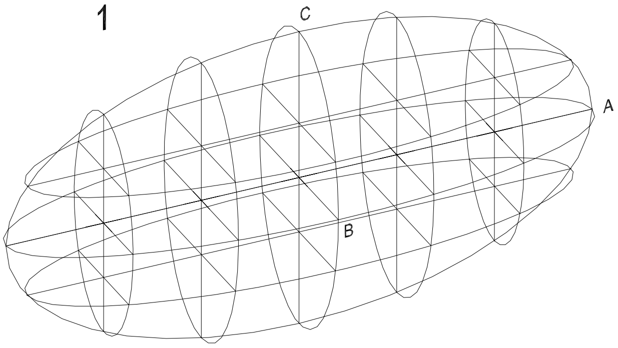
They have several variants; among them we find the round bodies (Point 19) that may be considered as degenerated quadrics. The basic quadrics are the ellipsoid, the hyperboloid and the paraboloid. From here on we shall study:

- Two types of ellipsoid, both interlocked laminar: the elliptic one, strictly speaking, and the other, also elliptic but deformable and made up of cyclic sections.
- The revolution ruled hyperboloid, considered as a virtual surface (see Point 17).
- The hyperbolic paraboloid, also in two versions: a virtual surface and as a deformable interlocked laminar construction.

21.1 ELLIPTIC ELLIPSOID

Fig.1 shows the wire-work version of the ellipsoid where one can see the three ellipses with vertices A, B; A, C; B, C that intersect orthogonally at the quadric center.

1



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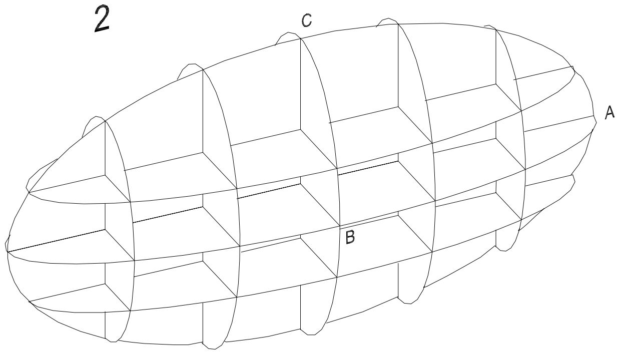
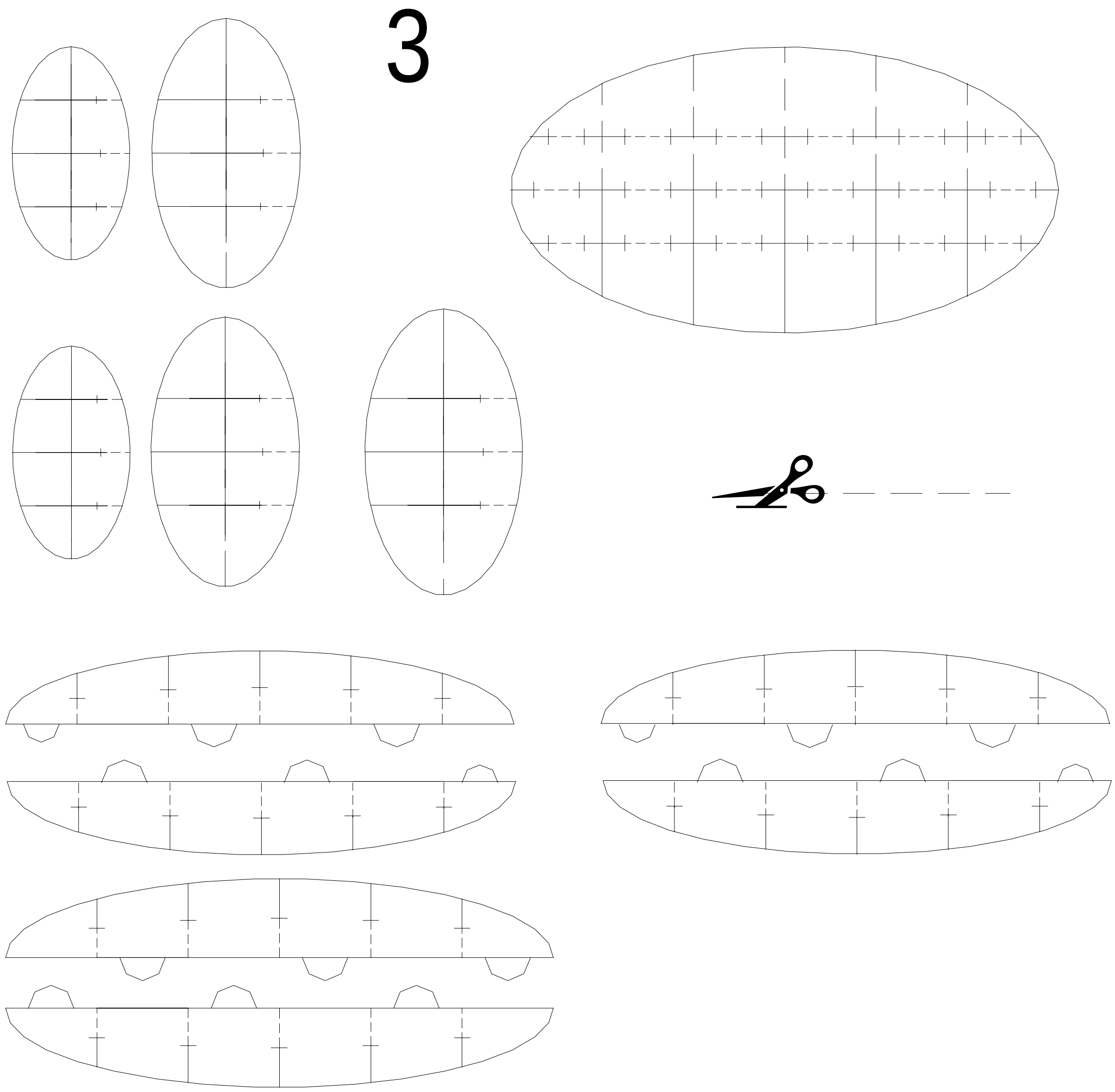


Note that the three axes have different length, hence the resultant ellipsoid is an elliptic one. Had two of the axes been equal to each other, one out of the three ellipses would had become a circumference and therefore the quadric would be an ellipsoid of revolution.

The ellipsoid generation takes place when the ellipses with vertices AC and BC act as directrices, and a horizontal one, acting as generatrix, moves in a parallel direction resting on the other two. The latter is shown in different positions: a centered one and two others symmetric to it.

Fig. 2 is a laminar vision of the paper-constructed ellipsoid. It consists in a total of 9 ellipses: the three main ones, two parallel to plane AB and four parallel to plane BC. The three horizontal ellipses have to be made in halves to permit assembly. Fig. 3 has all the ellipses or half-ellipses needed to build up the ellipsoid.



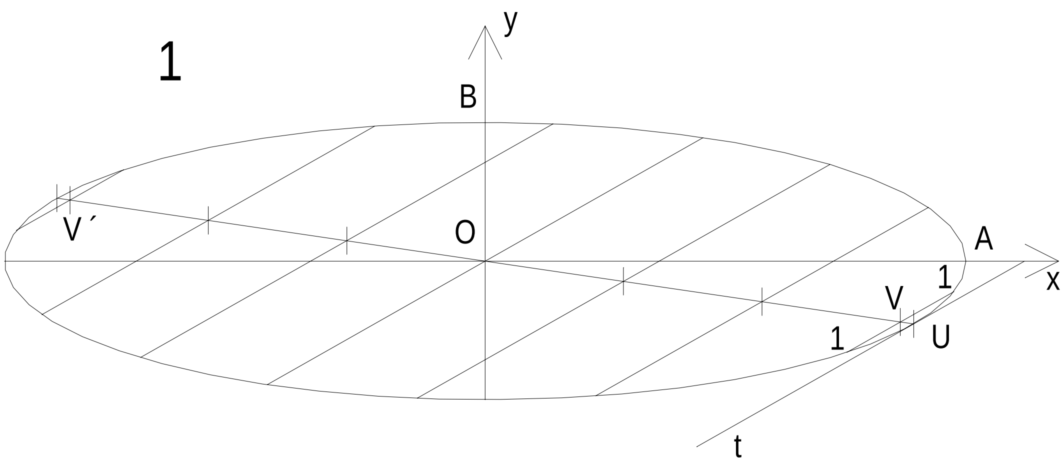
21.2 CYCLIC SECTIONS DEFORMABLE ELLIPSOID (CYCLIC ELLIPSOID)

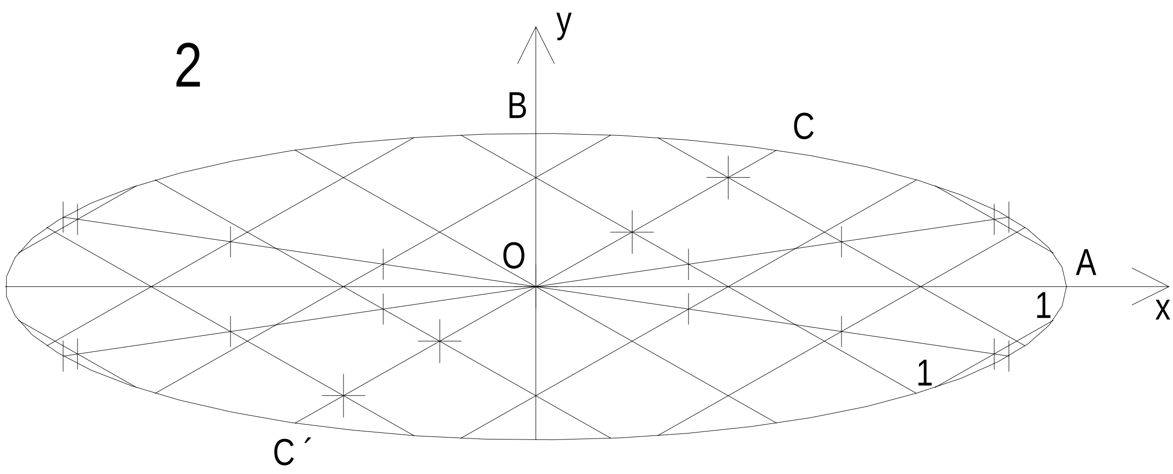
Cyclic sections of an ellipsoid are those produced in it by planes whose intersections are circumferences.

Fig. 1 shows the horizontal main elliptical section of the ellipsoid with its horizontal axes a = OA; b = OB. Let the steps to give, in order to get the third half-axis c, being c > b:

1. To draw any tangent t; its point of contact will be the umbilical point U.
2. To trace any secant 11 parallel to t, outside OA.
3. Within segment OU, divide OV in n equal parts. We have made n = 3 for the sake of simplicity.
4. To find the figure that is homothetic to OVU11 with center O and ratio 1.
5. To divide OV´ in three equal parts as done for segment OV.
6. Through these intersection points (besides O and V´), draw parallels to t within the ellipse.
7. Fig. 2 shows two pencils of parallel secants: the one obtained in step 6, plus its symmetric with respect to axis OX.

Now then, the 7 + 7 secants just obtained, are the intercepts in the main ellipse of plane XY, produced by the circles (the cyclic sections) parallel both, to axis OZ and to t (or to its symmetrical with respect to OX).





The plane tangent at U, as well as the three others, will produce cyclic sections of radius zero.

If CC´ is the trace of the cyclic section through the ellipsoid center, it means that CC´ is the diameter of a circumference whose points are all on the ellipsoid surface. Therefore OC is the radius equivalent to the half-axis c we are after: c = OC.

To obtain the 14 cyclic sections it´s enough to get the 4 corresponding to OV since the rest are equal or symmetrical.

To get rid of confusion when interlocking the circles between each other, Fig. 3 shows the traces of both pencils of planes: the circles should bear the corresponding number.

3

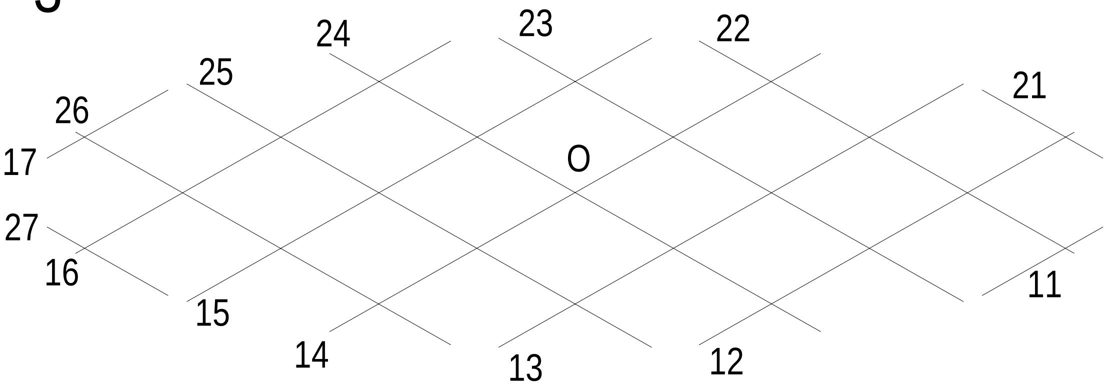


Fig. 4 shows what circle 14 looks like: CC´ is its diameter and it has the slits required to interlock the other 5 circles that intersect it. Same figure depicts also circles 12, 13 and 11. The dashed lines represent cuts.

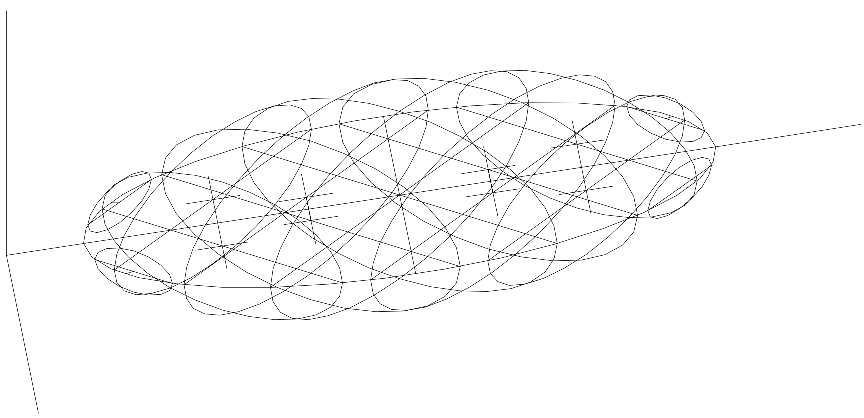
As said earlier, all the other circles come out of the just mentioned four ones: pencil 1 has the slits in its lower side, and pencil 2 in the upper one. We can observe that, looking at circles of Fig. 4.

We shall proceed with the construction process of the ellipsoid using all 14 circles, but at the same time we´ll advice that circles 11, 21, 16 and 26 may be ignored. These four circles are intercepted only by one circle (see Fig. 3), what means that they will not rest properly fixed, with the consequence of palling on the *ellipsoidal* vision of the whole assembly. Conversely, its omission does not lessen that vision´s quality.

Fig. 5 shows the ellipsoid´s deformation effect manifested in its horizontal main plane.

If keeping O in place, axis a is shortened by passing A_1 to A_2, axis b grows from OB_1 to OB_2. Third axis OC is always kept in its dimension, but cyclic planes´orientation changes from OC_1 to OC_2.

7



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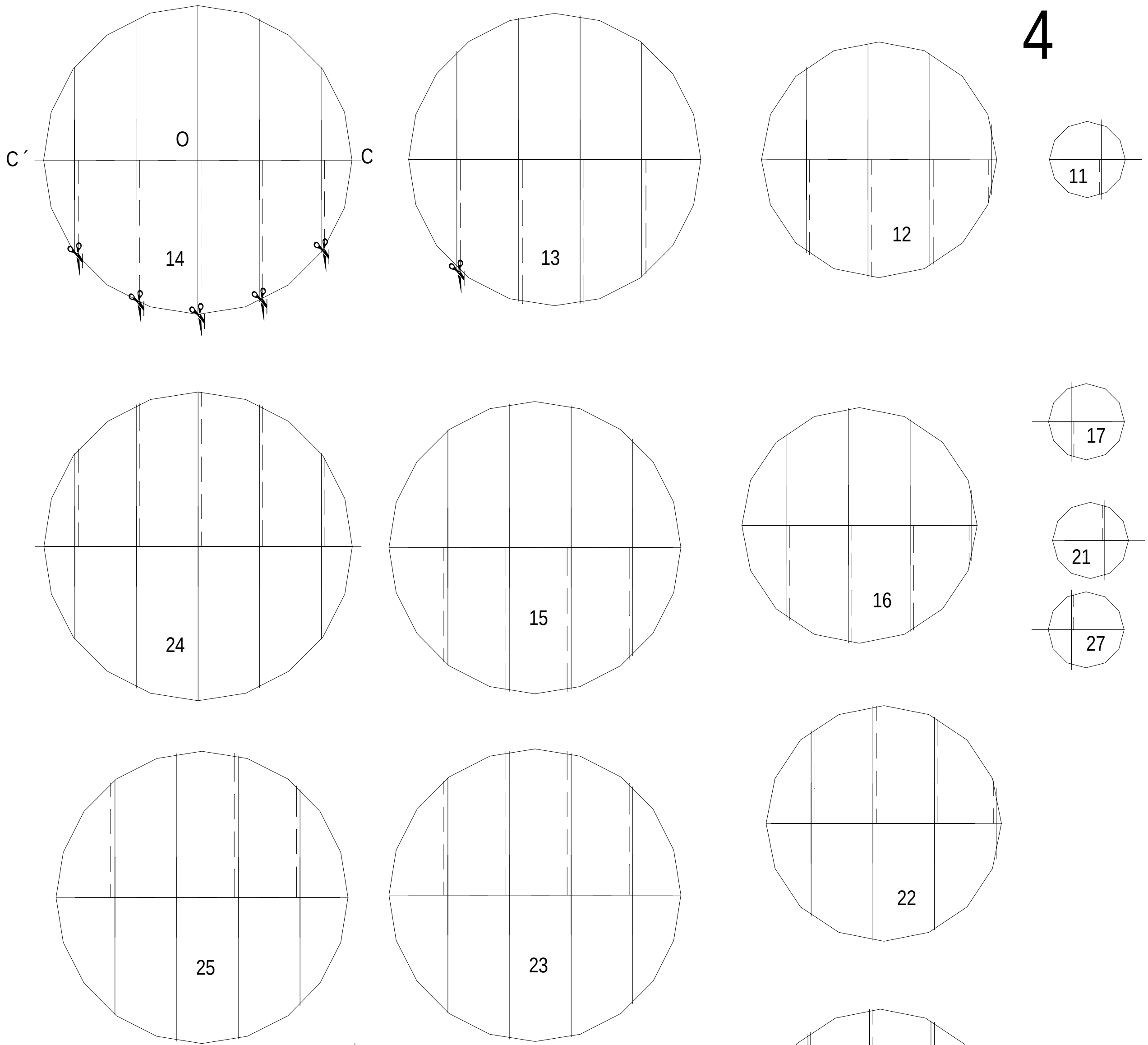

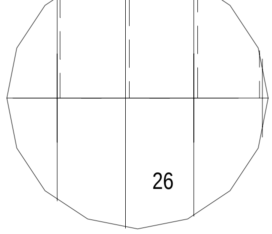

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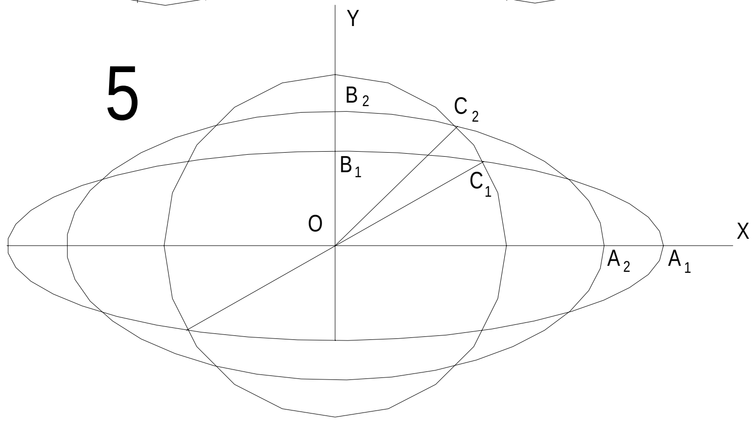
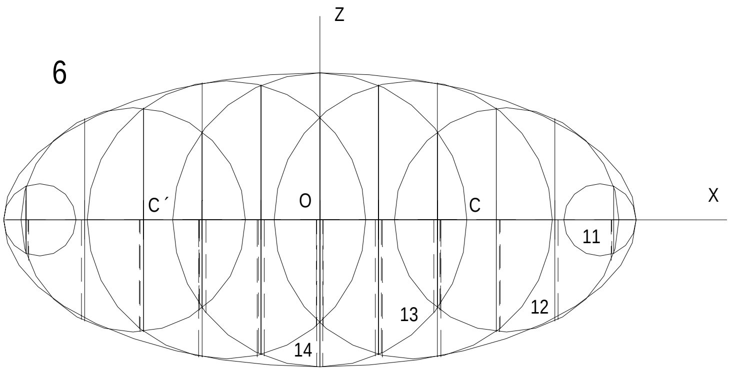


The spatial figure tends to collapse into the plane XZ as seen in Fig. 6: all the circles taking part in it appear surrounded by a new ellipse. The shortening of axis <u>a</u> requires an effort similar to that needed to actuate against a compression spring.

That´s why the resultant figure requires a sort of frame to stay steady with a certain degree of deformation.



If the produced deformation implies that both pencils in Fig. 2 become orthogonal, the result is a sphere.

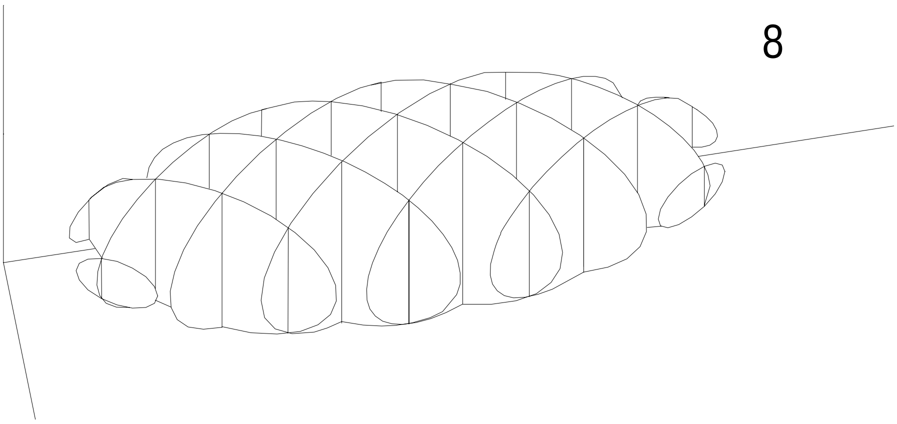
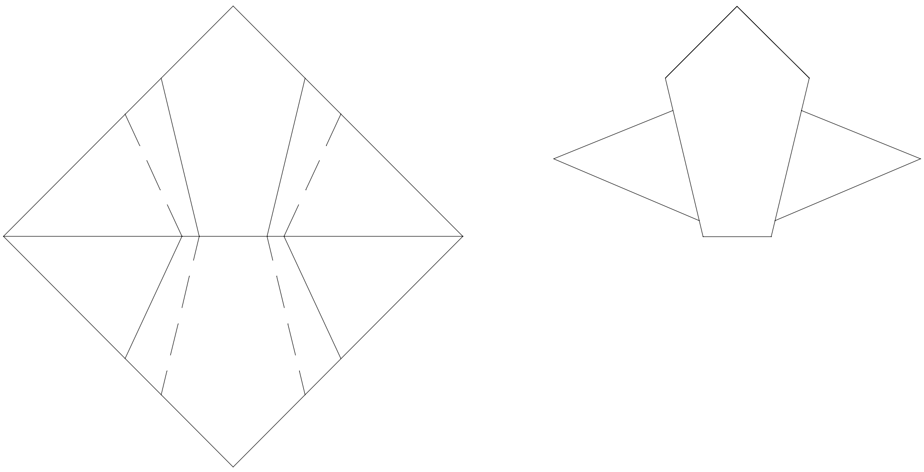


Fig. 7 is a wire-work view of the ellipsoid with its 14 cyclic circles. Fig. 8 is a perspective; in it, those cyclic circles look like ellipses just because of that perspective effect. In return of that deformation we can gaze at the peculiar aspect of a hive-like structure.

Interlude

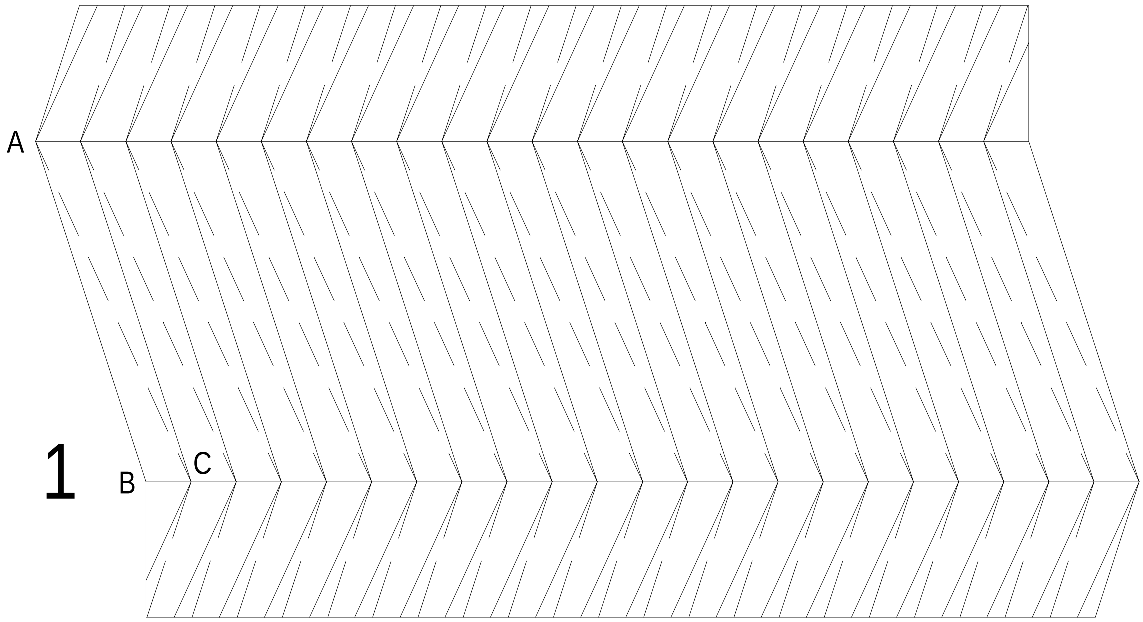


21.3 HYPERBOLOID

The quadric we are going to construct now is a warped hyperboloid of revolution, a one-sheet ruled surface.

Fig. 1 is its folding diagram. The whole figure is triangulated; both, upper and lower trapeziums serve to *self-pocket* the hyperboloid.

Note that this yperboloid structure reminds so much that of the pentagonal prismoid seen in Point 18.5.3. Now, though, we shall tend to augment the number of sides toward infinity.



1

Paper materialisation of this hyperboloid is not something easy or spontaneously stable, but yields an attractive result when achieved.

The optimal solution (self-stable) asks for these requirements:

- A paper both, resistant and docile, to guarantee that the generatrices will not collapse, whereas the folds may be easily produced.
- To fix with an adhesive tape, mountain as well as valley fold´s settlements in the vicinity of upper and lower polygons.
- * To pocket both polygons within the trapeziums to fasten the figure and hide the adhesive tape. That pocketing should apply to the perfect coupling of every four mountain / valley lines that coincide in the upper and lower polygonal vertices.
- Also to pocket the figure laterally: that constrains to have an extra pair of triangles (an extra parallelogram).

Most likely, it will be rather difficult to meet all those requirements (skill and patience to be added); that´s why I propose a practical advice:

- To use normal paper.
- Start with Fig. 1; the trapeziums will only serve to the purpose of reinforcing the upper and lower perimeter.
- To glue both extreme rhomboids to close the hyperboloid.

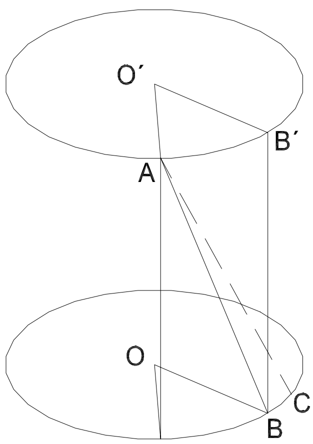
- To apply into both bases two opposed fine cardboard cones in order to conform the figure.
- Those cones will work as the coupling used in lathing: they adapt themselves to the *natural* shape of the quadric while allowing rotation to get its *extreme position.* Meanwhile the hyperboloid surface remains visible, and its stability may be guaranteed adding the adequate weight to the inside of the upper cone (e.g. a necklace, some loose beads, a little chain, etc.).
- We have to stick rigorously to demand * and pre-conform the hyperboloid according to the need of a good coupling of the overlapped folds.

As already pointed out, the main form in Fig. 1 is ΔABC that appears replicated in Figs. 3 and 4, though unnamed in the latter.

Angle in B must be obtuse (108º in our case) to produce an *insinuated* hyperboloid in its natural form (Fig. 3).

The base BC is the side of the upper and lower polygons that, in the limit, represent two circumferences (recall the end of Point 20.2). Sides AB and AC are, respectively, the mountain and valley fold creases.

2



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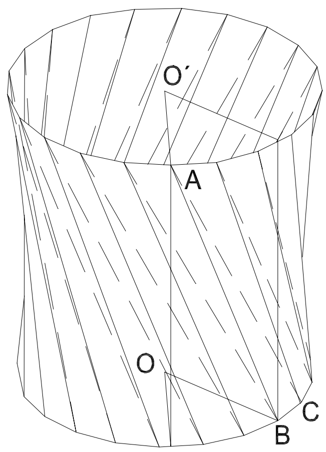

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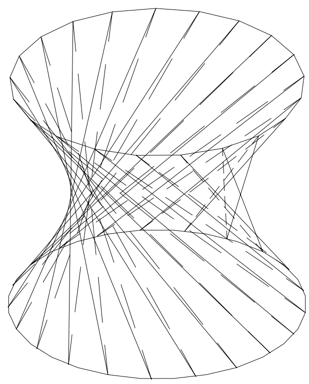


ΔABC is represented in Fig. 2 in connection with those two circumferences. It is essential to decide what the inclination of plane ABC will be with respect to the horizontal, for it determines the altitude of the hyperboloid. In our case we have taken 80º for that angle, as well in Fig. 2 as in Fig. 3. By so doing we get the so-called natural form of Fig. 3.

The hyperboloid we have just fabricated has 21 triangles with base AB (plus an extra one for lateral closing). Therefore, starting with Fig. 2 we get Fig. 3 by rotating successively said ΔABC around O, the value of the revolving angle being $\frac{360}{21} = 17.142857^\circ$. This contrivance permits to draw the figure and shows that the hyperboloid is a quadric of revolution.

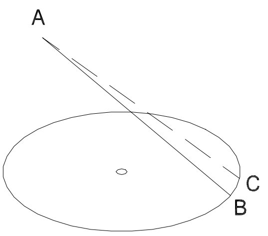
Fig. 4 derives from Fig. 5; in this, ΔABC forms with the horizontal an angle of 50º instead of 80º. To achieve that, the hyperboloid has to be revolved with the help of the cones referred to above, till the limit of torsion possibility.

We may observe then, two quite different sorts of rotation: the immaterial turning of ΔABC around the hyperboloid axis, and that of torsion, up to the possible limit. Fig. 3 is the paper hyperboloid showing its generatrices, whereas Fig. 4 is a wire-work vision. Note that the hyperbolas which are just visible laterally in Fig. 3, are heavily marked in Fig. 4.

It is recommended that the conic angle of the auxiliary cones will be close to that of the asymptotic cones not shown in Fig. 4. Fig. 6 shows both cones with the traces left in them by the hyperboloid, as well as the transversal section and development (to a different graphic scale). See Point 19 on how to construct a cone.

If the starting ΔABC has sides AB = 95; AC = 99,3658; BC = 12, the resulting cones will have these measures: generatrix = 65; altitude = 41,5331; conic angle = 100,57º. Thus the radius of the cone base is 50 against 40,2570 which is the extreme radius of the hyperboloid. This bears the consequence that both cones surpass the hyperboloid as can be seen in Fig. 7, in which the superposition of hyperboloid and lower cone is simplified.

5



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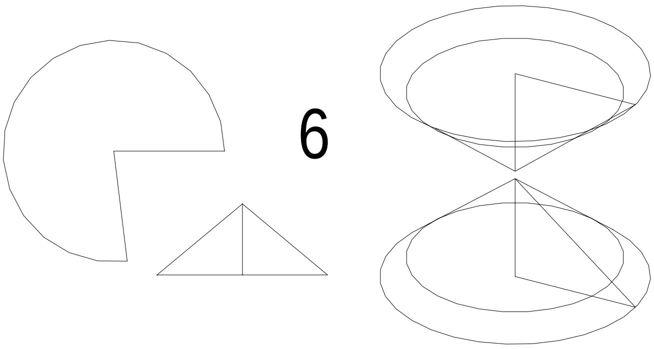

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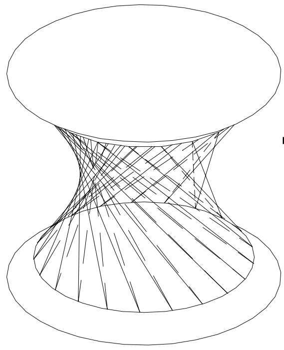


Let´s see now some geometric questions with regard to the hyperboloid we have just constructed. We could be interested, e.g., in the parameters of the outlined hyperbola of Fig. 4, from the measures of ΔABC and its inclination of 50º.

In the first place we should recall Point 17.2 (a conoid of paper) on ruled warped surfaces. Here we are in front of one of them; therefore Fig. 1 is not the unfolding of the hyperboloid surface, since it is undevelopable (it rather is a virtual surface consisting in straight generatrices): Fig. 1 is only the folding diagram that enables its construction.

When forming Fig. 4, in fact we have got two hyperboloids: one showing to the outside the mountain generatrices (convex paper from the exterior); the other exhibiting another set of generatrices, also mountain fold and also convex, but in this case, as seen from the interior of the hyperboloid: they are the valley creases of Fig. 1. For the time being we shall refer only to the former hyperboloid; the other set of generatrices has served just as ancillary to facilitate the hyperboloid construction.

A one-sheet ruled hyperboloid of revolution can be generated this way (Fig. 8): let the hyperbola h in a vertical plane with one focus at F, be the directrix curve, and a horizontal circumference c moving vertically and resting on h, be the generatrix (of course adopting the diameter that corresponds to each position). The locus of the centers of these circumferences is the vertical axis OZ that is also the axis of the hyperboloid and of the hyperbola.

The minimum radius of the generatrix circumference occurs when it rests on the hyperbola´s vertices; then it receives the name of neck circumference.

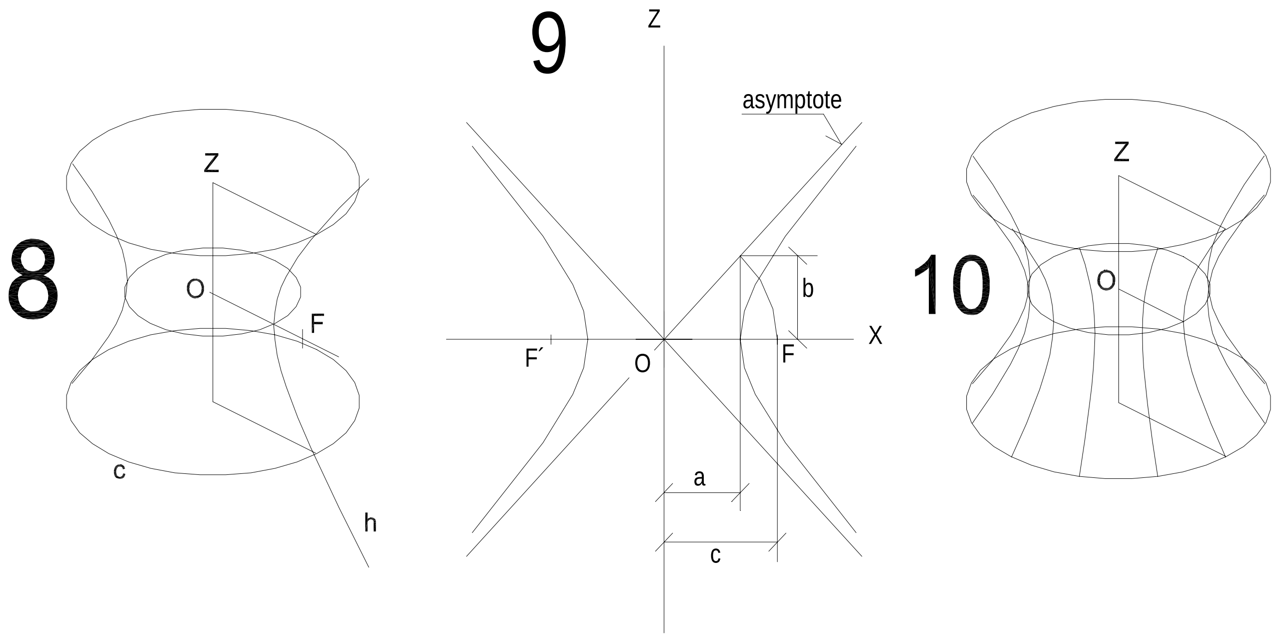
Conversely, rotating the directrix hyperbola h around axis OZ (Fig. 10) can also generate the hyperboloid: then, all the former circumferences (the neck one included) will rest on those various hyperbolas.

Here it is the equation of a hyperbola like that of Fig. 9 (two symmetrical branches, see Point 8.2.8.6):

$$\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1 \quad ; \quad c^2 = a^2 + b^2 \qquad (1)$$

hence, the equation of a hyperboloid of revolution (Fig. 8) is:

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{b^2} = 1 \qquad (2)$$



To get a and b we should have two points ($x_1y_1z_1$) ($x_2y_2z_2$) as given on the hyperboloid, then to substitute their co-ordinates in (2) and finally solve the resultant system of two equations with two unknowns (a and b).

If we make

$$A = x_1^2 + y_1^2 \quad ; \quad B = z_1^2 \quad ; \quad C = x_2^2 + y_2^2 \quad ; \quad D = z_2^2$$

we get:

$$a = \sqrt{\frac{A(B-D) - B(A-C)}{B-D}} \quad ; \quad b = \sqrt{\frac{A(B-D) - B(A-C)}{A-C}} \qquad (3)$$

¿Which pair of points could we chose? To answer we should recall that the hyperboloid is a ruled surface and therefore all its generatrices (mountain folds) are straight lines resting on it; in consequence, any point of these lines belong to the hyperboloid.

Off hand we have points B and A in ΔABC, but they induce division by zero in (3) because of the symmetry with respect to the co-ordinate plane Z = 0 that obviously contains the center of the hyperboloid.

Then we may keep A and look for another point in AB: the intersection point of AB with a horizontal plane distinct from the hyperboloid bases. The operation is easy when the program INTERPR.BAS that yields the intersection point of a plane and a straight line, helps CAD. By so doing we get

a = 24,2384 ; b = 26,5738

Substituting these values in (1) we obtain c which in turn determines the position of focus F in the hyperbola of Fig. 9 $c = \sqrt{a^2 + b^2} = 35{,}9676$

Only to add that to draw the hyperbolas of Figs. 8 and 9 we have to give values to x, take them to (1), obtain the corresponding z and then carry both (x,z) to the drawing.

From that drawing we get the angle of the hyperboloid´s asymptotic cone as

$$\alpha = 2\,\mathrm{arctg}\,\frac{a}{b} = 84{,}7369^{\circ}$$

Note that the angle is smaller than that of the cones in Fig. 6 (100,57º). This has being designed purposely to elude that the vertices of both cones will get in contact before their surfaces do on the hyperboloid bases.

To finish, let´s set forth the question of what the tangent plane on a point of the hyperboloid will be.

For that we shall do some simple changes in equation (2):

$$\left(\frac{x}{a}\right)^2 - \left(\frac{z}{b}\right)^2 = 1 - \left(\frac{y}{a}\right)^2 \qquad \text{(difference of squares)}$$

$$\left(\frac{x}{a}+\frac{z}{b}\right)\left(\frac{x}{a}-\frac{z}{b}\right) = \left(1+\frac{y}{a}\right)\left(1-\frac{y}{a}\right)$$

$$\frac{\frac{x}{a}-\frac{z}{b}}{1-\frac{y}{a}} = \frac{1+\frac{y}{a}}{\frac{x}{a}+\frac{z}{b}} = u \quad ; \quad \frac{\frac{x}{a}-\frac{z}{b}}{1+\frac{y}{a}} = \frac{1-\frac{y}{a}}{\frac{x}{a}+\frac{z}{b}} = v \qquad (4)$$

$$\left.\begin{aligned} &\frac{1}{a}x + \frac{u}{a}y - \frac{1}{b}z - u = 0 \\ &\frac{u}{a}x - \frac{1}{a}y + \frac{u}{b}z - 1 = 0 \end{aligned}\right\} (5) \qquad \left.\begin{aligned} &\frac{1}{a}x - \frac{v}{a}y - \frac{1}{b}z - v = 0 \\ &\frac{v}{a}x + \frac{1}{a}y + \frac{1}{b}z - 1 = 0 \end{aligned}\right\} (6)$$

Finally we get equations (5) and (6) that mean:

- The four of them are equations of planes.
- Either pair (5) or (6) represent the intersection lines of each of those two planes.
- Equations (5) are parametric in u, and (6) in v, what means that for each value of u we get a straight line (5) and for values v the right line we obtain is (6).
- Do not mix up lines u, v with the plückerian co-ordinates of Point 13.2. If we develop equations (5) and (6) (what we shall not do), we would arrive to the asymptotic or hiperboloidal co-ordinates that define parametrically the hyperboloids.
- Getting a new generatrix for each value of u recalls what was said earlier when each rotation of an angle of 17,142857º yielded, also, another generatrix.
- Generatrices v are the symmetric of u with respect to a plane containing the hyperboloid axis and the mid-point of a generatrix u. This way, Fig. 11 has been drawn from Fig. 4.
- Note that to get Fig. 11 as a paper construction, it suffices to begin with a Fig. 1 changed in such a way that the mountain / valley crease would be symmetric with

respect to the vertical. I leave to the ingenious paper-folders the task of getting a paper hyperboloid containing both, u and v generatrices.

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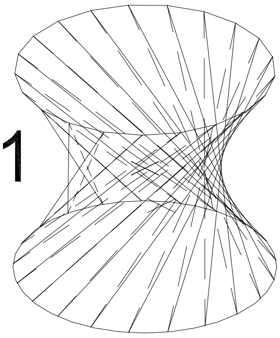
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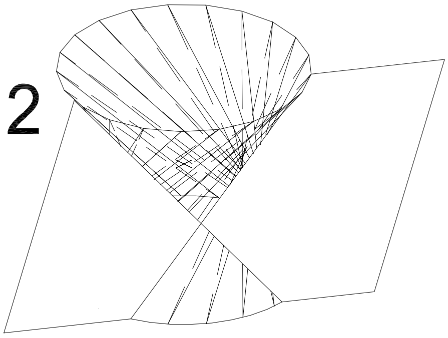
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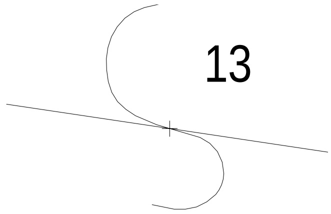
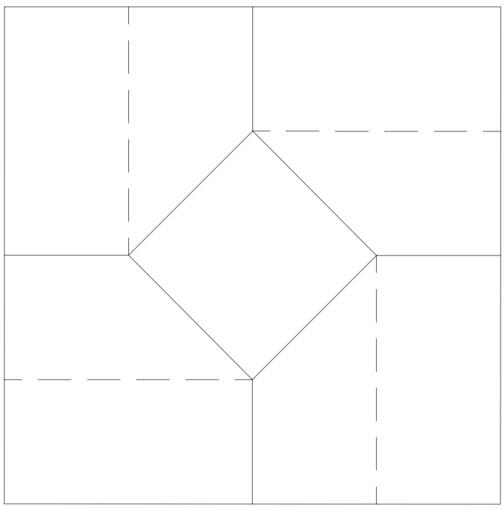
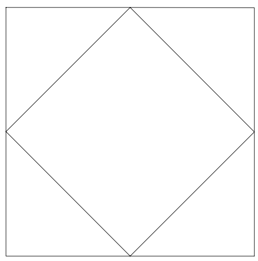


- All the u generatrices cross to each other; the same happens to the v ones. A generatrix u is parallel to the diametrically opposite v as can be easily *imagined* looking at the paper constructed hyperboloid.
- Any straight-line v intersects all the lines u it comes across between its two ends.
- As any two lines u / v intersect in one point, this point will belong to the hyperboloid and also to the plane formed by those two lines. In consequence, that plane will be tangent to the hyperboloid in the afore-said u / v intersection point.
- That plane of tangency (Fig. 12), paradoxically does intersect the hyperboloid. It is nothing extraordinary, though. Fig. 13 shows an antecedent: the tangent to a curve in one of its inflexion points, cuts the curve, too.

Interlude

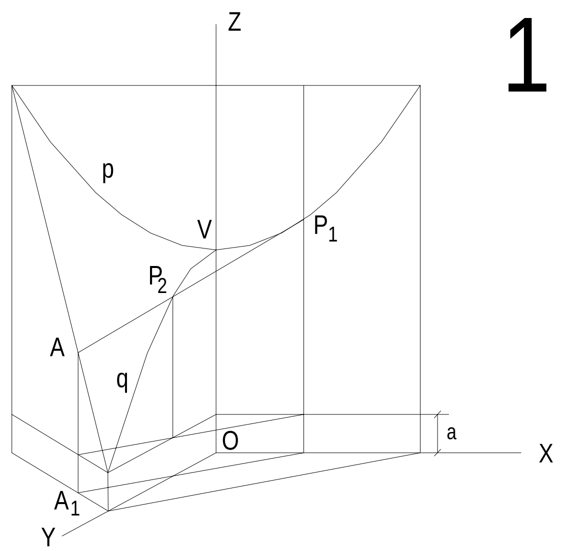



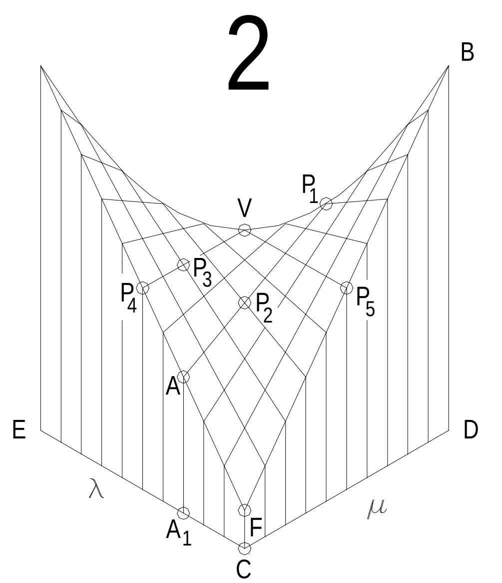
21.4 HYPERBOLIC PARABOLOID

It is a quadric (a second order surface associated to one or various conics), that may be generated this way (Fig.1):

- Begin with parabolas p and q that in turn may have equal or different parameters.
- They will have in common vertices and axes (OZ), while their foci are at either side of the common vertex.
- Their planes are at 90º.
- One of the parabolas, e.g. the p, will be the directrix; hence q will be the generatrix.
- Under these conditions, the hyperbolic paraboloid is generated when q moves parallel to itself while resting all the time its vertex on p.
- Both parabolas p and q are the main sections of the paraboloid. Their common axis is also the axis of the paraboloid and the common vertex is its vertex too.

As can be seen, the hyperbolic paraboloid is an unlimited surface. In Figs. 1,2 and associates is shown confined to a 90º sector (ECD of Fig. 2) of one of its two halves (the hyperbolic paraboloid, like the parabola, has bilateral symmetry). To facilitate the representation we have made p = q. Note that the altitude a in Fig. 1 is equal to FC in Fig. 2.





If we express as a vectorial relation the genesis of the hyperbolic paraboloid described before (generatrix moving in parallel resting on the directrix), we get the equation of the paraboloid:

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z$$

From now on we shall stick to the p = q simplification announced before. Hence, the equation of our paraboloid becomes:

$$x^2 - y^2 = 2pz \qquad (1)$$

This equation is referred to a set of co-ordinate axes like those in Fig. 1 but having its origin at V.

To properly justify the construction of the hyperbolic paraboloid we are dealing with, we shall cut it with several very special planes: they are vertical (i.e. parallel to axis Z) and parallel to each other:

$$x + y = 2\sqrt{p}\lambda \qquad \text{and} \qquad x - y = 2\sqrt{p}\mu \qquad (2)$$

When assigning values to parameters λ or μ we get different planes that are vertical (without z) and parallel to each other within either pencil of planes λ or μ because variables x,y have equal coefficients, again, within both pencils.

The intersection of the hyperbolic paraboloid (1) with the planes (2) are straight lines whose equations are:

$$\lambda \text{ system} \begin{cases} x + y = 2\sqrt{p}\lambda \\ \\ x - y = \dfrac{\sqrt{p}}{\lambda} z \end{cases} \qquad \mu \text{ system} \begin{cases} x + y = \dfrac{\sqrt{p}}{\mu} z \\ \\ x - y = 2\sqrt{p}\mu \end{cases} \qquad (3)$$

In turn, the first equation of λ system in (3) represents a secant vertical plane like A_1AP_1 that is a vertical one. The second equation represents a plane such as VP_1P_2 through the origin (it lacks of the independent term).

Hence, plane A_1AP_1 intersects the hyperbolic paraboloid along AP_1 that is one of the straight lines composing it. That´s why we can say that the hyperbolic paraboloid is a ruled surface as seen in Fig. 2.

Till now, all the cuts have been done according to the vertical λ system. Something alike happens with the parallel planes of the μ system (see Fig.2).

Summarising:

- Through every point P_3 of the hyperbolic paraboloid (Fig. 2) pass two straight generatrices belonging to the λ and μ systems respectively.
- Two generatrices of the same system cross to each other.
- Two generatrices of different systems meet at a point on the paraboloid.
- All the generatrices of the λ system (such as AP_1) are parallel to BCD. Something analogous happens with system μ.

21.4.1 LAMINAR VERSION

Fig. 3 shows all the elements needed to construct a hyperbolic paraboloid by means of their interlocking. The required slits will be performed from top to mid-point in Fig. 3.1, and from the bottom to the mid-point in the trapeziums. Two corresponding cuts are shown as a reference (see both pairs of scissors).

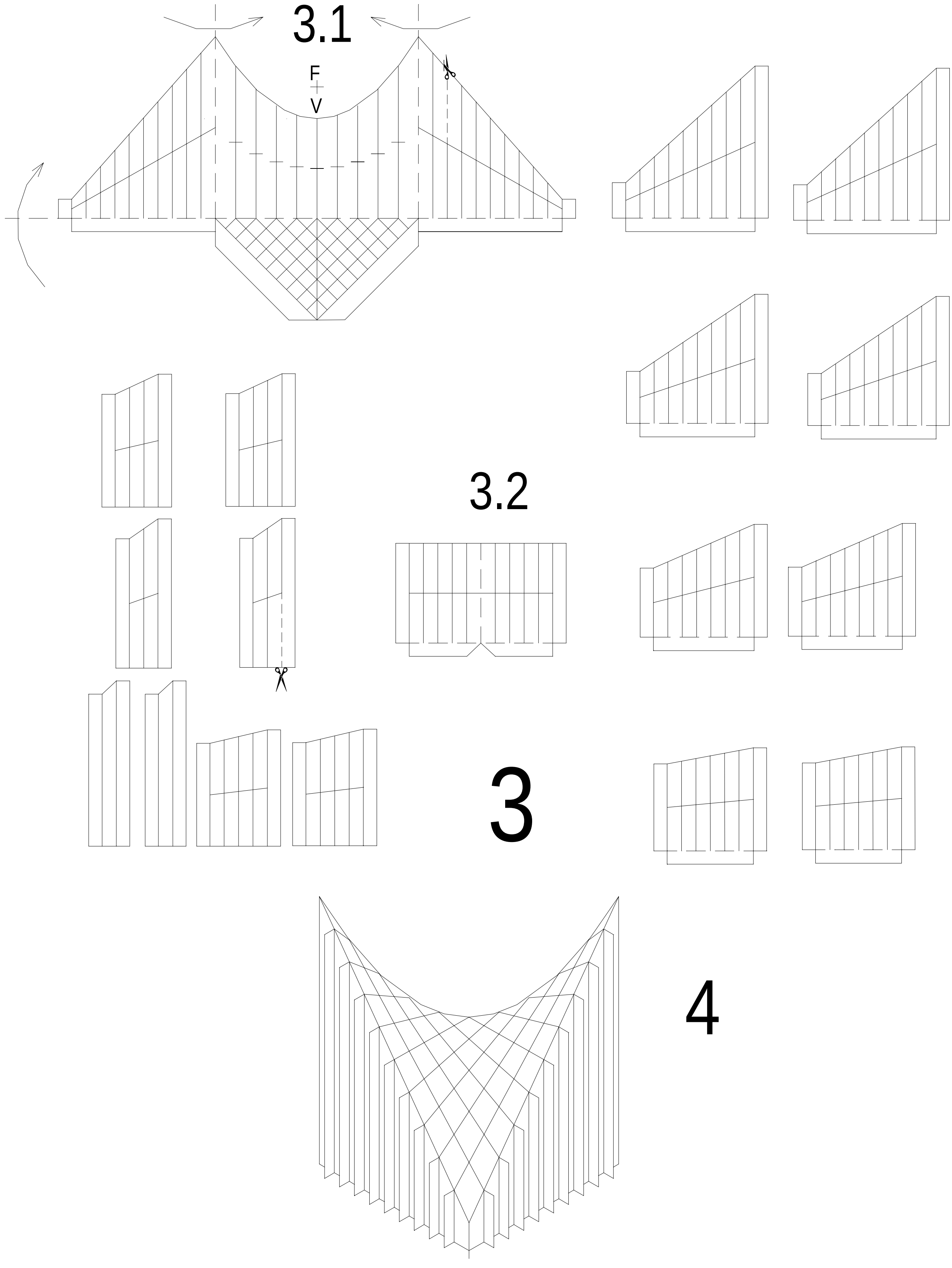
Note that each trapezium is duplicated: one is for the λ system, and the other (turned over), is for the μ one. We can count 16 trapeziums besides the double rectangle of Fig. 3.2.

The trapeziums have vertical lap joints to interlock them and, in some cases, also horizontal ones just to produce, whenever possible, a certain stiffness at the base of the secant planes.

The secant planes to the hyperbolic paraboloid of Fig. 2 are: BP_1V on parabola p; the two perpendicular planes produced by Fig. 3.2 whose intersects are VP_4 (parallel to CD) and VP5 (parallel to EC); the two viewed trapeziums BDCF and ECF; the rest of 16 trapeziums.

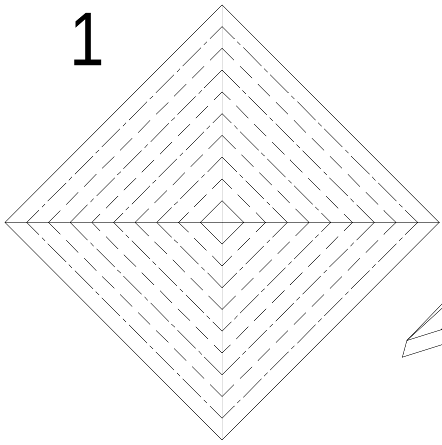
P_4VP_5 is a horizontal right angle coincident with the half-asymptotes of a hyperbola produced in the hyperbolic paraboloid when cut by a horizontal plane through V.

Fig. 4 shows the general appearance of the paraboloid though it lacks of some details. The construction may not turn out to be perfect because of the interference between slits and paper thickness. This handicap can be obviated when fine cardboard is used in connection with *wider* slits. Anyhow, the figure, which is a beautiful one, resembles once more a wasp hive.


3.1
F
V
3.2
3
4

21.4.2 PAPER-FOLDED VERSION

To start with a square of paper, pleat-folded to Fig. 1. The result is a collapsed flat figure in which the square´s diagonals end up in coincidence with its sides.



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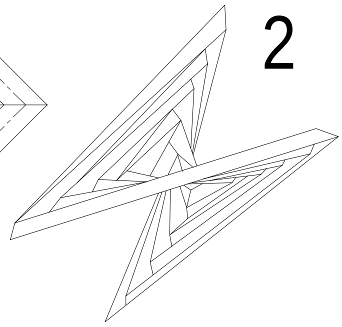

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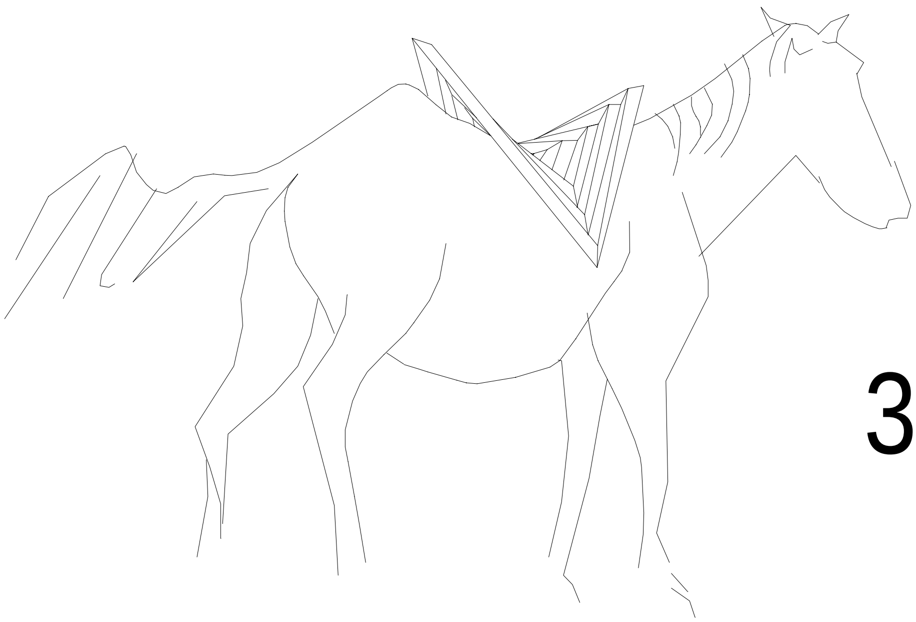
The unfolding of this figure yields a complete hyperbolic paraboloid with equal parabolas both, generatrix and directrix, though in each case their parameters differ depending on the degree of unfolding (Fig. 2).



3

Because of its shape, the hyperbolic paraboloid is familiarly called *the saddle*. If we suppose it set on horseback (Fig. 3) and cut by planes, we get the following conics (remind that the horse also has bilateral symmetry):

Parabolas:

- The planes parallel to the symmetry plane of the horse.
- The vertical planes that in turn are perpendicular to the aforesaid horse plane.

Hyperbolas:

- The planes parallel to the ground. This circumstance determines the adjective “hyperbolic” added to the noun paraboloid.

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